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Deformations of the Embedding of the SU(2) Monopole Solution in SU(3)

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Abstract. This paper is concerned with static Yang-Mills-Higgs fields, in the Prasad-Sommerfield limit of no Higgs self-interaction. One can obtain SU(3) multipole solutions from SU(2) solutions by embedding, in several different ways. In some of these cases, the embedding belongs to a family of SU(3) solutions that are not all embeddings; in other words, some embeddings can be deformed into non-embeddings. The simplest case, an embedding of the SU(2) spherically symmetric monopole, is studied with the aid of the twistor construction procedure. The family of axially symmetric SU(3) solutions to which it belongs is described.

1. Introduction

In recent years, there has been much progress towards understanding static magnetic multipoles in Yang-Mills-Higgs theories [1–9]. Most of this work has dealt with the case where the gauge group is SU(2), but the recently-developed techniques apply just as effectively to larger, more general gauge groups (although, of course, things become more complicated). In this paper, the twistor method is used to investigate the following rather curious phenomenon, which occurs in the case of larger gauge groups.

One can construct multipole solutions for (say) the gauge group SU(3) by embedding SU(2) solutions into SU(3). The general SU(2) solution, of charge n, depends on $4n-1$ parameters [7]. So the embedding will belong to an SU(3) family of at least $4n-1$ parameters; but in some cases (and this depends on the details of the embedding), the embedded SU(2) solutions belong to a family of *more* than $4n-1$ parameters. In other words, some embeddings can be continuously deformed into solutions that are no longer embeddings. More details of this will be described in Sect. 3, and the subsequent sections go on to investigate the simplest

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case, namely $n=1$. Here we will be able to see explicitly what the deformations of the embedded solution are.

The same sort of thing happens for gauge groups other than SU(3), but we shall restrict our attention in this paper to the SU(3) case, listing all the possible embeddings of SU(2) solutions in SU(3), and discussing deformations of these embeddings in the simplest case.

2. Topological Charges

This section is a summary of the topological classification of SU(3) multipoles; more details may be found in [7, 10, 11].

We suppose that the configuration is static and purely magnetic, and that the Higgs field Φ is in the adjoint representation and has vanishing self-interaction. So the fields in the problem are a gauge potential $A_j (j=1,2,3)$ and a Higgs scalar Φ; each of A_1, A_2, A_3 and Φ is a 3×3 tracefree Hermitian matrix, and is smooth on R^3. The field equation

$$F_{jk} = -\varepsilon_{jkl} D_l \Phi \tag{1}$$

is imposed, together with the boundary condition

$$\operatorname{tr}\Phi^2 = m_1 - m_2 r^{-1} + 0(r^{-2}) \quad \text{as} \quad r\to\infty\,, \tag{2}$$

where $F_{jk}=\partial_j A_k - \partial_k A_j + i[A_j, A_k]$, $D_j\Phi = \partial_j\Phi + i[A_j, \Phi]$; m_1 and m_2 are constants. The conditions (1) and (2) guarantee that the configuration has finite energy

$$E = \tfrac{1}{4}\int \operatorname{tr}\{\tfrac{1}{2}F^2 + (D\Phi)^2\} d^3x\,,$$

and that the energy achieves a local minimum; in fact, the value of E is easily seen to be $m_2\pi$ (use Stokes' theorem).

The topological classification can be described as follows. Choose a gauge in which the Higgs field on the positive z-axis has the form

$$\Phi = \Phi_0 - \Phi_1 z^{-1} + 0(z^{-2}) \quad \text{as} \quad z\to\infty\,, \tag{3}$$

where $\Phi_0 = \operatorname{diag}(\lambda_1, \lambda_2, \lambda_3)$ and $\Phi_1 = \frac{1}{2}\operatorname{diag}(n_1, n_2 - n_1, -n_2)$ are constant diagonal matrices, with $\lambda_1+\lambda_2+\lambda_3=0$ and $\lambda_1 \geqq \lambda_2 \geqq \lambda_3$. The numbers n_1 and n_2 appearing in Φ_1 always turn out to be integers [7]. (In reference [7], these integers appear in the z^{-2} term of the magnetic field rather than the z^{-1} term of the Higgs field, but the field equation (1) tells us that these two terms have the same coefficients.)

If the λ_i are all distinct, then the isotropy group of Φ_0 in SU(3) is $\mathrm{U}(1)\times\mathrm{U}(1)$, so this is the "residual" group down to which the symmetry is broken by the Higgs field. The other possibility is that two of the λ_i coincide, in which case the residual symmetry group is U(2). The Higgs field Φ determines an element of the homotopy group $\pi_2(\mathrm{SU}(3)/J)$, where J is the residual group (i.e., the isotropy group of Φ_0). In the $\lambda_1>\lambda_2>\lambda_3$ case, where $J=\mathrm{U}(1)\times\mathrm{U}(1)$, we get $\pi_2 = Z\oplus Z$; so the configurations are classified by two integers (*"topological charges"*) and these are precisely the integers n_1 and n_2 appearing in Φ_1, the coefficient of the z^{-1} term. If, on the other hand, $\lambda_1=\lambda_2$ or $\lambda_2=\lambda_3$, then $J=\mathrm{U}(2)$ and $\pi_2=Z$; so there is only one topological charge. For example, in the $\lambda_1 = \lambda_2$ case, the integer n_2 is the topological charge; the fact that $\lambda_1 = \lambda_2$ means that one cannot distinguish in a gauge-invariant way between n_1 and $n_2 - n_1$, and we say that n_1 (modulo the ambiguity $n_1 \mapsto n_2 - n_1$) is a *magnetic weight* [11]. Similarly, if $\lambda_2 = \lambda_3$, then n_1 is the topological charge, and $n_2 \sim n_1 - n_2$ is the magnetic weight.

The energy E depends only on the topological charge(s) and the asymptotic eigenvalues of Φ: it follows immediately from our earlier formulae that

$$E = (\lambda_1 n_1 + \lambda_2 n_2 - \lambda_2 n_1 - \lambda_3 n_2)\pi .$$

It is worth remarking that there is an existence theorem which says that for each of the two types of symmetry breaking, and for any positive value(s) of the topological charges(s), multipole solutions of (1) and (2) exist [6].

Finally, we come to the question of how many parameters' worth of solutions there are; in other words, given a solution with fixed $(\lambda_1, \lambda_2, \lambda_3)$ and (n_1, n_2), how many zero-frequency modes are there? If $J = \mathrm{U}(1) \times \mathrm{U}(1)$, the answer has been computed: it is $N = 4(n_1 + n_2) - 2 + k$, where k is a non-negative integer less than or equal to the number of n_a which vanish [7]. The conjecture is that equality holds; in other words, $k = 1$ if $n_1 = 0$ or $n_2 = 0$, and $k = 0$ otherwise [7].

If $J = \mathrm{U}(2)$, then the number of zero-modes is not known, in general (cf. [7]). But it is known in some special cases [7]: in particular, there are three zero-modes about an embedding of the spherically symmetric 1-monopole, of type II or IV (see Sect. 3 for the definition of what such an embedding is).

The term "zero-modes" refers here to actual non-trivial modes in the solution; that is, the number of parameters the solution depends on after pure-gauge modes have been removed.

Some more results on the subject of parameter-counting have recently been announced [16].

3. Embeddings

This section discusses the various different ways in which an SU(2) solution can be embedded in SU(3). To begin with, there are two inequivalent embeddings of the Lie algebra SU(2) into SU(3), namely the "maximal" one and the "minimal" one. The maximal (irreducible) embedding may be represented explicitly as

$$\begin{bmatrix} a & b \\ \bar{b} & -a \end{bmatrix} \mapsto \begin{bmatrix} 2a & \sqrt{2}b & 0 \\ \sqrt{2}\bar{b} & 0 & \sqrt{2}b \\ 0 & \sqrt{2}\bar{b} & -2a \end{bmatrix} . \tag{4}$$

The exponentiated version of this is the natural embedding of SO(3) in SU(3). From (4) one sees immediately that the maximal embedding of an SU(2) solution of charge n has asymptotic eigenvalues $(\lambda_1, \lambda_2, \lambda_3) = (\lambda, 0, -\lambda)$ for some λ, and the topological charges are $n_1 = n_2 = 2n$. Thus the residual symmetry group J is necessarily $\mathrm{U}(1) \times \mathrm{U}(1)$. In [12], it was shown that the embedding of the $n = 1$ (spherically symmetric) SU(2) monopole belongs to a family of spherically symmetric SU(3) monopoles having more general values of $(\lambda_1, \lambda_2, \lambda_3)$[1]. A limiting case of these is a solution with $J = \mathrm{U}(2)$, topological charge 2 and magnetic weight 1 [13]; some deformations of this solution are known [8].

Let us move on now to the minimal (reducible) embedding, which may be represented as

$$\begin{bmatrix} a & b \\ \bar{b} & -a \end{bmatrix} \mapsto \begin{bmatrix} a & b & 0 \\ \bar{b} & -a & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{5}$$

This case is discussed in [7], where the machinery of root systems is used to analyze minimal embeddings of SU(2) solutions into arbitrary simple groups. The following is a summary of the analysis, as it applies to our present problem.

The first observation is that if we take an SU(2) solution and embed it via (5) to obtain an SU(3) solution (A_j, Φ), then we can construct a new solution (A'_j, Φ'), according to

$$\begin{aligned} A'_j &= A_j, \\ \Phi' &= \Phi + \mathrm{diag}(k, k, -2k), \end{aligned} \tag{6}$$

where k is a real constant. (Clearly, the field equations (1) are preserved, since the extra bit added to Φ is constant and commutes with A_j.) Suppose the original SU(2) solution had charge n, and choose the gauge and scaling so that on the positive z-axis, its Higgs field behaves like $\mathrm{diag}(1 - \frac{1}{2}nz^{-1}, -1 + \frac{1}{2}nz^{-1}) + 0(z^{-2})$. Now embed it as in (5) and add on a k-term as in (6), permuting entries if necessary so as to achieve the ordering $\lambda_1 \geqq \lambda_2 \geqq \lambda_3$ in Φ_0 [cf. Eq. (3)]. Then we see that five different cases arise, namely:

I. $k > 1/3$, $J = \mathrm{U}(1) \times \mathrm{U}(1)$, $n_1 = n$, $n_2 = 0$;

II. $k = 1/3$, $J = \mathrm{U}(2)$, charge $n_1 = n$, weight $n_2 = 0 \sim n$;

III. $-1/3 < k < 1/3$, $J = \mathrm{U}(1) \times \mathrm{U}(1)$, $n_1 = n_2 = n$;

IV. $k = -1/3$, $J = \mathrm{U}(2)$, charge $n_2 = n$, weight $n_1 = 0 \sim n$;

V. $k < -1/3$, $J = \mathrm{U}(1) \times \mathrm{U}(1)$, $n_1 = 0$, $n_2 = n$.

Note that since $\mathrm{tr}(\Phi'^3) = -6k^3$ is a gauge-invariant quantity, any two solutions with different k are not gauge-equivalent. It is perhaps useful to visualize the situation as in Fig. 1. This is a graph of the Cartan subalgebra of su(3), i.e. the trace-free diagonal matrices. The crosses are the roots of su(3) (actually, these live in the dual of the Cartan subalgebra, but we are using the Killing metric to identify the space with its dual). And the thick line represents the various possibilities for Φ'_0, as k varies.

Applying the results mentioned in the previous section, we see that the number of zero-frequency modes for type III is $8n - 2$, and for all the others is $4n - 1$[2]. (The maximal embedding has $16n - 2$ zero-modes, but from now on we shall restrict our

1 This is not a family of "deformations" in the sense in which that word is being used in this paper, because it involves changing the asymptotic eigenvalues λ_i

2 For embeddings of type II or IV, this is known to be true if $n = 1$, but only conjectured if $n > 1$

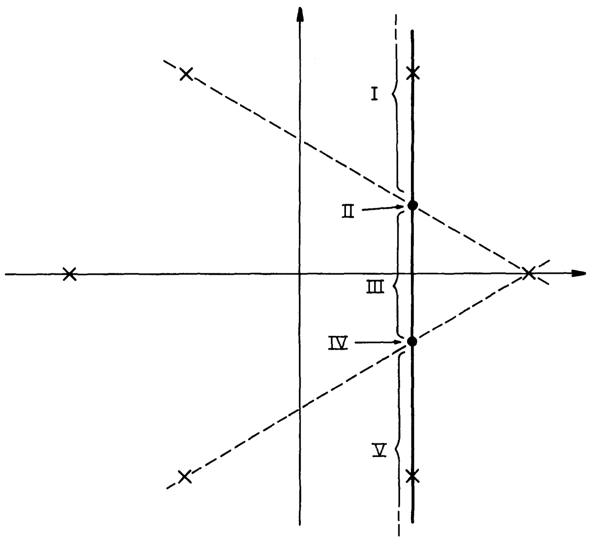

Fig. 1

attention to minimal embeddings.) Since the SU(2) n-monopole already depends on $4n-1$ parameters, we see that in case III, and only in that case, an embedding belongs to a larger family of solutions that are not embeddings. In the sections that follow, we shall investigate this phenomenon for the case $n=1$. We shall be able to see explicitly what the $8\times 1-2=6$ parameters' worth of solutions are.

4. The Twistor Construction

In the SU(2) case, the twistor method has proved to be particularly useful for understanding and constructing multipole solutions [1–5]. But the method applies just as effectively to other gauge groups, and in particular to SU(3). One can write down a general theorem (along the lines of the SU(2) theorem described in [5]) which says that SU(3) solutions correspond to certain holomorphic vector bundles. It leads to the following construction procedure.

Let g be a 3×3 matrix of functions of the two complex variables γ and ζ, satisfying

(i) g is analytic for all γ, and for ζ near $|\zeta|=1$;

(ii) $\det(g)=1$;

(iii) $g(\bar{\gamma}, -\bar{\zeta}^{-1})=g(\gamma,\zeta)^{\dagger}$. (7)

Now "split" g in the following way:

$$g(\xi\zeta-2z-\bar{\xi}\zeta^{-1},\zeta)=\hat{h}h^{-1}, \tag{8}$$

where $\xi=x+iy$ (x, y, z being the usual coordinates on R^3), $h=h(x,y,z,\zeta)$ is a 3×3 matrix analytic for $|\zeta|\leqq 1$, and $\hat{h}=\hat{h}(x,y,z,\zeta)$ is a matrix analytic for $|\zeta|\geqq 1$ (including $\zeta=\infty$). Then put

$$H(x,y,z)=h(x,y,z,0),$$
$$\hat{H}(x,y,z)=\hat{h}(x,y,z,\infty).$$

The final step, which defines the fields Φ and A_j, is to take

$$\begin{aligned}\Phi &= \tfrac{1}{2}H^{-1}H_z - \tfrac{1}{2}\hat{H}^{-1}\hat{H}_z,\\ A_z &= -\tfrac{1}{2}iH^{-1}H_z - \tfrac{1}{2}i\hat{H}^{-1}\hat{H}_z,\\ A_\xi &= -iH^{-1}H_\xi,\\ A_{\bar{\xi}} &= -i\hat{H}^{-1}\hat{H}_{\bar{\xi}},\end{aligned}$$

where the subscripts on the right-hand sides denote partial differentiation.

The fields (Φ, A_j) defined by this procedure will automatically give an SU(3) solution of the equations (1). So all we have to worry about is choosing g in such a way that the boundary condition (2) is satisfied, and that the splitting (8) is possible for all $(x, y, z)\in R^3$. (Neither of these two conditions is guaranteed *a priori.*)

The only difficult part of the procedure is the splitting (8). However, there is a special class of matrices for which one can write down explicit formulae for h and $\hat{h}$, namely the upper triangular ones:

$$\begin{bmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{bmatrix} \tag{9}$$

(for example, see [14, 15] for the SU(2) case). A matrix such as (9) cannot satisfy the reality condition (7) unless it is diagonal, and the class of diagonal matrices is too small to give any interesting solutions. But we can use the fact that there is a certain amount of freedom in g. In particular, we may multiply g on the right by a 3×3 matrix Λ of functions of γ and ζ, analytic for $|\zeta|\leqq 1$ and for all γ including $\gamma=\zeta^{-1}$. It is easy to check that g and $g\Lambda$ lead to the same (Φ, A_j).

Thus our requirement is that g should have the form (9), and that there should exist a Λ such that $g\Lambda$ satisfies (7). In the SU(2) case, it is known that all multipole solutions can be obtained from such matrices g [5]. As yet this analysis has not been extended to larger groups. But we shall see in the next section that upper triangular g's are sufficient to give us what we want in our particular problem.

5. Deformations of the Embedding

The upper triangular g-matrix which generates the SU(2) monopole of charge 1 is [1]

$$\begin{bmatrix} \zeta e^{\gamma} & \gamma^{-1}(e^{\gamma}-e^{-\gamma}) \\ 0 & \zeta^{-1}e^{-\gamma} \end{bmatrix}.$$

The type III embedding of this in SU(3) is generated by

$$\begin{bmatrix} \zeta e^{\lambda_1\gamma} & 0 & \gamma^{-1}(e^{\lambda_3\gamma}-e^{\lambda_1\gamma}) \\ 0 & e^{\lambda_2\gamma} & 0 \\ 0 & 0 & \zeta^{-1}e^{\lambda_3\gamma} \end{bmatrix}, \tag{10}$$

where $\lambda_1=1+k$, $\lambda_2=-2k$, and $\lambda_3=-1+k$; note that $\lambda_1+\lambda_2+\lambda_3=0$ and $\lambda_1>\lambda_2>\lambda_3$, since $-\frac{1}{3}<k<\frac{1}{3}$ for type III (cf. Sect. 3). We now look for deformations of this embedding among matrices of the form

$$g=\begin{bmatrix} \zeta e^{\lambda_1\gamma} & \Gamma_1 & \Gamma_2 \\ 0 & e^{\lambda_2\gamma} & \Gamma_3 \\ 0 & 0 & \zeta^{-1}e^{\lambda_3\gamma} \end{bmatrix}, \tag{11a}$$

where the Γ_i are suitable functions of γ and ζ. In fact, we require the Γ_i to be entire in γ, smooth for ζ near $|\zeta|=1$, and chosen in such a way that g is equivalent to a "real" matrix (in the sense described in Sect. 4).

One class of Γ_i's that satisfies these requirements is the following:

$$\Gamma_1=m\zeta\eta^{-1}(ae^{\lambda_1\gamma}-a^{-1}e^{\lambda_2\gamma}), \tag{11b}$$

$$\Gamma_2=a^{-1}\gamma^{-1}\eta^{-1}(-a^2\gamma e^{\lambda_1\gamma}+m^2e^{\lambda_2\gamma}+\eta e^{\lambda_3\gamma}), \tag{11c}$$

$$\Gamma_3=m\zeta^{-1}\gamma^{-1}(e^{\lambda_3\gamma}-e^{\lambda_2\gamma}), \tag{11d}$$

where m is a real parameter, $\eta=\gamma-m^2$ and $a=\{\exp(\lambda_2m^2-\lambda_1m^2)\}^{1/2}$. It is easy to check that these Γ_i are entire in γ, and that if g is multiplied on the right by

$$\Lambda=\begin{bmatrix} 0 & 0 & a \\ 0 & 1 & m \\ -a^{-1} & m\zeta & \gamma\zeta \end{bmatrix},$$

then the resulting matrix satisfies the reality condition (7). If $m=0$, then we regain the embedding (10), which is spherically symmetric; for general values of m, one finds, by examining $g\Lambda$, that the configuration is axially symmetric (cf. [9]).

Now one can carry out the procedure described in Sect. 4, and compute Φ and A_j. The details of this calculation may be found in the appendix. It turns out that the splitting (8) is indeed possible. Furthermore,

$$\operatorname{tr}\Phi^2=(\lambda_1^2+\lambda_2^2+\lambda_3^2)-\tfrac{1}{2}\nabla^2\log S, \tag{12}$$

where ∇^2 is the 3-dimensional Laplacian and S is a certain function on R^3, defined in Eq. (A4). This enables us to verify that the boundary condition (2) is satisfied. We can therefore conclude that for each value of the deformation parameter m, we do indeed have a solution.

6. Properties of the Deformations

As was mentioned before, the solutions are axially symmetric (about the z-axis, in our coordinate system). Examination of the expression for $\operatorname{tr}\Phi^2$ [Eqs. (12) and (A4)] shows that they are spherically symmetric only when $m=0$. So, regarding the asymptotic eigenvalues λ_1, λ_2 and λ_3 as fixed, the solutions depend on six parameters: m is one, and the other five correspond to rigid motions of R^3, ignoring rotations about the z-axis. This agrees with the zero-mode count described in Sect. 3.

We can get an idea of what the solutions look like by plotting $\operatorname{tr}\Phi^2$ along the axis of symmetry. This gives a graph as in Fig. 2. Here the scale along the z-axis is set by $c=\frac{1}{2}m^2$, and along the $\operatorname{tr}\Phi^2$-axis by $m_1=\lambda_1^2+\lambda_2^2+\lambda_3^2$. There is a single minimum, and the point $z=p$ at which it occurs lies between $-c$ and 0.

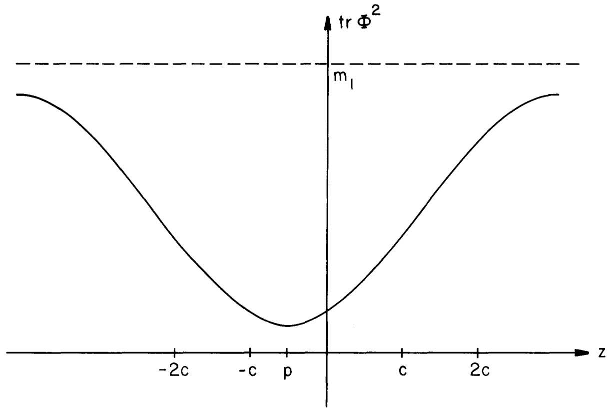

Fig. 2

(i) If $\lambda_1 \to \lambda_2$, then $p \to 0$. (The limit $\lambda_1 \to \lambda_2$ is the embedding of type II, with charge $n=1$. It is spherically symmetric, centered at $z=0$. The parameter m becomes redundant in the limit.)

(ii) If $\lambda_2 \to \lambda_3$, then $p \to -c$. (The limit $\lambda_2 \to \lambda_3$ is the type IV embedding, centered at $z=-c$.)

(iii) The half-way point $p=-\frac{1}{2}c$ occurs when $\lambda_1 - \lambda_2 = \lambda_2 - \lambda_3$.

Therefore we may provisionally interpret the solution as representing an axially symmetric magnetic pole, situated on the axis of symmetry. The deformation parameter m measures the deviation from spherical symmetry.

In the above, we have assumed that $\lambda_1 > \lambda_2 > \lambda_3$. The $\lambda_1 \to \lambda_2$ and $\lambda_2 \to \lambda_3$ limits exist, but what if we (say) increase λ_3 so that $\lambda_1 > \lambda_3 > \lambda_2$? In this case, one discovers that the solutions break down: in particular, the boundary condition (2) fails to hold, unless $m=0$. This is exactly what one expects, since $m \neq 0$ solutions would in this case be deformations of the type I embedding, and we know from the zero-mode count that there are no non-trivial deformations of type I.

7. Conclusion

We have constructed deformations of the minimal embedding of the SU(2) magnetic pole into SU(3). The fields Φ and A_j can, if desired, be written out explicitly; but many of their properties can be inferred without doing so.

Not all SU(3) solutions are deformations of an embedding. For example, the solutions described in [8], which include the Bais-Weldon solution [13], have topological charge 2 and magnetic weight 1; and the analysis in Sect. 3 shows that an embedding (and hence any continuous deformation of an embedding) can never have such a charge and weight.

The twistor method enables one to construct all solutions of the Bogomolny equations (1), for any gauge group. This paper has only investigated one somewhat special case. If one wanted to embark on a more general analysis of the problem, the procedure would be as follows.

First, one has to construct the patching matrices g corresponding to the class of solutions in question. This can either be done by trial and error, as in Sect. 5, or, for the SU(2) case, in [1, 3, 4]; or it can be done more systematically, as Hitchin did in the SU(2) case [5]. Complications can already arise at this first step; for example, the general anylysis of SU(2) g-matrices runs into transcendental equations [4, 5].

Next, one has to check the two things that are not guaranteed *a priori*, namely, the "splittability" of g and the boundary condition (2). This involves proving that a certain real-valued function S on R^3 is nowhere-zero and behaves suitably at infinity. In general, these things are hard to prove rigorously [2, 4]. But in any particular case, one has an explicit formula for S, and so one can convince oneself that they hold by evaluating S numerically on a sufficiently fine grid of points in R^3.

Finally, one can, if desired, obtain explicit expressions for the space-time fields Φ and A_j. These expressions are always more complicated than the matrix g which generates them: g contains all the information of the fields, in its simplest form.

In this sense, the problem of finding static magnetic multipoles is completely solvable. But what about a more general situation, for example, involving motion and scattering of poles? Not much is known about this; it appears to be a qualitatively different problem, in that the relevant equations of motion are far more difficult to solve than the Bogomolny equations (1). At present, no such non-static solution is known.

Appendix

This appendix describes how Φ and A_j are computed, starting from the matrix $g(\gamma, \zeta)$ defined in Eq. (11). The first step is to split g into $\hat{h}h^{-1}$.

To begin with, define $\hat{\mu} = -z - \xi\zeta^{-1}$, $\mu = z - \xi\zeta$ (so that $\gamma = \hat{\mu} - \mu$), $\hat{m} = \operatorname{diag}(e^{-\lambda_1\hat{\mu}}, e^{-\lambda_2\hat{\mu}}, e^{-\lambda_3\hat{\mu}})$ and $m = \operatorname{diag}(e^{\lambda_1\mu}, e^{\lambda_2\mu}, e^{\lambda_3\mu})$. Then $\hat{m}gm = \tilde{g}$, where

$$\tilde{g} = \begin{bmatrix} \zeta & \varrho_1 & \varrho_2 \\ 0 & 1 & \varrho_3 \\ 0 & 0 & \zeta^{-1} \end{bmatrix},$$

$$\varrho_1 = \Gamma_1 e^{-\lambda_1\hat{\mu} + \lambda_2\mu},$$

$$\varrho_2 = \Gamma_2 e^{-\lambda_1\hat{\mu} + \lambda_3\mu},$$

$$\varrho_3 = \Gamma_3 e^{-\lambda_2\hat{\mu} + \lambda_3\mu}.$$

If we now split $\tilde{g}$ into $\hat{k}k^{-1}$, then the matrices $\hat{h}$ and h which split g will be given by $\hat{h} = \hat{m}^{-1}\hat{k}$ and $h = mk$. So we may focus our attention on the matrix $\tilde{g}$, which has no $e^{\lambda\gamma}$ factors on its diagonal.

Write $k^{-1} = [\underline{\alpha}_1 \; \underline{\alpha}_2 \; \underline{\alpha}_3]$ and $\hat{k}^{-1} = [\underline{\beta}_1 \; \underline{\beta}_2 \; \underline{\beta}_3]$, where each of $\underline{\alpha}_1, \underline{\alpha}_2, \cdots$ denotes a column 3-vector. Then the equation $\tilde{g} = \hat{k}k^{-1}$ becomes

$$\zeta\underline{\beta}_1 = \underline{\alpha}_1, \tag{A1}$$

$$\varrho_1\underline{\beta}_1 = \underline{\alpha}_2 - \underline{\beta}_2, \tag{A2}$$

$$\varrho_2\underline{\beta}_1 + \varrho_3\underline{\beta}_2 = \underline{\alpha}_3 - \zeta^{-1}\underline{\beta}_3, \tag{A3}$$

and we can solve these equations one by one.

First, the general solution of (A1) is

$$\underline{\beta}_1=\underline{\beta}_{10}+\underline{\beta}_{11}\zeta^{-1},$$
$$\underline{\alpha}_1=\underline{\beta}_{10}\zeta+\underline{\beta}_{11},$$

where $\underline{\beta}_{10}$ and $\underline{\beta}_{11}$ are functions of z, ξ and $\bar{\xi}$. (Remember that the $\underline{\alpha}$'s must be analytic for $|\zeta|\leqq 1$ and the $\underline{\beta}$'s for $|\zeta|\geqq 1$.)

To solve (A2), first split $\varrho_1=\hat{\sigma}_1+\sigma_1$, where $\sigma_1=\sigma_1(z,\xi,\bar{\xi},\zeta)$ is analytic for $|\zeta|\leqq 1$ and $\hat{\sigma}_1$ for $|\zeta|\geqq 1$; so σ_1 is the "Taylor" part of ϱ_1 and $\hat{\sigma}_1$ the "Laurent" part. To fix σ_1 and $\hat{\sigma}_1$ uniquely, suppose that $\sigma_1=0$ at $\zeta=0$. Then the general solution of (A2) is

$$\underline{\alpha}_2=\sigma_1\underline{\beta}_1+\underline{\lambda},$$
$$\underline{\beta}_2=-\hat{\sigma}_1\underline{\beta}_1+\underline{\lambda},$$

where $\underline{\lambda}=\underline{\lambda}(z,\xi,\bar{\xi})$.

Finally, we come to Eq. (A3). To solve this, split

$$\varrho_2-\hat{\sigma}_1\varrho_3=\hat{\sigma}_2+\sigma_2 \quad \text{with} \quad \sigma_2=0 \quad \text{at} \quad \zeta=0,$$
$$\varrho_3=\hat{\sigma}_3+\sigma_3 \quad \text{with} \quad \hat{\sigma}_3=0 \quad \text{at} \quad \zeta=\infty,$$

where, as before, σ_2 and σ_3 are analytic inside $|\zeta|=1$, and $\hat{\sigma}_2$ and $\hat{\sigma}_3$ outside. Then the unique solution of (A3) is

$$\underline{\alpha}_3=\sigma_3\underline{\lambda}+\sigma_2\underline{\beta}_1+S\underline{\beta}_{10},$$
$$\underline{\beta}_3=-\zeta\hat{\sigma}_3\underline{\lambda}-\zeta\hat{\sigma}_2\underline{\beta}_1+\zeta S\underline{\beta}_{10},$$

where $S=\hat{\sigma}_2|_{\zeta=\infty}$.

This, then, gives us the matrices k and $\hat{k}$ which split $\tilde{g}$; but we still have to make sure that k and $\hat{k}$ are non-singular. It is clear that $\det k^{-1}=\det \hat{k}^{-1}=S\det[\underline{\beta}_{10}\underline{\beta}_{11}\underline{\lambda}]$, so as long as $S\neq 0$, we can choose the nine functions $\underline{\beta}_{10}$, $\underline{\beta}_{11}$ and $\underline{\lambda}$ in such a way that k and $\hat{k}$ are non-singular. In other words, the condition $S(z,\xi,\bar{\xi})\neq 0$ for all $z,\xi,\bar{\xi}$ is necessary and sufficient for the fields (A_j,Φ) that we will eventually obtain, to be smooth on R^3. And the choice of $\underline{\beta}_{10}$, $\underline{\beta}_{11}$, and $\underline{\lambda}$ corresponds to a choice of GL$(3,C)$ gauge.

From this point on, it is straightforward (although messy) to complete the calculation of A_j and Φ. Instead of writing out the answer in detail, let us content ourselves with the following remarks.

The function S turns out to be

$$S=r^{-1}R^{-1}\sinh(p_2R)\sinh(p_3r)\{R\coth(p_2R)+r\coth(p_3r)-c\}, \tag{A4}$$

where $p_2=\lambda_2-\lambda_1$, $p_3=\lambda_3-\lambda_2$, $r^2=z^2+\xi\bar{\xi}$, $R^2=r^2+2cz+c^2$, and $c=1/2m^2$. Note that S is an even function of both r and R. Simple estimates show that in each of the following four cases, S is nowhere zero:

(i) $p_2>0$ and $p_3>0$,

(ii) $p_3<0$ and $p_3<0$,

(iii) $p_2=0$ and $p_3\neq 0$,

(iv) $p_3=0$ and $p_2\neq 0$.

In particular, the deformations we are discussing have $p_2>0$ and $p_3>0$, so they satisfy the necessary condition $S\neq 0$ on R^3.

Let us choose the gauge

$$[\beta_{10} \quad \beta_{11} \quad \underline{\lambda}]=\begin{bmatrix} V & 0 & 0 \\ 0 & 0 & V \\ 0 & V & 0 \end{bmatrix},$$

where $V=-S^{-1/3}$. In this gauge, the Higgs field is given by

$$\Phi=S^{-1}\begin{bmatrix} \frac{1}{2}(\lambda_1+\lambda_3)S-\frac{1}{2}\partial_z S & S\partial_\xi\Delta_{1-1} & -\partial_\xi S \\ -\partial_{\bar\xi}\Delta_{31} & \lambda_2 & -\partial_\xi\Delta_{30} \\ -\partial_{\bar\xi}S & -S\partial_{\bar\xi}\Delta_{10} & \frac{1}{2}(\lambda_1+\lambda_3)S+\frac{1}{2}\partial_z S \end{bmatrix},$$

where the Δ's are Taylor-Laurent coefficients defined by

$$\begin{aligned} \varrho_1 &= \sum \Delta_{1r}\zeta^{-r}, \\ \varrho_2-\hat{\sigma}_1\varrho_3 &= \sum \Delta_{2r}\zeta^{-r}, \\ \varrho_3 &= \sum \Delta_{3r}\zeta^{-r}. \end{aligned}$$

Note that $S=\Delta_{20}$. The Δ's are linked together by equations (which they satisfy automatically as a consequence of the way they are defined). For example, one of the equations is

$$2\partial_\xi\Delta_{1r}=-\partial_z\Delta_{1(r+1)}+(\lambda_1-\lambda_2)\Delta_{1(r+1)}.$$

These equations, together with the expression for Φ, enables one to prove (with some tedious but straightforward algebra) that

$$\operatorname{tr}\Phi^2=(\lambda_1^2+\lambda_2^2+{}_3^2)-\tfrac{1}{2}V^2\log S.$$

Checking the boundary condition (2) is now easy: if $p_2>0$ and $p_3>0$, then

$$\log S=(\lambda_3-\lambda_1)r-2\log r+f,$$

where f is smooth and uniformly bounded as $r\to\infty$. So $\operatorname{tr}\Phi^2=(\lambda_1^2+\lambda_2^2+\lambda_3^2)-(\lambda_3-\lambda_1)r^{-1}+0(r^{-2})$ uniformly as $r\to\infty$.

If, on the other hand, $p_2>0$ and $p_3<0$, or $p_2<0$ and $p_3>0$, then $\log S$ behaves non-uniformly as $r\to\infty$, and the boundary condition fails on the z-axis.

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