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Derivations Commuting with Abelian Gauge Actions on Lattice Systems

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Abstract. Let τ be an action of a compact abelian group G on a C^*-algebra $\mathscr{A}$, and assume that the fixed-point subalgebra $\mathscr{A}^\tau$ is an AF-algebra. We show that if δ is a closed *-derivation on $\mathscr{A}$ commuting with τ, and the restriction of δ to $\mathscr{A}^\tau$ generates a one-parameter group of *-automorphisms, then δ itself is a generator. In particular, the result applies if τ is an infinite product action of G on a UHF algebra. Furthermore, if in this situation δ_1 and δ_2 are two derivations both satisfying the hypotheses on δ, and δ_1 and δ_2 have the same restriction to $\mathscr{A}^\tau$, then there exists a one-parameter subgroup of the action τ with generator δ_0 such that $D(\delta_1)\cap D(\delta_2)\cap D(\delta_0)$ is a joint core for the three derivations, and $\delta_2 = \delta_1 + \delta_0$ on this core.

1. Introduction

Let δ be a closed *-derivation with dense domain $D(\delta)$ in a C^*-algebra $\mathscr{A}$. Assume that δ commutes with a strongly continuous action τ of a compact abelian group G as *-automorphisms on $\mathscr{A}$. It was shown in [4] that if δ vanishes identically on the fixed-point algebra $\mathscr{A}^\tau = \{A \in \mathscr{A} : \tau(g)(A) = A,\ g \in G\}$, then δ is the infinitesimal generator of a strongly continuous one-parameter group of *-automorphisms of $\mathscr{A}$. Briefly, we say that δ is a generator. By a simple perturbation argument, it follows that the assumption $\delta|\mathscr{A}^\tau = 0$ may be weakened to the condition that $\delta|\mathscr{A}^\tau$ is inner. An example in [4] showed that it is not enough to assume that $\delta|\mathscr{A}^\tau$ is a generator on $\mathscr{A}^\tau$. In this example, $\mathscr{A}$ is abelian, and there is a geometric obstruction preventing δ from being a generator: Along the integral curves of the propagator, points burst into fibres, and conversely, fibres merge into points, in a finite time, [1].

On the other hand, Kishimoto and Robinson [15] showed that if one adds the assumption that $\mathscr{A}$ has an identity, and $\mathscr{A}^\tau(\gamma)^*\mathscr{A}^\tau(\gamma) = \mathscr{A}^\tau$ for all $\gamma \in \hat{G}$, where $\mathscr{A}^\tau(\gamma) =$

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$\{X \in \mathscr{A} : \tau(g)(X) = \langle \gamma, g \rangle X$ for all $g \in G\}$, then δ is a generator if $\delta|\mathscr{A}^\tau$ is so. This spectral condition is, however, not satisfied in interesting examples like the standard gauge action of $\mathbb{T}^1$ on the CAR-algebra, [5]. This paper was motivated by this example, but we prove the slightly more general result that Kishimoto–Robinson's spectral condition can be replaced by the condition that each of the ideals $\mathscr{A}^\tau(\gamma)^* \mathscr{A}^\tau(\gamma)$ in $\mathscr{A}^\tau$ has an approximate identity consisting of projections. If, in particular, $\mathscr{A}^\tau$ is an AF algebra, then this condition is satisfied for all the ideals, [5]. It is interesting to note that if $\mathscr{A}^\tau$ is abelian and AF, then the geometric obstruction referred to above is ruled out for trivial reasons: all the closed *-derivations of $\mathscr{A}^\tau$ are zero, [3]. Also note that a small modification of [4, Example 6.1], where the string in the pinched torus is contracted to a point, shows that Kishimoto–Robinson's condition cannot be replaced by the condition that each of the ideals $\mathscr{A}^\tau(\gamma)^* \mathscr{A}^\tau(\gamma)$ in $\mathscr{A}^\tau$ is essential in the sense that it intersects any other non-zero ideal. Thus, when $\mathscr{A}$ is abelian, it is the total disconnectedness of the spectrum of the fixed point algebra which prevents geometric obstructions and make our proof work.

Our results are applicable to a slightly larger class of C^*-dynamical systems than the CAR algebra with the standard gauge action. Let L be an index set, and for each $\iota \in L$, let M_ι be the full $n \times n$ matrix algebra. Let $\mathscr{A} = \bigotimes_{\iota \in L} M_\iota$, and let H be a compact abelian group acting on M_ι by an action τ' which is independent of ι. One can then associate a gauge action of first or second kind on $\mathscr{A}$. For the first kind, $G = H$ and $\tau(g) = \bigotimes_{\iota \in L} \tau'(g)$, for the second kind, G is the unrestricted direct product $\times_{\iota \in L} H$, and $\tau((g_\iota)_\iota) = \bigotimes_{\iota \in L} \tau'(g_\iota)$. In both cases, the canonical projection $P = \int_G dg\, \tau(g)$ onto the fixed point algebra $\mathscr{A}^\tau$ maps the finite sub-tensor products of $\mathscr{A}$ into themselves, and hence $\mathscr{A}^\tau$ is an AF-algebra.

For a gauge action of the second kind, Kishimoto–Robinson's (K–R) spectral condition is not satisfied unless it is satisfied on each factor M_ι for the group H, and this never happens if τ' is implemented by a unitary action, i.e. the condition cannot be fulfilled if $H = \mathbb{T}^d$. For a gauge action of the first kind the situation is more complicated. If the index set L is infinite, the fixed point algebra $\mathscr{A}^\tau$ is always prime, and if in addition G is finite, then $\mathscr{A}^\tau$ is simple, i.e. the K–R spectral condition is satisfied. If however G is connected, then $\mathscr{A}^\tau$ is never simple, and the K–R-condition is not valid [6], [19], see [21], [13] for further results in this direction.

This paper is organized as follows. In Sect. 2 we prove some results on smoothness of 1-cocycles. In Sect. 3 these results are combined with Kishimoto–Robinson's techniques from [15] to prove the main theorem. In Sect. 4, gauge actions of the first kind are analyzed in more detail, and Powers–Price's techniques from [17] are used to prove the result announced in the abstract.

2. Automatic Smoothness of 1-cocycles

Let δ be the generator of a strongly continuous one-parameter group $e^{t\delta}$ of *-automorphisms on a unital C^*-algebra $\mathscr{A}$. If P is a skew-adjoint element in $\mathscr{A}$, define

$$\delta^P(X) = \delta(X) + PX - XP = \delta(X) + \delta_P(X)$$

for all $X \in D(\delta)$. Then δ^P is the generator of a one-parameter group $e^{t\delta^P}$ of

*-automorphisms. There exists a map $t \to \Gamma_t^P$ from $\mathbb{R}$ into the unitary group of $\mathscr{A}$ such that

$$e^{t\delta^P}(X) = \Gamma_t^P e^{t\delta}(X) \Gamma_t^{P*}$$

for all $X \in \mathscr{A}$, $t \in \mathbb{R}$. The map can be taken to satisfy the 1-cocycle property $\Gamma_{t+s}^P = \Gamma_t^P e^{t\delta}(\Gamma_s^P)$. Here Γ^P can be taken to be the unique solution of the differential equation

$$\frac{d}{dt}\Gamma_t^P = \Gamma_t^P e^{t\delta}(P)$$

with initial condition $\Gamma_0^P = \mathbb{1}$. Then Γ^P also satisfy the equation

$$\frac{d}{dt}\Gamma_t^P = e^{t\delta^P}(P)\Gamma_t^P,$$

and Γ^P is given by the perturbation expansions

$$\begin{aligned}\Gamma_t^P &= \mathbb{1} + \sum_{n \geqq 1} \int_0^t dt_1 \int_0^{t_1} dt_2 \dots \int_0^{t_{n-1}} dt_n e^{t_n\delta}(P) \dots e^{t_1\delta}(P) \\ &= \mathbb{1} + \sum_{n \geqq 1} \int_0^t dt_1 \int_0^{t_1} dt_2 \dots \int_0^{t_{n-1}} dt_n e^{t_1\delta^P}(P) \dots e^{t_n\delta^P}(P),\end{aligned}$$

when $t \geqq 0$, see [8, Proposition 5.4.1.].

Each of the terms of these expansions involves a smoothing operation on P. This smoothing suffices to ensure that Γ_t^P is always contained in the domain $D(\delta)$ of the generator δ, even when P is not contained in this domain. More precisely

Lemma 2.1. *Adopt the assumptions and notation before this lemma. It follows that* $\Gamma_t^P \in D(\delta) = D(\delta^P)$ *and*

$$\begin{aligned}\delta(\Gamma_t^P) &= (e^{t\delta^P}(P) - P)\Gamma_t^P, \\ \delta^P(\Gamma_t^P) &= \Gamma_t^P(e^{t\delta}(P) - P).\end{aligned}$$

Hence one has

$$\begin{aligned}\frac{d\Gamma_t^P}{dt} &= \delta(\Gamma_t^P) + P\Gamma_t^P \\ &= \delta^P(\Gamma_t^P) + \Gamma_t^P P.\end{aligned}$$

Proof. By [15, Theorem A1], $\mathscr{A}$ may be represented on a Hilbert space $\mathscr{H}$ such that $e^{t\delta}$ is covariant, i.e. there exists a strongly continuous unitary group $t \mapsto e^{tH}$ such that $e^{t\delta}(X) = e^{tH} X e^{-tH}$ for all $X \in \mathscr{A}$. Here H is the skew-adjoint generator of the unitary group. The cocycle Γ_t^P is then given by $\Gamma_t^P = e^{t(H+P)}e^{-tH}$, [8, Corollary 5.4.2].

Assume first that H is bounded, and then $\delta = \mathrm{ad}(H)$ extends to all of $\mathscr{L}(\mathscr{H})$. One has

$$\begin{aligned}\delta(\Gamma_t^P) &= \delta(e^{t(H+P)})e^{-tH} \\ &= \int_0^t ds\, e^{s(H+P)}\delta(P)e^{(t-s)(H+P)}e^{-tH},\end{aligned}$$

see [18], or [7, Lemma 3.2.31]. Thus

$$\begin{aligned}\delta(\Gamma_t^P) &= \int_0^t ds\, e^{s(H+P)}\delta(P)e^{-s(H+P)}e^{t(H+P)}e^{-tH} \\ &= \int_0^t ds\, e^{s\delta^P}(\delta(P))\Gamma_t^P \\ &= \int_0^t ds\, e^{s\delta^P}(\delta^P(P))\Gamma_t^P.\end{aligned}$$

But $e^{s\delta^P}(\delta^P(P)) = (d/ds)\, e^{s\delta^P}(P)$, and hence $\delta(\Gamma_t^P) = (e^{t\delta^P}(P) - P)\Gamma_t^P$.

Assume next that $H = -H^*$ is unbounded. Using spectral theory, we may approximate H by a sequence (H_n) of bounded functions of H, i.e. each H_n is bounded and skew-adjoint, and $H_n\psi \to H\psi$ for all $\psi \in D(H)$. It follows from the Trotter–Kato theorem in Kurtz' form [7, Theorem 3.1.28], that e^{tH_n} and $e^{t(H_n+P)}$ converges strongly to e^{tH} and $e^{t(H+P)}$. Hence

$$(e^{t(H_n+P)}Pe^{-t(H_n+P)} - P)e^{t(H_n+P)}e^{-tH_n}$$

converges strongly to

$$(e^{t\delta^P}(P) - P)\Gamma_t^P.$$

It follows from the formula for $\delta(\Gamma_t^P)$ above that

$$[H_n, e^{t(H_n+P)}e^{-tH_n}]$$

converges strongly to $(e^{t\delta^P}(P) - P)\Gamma_t^P$, but at the same time it is clear from direct inspection that this expression converges as a bilinear form on $D(H) \times D(H)$ to $[H, \Gamma_t^P]$. It follows from [7, Proposition 3.2.55] that $\Gamma_t^P \in D(\delta)$ and

$$\delta(\Gamma_t^P) = (e^{t\delta^P}(P) - P)\Gamma_t^P.$$

The remaining assertions of the lemma follows from the formuli

$$\frac{d}{dt}\Gamma_t^P = e^{t\delta^P}(P)\Gamma_t^P = \Gamma_t^P e^{t\delta}(P),$$

and $\delta^P = \delta + \delta_P$.

3. Derivations Commuting with Compact Abelian Actions

Theorem 3.1. *Let $\mathscr{A}$ be a C^*-algebra, G a compact abelian group, and τ a continuous action of G as *-automorphisms of $\mathscr{A}$. Assume that the fixed point algebra $\mathscr{A}^\tau$ for this action is an* AF-*algebra.*

*Let δ be a closed *-derivation on $\mathscr{A}$ satisfying*

1. *$\delta\tau(g) = \tau(g)\delta$ for all $g \in G$.*
2. *The restriction δ_0 of δ to $\mathscr{A}^\tau$ is a generator.*

It follows that δ is a generator.

Remark. The assumption that $\mathscr{A}^\tau$ is an AF-algebra could be replaced by the hypothesis that each of the ideals $\overline{\mathscr{A}^\tau(\gamma)^*\mathscr{A}^\tau(\gamma)}$ in $\mathscr{A}^\tau$ has an approximate identity consisting of projections. This is the only property of AF-algebras used in the proof. In particular the proof applies if $\mathscr{A}^\tau$ is simple with unit, but this is already a consequence of [15, Theorem 1].

Proof. If $\mathscr{A}$ does not have an identity, adjoin one and extend δ and τ in the obvious manner. The hypotheses in Theorem 3.1 are then still fulfilled and we may assume from now that $\mathscr{A}$ has an identity.

The strategy of the proof is as follows: We prove that the restriction of δ to each of the spectral subspaces $\mathscr{A}^\tau(\gamma)$ is a generator of a group of isometries. Actually, we restrict δ to subspaces of $\mathscr{A}^\tau(\gamma)$ of the form $\mathscr{A}^\tau(\gamma)P$, where P are suitable smooth projections in an approximate identity for $\mathscr{A}^\tau(\gamma)^*\mathscr{A}^\tau(\gamma)$. We then perturb δ by an inner derivation $\delta_{-Q}=-\mathrm{ad}(Q)$ implemented by an element $Q\in\mathscr{A}^\tau$ to obtain a derivation $\delta^{-Q}=\delta+\delta_{-Q}$ such that $\delta^{-Q}(P)=0$, and hence δ^{-Q} leaves $\mathscr{A}^\tau(\gamma)P$ invariant. The derivation δ^{-Q} still satisfies the same hypotheses as δ, in particular $\delta^{-Q}|\mathscr{A}^\tau$ is a generator. Kishimoto–Robinson introduced in [15] an explicit isometry between $\mathscr{A}^\tau(\gamma)P$ and a closed subspace of $\mathscr{A}^\tau$. (Actually this is a simplified account, and we have to work in a tensor product.) Transporting δ^{-Q} by this isometry, we get an operator on the closed subspace of $\mathscr{A}^\tau$ which is a bounded perturbation of δ^{-Q}, and therefore is a generator there. It follows that $\delta^{-Q}\,|\mathscr{A}^\tau(\gamma)P$ is a generator. Perturbing back with a suitable 1-cocycle in $\mathscr{A}$, we obtain the group generated by δ.

Observation 1. If $\gamma\in\hat{G}$, there is a net (P_α) of projections in $\mathscr{A}^\tau(\gamma)^*\mathscr{A}^\tau(\gamma)$ such that:

1. Each $P\in(P_\alpha)$ is a finite sum $P=\sum\limits_k X_k^*X_k$, where $X_k\in\mathscr{A}^\tau(\gamma)\cap D(\delta)$.
2. If $X\in\mathscr{A}^\tau(\gamma)$, then $\lim\limits_\alpha XP_\alpha=X$, where the limit exists in norm.

Proof. $\overline{\mathscr{A}^\tau(\gamma)^*\mathscr{A}^\tau(\gamma)}$ is an ideal in $\mathscr{A}^\tau$, and as $\mathscr{A}^\tau$ is an AF-algebra, it follows that $\mathscr{A}^\tau(\gamma)^*\mathscr{A}^\tau(\gamma)$ has an approximate identity (P'_β) consisting of projections. There is a canonical projection from $\mathscr{A}$ onto $\mathscr{A}^\tau(\gamma)$ commuting with δ, and hence $\mathscr{A}^\tau(\gamma)\cap D(\delta)$ is dense in $\mathscr{A}^\tau(\gamma)$. It follows, [11], [16], that for any $\varepsilon>0$ there are a finite number of elements $Y_k\in\mathscr{A}^\tau(\gamma)\cap D(\delta)$ such that

$$\|P'_\beta-\sum_k Y_k^*Y_k\|<\varepsilon.$$

If $Y=\sum\limits_k Y_k^*Y_k$, then $Y\in\mathscr{A}^\tau\cap D(\delta)$, and as P'_β is a projection, the spectrum of Y is contained in a small neighbourhood of the set $\{0,1\}$. Let $f\in C^\infty(\mathbb{R})$ be a function which is 0 in a suitable neighbourhood of 0 and $f(x)=x^{-1/2}$ in a suitable neighbourhood of 1, such that $P_\alpha=P_{\beta,\varepsilon}=f(Y)Yf(Y)$ is a projection. Then $f(Y)\in D(\delta)$ by [20], or [7, Corollary 3.2.33] and hence $P_\alpha\in D(\delta)$. Furthermore, $\|P'_\beta-P_{\beta,\varepsilon}\|$ is dominated by a constant, only depending on ε, which vanishes as $\varepsilon\to0$.

Thus, if the set of (β,ε) is ordered by $(\beta,\varepsilon)<(\beta',\varepsilon')$ if $\beta<\beta'$ and $\varepsilon'\leqq\varepsilon$, the net P_α has the property that $\lim\limits_\alpha XP_\alpha=X$, for all $X\in\overline{\mathscr{A}^\tau(\gamma)^*\mathscr{A}^\tau(\gamma)}$. But then $\lim\limits_\alpha YXP_\alpha=YX$, for $X\in\mathscr{A}^\tau(\gamma)$, $Y\in\mathscr{A}^\tau(\gamma)\mathscr{A}^\tau(\gamma)^*$. But if g is a non-negative real function such that

$g(0)=0$ and $g(x)=1$ for $x \geqq \delta$, then $Y=g(XX^*)\in\mathscr{A}^\tau(\gamma)\mathscr{A}^\tau(\gamma)^*$ and

$$\begin{aligned}\|g(XX^*)X-X\| &\leqq \|(g(XX^*)X-X)(X^*g(XX^*)-X^*)\|^{1/2}\\ &=\|(g(XX^*)^2-2g(XX^*)+\mathbb{1})XX^*\|^{1/2}\\ &\leqq 2\delta^{1/2}.\end{aligned}$$

It follows from the relation above, by letting $\delta\to 0$, that $\lim_\alpha XP_\alpha = X$, for all $X\in\mathscr{A}^\tau(\gamma)$.

If $X_k=Y_kf(Y)$, then $X_k\in\mathscr{A}^\tau(\gamma)\cap D(\delta)$ and $P_\alpha=\sum_k X_k^*X_k$. This ends the proof of Observation 1.

From now we fix a $\gamma\in\hat{G}$ and a projection $P=\sum_{k=1}^N X_k^*X_k$, where each $X_k\in\mathscr{A}^\tau(\gamma)\cap D(\delta)$. Put $Q=\delta(P)P-P\delta(P)$, $\delta_Q(X)=QX-XQ$, and $\delta^{-Q}(X)=\delta(X)-\delta_Q(X)$ for $X\in D(\delta)$.

Then Q is a skew-adjoint element in $\mathscr{A}^\tau$, and it follows that δ^{-Q} is a closed $*$-derivation satisfying the same hypotheses as δ in Theorem 3.1. Furthermore $\delta^{-Q}(P)=0$, [14].

Let δ' be the restriction of δ^{-Q} to $D(\delta')=\mathscr{A}^\alpha(\gamma)P\cap D(\delta)$. As $\delta^{-Q}(P)=0$ it follows that δ' is a densely defined closed operator on $\mathscr{A}^\alpha(\gamma)P$.

Observation 2. The operator δ' is the generator of a strongly continuous one-parameter group S_t^P of bounded operators on $\mathscr{A}^\alpha(\gamma)P$.

Proof. We follow closely an argument from [15]. Recall that $P=\sum_{k=1}^N X_k^*X_k$. Consider the C^*-dynamical system $(\mathscr{A}_N=\mathscr{A}\otimes M_N, G, \bar{\tau})$, where M_N is the $N\times N$ matrix algebra, and $\bar{\tau}(g)=\tau(g)\otimes\iota$, where ι is the trivial action. Let $\bar{\delta}=\delta^{-Q}\otimes\iota$ with $D(\delta)=D(\delta)\otimes\iota$. Then $\bar{\tau}$ and $\bar{\delta}$ satisfy the same properties as τ and δ. Define

$$V=\begin{bmatrix}X_1 & 0 & \dots & 0\\ X_2 & 0 & \dots & 0\\ \vdots & & & \\ X_N & 0 & \dots & 0\end{bmatrix}\in\mathscr{A}_N^{\bar{\tau}}(\gamma)\cap D(\bar{\delta}).$$

Then

$$V^*V=\begin{bmatrix}P & 0 & \dots & 0\\ 0 & 0 & \dots & 0\\ \vdots & & & \vdots\\ 0 & 0 & \dots & 0\end{bmatrix},$$

and hence V is a partial isometry, and VV^* is a projection in $\mathscr{A}_N^{\bar{\tau}}\cap D(\bar{\delta})$. Furthermore

$$\mathscr{A}_N^{\bar{\tau}}(\gamma)V^*V=\begin{bmatrix}\mathscr{A}^\tau(\gamma)P & 0 & \dots & 0\\ \mathscr{A}^\tau(\gamma)P & 0 & \dots & 0\\ \vdots & \vdots & & \vdots\\ \mathscr{A}^\tau(\gamma)P & 0 & \dots & 0\end{bmatrix}$$

Thus Observation 2 follows once we can show that the restriction $\bar{\delta}'$, of $\bar{\delta}$ to $\mathscr{A}_N^{\bar{\tau}}(\gamma)V^*V$ is a generator.

To this end, we define an isometric isomorphism ϕ from the Banach space $\mathscr{A}_N^{\bar{\tau}}(\gamma)V^*V$ onto the Banach space $\mathscr{A}_N^{\bar{\tau}}VV^*$ by $\phi(A)=AV^*$, for $A\in\mathscr{A}_N^{\bar{\tau}}(\gamma)V^*V$. Then $\phi^{-1}(B)=BV$ for $B\in\mathscr{A}_N^{\bar{\tau}}VV^*$, and hence

$$\phi\bar{\delta}'\phi^{-1}(B)=\bar{\delta}'(BV)V^*=\bar{\delta}(BV)V^*.$$

But as $B=BVV^*$, we have by the derivation property $\bar{\delta}(B)=\bar{\delta}(BV)V^*+BV\bar{\delta}(V^*)$. Thus $\phi\bar{\delta}'\phi^{-1}(B)=\bar{\delta}(B)-BV\bar{\delta}(V^*)$, for all $B\in\mathscr{A}_N^{\bar{\tau}}VV^*$. Define an operator ρ on $\mathscr{A}_N^{\bar{\tau}}$, by $\rho(B)=\bar{\delta}(B)-BV\bar{\delta}(V^*)$, for $B\in D(\bar{\delta})\cap\mathscr{A}_N^{\bar{\tau}}$. Since $\bar{\delta}$ is the generator of a one-parameter group of automorphisms on $\mathscr{A}_N^{\bar{\tau}}$, and the operation of right multiplication by $-V\bar{\delta}(V^*)$ is a bounded operator on $\mathscr{A}_N^{\bar{\tau}}$, it follows that ρ is the generator of a strongly continuous one-parameter group, [7, Theorem 3.1.33]. The restriction $\phi\bar{\delta}'\phi^{-1}$ of ρ to $\mathscr{A}_N^{\bar{\tau}}VV^*$ is then a generator, and as ϕ is an isometry, $\bar{\delta}'$ is a generator. This completes the proof of Observation 2.

Next, define a unitary cocycle Γ_t^Q in $\mathscr{A}^\tau$ by

$$\Gamma_t^Q=\mathbb{1}+\sum_{n\geq 1}\int_0^t dt_1\int_0^{t_1}dt_2\cdots\int_0^{t_{n-1}}dt_n\, e^{t_n\delta^{-Q}}(Q)\cdots e^{t_1\delta^{-Q}}(Q),$$

where $e^{t\delta^{-Q}}$ here denotes the one-parameter group of *-automorphisms generated by $\delta^{-Q}=\delta-\delta_Q$ on $\mathscr{A}^\tau$. Then $t\to\Gamma_t^Q$ is a continuous map into the unitaries in $\mathscr{A}^\tau$ satisfying the cocycle relation

$$\Gamma_{t+s}^Q=\Gamma_t^Q e^{t\delta^{-Q}}(\Gamma_s^Q),$$

and the differential equation

$$\frac{d\Gamma_t^Q}{dt}=\Gamma_t^Q e^{t\delta^{-Q}}(Q),$$

see Sect. 2.

Define a strongly continuous one-parameter family T_t^P of maps from $\mathscr{A}^\tau(\gamma)P$ into $\mathscr{A}^\tau(\gamma)$ by

$$T_t^P(X)=\Gamma_t^Q S_t^P(X)\Gamma_t^{Q*}.$$

Observation 3. If $X\in\mathscr{A}^\tau(\gamma)P\cap D(\delta)$, then $T_t^P(X)\in D(\delta)$ for all t, $t\to T_t^P(X)$ is differentiable and

$$\frac{dT_t^P(X)}{dt}=\delta(T_t^P(X))$$

for all $t\in\mathbb{R}$.

Proof. As $X\in D(\delta)$ we have $X\in D(\delta^{-Q})$ and hence $X\in D(\delta')$ and

$$\frac{d}{dt}S_t^P(X)=\delta'(S_t^P(X))=(\delta-\delta_Q)(S_t^P(X)).$$

It follows from Lemma 2.1 that $\Gamma_t^Q\in D(\delta^{-Q})=D(\delta)$, and $(d/dt)\Gamma_t^Q=\delta^{-Q}(\Gamma_t^Q)+$

$Q\Gamma_t^Q = \delta(\Gamma_t^Q) + \Gamma_t^Q Q$. Thus $T_t^P(X) \in D(\delta)$, $t \to T_t^P(X)$ is differentiable and

$$\begin{aligned}\frac{dT_t^P(X)}{dt} &= \frac{d\Gamma_t^Q}{dt} S_t^P(X)\Gamma_t^{Q*} + \Gamma_t^Q \frac{dS_t^P(X)}{dt}\Gamma_t^{Q*} + \Gamma_t^Q S_t^P(X)\left(\frac{d\Gamma_t^Q}{dt}\right)^* \\ &= (\delta(\Gamma_t^Q) + \Gamma_t^Q Q)S_t^P(X)\Gamma_t^{Q*} + \Gamma_t^Q(\delta(S_t^P(X)) - QS_t^P(X) \\ &\quad + S_t^P(X)Q)\Gamma_t^{Q*}\Gamma_t^Q S_t^P(X)(\delta(\Gamma_t^{Q*}) - Q\Gamma_t^{Q*}) \\ &= \delta(\Gamma_t^Q)S_t^P(X)\Gamma_t^{Q*} + \Gamma_t^Q\delta(S_t^P(X))\Gamma_t^{Q*} + \Gamma_t^Q S_t^P(X)\delta(\Gamma_t^{Q*}) \\ &= \delta(T_t^P(X)).\end{aligned}$$

This proves Observation 3.

Observation 4. If $X \in \mathscr{A}^\tau(\gamma)P$, then $\| T_t^P(X)\| = \| X \|$ for all $t \in \mathbb{R}$.

Proof. Let $e^{t\delta}$ again denote the automorphism group generated by the restriction of δ to $\mathscr{A}^\tau$. We first argue that

$$T_t^P(X)^* T_t^P(X) = e^{t\delta}(X^* X)$$

for all $t \in \mathbb{R}$. Assume first $X \in \mathscr{A}^\tau(\gamma)P \cap D(\delta)$. It follows from Observation 3 and the derivation property that the following calculation is valid

$$\begin{aligned}&T_t^P(X)^* T_t^P(X) - e^{t\delta}(X^*X) \\ &= \int_0^t ds \frac{d}{ds}\{e^{(t-s)\delta}(T_s^P(X)^* T_s^P(X))\} \\ &= \int_0^t ds\, e^{(t-s)\delta}(-\delta(T_s^P(X)^* T_s^P(X)) + \delta(T_s^P(X)^*)T_s^P(X) \\ &\quad + T_s^P(X)^*\delta(T_s^P(X))) \\ &= \int_0^t ds\, e^{(t-s)\delta}(0) = 0.\end{aligned}$$

Thus

$$\| T_t^P(X)\|^2 = \| T_t^P(X)^* T_t^P(X)\| = \| e^{t\delta}(X^*X)\| = \| X^*X \| = \| X \|^2.$$

As $P \in D(\delta)$, $\mathscr{A}^\tau(\gamma)P \cap D(\delta)$ is dense in $\mathscr{A}^\tau(\gamma)P$, and as each T_t^P is bounded, Observation 4 follows.

Observation 5. The restriction δ_γ of δ to $\mathscr{A}^\tau(\gamma)$ is the generator of a strongly continuous one-parameter group of isometries on $\mathscr{A}^\tau(\gamma)$.

Proof. The operators $\pm\delta_\gamma$ are dissipative by [4, Lemma 4.3], and by the Lumer–Phillips theorem it suffices to prove that $(\lambda - \delta)(D(\delta_\gamma))$ is dense in $\mathscr{A}^\tau(\gamma)$ for a positive and a negative real number λ, see [7, Theorem 3.1.16]. This follows from the differential equation in Observation 3 as in [9, page 360–361], or as follows: Assume that $X \in \mathscr{A}^{\tau}(\gamma)P \cap D(\delta)$, where P is as in Observation 1, and assume that $\lambda > 0$. We will show that $X \in R(\lambda - \delta_{\gamma})$. Define

$$Y = \int_0^{\infty} dt\, e^{-\lambda t}\, T_t^P(X),$$

where T^P is defined before, Observation 3. The integral converges because of Observation 4, and Observation 3 implies that $Y \in D(\delta)$ with

$$\begin{aligned}\delta(Y) &= \int_0^{\infty} dt\, e^{-\lambda t}\delta(T_t^P(X)) = \int_0^{\infty} dt\, e^{-\lambda t}\frac{d}{dt}(T_t^P(X)) \\ &= \int_0^{\infty} dt \frac{d}{dt}(e^{-\lambda t} T_t^P(X)) + \lambda \int_0^{\infty} dt\, e^{-\lambda t} T_t^P(X) \\ &= -X + \lambda Y.\end{aligned}$$

Thus $X = (\lambda - \delta)Y$ and $X \in R(\lambda - \delta)$. It now follows from Observation 1 that $R(\lambda - \delta)$ is dense. This ends the proof of Observation 5.

Theorem 3.1 is now an immediate consequence of Observation 5 and [15, Proposition 2].

4. Gauge Actions of the First Kind

In this section, which is largely independent of the previous one, we will consider the following problem: Let $\mathscr{A}$ be a C^*-algebra, τ an action of a compact abelian group on $\mathscr{A}$, δ a generator on the fixed point algebra $\mathscr{A}^{\tau}$. Assume that δ has closed densely defined extensions to $\mathscr{A}$ which commute with τ, and let δ_1, δ_2 be two such extensions. What is the relation between δ_1 and δ_2?

We will actually consider this problem in a special setting.

Theorem 4.1. *Let $\mathscr{A}$ be a* UHF-*algebra of Glimm type n^{∞}*, i.e. $\mathscr{A} = \bigotimes_{\iota \in L} M_{\iota}$, *where the index set L is infinite and each M_{ι} is a full $n \times n$-matrix algebra M. Let G be a compact abelian group acting on M and let τ be the corresponding infinite tensor product action of G on $\mathscr{A}$. Let α be an automorphism of $\mathscr{A}$ such that*

$$\alpha\tau(g) = \tau(g)\alpha \qquad \text{for all } g \in G,$$

and

$$\alpha(X) = X \qquad \text{for all } X \in \mathscr{A}^{\tau}.$$

*It follows that there exists a $g \in G$ such that $\alpha = \tau(g)$. In particular, if δ_1, δ_2 are closed *-derivations of $\mathscr{A}$ such that $\delta_i\,\tau(g) = \tau(g)\delta_i$ for all $g \in G$, $i = 1, 2$, and $\delta_1 | \mathscr{A}^{\tau} = \delta_2 | \mathscr{A}^{\tau}$, and $\delta_1 | \mathscr{A}^{\tau}$ is a generator on $\mathscr{A}^{\tau}$, then there exists a one-parameter subgroup of the action τ with generator δ_0, such that $D(\delta_2) \cap D(\delta_1) \cap D(\delta_0)$ is a joint core for the three derivations and $\delta_2 = \delta_1 + \delta_0$ on this core.*

Remark. Actually we prove the slightly stronger result that the groups $e^{t\delta_i}$ mutually commute for $i = 0, 1, 2$, and the relation $e^{t\delta_2} = e^{t\delta_0} e^{t\delta_1}$ is valid for all $t \in \mathbb{R}$.

Lemma 4.2. *Adopt the same hypotheses on $\mathscr{A}, \tau$ as in Theorem 4.1, but G may be any subgroup of* Aut(M). *Let $(\mathscr{H}, \pi, \Omega)$ be the cyclic representation of $\mathscr{A}$ defined by the unique trace-state ω on $\mathscr{A}$, [7, Definition 2.3.18]. It follows that $\pi(\mathscr{A}^\tau)' \cap \pi(\mathscr{A})'' = \mathbb{C}\mathbb{1}$, and in particular $\pi(\mathscr{A}^\tau)''$ is a factor.*

Remark. It is known that the restriction of ω to $\mathscr{A}^\tau$ is a factor state; for a sharp result in this direction, see [21, Theorem 4 in Sect. III]. As the closed cyclic subspace generated by $\mathscr{A}^\tau$ in $\mathscr{H}$ is a proper subspace of $\mathscr{H}$, Lemma 4.2 requires a separate proof.

Proof. Although this probably could be proved using techniques from [12], we will use an argument based on an idea from [17]. Let $S(\infty)$ be the group of all finite permutations of the index set L. There is a canonical action α of $S(\infty)$ on $\mathscr{A} = \bigotimes_{\iota \in L} M_\iota$, and, as shown in [17], there is a unitary representation $s \in S(\infty) \to U(s) \in \mathscr{A}^\tau$ such that $\alpha(s) = \mathrm{Ad}(U(s))$ for all $s \in S(\infty)$.

As ω is invariant under all automorphisms of $\mathscr{A}$, ω is α-invariant. As ω is the product of the normalized traces on M_ι for all ι, it follows by approximating with finite tensors that ω is clustering with respect to the action α, i.e. for any $A, B \in \mathscr{A}$ and $\varepsilon > 0$, there exists an $s \in S(\infty)$ such that $|\omega(A\alpha(s)(B)) - \omega(A)\omega(B)| < \varepsilon$. It follows that ω is external among the α-invariant states [7, Theorems 4.3.22 and 4.3.20]. As ω is a trace, the cyclic vector Ω is separating for $\pi(\mathscr{A})''$. If U_ω is the unitary representation of $S(\infty)$ on $\mathscr{H}$, defined by

$$U_\omega(s)\pi(A)\Omega = \pi(\alpha(s)(A))\Omega$$

for all $A \in \mathscr{A}, s \in S(\infty)$, then it follows from extremal invariance of ω that

$$\pi(\mathscr{A})'' \cap U_\omega(S(\infty))' = \mathbb{C}\mathbb{1},$$

[7, Theorem 4.3.20]. As each $U_\omega(s)$ implements the same automorphism of $\pi(\mathscr{A})''$ as $\pi(U(s))$, and π is a factor representation, it follows that there exists a unitary operator $j(\pi(U(s))) \in \pi(\mathscr{A})'$ such that

$$U_\omega(s) = \pi(U(s))j(\pi(U(s))),$$

and hence

$$\pi(\mathscr{A})'' \cap \pi(U(S(\infty)))' = \pi(\mathscr{A})'' \cap U_\omega(S(\infty))' = \mathbb{C}\mathbb{1}.$$

But as $U(S(\infty)) \in \mathscr{A}^\tau$ it follows finally that

$$\pi(\mathscr{A})'' \cap \pi(\mathscr{A}^\tau)' = \mathbb{C}\mathbb{1}.$$

Proof of Theorem 4 1. Let α be an automorphism of $\mathscr{A}$ commuting with τ such that $\alpha|\mathscr{A}^\tau = \mathrm{id}$. As the trace state ω on $\mathscr{A}$ is α- and τ-invariant, α and τ extends by σ-weak closure to $\pi(\mathscr{A})''$, and as the projection $P(X) = \int_G dg\tau(g)(X)$ extends by σ-weak closure to $\pi(\mathscr{A})''$, we have that $\pi(\mathscr{A}^\tau)''$ is the fixed point algebra for the extended τ. Hence the extended α restricts to the identity on this fixed point algebra, and the extended α commutes with the extended τ. But Lemma 4.2 implies that $\pi(\mathscr{A}^\tau)''$ is a factor, and it follows from Robert's version of Pontryagins duality theorem that there exists a $g \in G$ such that $\alpha = \tau(g)$. (A general version of Roberts's theorem is proved in [2, Appendix C], the special version used here can also be found in [10].)

Finally, let δ_1, δ_2 be two closed *-derivations on $\mathscr{A}$ commuting with τ such that $\delta_1 | \mathscr{A}^\tau = \delta_2 | \mathscr{A}^\tau$ is a generator on $\mathscr{A}^\tau$. It follows from Theorem 3.1 that δ_1, δ_2 generates groups $e^{t\delta_1}, e^{t\delta_2}$ on $\mathscr{A}$, and then $e^{t\delta_1} | \mathscr{A}^\tau = e^{t\delta_2} | \mathscr{A}^\tau$. Hence, by the first part of the theorem, there exists for each $t \in \mathbb{R}$ an element $g(t) \in G$ such that $e^{t\delta_2} = \tau(g(t)) e^{t\delta_1}$. By [2, Appendix B] we may assume that G acts faithfully, and hence $g(t)$ is unique. Here $\tau(g(s))$ commutes with $e^{t\delta_1}$ and $e^{t\delta_2}$ by assumption, for $s, t \in \mathbb{R}$ and hence $t \rightarrow g(t)$ is a one-parameter group. Also $t \rightarrow \tau(g(t)) = e^{t\delta_2} e^{-t\delta_1}$ is strongly continuous and it has a generator δ_0 which is a closed densely defined *-derivation. As $e^{s\delta_0}$ and $e^{t\delta_1}$ commute for all s, t, we get a strongly continuous representation by $\mathbb{R}^2$ by $(s, t) \rightarrow e^{s\delta_0} e^{t\delta_1}$. The C^1-vectors $D(\delta_0) \cap D(\delta_1)$ for this representation are invariant under both $e^{s\delta_0}$ and $e^{t\delta_1}$. It follows from $e^{t\delta_2} = e^{t\delta_0} e^{t\delta_1}$ that these vectors are contained in $D(\delta_2)$ and are invariant under $e^{t\delta_2}$. Hence these vectors are a core for both δ_0, δ_1 and δ_2, [7, Corollary 3.1.7]. Thus $D(\delta_0) \cap D(\delta_1) \cap D(\delta_2)$ is a joint core for δ_0, δ_1 and δ_2.

It follows from $e^{t\delta_2} = e^{t\delta_0} e^{t\delta_1}$ that $\delta_2 = \delta_1 + \delta_0$ on this core.

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