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Phase Diagrams and Cluster Expansions for Low Temperature $\mathscr{P}(\phi)_2$ Models*

II. The Schwinger Functions

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Abstract. We give a cluster expansion for the Schwinger functions of the stable phases found in Part I. The Wightman axioms, the mass gap, and asymptoticity of perturbation theory follow.

In Part I the phase diagram of a generic low temperature $\mathscr{P}(\phi)_2$ quantum field model was mapped out. At each point in the diagram a number of stable phases q_0 were found such that

$$\frac{Z(\mathbb{V}^q)}{Z(\mathbb{V}^{q_0})} \leqq e^{2\lambda^{1/2}|\partial\mathbb{V}|}$$

for every q. We now use this information with some other Part I machinery to give a cluster expansion for the Schwinger functions in the stable phases. We also prove the convergence estimates needed in Parts I and II. The reader is referred to the list of references in Part I.

4. An Expansion for the Schwinger Functions

4.1. Constrained Expansions

In this chapter we derive a convergent expansion for the Schwinger functions from bounds on ratios of partition functions. The presence of clusters containing field monomials introduces constraints on partition function sums. The constraints must be handled in such a way that the phase structure of Chap. 3 is not destroyed.

So far we have always multiplied clusters by ratios of interior partition functions. This procedure must be altered for clusters surrounding squares containing field monomials. In [20], *a priori* bounds on ratios of partition functions in non-simply-connected regions were available. Thus it was possible to multiply $\tilde{\varrho}(\mathbb{Y})$ by a ratio of partition functions in $(\mathbb{I}\mathrm{nt}\,\mathbb{Y})\backslash\mathbb{X}$, where $\mathbb{X}$ is a cluster containing field monomials.

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In order to avoid an uncontrollable accumulation of surface effects on $\mathbb{X}$, we must alter the procedure of [20] by considering partition function sums with the constraint that no cluster shall surround $\mathbb{X}$. If one fixes all the clusters surrounding $\mathbb{X}$, then the resummation between the clusters yields partition function sums of this type. (This procedure was not necessary in [20] because surface effects were always favorable.)

We bound ratios of constrained partition functions in Sect. 4.2, using some of the tools of Chap. 3. The ratios transform the expansion into an explicit sum over clusters surrounding $\mathbb{X}$'s and a constrained sum over other clusters. As the constraints are ultimately connected with the presence of $\mathbb{X}$'s, the techniques of Sect. 3.2 can be applied to factor out the normalization $\int e^{-V_q} d\mu_{m_q^2}(\psi_q)$ exactly. The result is the expansion for $\langle R\rangle_{\Lambda,q}$.

4.2. More General Ratios of Partition Functions

This section is devoted to obtaining bounds on ratios of constrained partition functions in regions that are not simply connected. The constraint will be that all clusters $\mathbb{Y}$ contributing to the partition function in a region $\mathbb{V}$ must have $\mathbb{Y} \cup \mathrm{Int}\,\mathbb{Y} \subseteqq \mathbb{V}$. When $\mathbb{V}$ is not simply connected, this is nontrivial constraint.

The phase structure of the theory has already been determined from the considerations of Chap. 3. Thus, our task will be to show that the constraints produce at most surface effects, so that convergence factors from boundary clusters can control the expansion.

We assume a solution to the equation $L = \mathcal{N}(L)$ and we use the associated objects F_L, $s(F_L^q)$, and $a^q(F_L)$. The shorthand notation F, s^q, and a^q will be used. We hold to the convention of Chap. 3 that $|\mathbb{Y}| > 1$ for all clusters $\mathbb{Y}$. In this section $\mathbb{V}$ will denote a connected, but not necessarily simply connected, region with boundary condition $p(\mathbb{V})$. The boundary condition is the same on all boundary loops of $\mathbb{V}$, including interior boundaries. Define constrained partition functions in $\mathbb{V}$ as follows:

$$\Omega_c^a(F, \mathbb{V}) = \sum_{\substack{\{\mathbb{Y}_s\} : p(\mathbb{Y}_s) = p(\mathbb{V}) \\ \mathbb{Y}_s \cup \mathrm{Int}\, \mathbb{Y}_s \subseteqq \mathbb{V}}} \prod_s F(\mathbb{Y}_s) e^{a^{p(\mathbb{V})} |\bigcup_s \mathbb{Y}_s^{\mathrm{int}}|}, \tag{4.2.1}$$

$$\Omega_c(F, \mathbb{V}) = \sum_{\substack{\{\mathbb{Y}_s\} : p(\mathbb{Y}_s) = p(\mathbb{V}) \\ \mathbb{Y}_s \cup \mathrm{Int}\, \mathbb{Y}_s \subseteqq \mathbb{V}}} \prod_s F(\mathbb{Y}_s). \tag{4.2.2}$$

Here $p(\mathbb{Y}_s)$ denotes the external boundary condition of $\mathbb{Y}_s$. The $\mathbb{Y}_s$'s are nonoverlapping. When $\mathbb{V}$ is simply connected, this definition agrees with (3.4.7), so that $\Omega_c^{(a)}(F, \mathbb{V}) = \Omega^{(a)}(F, \mathbb{V})$.

Proposition 4.2.1. *Suppose $\lambda \ll 1 \ll l$. The following expressions for $\Omega_c^{(a)}(F, \mathbb{V})$ are valid:*

$$\Omega_c^a(F, \mathbb{V}) = \frac{\sum_{\Sigma \cap \mathbb{V} : \partial\mathbb{V} \cap \bigcup_i R_i = \emptyset} Z_\Sigma(\mathbb{V})}{\prod_{\Delta \subseteqq \mathbb{V}} Z_{\Delta p(\mathbb{V})}} \tag{4.2.3}$$

$$\Omega_c(F, \mathbb{V}) = \frac{\sum_{\Sigma \cap \mathbb{V} : \partial\mathbb{V} \cap \bigcup_i R_i = \emptyset} Z_\Sigma(\mathbb{V}) e^{-a^{p(\mathbb{V})} |\bigcup_i R_i|}}{\prod_{\Delta \subseteqq \mathbb{V}} Z_{\Delta p(\mathbb{V})}}. \tag{4.2.4}$$

Here $\{R_i\}$ are the regions associated with $\Sigma \cap \mathbb{V}$. In addition, the following bounds hold:

$$\frac{Z_{\Sigma\equiv p(\mathbb{V})}(\mathbb{V})}{\prod\limits_{\Delta \subseteqq \mathbb{V}} Z_{\Delta p(\mathbb{V})}} \leqq \Omega_c^a(F,\mathbb{V}) \leqq \Omega_c(F,\mathbb{V}) e^{a^{p(\mathbb{V})}|\mathbb{V}|}$$

$$= e^{(a^{p(\mathbb{V})}+s^{p(\mathbb{V})})|\mathbb{V}|} e^{\Delta_c(F,\mathbb{V})} \tag{4.2.5}$$

$$|\Delta_c(F,\mathbb{V})| \leqq 2\lambda^{1/2}|\partial\mathbb{V}|\,. \tag{4.2.6}$$

Proof. Recall from (3.4.1) and (3.4.10)–(3.4.12) that

$$F(\mathbb{Y}) = \tilde{\varrho}(\mathbb{Y}) \prod_{\mathbb{V} \subseteqq \mathrm{Int}\,\mathbb{Y}} \left[\frac{\Omega^a(F,\mathbb{V})}{\Omega(F,\mathbb{V})} e^{(-a^{p(\mathbb{V})} + \log Z_{\Delta p(\mathbb{V})} - \log Z_{\Delta p(\mathbb{Y})})|\mathbb{V}|}\right]$$

$$\cdot \prod_i e^{(-a^{p(\mathbb{Y})} + \log Z_{\Delta p(R_i)} - \log Z_{\Delta p(\mathbb{Y})})|R_i \cap \mathbb{Y}|}, \tag{4.2.7}$$

where $\{R_i\}$ are the regions associated with $\Sigma_{\mathbb{Y}\mathrm{int}}$. Call a $\mathbb{Y}_s$ outer if it is not contained in any $\mathrm{Int}\,\mathbb{Y}_{s'}$. With the outer clusters in (4.2.1) fixed, the sums over the others produce a factor $\prod\limits_{\mathbb{W}} \Omega(F,\mathbb{W})$, where $\mathbb{W}$ runs over the components of $\mathrm{Int}\,\mathbb{Y}_s$, $\mathbb{Y}_s$ outer. We obtain

$$\Omega_c^a(F,\mathbb{V}) = \sum_{\{\mathbb{Y}_s\}\text{ outer}} \prod_s \left[\tilde{\varrho}(\mathbb{Y}_s) \prod_i e^{(\log Z_{\Delta p(R_i)} - \log Z_{\Delta p(\mathbb{V})})|R_i \cap \mathbb{Y}_s|}\right]$$

$$\cdot \prod_{\mathbb{W}} \left[\Omega^a(F,\mathbb{W}) \prod_{\Delta \subseteqq \mathbb{W}} \frac{Z_{\Delta p(\mathbb{W})}}{Z_{\Delta p(\mathbb{V})}}\right] e^{-a^{p(\mathbb{V})}\left|\bigcup\limits_i R_i\right|} e^{a^{p(\mathbb{V})}\left|\bigcup\limits_s \mathbb{Y}_s^{\mathrm{int}}\right|}. \tag{4.2.8}$$

The last two factors cancel, and each term in $\prod\limits_s$ is equal to

$$\varrho(\mathbb{Y}_s) e^{(E_c^{p(\mathbb{Y}_s)} - E_w^{p(\mathbb{Y}_s)})l^2|\mathbb{Y}_s \cup \mathrm{Int}\,\mathbb{Y}_s|} \prod_{\mathbb{W} \subseteqq \mathrm{Int}\,\mathbb{Y}_s} e^{(-E_c^{p(\mathbb{W})} + E_w^{p(\mathbb{W})})l^2|\mathbb{W}|} \prod_{\Delta \subseteqq \mathbb{Y}_s} (Z_{\Delta p(\mathbb{V})})^{-1}. \tag{4.2.9}$$

Moreover, by (3.5.2), $\Omega^a(F,\mathbb{W}) = Z(\mathbb{W}) \Big/ \prod\limits_{\Delta \subseteqq \mathbb{W}} Z_{\Delta p(\mathbb{W})}$ so that

$$\Omega_c^a(F,\mathbb{V}) = \sum_{\{\mathbb{Y}_s\}\text{ outer}} \prod_s \left[\frac{\varrho(\mathbb{Y}_s) e^{(E_c^{p(\mathbb{Y}_s)} - E_w^{p(\mathbb{Y}_s)})l^2|\mathbb{Y}_s \cup \mathrm{Int}\,\mathbb{Y}_s|}}{\prod\limits_{\Delta \subseteqq \mathbb{Y}_s \cup \mathrm{Int}\,\mathbb{Y}_s} Z_{\Delta p(\mathbb{V})}}\right]$$

$$\cdot \prod_{\mathbb{W}} e^{(-E_c^{p(\mathbb{W})} + E_w^{p(\mathbb{W})})l^2|\mathbb{W}|} Z(\mathbb{W})\,. \tag{4.2.10}$$

Expand $Z(\mathbb{W})$ into spin configurations. In the resulting sum over $\{\mathbb{Y}_s\}$ and $\{\Sigma_{\mathbb{W}}\}$, fix $\Sigma = \bigcup\limits_s \Sigma_{\mathbb{Y}_s} \cup \bigcup\limits_{\mathbb{W}} \Sigma_{\mathbb{W}}$ and sum over $\mathbb{Y}_s$'s compatible with Σ. If Σ is such that $\partial\mathbb{V} \cap \bigcup\limits_i R_i \neq \emptyset$, then there is no compatible $\{\mathbb{Y}_s\}$: If there were, then some $\mathbb{Y}_s$ would have $\mathrm{Int}\,\mathbb{Y}_s \nsubseteq \mathbb{V}$, contrary to construction. (Recall that $\bigcup\limits_i R_i$ does not contain the sea of constant phase at the outer boundary of $\mathbb{V}$. If interior boundaries of $\mathbb{V}$ are in other seas, then some clusters would have to effect the transition, resulting in some $\mathrm{Int}\,\mathbb{Y}_s \nsubseteq \mathbb{V}$.) If Σ is such that $\partial\mathbb{V} \cap \bigcup\limits_i R_i = \emptyset$, then the restriction $\mathrm{Int}\,\mathbb{Y}_s \subseteqq \mathbb{V}$ is vacuous.

Thus

$$\Omega_c^a(F,\mathbb{V})=\sum_{\Sigma:\partial\mathbb{V}\cap\bigcup_i R_i=\emptyset}\ \sum_{\substack{\{\mathbb{Y}_s\}\text{ outer}\\ \text{compatible with }\Sigma}}$$

$$\cdot\prod_s\left[\frac{\varrho(\mathbb{Y}_s)e^{(E_c^{p(\mathbb{Y}_s)}-E_w^{p(\mathbb{Y}_s)})l^2|\mathbb{Y}_s\cup\mathbb{I}\mathrm{nt}\,\mathbb{Y}_s|}}{\prod\limits_{\Delta\subseteqq\mathbb{Y}_s\cup\mathbb{I}\mathrm{nt}\,\mathbb{Y}_s}Z_{\Delta^{p(\mathbb{V})}}}\prod_{\mathbb{W}\subseteqq\mathbb{I}\mathrm{nt}\,\mathbb{Y}_s}[e^{(-E_c^{p(\mathbb{W})}+E_w^{p(\mathbb{W})})l^2|\mathbb{W}|}Z_{\Sigma\cap\mathbb{W}}(\mathbb{W})]\right].\tag{4.2.11}$$

Equation (3.4.33) can now be applied to yield (4.2.3).

The same set of manipulations can be performed for $\Omega_c(F,\mathbb{V})$, except that the factor $e^{-a^{p(\mathbb{V})}|\bigcup_i R_i|}$ in (4.2.8) is not cancelled. Since $\left|\bigcup_i R_i\right|$ is independent of $\Sigma\cap\mathbb{W}$, the factor agrees with the one in (4.2.4). This completes the proof of (4.2.3) and (4.2.4).

The first two inequalities in (4.2.5) are immediate consequences of (4.2.3) and (4.2.4). The last step defines $\Delta_c(F,\mathbb{V})$. As in (3.3.7), we have

$$\log\Omega_c(F,\mathbb{V})=\sum_k\frac{1}{k!}\sum_{\substack{(\mathbb{Y}_1,\ldots,\mathbb{Y}_k)\\ \mathbb{Y}_s\cup\mathbb{I}\mathrm{nt}\,\mathbb{Y}_s\subseteqq\mathbb{V}}}\sum_{G_c}\prod_{\mathscr{L}\in G_c}A(\mathscr{L})\prod_{s=1}^k F(\mathbb{Y}_s).\tag{4.2.12}$$

We can obtain an expansion for $\Delta_c(F,\mathbb{V})$ from this formula and (3.3.8), as in the derivation of (3.3.9):

$$\Delta_c(F,\mathbb{V})\equiv\log\Omega_c(F,\mathbb{V})-s^{p(\mathbb{V})}|\mathbb{V}|=\sum_k\frac{1}{k!}\sum_{\substack{(\mathbb{Y}_1,\ldots,\mathbb{Y}_k)\\ \mathbb{V}\cap\bigcup_s\mathbb{Y}_s\neq\emptyset\neq\sim\mathbb{V}\cap\bigcup_s(\mathbb{Y}_s\cup\mathbb{I}\mathrm{nt}\,\mathbb{Y}_s)}}$$

$$-\frac{\left|\mathbb{V}\cap\bigcup_s\mathbb{Y}_s\right|}{\left|\bigcup_s\mathbb{Y}_s\right|}\sum_{G_c}\prod_{\mathscr{L}\in G_c}A(\mathscr{L})\prod_{s=1}^k F(\mathbb{Y}_s).\tag{4.2.13}$$

Apply Lemma 3.3.1 to every boundary square of $\mathbb{V}$ to bound sums over clusters intersecting both $\mathbb{V}$ and $\sim\mathbb{V}$. With $\|F\|\leqq\lambda^{1/2}$, we obtain a term $\lambda^{1/2}|\partial\mathbb{V}|$, as in (3.3.12). It remains for us to bound the sum over clusters contained in $\mathbb{V}$ with $\bigcup_s\mathbb{I}\mathrm{nt}\,\mathbb{Y}_s\nsubseteqq\mathbb{V}$. Single out one such cluster:

$$|\Delta_c(F,\mathbb{V})|\leqq\lambda^{1/2}|\partial\mathbb{V}|+\sum_k k\sum_{\mathbb{Y}\subseteqq\mathbb{V}:\mathbb{I}\mathrm{nt}\,\mathbb{Y}\nsubseteqq\mathbb{V}}|F(\mathbb{Y})|\frac{1}{k!}\sum_{(\mathbb{Y}_1,\ldots,\mathbb{Y}_{k-1})}$$

$$\cdot\left|\sum_{G_c}\prod_{\mathscr{L}\in G_c}A(\mathscr{L})\prod_{s=1}^{k-1}F(\mathbb{Y}_s)\right|.\tag{4.2.14}$$

Another lemma is needed to control this sum.

Lemma 4.2.2. *Suppose $\lambda\ll1\ll l$. If F^q is any q-contour model with $\|F^q\|\leqq1$, then for any $\mathbb{Y}$,*

$$\sum_{\substack{(\mathbb{Y}_1,\ldots,\mathbb{Y}_k)\\ \sum_s|\mathbb{Y}_s|=N,\ p(\mathbb{Y})=q}}\left|\sum_{G_c}\prod_{\mathscr{L}\in G_c}A(\mathscr{L})\prod_{s=1}^k F^q(\mathbb{Y}_s)\right|\leqq k!\,e^{-\tau_1 l(k+N)/4}e^{|\mathbb{Y}|}.\tag{4.2.15}$$

Here G_c is a connected graph involving $\mathbb{Y}$ and all $\mathbb{Y}_s$'s.

The lemma will be proved in Sect. 4.4. It yields the bound

$$|\Delta_c(F,\mathbb{V})| \leqq \lambda^{1/2}|\partial\mathbb{V}| + \sum_{k\geqq 1}\sum_{\mathbb{Y}} |F(\mathbb{Y})| e^{-\tau_1 l(k-1)/4} e^{|\mathbb{Y}|}. \tag{4.2.16}$$

We fix Y, the region in $\mathbb{R}^2$ covered by $\mathbb{Y}$, and sum over $\Gamma\cap\mathbb{Y}$ and $\Sigma_{\mathbb{Y}}$. The first sum produces a factor $2^{2|Y|}$, and the second a factor

$$\sum_{|\Sigma|=0}^{2l^2|Y|} O(1)^{|\Sigma|}\binom{2l^2|Y|}{|\Sigma|} e^{-\tau_2\lambda^{-2}|\Sigma|} \leqq (1+e^{-\tau_2\lambda^{-2}/2})^{2l^2|Y|} \leqq e^{\lambda|Y|}. \tag{4.2.17}$$

The number of Y's with $|Y|=N$ surrounding a given component of $\sim\mathbb{V}$ is less than $e^{O(1)|Y|}$. Applying $\|F\|\leqq\lambda^{1/2}$, we obtain

$$|\Delta_c(F,\mathbb{V})| \leqq |\partial\mathbb{V}|\left(\lambda^{1/2} + 2\sum_{N\geqq 8} e^{O(1)N} e^{-\tau_1 lN}\lambda^{1/2}\right) \leqq 2\lambda^{1/2}|\partial\mathbb{V}|. \tag{4.2.18}$$

Proposition 4.2.1 is proven. □

4.3. Exponential Clustering and Asymptoticity of the Perturbation Series

We are now prepared to give a convergent cluster expansion for the Schwinger functions, with bounds independent of the interaction volume Λ. We must use boundary conditions corresponding to a stable phase q, that is, we must have $a^q=0$. In Chap. 3 and in Sect. 4.2 we have always taken Λ to be larger than $\mathbb{Z}$ to define $\varrho(\mathbb{Z})$ and $\tilde{\varrho}(\mathbb{Z})$. We now fix Λ and return to the original objects $\varrho_{\Lambda,q}(\mathbb{Z})$. We no longer require $|\mathbb{Y}|>1$, except where specifically indicated. $\varrho_{\Lambda,q}(\mathbb{Z})$ is not invariant under translations of $\mathbb{Z}$. We will eventually find a formula for the infinite volume Schwinger functions that involves $\varrho(\mathbb{Z})$ and $\tilde{\varrho}(\mathbb{Z})$, with no subscripts Λ, q.

A standard argument [6, 19] expresses a Schwinger function as its first several orders of perturbation theory plus a remainder. The remainder is a power of λ times other (generalized) Schwinger functions. Thus bounds on the generalized Schwinger functions yield asymptoticity of perturbation theory. Asymptoticity immediately implies that the phases we construct at the coexistence hypersurfaces are all distinct.

Normally, the argument shows for example that for all μ and all $\lambda\in[0,\lambda_0]$, $|S(\lambda)-S(0)|=O(\lambda)$. We obtain a slightly weaker result, due to phase transitions which will occur as λ is varied. We prove that if q is one of the stable phases at λ, then $S(q,\lambda)$ differs from the first n orders of perturbation theory about the q^{th} minimum by $O(\lambda^{n+1})$. C^∞ properties will not in general be uniform in parameter space. If we choose μ near a classical phase transition hypersurface $\left(\text{i.e. } E_c^q(\mu,\lambda=1) - \inf_q E_c^q(\mu,\lambda=1)\ll 1 \text{ for more than one } q\right)$ then one must take λ exceptionally small to get the theory into the $\lambda=0$ phase. Reparametrizing the interaction to avoid phase transitions as $\lambda\to 0$ would not solve the problem because we only know Lipschitz continuity of the phase transition hypersurfaces. By using the perturbation expansion for the vacuum energies, we could give a description of the phase diagram that is much more precise than the one in Theorem 3.7.2. However, it seems that Lipschitz continuity is an intrinsic limitation of the construction, at least at the $e^{-O(\lambda^{-2})}$ level. Whether this is a real effect or an artifact of the construction is an interesting open question, even for lattice systems.

We use the field $\psi_q = \phi - \xi_q$ to generate the perturbation series. We have

$$\mathscr{P}_{\lambda,\mu}(\psi_q + \xi_q) - \frac{m_q^2}{2}\psi_q^2 - \mathscr{P}_{\lambda,\mu}(\xi_q) = \sum_{n=3}^{\deg\mathscr{P}} a_{n,q}\lambda^{n-2}\psi_q^n,$$

so that all coefficients in the interaction

$$V_q = \int_\Lambda \left[:\mathscr{P}_{\lambda,\mu}(\psi_q(x) + \xi_q): - \frac{m_q^2}{2} :\psi_q(x)^2: - \mathscr{P}_{\lambda,\mu}(\xi_q) \right] dx \tag{4.3.1}$$

are $O(\lambda)$ or smaller. We defined the finite volume measure to be $e^{-V_q} d\mu_{m_q^2}(\psi_q)$.

We give an asymptotic expansion for

$$\langle R \rangle_{\Lambda,q} = \int \mathrm{R} e^{-V_q} d\mu_{m_q^2}(\psi_q) / \int e^{-V_q} d\mu_{m_q^2}(\psi_q).$$

The R's we consider have the form

$$R = \int w(x) \prod_{i=1}^n :\psi_q(x_i)^{p_i}: dx, \tag{4.3.2}$$

where $w(x_1, \ldots, x_n) \in L^p\left(\prod_{i=1}^n \Delta_i\right)$ is supported in a product of l-lattice squares, and $p > 1$ is fixed.

In the integration by parts formula

$$\begin{aligned} &\int :\psi_q(x)^m: R(\psi_q) e^{-V_q} d\mu_{m_q^2}(\psi_q) \\ &= \int\int dy C_{m_q^2}(x-y) \left[:\psi_q(x)^{m-1}: \frac{\delta R}{\delta\psi_q(y)} - R\frac{\delta V_q}{\delta\psi_q(y)} \right] e^{-V_q} d\mu_{m_q^2}(\psi_q), \end{aligned} \tag{4.3.3}$$

we have

$$\frac{\delta V_q}{\delta\psi_q(x)} = \sum_{n=3}^{\deg\mathscr{P}} a_n \lambda^{n-2} n :\psi_q(x)^{n-1}:, \tag{4.3.4}$$

so that each derivative of V produces factors of λ. We integrate by parts each factor of ψ in (4.3.2), and the factors of $\frac{\delta V_q}{\delta\psi_q}$ that result. We continue inductively until all terms either have the form $\int(\mathrm{const}) e^{-V_q} d\mu_{m_q^2}(\psi_q)$, or else they have an explicit factor of λ^{n+1}. Dividing by $\int e^{-V_q} d\mu_{m_q^2}(\psi_q)$, we obtain

$$\langle R \rangle_{\Lambda,q} = \sum_{j=0}^n \alpha_j \lambda^j + \lambda^{n+1} \sum_k \langle R_k \rangle_{\Lambda,q}. \tag{4.3.5}$$

The α_j's form the usual perturbation series for $\langle R \rangle$ through order n. The other terms are the remainder, and the R_k's have the form (4.3.2).

Theorem 4.3.1. *Let $p > 1$ be given and suppose $a^q = 0$. For $\lambda \ll 1 \ll l$, there exist positive constants K, τ_2 depending on p, η, C such that for R of the form* (4.3.2),

$$|\langle R \rangle_{\Lambda,q}| \leqq \|w\|_{L_p} \prod_\Delta (N(\Delta)!)^{1/2} e^{Kl \deg R} (1 + \lambda^{-\deg R} e^{-\tau_2 \lambda^{-2}}). \tag{4.3.6}$$

Here $N(\Delta)$ is the degree of R in l-lattice square Δ.

The theorem bounds the remainder in (4.3.5) by $O(\lambda^{n+1})$. Note that l does not diverge with λ as in [19]. The factor $\lambda^{-\deg R} e^{-\tau_2 \lambda^{-2}}$ in (4.3.6) arises from the possibility of fluctuations into minima other than q. Since $|\xi_{q_1} - \xi_{q_2}| = O(\lambda^{-1})$, we have contributions of size $\lambda^{-\deg R}$, but with small probability $e^{-\tau_2 \lambda^{-2}}$.

The theorem yields bounds uniform in Λ, so that when we take Λ to infinity we will obtain asymptoticity of perturbation theory for the infinite volume Schwinger functions. In particular, we will have

$$\langle \phi \rangle_q = \xi_q + O(\lambda), \tag{4.3.7}$$

where $\langle \cdot \rangle_q$ denotes the infinite volume expectation obtained as a limit of expectations with boundary condition q. This distinguishes the different states that we construct on coexistence hypersurfaces.

Proof of Theorem 4.3.1. Perform the mean field expansion on $\int R e^{-V_q} d\mu_{m_q^2}(\psi_q) \equiv F_R$ (see Chap. 2). We defer integration against the test function w and for the moment take R to be a product of factors $:\psi_q(x_i)^{p_i}:$.

$$F_R = \sum_{\substack{\{\mathbb{Z}_\kappa\} \text{ nonoverlapping, filling } \mathbb{R}^2 \\ \text{agreeing on common boundaries, } \Sigma_{\mathbb{Z}_\kappa} = q \text{ in } Z_\kappa \setminus \Lambda \\ \text{only finitely many } \mathbb{Z}_\kappa \text{ have } |\mathbb{Z}_\kappa| > 1}} \prod_\kappa \varrho_{\Lambda,q}(\mathbb{Z}_\kappa). \tag{4.3.8}$$

We make some notational conventions. Clusters that contain field monomials will be denoted with the letter $\mathbb{X}$ and called nonvacuum clusters. Other clusters will be denoted with the letter $\mathbb{Y}$ and called vacuum clusters. There are two types of $\mathbb{Y}$'s: those such that $\mathbb{X} \subseteqq \mathbb{I}\mathrm{nt}\mathbb{Y}$ for some $\mathbb{X}$, and those such that no $\mathbb{X}$ is contained in $\mathbb{I}\mathrm{nt}\mathbb{Y}$. The first type will be denoted $\bar{\mathbb{Y}}$, the second simply $\mathbb{Y}$. We shall have occasion to extend sums over $\mathbb{Y}$'s to clusters overlapping or surrounding $\mathbb{X}$'s or $\mathbb{Y}$'s. When this happens, $\varrho_{\Lambda,q}(\mathbb{Y})$ will be defined as in the expansion for the partition function $F_{R=1}$. Thus $\varrho_{\Lambda,q}(\mathbb{Y})$ is independent of R. In this respect it differs from $\varrho_{\Lambda,q}(\mathbb{X})$, which does contain monomials from R. We use the letter $\mathbb{T}$ to denote either an $\mathbb{X}$ or a $\bar{\mathbb{Y}}$. The letter $\mathbb{Z}$ will be used for all types of clusters – $\mathbb{X}$, $\bar{\mathbb{Y}}$ or $\mathbb{Y}$.

In (4.3.8), fix all $\mathbb{X}$'s, $\bar{\mathbb{Y}}$'s, and all $\mathbb{Y}$'s such that $\mathbb{Y} \nsubseteqq \mathbb{I}\mathrm{nt}\mathbb{Z}_\kappa$ for any κ. The external boundary condition of the $\mathbb{Y}$'s is q, because $\Sigma = q$ in $\sim \Lambda$. Let $\mathbb{V}_\alpha$ be the components of the region complementary to all the fixed clusters. At this stage, clusters can have $|\mathbb{Z}| = 1$, so exterior regions are completely filled with fixed clusters. Resumming the expansion inside each $\mathbb{V}_\alpha$ yields a partition function in case $\mathbb{V}_\alpha$ is simply connected, and a constrained partition function if not. If $\mathbb{V}_\alpha \subseteqq \mathbb{I}\mathrm{nt}\mathbb{Y}$, then $\mathbb{V}_\alpha$ is simply connected. The result of resummation is

$$\begin{aligned} F_R = & \sum_{\substack{\{\mathbb{X}_r, \bar{\mathbb{Y}}_t, \mathbb{Y}_s\}, \mathbb{Y}_s \nsubseteqq \mathbb{I}\mathrm{nt}\mathbb{Z} \\ p(\mathbb{Y}_s) = q}} \prod_r \varrho_{\Lambda,q}(\mathbb{X}_r) \prod_t \varrho_{\Lambda,q}(\bar{\mathbb{Y}}_t) \\ & \cdot \prod_s \left[\varrho_{\Lambda,q}(\mathbb{Y}_s) \prod_{\mathbb{V}_\alpha \subseteqq \mathbb{I}\mathrm{nt}\mathbb{Y}_s} [Z(\mathbb{V}_\alpha) e^{(E_c^q - E_c^{p(\mathbb{V}_\alpha)} + E_w^{p(\mathbb{V}_\alpha)})|\mathbb{V}_\alpha| l^2}] \right] \\ & \cdot \prod_{\mathbb{V}_\alpha \nsubseteqq \bigcup_s \mathbb{I}\mathrm{nt}\mathbb{Y}_s} \left[\sum_{\Sigma \cap \mathbb{V}_\alpha : \bigcup_i R_i \cap \partial \mathbb{V}_\alpha = \emptyset} Z_\Sigma(\mathbb{V}_\alpha) e^{(E_c^q - E_c^{p(\mathbb{V}_\alpha)} + E_w^{p(\mathbb{V}_\alpha)})|\mathbb{V}_\alpha| l^2} \right]. \end{aligned} \tag{4.3.9}$$

Note that $\mathbb{V}_\alpha \subseteqq \Lambda$ if $\mathbb{V}_\alpha \subseteqq \mathbb{I}\mathrm{nt}\mathbb{Y}_s$. The $\mathbb{Y}_s$'s cannot surround $\mathbb{X}$'s and $\bar{\mathbb{Y}}$'s, and the outermost $\mathbb{X}$'s and $\bar{\mathbb{Y}}$'s have external boundary condition q.

Divide F_R by $\prod_{\Delta \subseteq \Lambda} (Z_{\Delta^q} e^{E^q_w l^2})$ and denote the result by $\tilde{F}_R$. This cancels all $\varrho_{\Lambda,q}(\mathbb{Y})$'s with $|\mathbb{Y}|=1$ in (4.3.9). [Recall that $\varrho_{\Lambda,q}(\Delta)=1$ for $\Delta \nsubseteq \Lambda$.] It also cancels the factors $e^{E^q_c|\mathbb{V}|l^2}$. By (3.5.2) and (4.2.3),

$$Z(\mathbb{V}_\alpha)=\Omega^a(F,\mathbb{V}_\alpha)\prod_{\Delta \subseteqq \mathbb{V}_\alpha} Z_{\Delta^{p(\mathbb{V}_\alpha)}}$$

and

$$\sum_{\Sigma \cap \mathbb{V}:\, \bigcup_i R_i \cap \partial \mathbb{V}=\emptyset} Z_\Sigma(\mathbb{V}_\alpha)=\Omega^a_c(F,\mathbb{V}_\alpha)\prod_{\Delta \subseteqq \mathbb{V}_\alpha} Z_{\Delta^{p(\mathbb{V}_\alpha)}}\,.$$

We substitute in $\tilde{F}_R$

$$\begin{aligned}&\varrho_{\Lambda,q}(\mathbb{Z})(Z_{\Delta^q}e^{E^q_w l^2})^{-|\mathbb{Z}\cap\Lambda|}\\ &=\tilde{\varrho}_{\Lambda,q}(\mathbb{Z})e^{(E^q_c-E^q_w-E^{p(\mathbb{Z})}_c+E^{p(\mathbb{Z})}_w)l^2|\mathbb{Z}\cap\Lambda|}e^{(\log Z_{\Delta^{p(R_i)}}-\log Z_{\Delta^q})|R_i\cap\mathbb{Z}|}\\ &\quad\cdot\prod_m e^{(E^m_c-E^m_w-E^{p(\mathbb{Z})}_c+E^{p(\mathbb{Z})}_w)l^2|\mathrm{Int}_m\mathbb{Z}|}\,,\end{aligned}$$

where the regions R_i are those associated with the spin configuration that agrees with $\{\mathbb{X}_r, \bar{\mathbb{Y}}_t, \mathbb{Y}_s\}$ and that is constant in each $\mathbb{V}_\alpha$. Observe that each $e^{(E^m_c-E^m_w)l^2|\mathrm{Int}_m\mathbb{Z}|}$ factor cancels with

$$e^{(-E^{p(\bar{\mathbb{Y}})}_c+E^{p(\bar{\mathbb{Y}})}_w)l^2(|\bar{\mathbb{Y}}\cap\Lambda|+|\mathrm{Int}\,\bar{\mathbb{Y}}|)}$$

from outermost $\bar{\mathbb{Y}}$'s in $\mathrm{Int}_m\mathbb{Z}$ or with $e^{(-E^{p(\mathbb{V}_\alpha)}_c+E^{p(\mathbb{V}_\alpha)}_w)l^2|\mathbb{V}_\alpha|}$ from outermost $\mathbb{V}_\alpha$'s in $\mathrm{Int}_m\mathbb{Z}$. The

$$e^{(-E^{p(\mathbb{Z})}_c+E^{p(\mathbb{Z})}_w)l^2(|\mathbb{Z}\cap\Lambda|+|\mathrm{Int}\,\mathbb{Z}|)}$$

factors for $\mathbb{Z}$'s that are not $\bar{\mathbb{Y}}$'s cancel the $e^{(E^q_c-E^q_w)l^2|\mathbb{Z}\cap\Lambda|}$ factors, the $e^{E^q_c l^2|\mathbb{V}_\alpha|}$ factors, and the $e^{-E^q_w l^2}$ factors from $(Z_{\Delta^q}e^{E^q_w l^2})^{-1}$, $\Delta \subseteqq \mathbb{V}_\alpha$. All of the energy factors have disappeared, and (4.3.9) becomes

$$\begin{aligned}\tilde{F}_R=&\sum_{\{\mathbb{X}_r,\bar{\mathbb{Y}}_t,\mathbb{Y}_s\}}\prod_r\tilde{\varrho}_{\Lambda,q}(\mathbb{X}_r)\prod_t\tilde{\varrho}_{\Lambda,q}(\bar{\mathbb{Y}}_t)\\ &\cdot\prod_{s:|\mathbb{Y}_s|>1}\left[\tilde{\varrho}_{\Lambda,q}(\mathbb{Y}_s)\prod_{\mathbb{V}_\alpha\subseteqq\mathrm{Int}\,\mathbb{Y}_s}\Omega^a(F,\mathbb{V}_\alpha)\right]\\ &\cdot\prod_{\mathbb{V}_\alpha\nsubseteq\bigcup_s\mathrm{Int}\,\mathbb{Y}_s}\Omega^q_c(F,\mathbb{V}_\alpha)\prod_i e^{(\log Z_{\Delta^{p(R_i)}}-\log Z_{\Delta^q})|R_i|}\,.\end{aligned}\tag{4.3.10}$$

Define

$$F_\Lambda(\mathbb{Y})=\tilde{\varrho}_{\Lambda,q}(\mathbb{Y})e^{L(\mathbb{Y})}\tag{4.3.11}$$

so that by (3.4.10)–(3.4.12) and $a^q=0$,

$$\begin{aligned}&\tilde{\varrho}_{\Lambda,q}(\mathbb{Y})\prod_{\mathbb{V}_\alpha\subseteqq\mathrm{Int}\,\mathbb{Y}}\Omega^a(F,\mathbb{V}_\alpha)\prod_{i:R_i\cap\mathbb{Y}\neq\emptyset}e^{(\log Z_{\Delta^{p(R_i)}}-\log Z_{\Delta^q})|R_i|}\\ &=F_\Lambda(\mathbb{Y})\prod_{\mathbb{V}_\alpha\subseteqq\mathrm{Int}\,\mathbb{Y}}\Omega(F,\mathbb{V}^q_\alpha)\\ &=F_\Lambda(\mathbb{Y})\sum_{\substack{\{\mathbb{Y}_s\}:\,\mathbb{Y}_s\subseteqq\mathrm{Int}\,\mathbb{Y}\\ p(\mathbb{Y}_s)=q,\,|\mathbb{Y}_s|>1}}\prod_s F_\Lambda(\mathbb{Y}_s)\,.\end{aligned}\tag{4.3.12}$$

For interior $\mathbb{Y}$'s, F and F_Λ are the same. Multiply and divide by $\Omega_c(F,\mathbb{V}_\alpha)$ for $\mathbb{V}_\alpha \subsetneqq \bigcup_s \mathrm{Int}\,\mathbb{Y}_s$ to obtain

$$\tilde{F}_R = \sum_{\{\mathbb{X}_r,\,\bar{\mathbb{Y}}_t\}} \sum_{\substack{\{\mathbb{Y}_s\}:\,\mathbb{Y}_s \cup \mathrm{Int}\,\mathbb{Y}_s \subseteqq \mathbb{R}^2 \setminus \bigcup_r \mathbb{X}_r \setminus \bigcup_t \bar{\mathbb{Y}}^t \\ p(\mathbb{Y}_s)=q,\, |\mathbb{Y}_s|>1}} \prod_r \tilde{\varrho}_{\Lambda,q}(\mathbb{X}_r)$$
$$\cdot \prod_t \tilde{\varrho}_{\Lambda,q}(\bar{\mathbb{Y}}_t) \prod_{\mathbb{V}_\alpha \subsetneqq \bigcup_s \mathrm{Int}\,\mathbb{Y}_s} \frac{\Omega_c^q(F,\mathbb{V}_\alpha)}{\Omega_c(F,\mathbb{V}_\alpha^q)}$$
$$\cdot \prod_{i:\,R_i \cap \left(\bigcup_r \mathbb{X}_r \cup \bigcup_t \bar{\mathbb{Y}}_t\right) \neq \emptyset} e^{(\log Z_{\Delta^{p(R_i)}} - \log Z_{\Delta^q})|R_i|} \prod_s F_\Lambda(\mathbb{Y}_s). \qquad (4.3.13)$$

We have expanded $\Omega_c(F,\mathbb{V}_\alpha^q)$ as in (4.2.2), and put all the sums over $\mathbb{Y}_s$'s together.

With $\{\mathbb{T}_u\} = \{\mathbb{X}_r, \bar{\mathbb{Y}}_t\}$, define

$$\Xi_\Lambda^q(\{\mathbb{T}_u\}) = \prod_u \tilde{\varrho}_{\Lambda,q}(\mathbb{T}_u) \prod_{\mathbb{V}_\alpha \subseteqq \bigcup_u \mathrm{Int}\,\mathbb{T}_u \setminus \bigcup_u \mathbb{T}_u} \frac{\Omega_c^q(F,\mathbb{V}_\alpha)}{\Omega_c(F,\mathbb{V}_\alpha^q)}$$
$$\cdot \prod_i e^{(\log Z_{\Delta^{p(R_i)}} - \log Z_{\Delta^q})|R_i|}. \qquad (4.3.14)$$

If not all external $\mathbb{T}$'s have boundary condition q, then $\Xi_\Lambda^q(\{\mathbb{T}_u\})=0$. (External means not contained in $\mathrm{Int}\,\mathbb{T}_u$ for any u.)

We wish to extend the sum over $\{\mathbb{Y}_s\}$ to an unrestricted sum over $k, (\mathbb{Y}_1, \ldots, \mathbb{Y}_k)$ as in Sect. 3.3. The extra terms will be eliminated with projections $U(\mathbb{Z}_1,\mathbb{Z}_2)$. Define

$$U(\mathbb{T},\mathbb{Y}) = \begin{cases} 0 & \text{if } \mathbb{T} \subseteqq \mathrm{Int}\,\mathbb{Y} \text{ or if } \mathbb{T} \text{ and } \mathbb{Y} \text{ overlap} \\ 1 & \text{otherwise}, \end{cases}$$
$$U(\mathbb{Y}_1,\mathbb{Y}_2) = \begin{cases} 0 & \text{if } \mathbb{Y}_1 \text{ and } \mathbb{Y}_2 \text{ overlap} \\ 1 & \text{otherwise}. \end{cases} \qquad (4.3.15)$$

Then

$$\tilde{F}_R = \sum_{\{\mathbb{T}_u\}} \sum_k \frac{1}{k!} \sum_{(\mathbb{Y}_1,\ldots,\mathbb{Y}_k),\, |\mathbb{Y}_s|>1} \Xi_\Lambda^q(\{\mathbb{T}_u\}) \prod_{u,s} U(\mathbb{T}_u,\mathbb{Y}_s)$$
$$\cdot \prod_{s_1<s_2} U(\mathbb{Y}_{s_1},\mathbb{Y}_{s_2}) \prod_{s=1}^k F_\Lambda(\mathbb{Y}_s). \qquad (4.3.16)$$

The sum over $(\mathbb{Y}_1,\ldots,\mathbb{Y}_k)$ is over ordered families of $\mathbb{Y}_s$'s, including overlapping clusters and clusters surrounding or overlapping $\mathbb{T}$'s.

Expand $U=1+A$ as in Sect. 3.3 to obtain

$$\tilde{F}_R = \sum_{\{\mathbb{T}_u\}} \sum_k \frac{1}{k!} \sum_{\substack{(\mathbb{Y}_1,\ldots,\mathbb{Y}_k) \\ |\mathbb{Y}_s|>1}} \sum_G \Xi_\Lambda^q(\{\mathbb{T}_u\}) \prod_{\mathscr{L}\in G} A(\mathscr{L}) \prod_{s=1}^k F_\Lambda(\mathbb{Y}_s). \qquad (4.3.17)$$

G is a graph of unordered pairs (or lines) $\{\mathbb{T}_u,\mathbb{Y}_s\}$ or $\{\mathbb{Y}_{s_1},\mathbb{Y}_{s_2}\}$. Let G_c be the part of G that contains lines connected directly or indirectly to some $\mathbb{T}_u$. Let $G_0 = G\setminus G_c$.

G is said to be connected with respect to $\{\mathbb{X}_r\}$ if $G_0=\emptyset$. We sum separately over the $\mathbb{Y}_s$'s in G_c and the $\mathbb{Y}_s$'s in G_0, using

$$\sum_k \frac{1}{k!} \sum_{(\mathbb{Y}_1,\ldots,\mathbb{Y}_k)} \sum_G = \sum_{k_c} \frac{1}{k_c!} \sum_{(\mathbb{Y}_1',\ldots,\mathbb{Y}_{k_c}')} \sum_{G_c} \sum_{k_0} \frac{1}{k_0!} \sum_{(\mathbb{Y}_1'',\ldots,\mathbb{Y}_{k_0}'')} \sum_{G_0}.$$

Here $\mathbb{Y}_s'$ is a cluster for G_c, and $\mathbb{Y}_s''$ is a cluster for G_0.

$$\tilde{F}_R = \left(\sum_{\{\mathbb{T}_u\}} \sum_{k_c} \frac{1}{k_c!} \sum_{\substack{(\mathbb{Y}_1',\ldots,\mathbb{Y}_{k_c}') \\ |\mathbb{Y}_s|>1}} \sum_{G_c} \Xi_\Lambda^q(\{\mathbb{T}_u\}) \prod_{\mathscr{L}\in G_c} A(\mathscr{L}) \prod_{s=1}^{k_c} F_\Lambda(\mathbb{Y}_s') \right)$$
$$\cdot \left(\sum_{k_0} \frac{1}{k_0!} \sum_{\substack{(\mathbb{Y}_1'',\ldots,\mathbb{Y}_{k_0}'') \\ |\mathbb{Y}_s|>1}} \sum_{G_0} \prod_{\mathscr{L}\in G_0} A(\mathscr{L}) \prod_{s=1}^{k_0} F_\Lambda(\mathbb{Y}_s'') \right). \tag{4.3.18}$$

The second factor is just what we would have obtained for $R=1$. With $\langle R\rangle = \tilde{F}_R/\tilde{F}_{R=1}$, we have

$$\langle R\rangle_{\Lambda,q} = \sum_{\{\mathbb{T}_u\}} \sum_k \frac{1}{k!} \sum_{\substack{(\mathbb{Y}_1,\ldots,\mathbb{Y}_k) \\ |\mathbb{Y}_s|>1}} \sum_{G_c} \Xi_\Lambda^q(\{\mathbb{T}_u\}) \prod_{\mathscr{L}\in G_c} A(\mathscr{L}) \prod_{s=1}^{k} F_\Lambda(\mathbb{Y}_s). \tag{4.3.19}$$

This is the final form of the expansion.

To make the transformations leading to (4.3.19) completely well-defined, we should have placed Dirichlet boundary conditions on the boundary of a square much larger than Λ. All clusters would then be constrained to lie in some large $\mathbb{V}$ with $p(\mathbb{V})=q$. We show below that the first three sums in (4.3.19) converge absolutely, so that the right-hand side of (4.3.14) converges as $\mathbb{V}$ tends to infinity. The left-hand side also converges by virtue of the "regularity at infinity" of [17], so that (4.3.19) is valid in the limit.

We require some exponential decay of $\Xi_\Lambda^q(\{\mathbb{T}_u\})$ with $\left|\bigcup_u \mathbb{T}_u\right|$ and with $\left|\bigcup_u \Sigma_{\mathbb{T}_u}\right|$. From (4.2.5)–(4.2.6) we have

$$\begin{aligned} 0 &\leqq \frac{\Omega_c^q(F,\mathbb{V})}{\Omega_c(F,\mathbb{V}^q)} e^{(\log Z_{\Lambda^{p(\mathbb{V})}} - \log Z_{\Lambda^q})|\mathbb{V}|} \\ &\leqq e^{(a^{p(\mathbb{V})} + s^{p(\mathbb{V})} - s^q + \log Z_{\Lambda^{p(\mathbb{V})}} - \log Z_{\Lambda^q})|\mathbb{V}|} e^{(\Delta_c(F,\mathbb{V}) - \Delta_c(F,\mathbb{V}^q))} \\ &\leqq e^{4\lambda^{1/2}|\partial\mathbb{V}|}. \end{aligned} \tag{4.3.20}$$

The volume coefficients have canceled exactly, by virtue of $a^q=0$. In addition,

$$\log Z_{\Lambda^{p(R_i)}} - \log Z_{\Lambda^q} \leqq s^q - s^{p(R_i)} \leqq 2\lambda^{1/2}$$

so that

$$\prod_i e^{(\log Z_{\Lambda^{p(R_i)}} - \log Z_{\Lambda^q})\left|R_i \cap \bigcup_u \mathbb{T}_u\right|} \leqq e^{2\lambda^{1/2}\left|\bigcup_u \mathbb{T}_u\right|}. \tag{4.3.21}$$

Thus

$$\begin{aligned} |\Xi_\Lambda^q(\{\mathbb{T}_u\})| &\leqq \prod_u |\tilde{\varrho}_{\Lambda,q}(\mathbb{T}_u)| e^{18\lambda^{1/2}\left|\bigcup_u \mathbb{T}_u\right|} \\ &\leqq \|w\|_{L^p} \prod_{\Delta: N(\Delta)>0} [(N(\Delta)!)^{1/2} e^{-\tau_1 l}] e^{Kl \deg R} (1+\lambda^{-\deg R} e^{-\tau_2\lambda^{-2}}) \\ &\quad \cdot e^{-\tau_1 l \left|\bigcup_u \mathbb{T}_u\right|} e^{-\tau_2\lambda^{-2}\left|\bigcup_u \Sigma_{\mathbb{T}_u}\right|}. \end{aligned} \tag{4.3.22}$$

In the second line we have used (2.5.11) and combined factors associated with R. The test function w has been reintroduced. The external boundary condition of each $\mathbb{T}$ agrees with the boundary loop of the $\mathbb{T}$ immediately surrounding it. Since external loops all have boundary condition q, any factor of $\lambda^{-\deg R_{\mathbb{X}}}$ coming from an $\mathbb{X}$ with $\Sigma \equiv m \neq q$ is compensated by a factor $e^{-\tau_2 \lambda^{-2}}$ coming from a phase boundary in some other $\mathbb{T}$. Hence the factor $(1+\lambda^{-\deg R} e^{-\tau_2 \lambda^{-2}})$ in (4.3.22).

The sums over k, $(\mathbb{Y}_1, \ldots, \mathbb{Y}_k)$, and G_c are controlled by a lemma proven in Sect. 4.4. Define

$$\Phi_F(\mathbb{Z}_1, \ldots, \mathbb{Z}_j; \mathbb{Y}_1, \ldots, \mathbb{Y}_k) = \sum_{G_c} \prod_{\mathscr{L} \in G_c} A(\mathscr{L}) \prod_{s=1}^{k} F(\mathbb{Y}_s), \tag{4.3.23}$$

where each $\mathbb{Z}_r$ is either a $\mathbb{T}$ or a $\mathbb{Y}$. G_c is any graph that is connected with respect to $\mathbb{Z}_1, \ldots, \mathbb{Z}_j$, that involves all $\mathbb{Y}$'s, and that does not contain lines $\{\mathbb{Z}_r, \mathbb{Z}_{r'}\}$, $1 \leqq r, r' \leqq j$. For $k=0$ we have $G_c = \emptyset$ and $\Phi_F = 1$.

Lemma 4.3.2. *Suppose $\lambda \ll 1 \ll l$. If F^q is any q-contour model with $\|F^q\| \leqq 1$, then for all $\{\mathbb{Z}_1, \ldots, \mathbb{Z}_j\}$, $j \geqq 1$,*

$$\begin{aligned} &\sum_{\substack{(\mathbb{Y}_1, \ldots, \mathbb{Y}_k) \\ \sum_s |\mathbb{Y}_s| = N,\ p(\mathbb{Y}_s) = q}} |\Phi_F(\mathbb{Z}_1, \ldots, \mathbb{Z}_j; \mathbb{Y}_1, \ldots, \mathbb{Y}_k)| \\ &\quad \leqq k!\, e^{-\tau_1 l(k+N)/4} e^{\sum_r |\mathbb{Z}_r|}. \end{aligned} \tag{4.3.24}$$

Together with (4.3.22), the lemma yields the following bound on (4.3.19):

$$\begin{aligned} |\langle R \rangle_{\Lambda, q}| \leqq & \sum_{\{\mathbb{T}_u\}} 2\|w\|_{L^p} \prod_{\Delta : N(\Delta) > 0} [(N(\Delta)!)^{1/2} e^{-\tau_1 l}] \\ & \cdot e^{Kl \deg R} (1 + \lambda^{-\deg R} e^{-\tau_2 \lambda^{-2}}) \\ & \cdot e^{-\frac{3}{4}\tau_1 l \left|\bigcup_u \mathbb{T}_u\right|} e^{-\tau_2 \lambda^{-2} \left|\bigcup_u \Sigma_{\mathbb{T}_u}\right|}. \end{aligned} \tag{4.3.25}$$

Fix $\{T_u\}$, the regions containing $\{\mathbb{T}_u\}$. The sum over $\Gamma \cap \bigcup_u \mathbb{T}_u$ and over $\Sigma \cap \bigcup_u \mathbb{T}_u$ are controlled as in (4.2.17), yielding factors of $e^{O(1)\left|\bigcup_u T_u\right|}$.

The number of connected regions of size m overlapping or surrounding a square is less than $e^{O(1)m}$. If a number of regions with total size N all overlap or surround a square, then there are 2^N ways of distributing the size into connected regions. Given the distribution $N = m_1 + m_2 + \ldots + m_r$, there are $\prod_i e^{O(1)m_i} = e^{O(1)N}$ possibilities, or $e^{O(1)N}$ in all. Hence

$$\sum_{\{T_u\}} e^{-\frac{1}{2}\tau_1 l \left|\bigcup_u T_u\right|} e^{O(1)\left|\bigcup_u T_u\right|} \leqq \prod_{\Delta : N(\Delta) > 0} \left(\sum_{N=0}^{\infty} e^{(-\frac{1}{2}\tau_1 l + O(1))N} \right) \leqq \prod_{\Delta : N(\Delta) > 0} O(1). \tag{4.3.26}$$

The possibility $N=0$ is included because some T's might contain more than one Δ. The theorem now follows from (4.3.25) and (4.3.26). $\square$

We have shown that the first three sums in (4.3.19) converge absolutely, with a bound uniform in Λ given by (4.3.6). Each term converges as $\Lambda \to \infty$. In fact, $\tilde{\varrho}_{\Lambda, q}(\mathbb{Z}) = \tilde{\varrho}(\mathbb{Z})$ for $\mathbb{Z} \subseteqq \Lambda$, so that $F_\Lambda(\mathbb{Y}) = F(\mathbb{Y})$ for $\mathbb{Y} \subseteqq \Lambda$ and $\Xi^q_\Lambda(\{\mathbb{T}_u\}) = \Xi^q(\{\mathbb{T}_u\})$ for

$\bigcup_u \mathbb{T}_u \subseteqq \Lambda$. Here $\Xi^q(\{\mathbb{T}_u\})$ is given by (4.3.14) but with $\tilde{\varrho}$ replacing $\tilde{\varrho}_{\Lambda,q}$. Therefore, $\langle R\rangle_{\Lambda,q}$ converges as $\Lambda\to\infty$ to $\langle R\rangle_q$ given by

$$\langle R\rangle_q=\sum_{\{\mathbb{T}_u\}}\sum_k\frac{1}{k!}\sum_{\substack{(\mathbb{Y}_1,\dots,\mathbb{Y}_k)\\|\mathbb{Y}_s|>1}}\sum_{G_c}\Xi^q(\{\mathbb{T}_u\})\prod_{\mathscr{L}\in G_c}A(\mathscr{L})\prod_{s=1}^k F(\mathbb{Y}_s) \tag{4.3.27}$$

and satisfying the bound of Theorem 4.3.1. Except for clustering, all of the Osterwalder-Schrader axioms [22] are immediate consequences.

We now show that truncated expectations

$$\langle R_1;R_2\rangle=\langle R_1R_2\rangle-\langle R_1\rangle\langle R_2\rangle \tag{4.3.28}$$

display exponential clustering. Let R_1 and R_2 be of the form $\prod_i :\psi_q(x_i)^{p_i}:$, and let w be an L^p function of all the variables, supported in a product of l-lattice squares. Let D be the distance (in ordinary units) between the R_1-squares and the R_2-squares.

Theorem 4.3.3. *Let $p>1$ be given and suppose $a^q=0$. For $\lambda\ll 1\ll l$ there exist positive constants K,τ_1,τ_2 depending on p,η,C such that*

$$|\textstyle\int dxw(x)\langle R_1;R_2\rangle(x)|\leqq\|w\|_{L^p}\prod_\Delta(N(\Delta)!)^{1/2}e^{Kl\deg R_1R_2}$$
$$\cdot(1+\lambda^{-\deg R_1R_2}e^{-\tau_2\lambda^{-2}})e^{-\tau_1D/5}. \tag{4.3.29}$$

Here $\langle\cdot\rangle$ refers either to the finite volume expectation or to the infinite volume one.

Proof. Define

$$\langle R\rangle_M=\sum_{\{\mathbb{T}_u\}}\sum_k\sum_{\substack{(\mathbb{Y}_1,\dots,\mathbb{Y}_k):|\mathbb{Y}_s|>1\\ \sum_u|\mathbb{T}_u|+\sum_s|\mathbb{Y}_s|=M}}\sum_{G_c}\Xi^q_{(\Lambda)}(\{\mathbb{T}_u\})$$
$$\cdot\prod_{\mathscr{L}\in G_c}A(\mathscr{L})\prod_{s=1}^k F_{(\Lambda)}(\mathbb{Y}_s), \tag{4.3.30}$$

so that

$$\langle R\rangle=\sum_{M=1}^\infty\langle R\rangle_M \tag{4.3.31}$$

and

$$\langle R_1;R_2\rangle=\sum_{M=1}^\infty\left[\langle R_1R_2\rangle_M-\sum_{K=1}^{M-1}\langle R_1\rangle_K\langle R_2\rangle_{M-K}\right]. \tag{4.3.32}$$

As long as $M<D/l$, each term in the sums over clusters in $\langle R_1R_2\rangle_M$ has a factorization property:

$$\sum_{G_c}\Xi^q_{(\Lambda)}(\{\mathbb{T}_u\})\prod_{\mathscr{L}\in G_c}A(\mathscr{L})\prod_{s=1}^k F_{(\Lambda)}(\mathbb{Y}_s)$$
$$=\Big(\sum_{G_{1c}}\Xi^q_{(\Lambda)}(\{\mathbb{T}_u:\operatorname{suppt}R_1\cap(\mathbb{T}_u\cup\operatorname{Int}\mathbb{T}_u)\neq\emptyset\})\prod_{\mathscr{L}\in G_{1c}}A(\mathscr{L})\prod_{\mathbb{Y}_s\in G_{1c}}F_{(\Lambda)}(\mathbb{Y}_s)\Big)$$
$$\cdot\Big(\sum_{G_{2c}}\Xi^q_{(\Lambda)}(\{\mathbb{T}_u:\operatorname{suppt}R_2\cap(\mathbb{T}_u\cup\operatorname{Int}\mathbb{T}_u)\neq\emptyset\})\prod_{\mathscr{L}\in G_{2c}}A(\mathscr{L})\prod_{\mathbb{Y}_s\in G_{2c}}F_{(\Lambda)}(\mathbb{Y}_s)\Big). \tag{4.3.33}$$

Here G_{1c} is a graph connected with respect to the first set of $\mathbb{T}$'s, and G_{2c} is connected with respect to the second set. Equation (4.3.33) is a consequence of the fact that $U \neq 1$ $(A \neq 0)$ only when one cluster overlaps or surrounds another. With the restriction on total size of clusters, a graph connected with respect to all the $\mathbb{T}$'s breaks into independent parts. The splitting of $(\mathbb{Y}_1, \ldots, \mathbb{Y}_k)$ into $(\mathbb{Y}_s \in G_{1c}) \cup (\mathbb{Y}_s \in G_{2c})$ is independent of the graph.

Factorization implies that the terms of (4.3.32) with $M < D/l$ cancel, as there is a one-to-one correspondence between nonvanishing terms of $\langle R_1 R_2 \rangle_M$ and of $\sum_{K=1}^{M-1} \langle R_1 \rangle_K \langle R_2 \rangle_{M-K}$.

We estimate $\langle R \rangle_M$ as before. The sum over $(\mathbb{Y}_1, \ldots, \mathbb{Y}_k)$ involves only sets of clusters with $\sum_{s=1}^{k} |\mathbb{Y}_s| = M - \sum_u |\mathbb{T}_u|$. Lemma 4.3.2 insures that we have a convergent factor $e^{-\tau_1 l \left(M - \sum_u |\mathbb{T}_u|\right)/4}$ left after summing over k and $(\mathbb{Y}_1, \ldots, \mathbb{Y}_k)$. Hence (4.3.25) is valid for $|\langle R \rangle_M|$ with an extra factor $e^{-\tau_1 l M/4}$ if $e^{-\frac{3}{4}\tau_1 l \left|\bigcup_u \mathbb{T}_u\right|}$ is replaced by $e^{-\frac{1}{2}\tau_1 l \left|\bigcup_u \mathbb{T}_u\right|}$. The rest of the estimate is identical. Putting the resulting bound into (4.3.32), we obtain

$$|\int dx|\, w(x) \langle R_1 ; R_2 \rangle (x)| \leqq \|w\|_{L^p} \prod_{\Delta} (N(\Delta)!)^{1/2} e^{Kl \deg R_1 R_2} \cdot (1 + 3\lambda^{-\deg R_1 R_2} e^{-\tau_2 \lambda^{-2}}) \sum_{M \geqq D/l} M e^{-\tau_1 l/4} \tag{4.3.34}$$

and the theorem follows. □

Theorem 4.3.3 establishes the remaining Osterwalder-Schrader axiom and shows that all the states we have constructed are pure states. Moreover, the Wightman field theory associated to the Schwinger functions has a positive mass gap. The mass gap is uniform as λ tends to zero, and it is uniform in the parameters $\{\mu^i\}$. When combined with the Chap. 3 results on the existence of phases with $a^q = 0$, Theorems 4.3.1 and 4.3.3 establish Theorem 1.1.1.

4.4. The Convergence Lemmas

In Sects. 3.3, 4.2, and 4.3 we have made use of lemmas which proved convergence of expansions involving U- and A-operations. We prove the lemmas here. The proof of the main result, Lemma 4.3.2, is essentially contained in [1]. We include it here for completeness.

Proof of Lemma 3.3.1, Assuming Lemma 4.3.2. Fix a cluster $\mathbb{Y}$ containing Δ and sum over the others in (3.3.10) before summing over $\mathbb{Y}$:

$$\sum_{\substack{(\mathbb{Y}_1, \ldots, \mathbb{Y}_k) : \bigcup_s \mathbb{Y}_s \supseteq \Delta \\ \sum_s |\mathbb{Y}_s| = N,\ p(\mathbb{Y}_s) = q}} \left| \sum_{G_c} \prod_{\mathscr{L} \in G_c} A(\mathscr{L}) \prod_{s=1}^{k} F^q(\mathbb{Y}_s) \right|$$
$$\leqq k \sum_n \sum_{\substack{\mathbb{Y} \supseteq \Delta \\ |\mathbb{Y}| = n}} |F(\mathbb{Y})| \sum_{\substack{(\mathbb{Y}_1, \ldots, \mathbb{Y}_{k-1}) \\ \sum_s |\mathbb{Y}_s| = N - n}} |\Phi_F(\mathbb{Y}; \mathbb{Y}_1, \ldots, \mathbb{Y}_{k-1})|$$
$$\leqq k! \sum_n \sum_{\substack{\mathbb{Y} \supseteq \Delta \\ |\mathbb{Y}| = n}} \|F\| e^{-\tau_1 l |\mathbb{Y}| - \tau_2 \lambda^{-2} |\Sigma_{\mathbb{Y}}|} e^{-\tau_1 l (k-1+N-n)/4} e^{|\mathbb{Y}|}. \tag{4.4.1}$$

A connected graph is connected with respect to any one of its clusters, so the substitution of (4.3.23) into (4.4.1) is valid. In the last step we have applied Lemma 4.3.2 and used $\|F\| \leqq 1$. The sum over $\mathbb{Y}$ is controlled as in (4.2.17). Using $k \geqq 1$, $n \geqq 2$ we bound (4.4.1) by

$$k!\,\|F\| \sum_n e^{-\tau_1 l n} e^{O(1)n} e^{-\tau_1 l(k+N)/4} e^{\tau_1 l n/4} \leqq k!\,\|F\| e^{-\tau_1 l(k+N)/4}. \tag{4.4.2}$$

This completes the proof. □

Lemma 4.2.2 is just Lemma 4.3.2 in the case $\{\mathbb{Z}_1, \ldots, \mathbb{Z}_j\} = \{\mathbb{Y}\}$.

Proof of Lemma 4.3.2. Our first task will be to find a Kirkwood-Salzburg type equation expressing $\Phi(\mathbb{Z}_1, \ldots, \mathbb{Z}_j; \mathbb{Y}_1, \ldots, \mathbb{Y}_k)$ as a sum of terms involving Φ's with smaller $j+k$. For each G_c in (4.3.23) let Ω be the set of all s such that $\{\mathbb{Z}_1, \mathbb{Y}_s\} \in G_c$. Ω may be empty, but only if $j \geqq 2$. Let G_c' denote the subset of G_c composed of lines $\{\mathbb{Z}_r, \mathbb{Y}_s\}$ and $\{\mathbb{Y}_s, \mathbb{Y}_{s'}\}$, with $s, s' \in \Omega$ and $r > 1$. Let G_c'' be composed of all remaining lines except the lines $\{\mathbb{Z}_1, \mathbb{Y}_s\}$. If we fix Ω, the summation over G_c' is unconstrained but G_c'' must be a graph connected with respect to $\{\mathbb{Z}_2, \ldots, \mathbb{Z}_j, (\mathbb{Y}_s)_{s\in\Omega}\}$. Since

$$\sum_{G_c'} \prod_{\mathscr{L}\in G_c'} A(\mathscr{L}) = \prod_{r=2}^{j} \prod_{s\in\Omega} U(\mathbb{Z}_r, \mathbb{Y}_s) \prod_{s_1 < s_2;\, s_1, s_2 \in \Omega} U(\mathbb{Y}_{s_1}, \mathbb{Y}_{s_2}), \tag{4.4.3}$$

Equation (4.3.23) becomes

$$\begin{aligned}\Phi_F(\mathbb{Z}_1, \ldots, \mathbb{Z}_j; \mathbb{Y}_1, \ldots, \mathbb{Y}_k) = &\sum_{\Omega \subseteq \{1, \ldots, k\}} \prod_{s\in\Omega} A(\mathbb{Z}_1, \mathbb{Y}_s) \\ &\cdot \prod_{r=2}^{j} \prod_{s\in\Omega} U(\mathbb{Z}_r, \mathbb{Y}_s) \prod_{s_1 < s_2;\, s_1, s_2 \in \Omega} U(\mathbb{Y}_{s_1}, \mathbb{Y}_{s_2}) \\ &\cdot \prod_{s\in\Omega} F(\mathbb{Y}_s) \sum_{G_c''} \prod_{\mathscr{L}\in G_c''} A(\mathscr{L}) \prod_{s\notin\Omega} F(\mathbb{Y}_s).\end{aligned} \tag{4.4.4}$$

Hence,

$$\begin{aligned}\Phi_F(\mathbb{Z}_1, \ldots, \mathbb{Z}_j; \mathbb{Y}_1, \ldots, \mathbb{Y}_k) = &\sum_{\Omega} \prod_{s\in\Omega} A(\mathbb{Z}_1, \mathbb{Y}_s) \\ &\cdot \prod_{r=2}^{j} \prod_{s\in\Omega} U(\mathbb{Z}_r, \mathbb{Y}_s) \prod_{s_1 < s_2;\, s_1, s_2 \in \Omega} U(\mathbb{Y}_{s_1}, \mathbb{Y}_{s_2}) \\ &\cdot \prod_{s\in\Omega} F(\mathbb{Y}_s) \Phi_F(\mathbb{Z}_2, \ldots, \mathbb{Z}_j, (\mathbb{Y}_s)_{s\in\Omega}; (\mathbb{Y}_s)_{s\notin\Omega}).\end{aligned} \tag{4.4.5}$$

This equation will enable us to prove the lemma by induction on $j+k$. To start the induction, notice that for $k=0$ we have $\Phi_F(\mathbb{Z}_1, \ldots, \mathbb{Z}_j; \emptyset) = 1$, by definition. The lemma holds in this case. We define $\Phi_F(\emptyset; \mathbb{Y}_1, \ldots, \mathbb{Y}_k) = 0$.

For $j+k \geqq 2$, $j \geqq 1$, $k \geqq 1$, assume the lemma for smaller $j+k$. Since $A(\mathbb{Z}_1, \mathbb{Y}_s) = 0$ unless $\mathbb{Y}_s$ overlaps or surrounds $\mathbb{Z}_1$, and since the U's and A's are either ± 1 or 0, (4.4.5) yields

$$\sum_{\substack{(\mathbb{Y}_1,\ldots,\mathbb{Y}_k)\\ \sum_s|\mathbb{Y}_s|=N}} |\Phi_F(\mathbb{Z}_1,\ldots,\mathbb{Z}_j;\mathbb{Y}_1,\ldots,\mathbb{Y}_k)|$$

$$\leqq \sum_{\Omega} \sum_{\substack{(\mathbb{Y}_1,\ldots,\mathbb{Y}_k)\\ \sum_s|\mathbb{Y}_s|=N,\ \mathbb{Y}_s \text{ overlaps or surrounds } \mathbb{Z}_1 \text{ for } s\in\Omega}} \prod_{s\in\Omega}|F(\mathbb{Y}_s)||\Phi_F(\mathbb{Z}_2,\ldots,\mathbb{Z}_j,(\mathbb{Y}_s)_{s\in\Omega};(\mathbb{Y}_s)_{s\notin\Omega})|$$

$$\leqq k!\sum_{|\Omega|=1}^{k-1}\frac{1}{|\Omega|!}\sum_{M=|\Omega|}^{N-1}\sum_{\substack{(\mathbb{Y}'_1,\ldots,\mathbb{Y}'_{|\Omega|}):\sum|\mathbb{Y}_s|=M\\ \mathbb{Y}_s \text{ overlaps or surrounds } \mathbb{Z}_1}}\prod_{s=1}^{|\Omega|}|F(\mathbb{Y}'_s)|$$

$$\cdot\frac{1}{(k-|\Omega|)!}\sum_{\substack{(\mathbb{Y}''_1,\ldots,\mathbb{Y}''_{k-|\Omega|})\\ \sum_s|\mathbb{Y}_s|=N-M}}|\Phi_F(\mathbb{Z}_2,\ldots,\mathbb{Z}_j,\mathbb{Y}'_1,\ldots,\mathbb{Y}'_{|\Omega|};\mathbb{Y}''_1,\ldots,\mathbb{Y}''_{k-|\Omega|})|$$

$$+\sum_{\substack{(\mathbb{Y}_1,\ldots,\mathbb{Y}_k)\\ \sum_s|\mathbb{Y}_s|=N,\ \mathbb{Y}_s \text{ overlaps or surrounds } \mathbb{Z}_1}}\prod_{s=1}^{k}|F(\mathbb{Y}_s)|$$

$$+\sum_{\substack{(\mathbb{Y}_1,\ldots,\mathbb{Y}_k)\\ \sum_s|\mathbb{Y}_s|=N}}|\Phi_F(\mathbb{Z}_2,\ldots,\mathbb{Z}_j;\mathbb{Y}_1,\ldots,\mathbb{Y}_k)|\,. \tag{4.4.6}$$

We control the sums over $\mathbb{Y}_s$, $s\in\Omega$ as follows. Given Y_s, the region covered by $\mathbb{Y}_s$, the sum over $\Sigma_{\mathbb{Y}_s}$ and $\Gamma\cap\mathbb{Y}_s$ are controlled as usual, using $\|F\|\leqq 1$. This produces a factor $e^{O(1)|Y_s|}$. There are at most 2^M ways of expressing M as $m_1+m_2+\ldots+m_{|\Omega|}$, and there are at most $|\mathbb{Z}_1|e^{O(1)|m_i|}$ connected regions overlapping or surrounding $\mathbb{Z}_1$. Altogether, there is a factor $e^{O(1)M}|\mathbb{Z}_1|^{|\Omega|}$ from the sum over $(\mathbb{Y}'_1,\ldots,\mathbb{Y}'_{|\Omega|}):\sum_s|\mathbb{Y}_s|=M$. Apply the induction hypothesis to the sums over $\mathbb{Y}_s$, $s\notin\Omega$:

$$\sum_{\substack{(\mathbb{Y}_1,\ldots,\mathbb{Y}_k)\\ \sum_s|\mathbb{Y}_s|=N}}|\Phi_F(\mathbb{Z}_1,\ldots,\mathbb{Z}_j;\mathbb{Y}_1,\ldots,\mathbb{Y}_k)|$$

$$\leqq k!\sum_{|\Omega|=1}^{k-1}\frac{1}{|\Omega|!}\sum_{M=|\Omega|}^{N-1}e^{-\frac{3}{4}\tau_1 lM}|\mathbb{Z}_1|^{|\Omega|}e^{-\tau_1 l(k-|\Omega|+N-M)/4}e^{\sum_{r=2}^{j}|\mathbb{Z}_r|}$$

$$+e^{-\frac{3}{4}\tau_1 lN}|\mathbb{Z}_1|^k+k!\,e^{-\tau_1 l(N+k)/4}e^{\sum_{r=2}^{j}|\mathbb{Z}_r|}$$

$$\leqq k!\,e^{-\tau_1 l(N+k)/4}e^{\sum_{r=2}^{j}|\mathbb{Z}_r|}\left(\sum_{|\Omega|=0}^{k}\frac{|\mathbb{Z}_1|^{|\Omega|}}{|\Omega|!}\right)$$

$$\leqq k!\,e^{-\tau_1 l(N+k)/4}e^{\sum_{r=1}^{j}|\mathbb{Z}_r|}\,. \tag{4.4.7}$$

We have used $M\geqq|\Omega|$, $N\geqq k$ in the second step above. This completes the proof. □

5. Converge Estimates

5.1. Structure of the Estimates

Chapter 5 is devoted to the proofs of Propositions 2.5.1–2.5.6, which are the essential input to the analysis of Chaps. 3 and 4. The starting point for all the estimates is the vacuum energy bound, Proposition 2.5.1. The proof begins with the Wick ordering lower bound

$$:\mathscr{P}_{\lambda,\mu}(\phi_\kappa(x)): -\tfrac{1}{2}\eta :(\phi_\kappa(x)-\xi_q)^2: -\mathscr{P}_{\lambda,\mu}(\xi_q) - \log\chi_q(\bar{\phi}(\Delta)) + \zeta :\delta\phi_\kappa(x)^2: \\ \geqq -b(\log\kappa)^{\deg\mathscr{P}/2}. \tag{5.1.1}$$

Here ϕ_κ is the momentum cutoff field and $\delta\phi_\kappa = \phi_\kappa - \phi$. This bound assures us that the $O(\lambda^{-2})$ differences in classical energies and the $O(\lambda^{-2})$ effect from the term $-\frac{1}{2}\eta(\phi_\kappa-\xi_q)^2$ can be controlled by the spin localization factor χ_q and by estimates on the fluctuation field $\delta\phi_\kappa$. [The term $\frac{1}{2}\eta :(\phi_\kappa-\xi_q)^2:$ is subtracted from $\mathscr{P}$ in order to leave a small mass in the Gaussian measure when doing the vacuum energy bound.]

The proof of (5.1.1) involves showing that $-\log\chi_q(\bar{\phi})$ is large unless $\bar{\phi}$ is in the range $[\frac{1}{2}(\xi_{q-1}+\xi_q), \frac{1}{2}(\xi_q+\xi_{q+1})]$. The term $\zeta\delta\phi_\kappa^2$ is large unless ϕ_κ is close to $\bar{\phi}$. Thus ϕ_κ is localized near ξ_q, where we have a quadratic lower bound on $\mathscr{P}$ [condition (vii), Sect. 2.1].

After proving the Wick bounds, the proof follows [19] with a few modifications arising from differences in classical masses. In each phase p the vacuum energy relative to the mass m_p ground state is bounded below by $E_c^p - O(\lambda)$. Phase boundaries produce strong convergence $e^{-3\tau_2\lambda^{-2}|\Sigma|}$ arising from the gradient term in the Euclidean action. The field changes by $O(\lambda^{-1})$ in a distance $O(1)$, so that $\int\frac{1}{2}|\nabla\phi|^2 \geqq O(\lambda^{-2})$. The other important ingredient for the vacuum energy bound is a bound on the fluctuation field

$$\int \exp(\zeta :\delta\phi_\kappa^2(\Delta^1):) d\mu_\eta(\phi) \leqq O(1), \tag{5.1.2}$$

uniform as $\eta\to 0$.

The lower bound on Z_{Δ^p} (Sect. 5.4) puts an upper bound $E_w^p + O(\lambda)$ on the vacuum energy (relative to mass $m_{\bar{q}}$). The lower bound

$$Z_\Sigma(\mathbb{V}) \geqq \exp[(-E_w^{p(\mathbb{V})} - O(\lambda^{-2}))l^2|\mathbb{V}|]$$

is needed in Sect. 5.7 to prove smoothness of Z_Σ in μ. Bounds of this type have not previously been needed in low temperature expansions. It is ordinarily sufficient to support the measure on uniform spin configurations, leaving phase boundary terms to be as small as they like. In our case the smoothness in μ is needed for all configurations if we are to construct hypersurfaces with $2, 3, \ldots, r-1$ coexisting phases. The proof involves bounding the expectation of $:\psi_p^j(\mathbb{V}):$ in the measure $\chi_\Sigma e^{-V_p(\mathbb{V})} d\mu_{m_p^2}(\psi_p)$ by $O(\lambda^{-2})l^2|\mathbb{V}|$. Cluster expansion techniques cannot be used for such expectations; we must control the error involved in considering $:\psi_p^j(\mathbb{V}):$ as a bounded variable.

Section 5.5 is concerned with the mass-shift normalization factors arising in the decoupling expansion. Just as in Sect. 3.3, it is important to obtain the exact volume dependence of $\log Z_{\omega_1\omega_2}(s)$ and to bound the remainder as a surface effect.

We also need smoothness in μ. Armed with these estimates, we bound the terms of the decoupling expansion in Sect. 5.6 and we obtain their smoothness in μ in Sect. 5.8.

5.2. *Wick Ordering Lower Bounds*

In this section we prove lower bounds on the Wick ordered interactions. These bounds diverge slowly as the momentum cutoff κ tends to infinity. They are an important ingredient in the vacuum energy bounds.

We use the momentum cutoff of [19]. It consists of a smoothing operator $\varrho_\kappa : f \to f_\kappa$ which preserves localization in unit squares and acts as the identity on characteristic functions of unit squares. If we define the fluctuation field $\delta\phi(x) = \phi(x) - \bar{\phi}(x) = \phi(x) - \int_{\Delta \ni x} \phi(x)dx$, we have $(\delta\phi)_\kappa(x) = \phi_\kappa(x) - \bar{\phi}(x)$.

We shall require certain lower bounds on the polynomial $\mathscr{P}_1(\xi)$. The Wick bounds then apply to $\mathscr{P}_\lambda(\xi) = \lambda^{-2}\mathscr{P}_1(\lambda\xi)$ for $\lambda \leqq 1$. If $\xi_1, \ldots, \xi_r$ are a number of relative minima of $\mathscr{P}_1$, then $\xi_q(\lambda) = \lambda^{-1}\xi_q$ are relative minima of $\mathscr{P}_\lambda$. With $\mathscr{P}_1(\xi + \xi_q) = \sum_{j=0}^{d} a_{j,q}\xi^j$, put $\xi_0 = -\infty$, $\xi_{r+1} = \infty$ and take $\xi_q < \xi_{q+1}$. Recall the definition of the spin localization functions:

$$\chi_q(\xi, \lambda) = \pi^{-1/2} \int_{(\xi_{q-1}(\lambda) + \xi_q(\lambda))/2}^{(\xi_q(\lambda) + \xi_{q+1}(\lambda))/2} e^{-(\xi - z)^2} dz, \tag{5.2.1}$$

where $q = 1, \ldots, r$. The argument λ in ξ_q and χ_q will often be omitted. Let $\chi_q^{(n)}(\xi) = \frac{d^n}{d\xi^n}\chi_q(\xi)$.

Proposition 5.2.1. *Suppose $\zeta \in (0, \frac{1}{3}]$, $\eta \in (0, \zeta/4]$, and $C > 2$. Let $\mathscr{P}_1$ be any polynomial with $d \leqq C$ even. Suppose $|\xi_q - \xi_{q+1}| \geqq C^{-1}$, $|a_{d,q}| \geqq C^{-1}$, and $|a_{j,q}| \leqq C$ for $j = 1, \ldots, d$. Suppose further that for $q = 1, \ldots, r$*

$$\mathscr{P}_1(\xi) - \mathscr{P}_1(\xi_q) \geqq \begin{cases} \eta(\xi - \xi_q)^2, & \xi \in (\frac{1}{2}(\xi_{q-1} + \xi_q), \frac{1}{2}(\xi_q + \xi_{q+1})) \\ \eta(\xi - \xi_q)^2 - \frac{\zeta}{4}(\xi - \frac{1}{2}(\xi_q + \xi_{q+1}))^2, & \xi \geqq \frac{1}{2}(\xi_q + \xi_{q+1}) \\ \eta(\xi - \xi_q)^2 - \frac{\zeta}{4}(\xi - \frac{1}{2}(\xi_q + \xi_{q-1}))^2, & \xi \leqq \frac{1}{2}(\xi_q + \xi_{q-1}) . \end{cases} \tag{5.2.2}$$

Then there exist constants $b(C)$, $a(\eta, C) > 0$, and K (depending only on their respective arguments) such that for all κ, all $x \in \Delta$, all $\lambda \in (0, 1]$, and $q = 1, \ldots, r$

$$\begin{aligned} &:\mathscr{P}_\lambda(\phi_\kappa(x)): - \tfrac{1}{2}\eta :(\phi_\kappa(x) - \xi_q)^2: - \mathscr{P}_\lambda(\xi_q) - \log \chi_q(\bar{\phi}(\Delta)) + \zeta :\delta\phi_\kappa(x)^2: \\ &\geqq -b(C)(\log\kappa)^{d/2} . \end{aligned} \tag{5.2.3}$$

Under the same conditions,

$$\begin{aligned} &:\mathscr{P}_\lambda(\phi_\kappa(x)): - \tfrac{1}{2}\eta :(\phi_\kappa(x) - \xi_q)^2: - \mathscr{P}_\lambda(\xi_q) - \log|\chi_q^{(n(\Delta))}(\bar{\phi}(\Delta))| + \zeta :\delta\phi_\kappa(x)^2: \\ &\geqq a(\eta, C)\lambda^{-2} - \log K n(\Delta)! - b(C)(\log\kappa)^{d/2} \end{aligned} \tag{5.2.4}$$

for any $n(\Delta) \geqq 1$.

Proof. We begin with some upper bounds on $\log|\chi_q^{(n)}(\bar{\phi}(\Delta))|$. Put $A=\frac{1}{2}(\xi_q+\xi_{q-1})$, $B=\frac{1}{2}(\xi_q+\xi_{q+1})$. The following bounds are valid:

$$
\begin{aligned}
&0\leqq \chi_q(\bar{\phi}(\Delta))\leqq 1\,,\\
&\chi_q(\bar{\phi}(\Delta))\leqq \tfrac{1}{2}, \qquad \bar{\phi}(\Delta)\leqq A \quad \text{or} \quad \bar{\phi}(\Delta)\geqq B\,,\\
&\chi_q(\bar{\phi}(\Delta))\leqq \sqrt{2}e^{-\frac{1}{2}(\bar{\phi}(\Delta)-A)^2}, \qquad \bar{\phi}(\Delta)\leqq A\,,\\
&\chi_q(\bar{\phi}(\Delta))\leqq \sqrt{2}e^{-\frac{1}{2}(\bar{\phi}(\Delta)-B)^2}, \qquad \bar{\phi}(\Delta)\geqq B\,,\\
&|\chi_q^{(n)}(\bar{\phi}(\Delta))|\leqq \tfrac{1}{2}Kn!\,(e^{-\frac{1}{3}(\bar{\phi}(\Delta)-A)^2}+e^{-\frac{1}{3}(\bar{\phi}(\Delta)-B)^2}), \qquad n\geqq 1\,.
\end{aligned}
\tag{5.2.5}
$$

The first two bounds are easy consequences of (5.2.1), and the others are proven in [19]. We can combine the second, third, and fourth bounds to yield

$$
\begin{aligned}
&\log\chi_q(\bar{\phi}(\Delta))\leqq -\tfrac{1}{3}(\bar{\phi}(\Delta)-A)^2, \qquad \bar{\phi}(\Delta)\leqq A\,,\\
&\log\chi_q(\bar{\phi}(\Delta))\leqq -\tfrac{1}{3}(\bar{\phi}(\Delta)-B)^2, \qquad \bar{\phi}(\Delta)\geqq B\,.
\end{aligned}
\tag{5.2.6}
$$

Furthermore, for $n\geqq 1$, (5.2.5) implies

$$
\begin{aligned}
&\log|\chi_q^{(n)}(\bar{\phi}(\Delta))|\leqq Kn!-\tfrac{1}{3}(\bar{\phi}(\Delta)-A)^2, \qquad \bar{\phi}(\Delta)\leqq \xi_q\,,\\
&\log|\chi_q^{(n)}(\bar{\phi}(\Delta))|\leqq Kn!-\tfrac{1}{3}(\bar{\phi}(\Delta)-B)^2, \qquad \bar{\phi}(\Delta)\geqq \xi_q\,.
\end{aligned}
\tag{5.2.7}
$$

Define $X=\lambda|\phi_\kappa-\xi_q|$ and break the proof into two cases, depending on whether $|\phi_\kappa|$ is very large or not.

Case 1. $X>4C^3$. We use the fact that the leading term $a_{d,q}\lambda^{d-2}(\phi-\xi_q)^d$ of $\mathscr{P}_\lambda$ dominates everything else, including the Wick counterterms. The Wick constants are $O(\log\kappa)$, so the following bound holds:

$$
\begin{aligned}
P&\equiv :\mathscr{P}_\lambda(\phi_\kappa(x)):-\tfrac{1}{2}\eta:(\phi_\kappa(x)-\xi_q)^2:-\mathscr{P}_\lambda(\xi_q)+\zeta:\delta\phi_\kappa(x)^2:\\
&\geqq a_{d,q}\lambda^{d-2}(\phi_\kappa-\xi_q)^d-\sum_{j=1}^{d-1}|a_{j,q}|\,|\phi_\kappa-\xi_q|^j\lambda^{j-2}-\tfrac{1}{2}\eta(\phi_\kappa-\xi_q)^2\\
&\quad-\sum_{j=2}^{d}\sum_{k=1}^{[j/2]}|\beta(C)|(\log\kappa)^k|\phi_\kappa-\xi_q|^{j-2k}\lambda^{j-2}\,.
\end{aligned}
\tag{5.2.8}
$$

The last term contains all the Wick counterterms. The index j runs over the degree of monomials $(\phi_\kappa-\xi_q)^j$ in $\mathscr{P}_\lambda$. Since $|a_{j,q}|\leqq C$ and $j\leqq d\leqq C$, the coefficients in the Wick counterterms all satisfy a bound depending only on C. The proposition will follow in Case 1 if we can show

$$P\geqq a(\eta,C)\lambda^{-2}-b(C)(\log\kappa)^{d/2}\,, \tag{5.2.9}$$

because $\log\chi_q(\bar{\phi}(\Delta))\leqq 0$, $\log|\chi_q^{(n)}(\bar{\phi}(\Delta))|\leqq Kn!$.

We prove (5.2.9) by establishing the following two bounds:

$$\frac{1}{2C}\lambda^{-2}X^d\geqq a(\eta,C)\lambda^{-2}+\sum_{j=1}^{d-1}CX^j\lambda^{-2}+\tfrac{1}{2}\eta X^2\lambda^{-2}\,, \tag{5.2.10}$$

$$\frac{1}{2C^3}\lambda^{-2}X^d\geqq |\beta(C)|\sup_{j,k}(\log\kappa)^kX^{j-2k}\lambda^{2k-2}-b(C)(\log\kappa)^{d/2}\,. \tag{5.2.11}$$

Cancel the common factor λ^{-2} in (5.2.10). Since $X>4C^3$, we have $\frac{1}{4C}X^d\geqq 1 \geqq a(\eta,C)$. Furthermore, $\frac{1}{4C}X^d\geqq C^2X^{d-1}\geqq \sum_{j=1}^{d-1} CX^j+\frac{1}{2}\eta X^2$, so (5.2.10) is valid.

For $k=j/2=d/2$, (5.2.11) is immediate. Otherwise, note that $X^d-M(\log\kappa)^kX^{d-2k}$ is minimized at $X\leqq M^{1/2k}(\log\kappa)^{1/2}$ so that $X^d\geqq M(\log\kappa)^kX^{d-2k}-M^{d/2k}(\log\kappa)^{d/2}$. Therefore,

$$\begin{aligned}\frac{1}{2C^3}\lambda^{-2}X^d&\geqq\frac{1}{2C^3}X^d\geqq|\beta(C)|(\log\kappa)^kX^{d-2k}-b(C)(\log\kappa)^{d/2}\\&\geqq|\beta(C)|(\log\kappa)^kX^{j-2k}\lambda^{2k-2}-b(C)(\log\kappa)^{d/2},\end{aligned}\tag{5.2.12}$$

using $X\geqq 1$, $\lambda\leqq 1$. This completes the proof of (5.2.11).

Case 2. $X\leqq 4C^3$. In this region we have a lower bound $-b(C)(\log\kappa)^{d/2}$ on the Wick counterterms, so we can work freely with unordered polynomials. We consider only $\bar\phi(\Delta)\geqq\xi_q$, as the case $\bar\phi(\Delta)\leqq\xi_q$ is essentially the same. Consider three subcases.

Case 2A. $B<\infty$, and either $\bar\phi(\Delta)>B$ or $n(\Delta)>0$. Let $\bar L=\log Kn(\Delta)!$ if $n(\Delta)>0$ or 0 if $n(\Delta)=0$. The proposition will follow from

$$\begin{aligned}&\mathscr{P}_\lambda(\phi_\kappa)-\tfrac{1}{2}\eta(\phi_\kappa-\xi_q)^2-\mathscr{P}_\lambda(\xi_q)-\log|\chi_q^{(n(\Delta))}(\bar\phi(\Delta))|+\zeta\delta\phi_\kappa^2\\&\quad\geqq a(\eta,C)\lambda^{-2}-\bar L,\end{aligned}\tag{5.2.13}$$

by virtue of the lower bound on the Wick counterterms.

Substitute $\xi_q=\lambda\xi_q(\lambda)$, $\xi=\lambda\xi'$ in (5.2.2) and divide both sides by λ^2. The left-hand side becomes $\lambda^{-2}\mathscr{P}_1(\lambda\xi')-\lambda^{-2}\mathscr{P}_1(\lambda\xi_q)=\mathscr{P}_\lambda(\xi')-\mathscr{P}_\lambda(\xi_q)$, and the right-hand side is invariant, except that ξ is replaced by ξ'. Thus (5.2.2) holds for $\mathscr{P}_\lambda$, $\xi_q(\lambda)$, and

$$\mathscr{P}_\lambda(\phi_\kappa)-\tfrac{1}{2}\eta(\phi_\kappa-\xi_q)^2-\mathscr{P}_\lambda(\xi_q)\geqq\tfrac{1}{2}\eta(\phi_\kappa-\xi_q)^2-\frac{\zeta}{4}(\phi_\kappa-B)^2.\tag{5.2.14}$$

We have used the fact that $(\phi_\kappa-B)^2>(\phi_\kappa-A)^2$ if $\phi_\kappa\leqq A$. For $\phi_\kappa\in[A,B]$, the last term could have been omitted.

From (5.2.6) and (5.2.7) we have

$$-\log|\chi_q^{(n(\Delta))}(\bar\phi(\Delta))|\geqq-\bar L+\tfrac{1}{3}(\bar\phi(\Delta)-B)^2.\tag{5.2.15}$$

Thus (5.2.13) reduces to the inequality

$$\tfrac{1}{2}\eta(\phi_\kappa-\xi_q)^2-\frac{\zeta}{4}(\phi_\kappa-B)^2+\tfrac{1}{3}(\bar\phi(\Delta)-B)^2+\zeta(\phi_\kappa-\bar\phi(\Delta))^2\geqq a(\eta,C)\lambda^{-2}.\tag{5.2.16}$$

Using $u^2+v^2\geqq\frac{1}{2}(u+v)^2$ and $\zeta\leqq\frac{1}{3}$, we have

$$\frac{\zeta}{4}(\phi_\kappa-B)^2\leqq\tfrac{1}{6}(\bar\phi(\Delta)-B)^2+\frac{\zeta}{2}(\phi_\kappa-\bar\phi(\Delta))^2.\tag{5.2.17}$$

Thus the left-hand side of (5.2.16) is bounded below by

$$\begin{aligned}&\tfrac{1}{2}\eta(\phi_\kappa-\xi_q)^2+\tfrac{1}{6}(\bar{\phi}(\Delta)-B)^2+\frac{\zeta}{2}(\phi_\kappa-\bar{\phi}(\Delta))^2\\ &\quad\geqq\tfrac{1}{2}\eta(\phi_\kappa-\xi_q)^2+\frac{\zeta}{4}(\phi_\kappa-B)^2\\ &\quad\geqq\tfrac{1}{4}\eta(\xi_q-B)^2\geqq\tfrac{1}{16}\eta C^{-2}\lambda^{-2}.\end{aligned}\tag{5.2.18}$$

We have used $\eta\leqq\frac{\zeta}{4}$ and

$$|\xi_q-B|=\tfrac{1}{2}|\xi_q-\xi_{q+1}|=\tfrac{1}{2}\lambda^{-1}|\xi_q(\lambda=1)-\xi_{q+1}(\lambda=1)|\geqq\tfrac{1}{2}\lambda^{-1}C^{-1}.\tag{5.2.19}$$

This completes Case 2A.

Case 2B. $\bar{\phi}(\Delta)\leqq B$ and $n(\Delta)=0$. The proposition will follow from

$$\mathscr{P}_\lambda(\phi_\kappa)-\tfrac{1}{2}\eta(\phi_\kappa-\xi_q)^2-\mathscr{P}_\lambda(\xi_q)+\zeta\delta\phi_\kappa^2\geqq 0.\tag{5.2.20}$$

As in (5.2.15), (5.2.2) reduces the inequality to proving the positivity of

$$\begin{aligned}&\tfrac{1}{2}\eta(\phi_\kappa-\xi_q)^2-\frac{\zeta}{4}(\phi_\kappa-B)^2+\zeta(\bar{\phi}(\Delta)-\phi_\kappa)^2, &&\phi_\kappa>B\\ &\tfrac{1}{2}\eta(\phi_\kappa-\xi_q)^2-\frac{\zeta}{4}(\phi_\kappa-A)^2+\zeta(\bar{\phi}(\Delta)-\phi_\kappa)^2, &&\phi_\kappa<A\\ &\tfrac{1}{2}\eta(\phi_\kappa-\xi_q)^2+\xi(\bar{\phi}(\Delta)-\phi_\kappa)^2, &&\phi_\kappa\in[A,B].\end{aligned}\tag{5.2.21}$$

In the first case, $|\bar{\phi}-\phi_\kappa|\geqq|\phi_\kappa-B|$ proves positivity. The third case is positive as it stands. In the second case, use $\bar{\phi}(\Delta)\geqq\xi_q$ to show $|\bar{\phi}(\Delta)-\phi_\kappa|\geqq|\phi_\kappa-A|$ and prove positivity. This completes Case 2B.

Case 2C. $B=\infty$ and $n(\Delta)\geqq 1$. We have

$$\begin{aligned}-\log|\chi_q^{(n)}(\bar{\phi}(\Delta))|&\geqq -Kn!+\tfrac{1}{3}(\bar{\phi}(\Delta)-A)^2\\ &\geqq -Kn!+\tfrac{1}{12}|\xi_q-\xi_{q-1}|^2\\ &\geqq -Kn!+\tfrac{1}{12}C^{-2}\lambda^{-2}.\end{aligned}\tag{5.2.22}$$

Thus Case 2C follows from (5.2.20). This completes the proof of Proposition 5.2.1. □

5.3. *Vacuum Energy Bound*

We begin the proof of Proposition 2.5.1 by stating a number of lemmas bounding $F_1,\dots,F_4$ and the fluctuation field. The proofs are as in [19], with only slight modifications arising from masses not equal to unity.

Lemma 5.3.1. *For any* $C>2$, $\eta\in(0,1]$ *there exists* $\tau_2(\eta,C)>0$ *such that*

$$F_1(\mathbb{Y})\geqq 6\tau_2\lambda^{-2}|\Sigma_{\mathbb{Y}}|.\tag{5.3.1}$$

The proof uses the fact that the minima are separated by at least $C^{-1}\lambda^{-1}$ to obtain $O(\lambda^{-2})$ terms from $|\nabla g|^2$ and $(g-h)^2$.

Let $\omega(x)\in[\eta/2,\bar{m}^2]$ for all x and suppose $m_q^2-\omega(x)$ is compactly supported.

Lemma 5.3.2. *For any* $C>2$, $\eta\in(0,1]$, *there exists a* $\lambda_0(\eta,C)>0$ *such that for* $\lambda\in(0,\lambda_0]$ *and* $p\in[1,O(\bar{m}/\eta)]$,

$$\int e^{-pF_3(\mathbb{Y})}d\mu_{\omega,s}(\psi)\leqq e^{\lambda F_1(\mathbb{Y})}. \tag{5.3.2}$$

Lemma 5.3.3. *There exists a constant* K_1 *such that for any* $C>2$, $\zeta\in(0,\pi^2/2^7)$, $\eta\in(0,\zeta/4]$ *there exists* $\lambda_0(\eta,C)>0$ *such that for* $\lambda\in(0,\lambda_0]$, $p\geqq 1$, *and any* κ, s, ω, Y_{ct} *with* $\operatorname{suppt}(m_q^2-\omega(x))\subseteqq\Lambda$, $Y_{ct}\subseteqq\mathbb{Y}\cap\Lambda$,

$$\begin{aligned}&\int\exp\left[\int_{Y_{ct}}\zeta:\delta\phi_\kappa^2(x):dx-pF_4(\mathbb{Y})\right]d\mu_{\omega,s}(\psi)\\&\leqq e^{K_1(|Y_{ct}|^{2}+\zeta F_1(\mathbb{Y}))}e^{p^2(1-\eta/m_q^2)F_2(\mathbb{Y})}.\end{aligned} \tag{5.3.3}$$

We next prove some estimates on the coefficients occurring in $Q_{\omega_n}(\mathbb{Y})$. Let $\Delta^1\subseteqq Y\cap\Lambda$, $h(\Delta^1)=\xi_m$ and define

$$U(\Delta^1)=\int_{\Delta^1}[:\mathscr{P}_{\lambda,\mu}(\phi(x)):-E_c^q-\tfrac{1}{2}\eta:(\phi(x)-h(x))^2:-\tfrac{1}{2}(\omega_n(x)-\eta):\psi(x)^2:]dx \tag{5.3.4}$$

$$W(t,\Delta^1)=t(U(\Delta^1)+E_c^q-E_c^m)+\int_{\Delta^1}\tfrac{1}{2}(\omega_n(x)-\eta):\psi(x)^2:dx. \tag{5.3.5}$$

See (2.3.8) and (2.4.3). Write

$$U(\Delta^1)=\sum_{j=0}^{d}\int_{\Delta^1}k_j(x):\psi(x)^j:dx. \tag{5.3.6}$$

Lemma 5.3.4. *The following bounds hold for* $\operatorname{dist}(x,\Sigma)\geqq L/2$:

$$\begin{aligned}&|k_j(x)|\leqq C\lambda^{j-2},\qquad 3\leqq j\leqq d,\\&k_2(x)=k_1(x)=0,\\&k_0(x)=E_c^m-E_c^q=O(\lambda^{-2}),\qquad h(x)=\xi_m.\end{aligned} \tag{5.3.7}$$

If $\operatorname{dist}(x,\Sigma)<L/2$, *then*

$$|k_j(x)|\leqq O(1)\lambda^{j-2}. \tag{5.3.8}$$

Proof. For $\operatorname{dist}(x,\Sigma)\geqq L/2$ we have $\psi=\psi_m=\phi-h$ for some m. By condition (iii), Sect. 2.1 we have

$$\begin{aligned}\mathscr{P}_\lambda(\psi+\xi_m)&=\lambda^{-2}\mathscr{P}_1(\lambda(\psi+\xi_m(\lambda)))=\lambda^{-2}\mathscr{P}_1(\lambda\psi+\xi_m(1))\\&=\sum_{j=3}^{d}a_{j,m}\psi^j\lambda^{j-2}+\tfrac{1}{2}m_m^2\psi^2+E_c^m\end{aligned} \tag{5.3.9}$$

and the first bound follows. The last two terms in (5.3.4) sum to $-\frac{1}{2}\omega_n(x):\psi_m(x)^2:$ $=-\frac{1}{2}m_m^2:\psi(x)^2:$, since $\omega_n(x)=m_{\sigma(x)}^2$ whenever $\operatorname{dist}(x,\Sigma)\geqq L/2$. Thus the $j=2$ and $j=1$ terms vanish. The bound on $|E_c^m-E_c^q|$ comes from (v) and the restriction $|\mu^i|\leqq C^{-1}$.

The minima of $\mathscr{P}_\lambda$ are separated by $O(\lambda^{-1})$, and hence $|g-\xi_q|$ is $O(\lambda^{-1})$. Thus

$$\mathscr{P}_\lambda(\phi)-E_c^q=\lambda^{-2}\mathscr{P}_1(\lambda\psi+O(1)+\xi_q(1))-E_c^q.$$

Expanding in terms of ψ, we find that each monomial ψ^j has a coefficient $O(\lambda^{j-2})$. The same is true for the other terms in U. This completes the proof. $\square$

Define for $\Delta^1 \subsetneqq Y \cap \Lambda$

$$W_\kappa(t, \Delta^1) = t \int_{\Delta^1} [:\mathscr{P}_{\lambda,\mu}(\phi_\kappa(x)): - E_c^m - \tfrac{1}{2}\eta :(\phi_\kappa(x) - h(x))^2:]dx$$
$$+ \int_{\Delta^1} \tfrac{1}{2}(1-t)(\omega_n(x) - \eta) :\psi_\kappa(x)^2: dx, \tag{5.3.10}$$

and let $\delta W_\kappa(t, \Delta^1) = W(t, \Delta^1) - W_\kappa(t, \Delta^1)$. Then with

$$\begin{aligned} \hat{k}_j(x) &= k_j(x), \quad j \neq 2, \\ \hat{k}_2(x) &= -\tfrac{1}{2}(\omega_n(x) - \eta), \end{aligned} \tag{5.3.11}$$

we have that δW_κ is a sum of terms $\hat{k}_j(:\psi^j: - :(\psi_\kappa + g_\kappa - g)^j:)$. Expanding these yields

$$\delta W_\kappa(t, \Delta^1) = t \sum_{j=1}^{d} \int_{\Delta^1} \hat{k}_j(x)(:\psi(x)^j: - :\psi_\kappa(x)^j:)dx$$
$$+ t \sum_{j=0}^{d-1} \int_{\Delta^1} \delta\hat{k}_j(x, \kappa) :\psi_\kappa(x)^j: dx$$
$$+ \int_{\Delta^1} \tfrac{1}{2}(1-t)(\omega_n(x) - \eta)(:\psi(x)^2: - :\psi_\kappa(x)^2:)dx, \tag{5.3.12}$$

where

$$\delta\hat{k}_n(x, \kappa) = - \sum_{j=n+1}^{d} \hat{k}_j(x) \binom{j}{n} (g_\kappa(x) - g(x))^{j-n}. \tag{5.3.13}$$

Lemma 5.3.5. *For all $p < \infty$ there exists $\varepsilon(p, C) > 0$ such that the following bounds hold:*

$$\begin{aligned} |\hat{k}_j(x)| &\leqq O(1), \quad j > 1 \quad \text{and} \quad \operatorname{dist}(x, \Sigma) \geqq L/2, \\ \delta\hat{k}_j(x) &= \hat{k}_1(x) = 0, \quad \operatorname{dist}(x, \Sigma) \geqq L/2, \\ |\hat{k}_j(x)| &\leqq O(1)\lambda^{j-2}, \\ \|\delta\hat{k}_j\|_{L^p(\Delta^1)} &\leqq O(1)\lambda^{-2}\kappa^{-\varepsilon}. \end{aligned} \tag{5.3.14}$$

Proof. The bounds on $\hat{k}_j$ follow from Lemma 5.3.4. Since $g(x) = \text{const}$ for $\operatorname{dist}(x, \Sigma) \geqq L/2$, $\varrho_\kappa \chi_{\Delta^1} = \chi_{\Delta^1}$ implies $g_\kappa = g$ and $\delta\hat{k}_j = 0$. The last bound follows from

$$\|\hat{k}_j(g_\kappa - g)^{j-n}\|_{L^p} \leqq |\hat{k}_j| \lambda^{-(j-n)} \|g_\kappa(\lambda = 1) - g(\lambda = 1)\|_{L^{p(j-n)}}^{j-n} \leqq O(1)\lambda^{n-2}\kappa^{-\varepsilon}. \tag{5.3.15}$$

The bound on $\|g_\kappa - g\|$ follows from properties of the momentum cutoff, see [19]. $\square$

Lemma 5.3.6. *There exists $K_2(\omega, C)$ and $\delta(C) > 0$ with the following property. Let $\{m(\Delta^1) : \Delta^1 \subsetneqq Y \cap \Lambda\}$ be a set of nonnegative integers and let $\{\kappa(\Delta^1) : \Delta^1 \subsetneqq Y \cap \Lambda\}$ be a set of positive numbers with $\kappa(\Delta^1)^\delta \geqq \lambda^{-2}$ for* $\operatorname{dist}(\Delta^1, \Sigma) \leqq L/2$. *Then*

$$\left| \int \prod_{\Delta^1 \subsetneqq Y \cap \Lambda} \delta W_{\kappa(\Delta^1)}(t, \Delta^1)^{m(\Delta^1)} d\mu_{\omega,s}(\psi) \right|$$
$$\leqq \prod_{\Delta^1 \subsetneqq Y \cap \Lambda} [(dm(\Delta^1))! (K_2 \kappa(\Delta^1)^{-\delta})^{m(\Delta^1)}]. \tag{5.3.16}$$

Proof. We require $C_\omega(s)$ to have fractional derivatives in some L^q. This follows from the Neumann series representation for (2.4.1)

$$C_\omega(s)=C_{\bar{m}^2}(s)\sum_{n=0}^{\infty}[(\bar{m}^2-\omega)C_{\bar{m}^2}(s)]^n \tag{5.3.17}$$

as in [19]. Convergence follows from the fact that $0<\omega(x)\leqq\bar{m}^2$ and $\|C_{\bar{m}^2}(s)\|\leqq\frac{1}{\bar{m}^2}$. This representation also shows that both the kernel and the operator $C_\omega(s)$ are positive, increasing in s, and decreasing in ω. The bounds of Lemma 5.3.5 suffice to complete the proof as in [7]. □

We establish a weaker version of Proposition 2.5.1 and then recover the full version using a perturbation argument. Let Y_d be the union of the unit lattice squares of $Y\cap\Lambda$ that satisfy $\mathrm{dist}(\Delta^1,\Sigma)\geqq L/2$ and $n(\Delta^1)=0$. For each $\Delta^1\subseteqq Y_d$, introduce a parameter $t(\Delta^1)\in[0,1]$. Let Y_t be the union of all $\Delta^1\subseteqq Y_d$ with $t(\Delta^1)\neq 0$ and let $Y_c=(Y\cap\Lambda)\backslash Y_d$. Define

$$\begin{aligned}&\tilde{U}(\Delta^1)=U(\Delta^1)+E_c^q-E_c^{p(\Delta^1)},\\&\tilde{U}(t,Y)=\sum_{\Delta^1\subseteqq Y_d}t(\Delta^1)\tilde{U}(\Delta^1)+\sum_{\Delta^1\subseteqq Y_c}\tilde{U}(\Delta^1).\end{aligned}\tag{5.3.18}$$

Proposition 5.3.7. *Under the hypotheses of Proposition 2.5.1, but with* $p\in\left[1,1+\frac{\eta}{10\bar{m}}\right]$,

$$\left\|\chi^{(\,)}_{\Sigma\cap Y}e^{-\tilde{U}(t,Y)-\sum_\iota F_\iota(Y)}\right\|_{L^p(d\mu_{\omega_n,s}(\psi))}\leqq\prod_{\Delta^1}n(\Delta^1)!\,e^{-3\tau_2\lambda^{-2}(|\Sigma|+|\Sigma'|)}e^{al^2|Y_t|}. \tag{5.3.19}$$

Proof. We follow [19] closely. Let $Y_{ct}=Y_c\cup Y_t$, put $t(\Delta)=1$ for $\Delta\subseteqq Y_c$, and define

$$W(t,Y)=\tilde{U}(t,Y)+\frac{1}{2}\int_{Y_{ct}}(\omega_n(x)-\eta):\psi(x)^2:dx=\sum_{\Delta}W(t,\Delta). \tag{5.3.20}$$

Then with $\omega=\omega_n-p(\omega_n-\eta)\chi_{Y_{ct}}$,

$$\begin{aligned}&\left\|\chi^{(\,)}e^{-\tilde{U}(t,Y)-\sum_i F_\iota}\right\|_{L^p(d\mu_{\omega_n,s}(\psi))}\\&=Z_{\omega_n\omega}(s)^{1/p}\left\|\chi^{(\,)}e^{-W(t,Y)-\sum_\iota F_i}\right\|_{L^p(d\mu_\omega,s(\psi))}\\&\leqq Z_{\omega_n\omega}(s)^{1/p}e^{-(F_1+F_2)}\|e^{-F_3}\|_{L^{q'p}}\|\chi^{(\,)}e^{-W(t,Y)-F_4}\|_{L^{pq}}.\end{aligned}\tag{5.3.21}$$

Here $q\lesssim 1+\frac{\eta}{10\bar{m}}$, q' is even, and $q'=q/(q-1)\leqq pq'\leqq O(\eta/\bar{m})$. The F_3 integral is bounded by Lemma 5.3.2 and $Z_{\omega_n\omega}(s)$ is bounded in Sect. 5.5, yielding the estimate

$$e^{-(1-\lambda)F_1-F_2}e^{K(\eta,\bar{m})|Y_{ct}|}\|e^{-(W(t,Y)+F_4-\log|\chi^{(\,)}|)}\|_{L^{pq}}. \tag{5.3.22}$$

The last factor can be bounded using the Wick bounds of Sect. 5.2. For each $\Delta^1\subseteqq Y_{ct}$ we specify a positive integer $i(\Delta^1)$. We take $\kappa_{i(\Delta)}=2^{i(\Delta)}$ and sum over all sets

$\{i(\Delta)\}$ subject to the restriction that $\kappa_{i(\Delta)}^{\delta} \geqq \lambda^{-2}$ for $\operatorname{dist}(\Delta, \Sigma) \leqq L/2$. For each $\{i(\Delta)\}$ we find a bound that applies only to the subset of $\mathscr{S}'$ such that for all $\Delta \subseteqq Y_{ct}$

$$\begin{aligned}
&(W(t,\Delta)+\zeta:\delta\phi_\kappa^2(\Delta):-\log|\chi^{(\prime)}(\phi(\Delta))|)\in \\
&\quad[-b(\log\kappa_{i(\Delta)+1})^{d/2}-1,\,-b(\log\kappa_{i(\Delta)})^{d/2}-1]\,, \qquad n(\Delta)=0\,,\\
&\quad[-b(\log\kappa_{i(\Delta)+1})^{d/2}+a\lambda^{-2}-\log Kn(\Delta)!-1,\\
&\quad\ -b(\log\kappa_{i(\Delta)})^{d/2}+a\lambda^{-2}-\log Kn(\Delta)!-1]\,, \qquad n(\Delta)>0\,.
\end{aligned}\tag{5.3.23}$$

When $i(\Delta)$ assumes its minimum value, we omit the upper limit.

On this subset, and for $i(\Delta)$ not minimal, we have

$$\begin{aligned}
&W(t,\Delta)+\zeta:\delta\phi_\kappa^2(\Delta):-\log|\chi^{(\prime)}(\phi(\Delta))|\\
&\quad\leqq -b(\log\kappa_{i(\Delta)})^{d/2}-1+(a\lambda^{-2}-\log Kn(\Delta)!,\ n(\Delta)>0)\\
&\quad\leqq W_\kappa(t,\Delta)+\zeta:\delta\phi_\kappa^2(\Delta):-\log|\chi^{(\prime)}(\phi(\Delta))|-1\,,
\end{aligned}\tag{5.3.24}$$

where we have used the Wick bound

$$\begin{aligned}
&t(\Delta)(:\mathscr{P}(\phi_\kappa):-\tfrac{1}{2}\eta:(\phi_\kappa-\xi)^2:-E_c^m)+\zeta:\delta\phi_\kappa^2:-\log|\chi_m^{(\prime)}(\phi(\Delta))|\\
&\quad+\tfrac{1}{2}(1-t(\Delta))(\omega_n-\eta):\psi^2:\\
&\geqq -b(\log\kappa_{i(\Delta)})^{d/2}+(a\lambda^{-2}-\log Kn(\Delta)!,\ n(\Delta)>0)\,.
\end{aligned}\tag{5.3.25}$$

For $n(\Delta)>0$ we have $t(\Delta)=1$ and this is Proposition 5.2.1. For $n(\Delta)=0$ the additional terms satisfy

$$(1-t(\Delta))(:\delta\phi_\kappa^2:-\log\chi_m(\phi(\Delta))+\tfrac{1}{2}(\omega_n-\eta):\psi^2:)\geqq -b'\log\kappa$$

and (5.3.25) follows using $b=b(C)+b'$.

Equation (5.3.24) implies that

$$1\leqq|\delta W_\kappa(t,\Delta)|\leqq\delta W_\kappa(t,\Delta)^{m(\Delta)}\tag{5.3.26}$$

for m any positive even integer. We choose $m(\Delta)\gtrsim\kappa_{i(\Delta)}^{\delta/2d}/q'$, where δ is given by Lemma 5.3.6. For $i(\Delta)$ minimal, put $m(\Delta)=0$. Applying the lower bounds in (5.3.23), we obtain

$$\begin{aligned}
&\|e^{-W(t,Y)-\log|\chi^{(\prime)}|-F_4}\|_{L^{pq}}^{pq}\leqq\prod_{\Delta:n(\Delta)>0}(Kn(\Delta)!e^{-a\lambda^{-2}})^{pq}\\
&\quad\cdot\sum_{\{i(\Delta)\}}\left|\int\prod_{\Delta\subseteqq Y_{ct}}[\delta W_{\kappa_{i(\Delta)}}(t,\Delta)^{m(\Delta)}e^{pq(b(\log\kappa_{i(\Delta)+1})^{d/2}+1+\zeta:\delta\phi_\kappa^2(\Delta):)}e^{-pqF_4}]d\mu_{\omega,s}(\psi)\right|\\
&\leqq\left(\prod_\Delta n(\Delta)!e^{-a'\lambda^{-2}|\Sigma'|}\right)^{pq}\left\|e^{\zeta\int_{Y_{ct}}:\delta\phi_\kappa^2(x):dx-F_4(Y)}\right\|_{L^{pq^2}}^{pq}\\
&\quad\cdot\sum_{\{i(\Delta)\}}\left\|\prod_{\Delta\subseteqq Y_{ct}}\delta W_{\kappa_{i(\Delta)}}(t,\Delta)^{m(\Delta)}\right\|_{L^{q'}}\prod_{\Delta\subseteqq Y_{ct}}e^{b_1(\log\kappa_{i(\Delta)})^{d/2}}\\
&\leqq\left(\prod_\Delta n(\Delta)!e^{-a'\lambda^{-2}|\Sigma'|}\right)^{pq}e^{K_1(|Y_{ct}|l^2+pq^2\zeta F_1)/q}e^{p^2q^3(1-\eta)F_2}\\
&\quad\cdot\prod_{\Delta\subseteqq Y_{ct}}\Bigg[\sum_{i(\Delta)>i_{\min}(\Delta)}[(d\kappa_{i(\Delta)}^{\delta/2d}+2)!(K_2\kappa_{i(\Delta)}^{-\delta})^{\kappa_{i(\Delta)}^{\delta/2d}}]^{1/q'}\\
&\quad\cdot e^{b_1(\log\kappa_{i(\Delta)})^{d/2}}+e^{b_1(\log\kappa_{i(\Delta)})^{d/2}}\Bigg]\,.
\end{aligned}\tag{5.3.27}$$

We have applied Lemma 5.3.3 to the F_4 integral and Lemma 5.3.6 to the δW_κ integrals.

The $i(\Delta)$ sum converges, and the term $i(\Delta)=i_{\min}(\Delta)$ is less than $e^{K_5(C)|\log\lambda|^{d/2}}$ for $\mathrm{dist}(\Delta,\Sigma)\leqq L/2$. Thus the product in (5.3.27) is bounded by

$$O(1)^{|Y_{ct}|l^2}\prod_{\Delta:\,\mathrm{dist}(\Delta,\Sigma)\leqq L/2} e^{K_5|\log\lambda|^{d/2}}\leqq e^{O(1)|Y_{ct}|l^2}e^{\lambda F_1}\,. \tag{5.3.28}$$

We have used Lemma 5.3.1 to show that

$$L^2K_5|\log\lambda|^{d/2}|\Sigma|\leqq 6\tau_2\lambda^{-1}|\Sigma|\leqq\lambda F_1\,.$$

Combining (5.3.27) and (5.3.22), we obtain

$$\begin{aligned}&\left\|\chi^{(\cdot)}e^{-U(t,Y)-\sum_i F_i}\right\|_{L^p}\\&\leqq\prod_\Delta n(\Delta)!\,e^{-a'\lambda^{-2}|\Sigma'|}e^{O(1)|Y_{ct}|l^2}e^{-(1-2\lambda-\zeta K_1)F_1}e^{-(1-pq^2(1-\eta))F_2}\,.\end{aligned}$$

The difference $|Y_{ct}|l^2-|Y_t|l^2$ can be absorbed with a decrease in a' and the loss of another factor λF_1. Take ζ less than $\pi^2/2^7pq$ and small enough so that $1-3\lambda-\zeta K_1\geqq\frac{2}{3}$. Since $pq^2<1+\dfrac{\eta}{2}$, the coefficient of F_2 is negative. Since $F_2>0$, we may drop the F_2 factor. This completes the proof. □

Proof of Proposition 2.5.1. The change from $\tilde U(1,Y)$ to $Q_{\omega_n}(Y)$ produces the factor $\exp\left(\sum_m l^2(E_c^q-E_c^m)|Y|_m\right)$ in (2.5.3), see (5.3.18). To obtain a factor λ in front of the volume, apply the identity

$$e^{-p\tilde U(\Delta)}=1-p\tilde U(\Delta)\int_0^1 e^{-pt(\Delta)\tilde U(\Delta)}dt \tag{5.3.29}$$

for each $\Delta\subseteqq Y_d$, as in [19]. In each term of the resulting sum, separate the $p\tilde U(\Delta)$ factors from the exponential with Hölder's inequality. Lemma 5.3.4 bounds the coefficients in $\tilde U$ $(\Delta\subseteqq Y_d)$ by $O(\lambda)$, so the $\tilde U$-integral is bounded by $O(\lambda)^{l^2|Y_t|}$. The preceding proposition bounds the other integral. Thus for $p\in\left[1,1+\dfrac{\eta}{30\bar m}\right]$,

$$\begin{aligned}&\|\chi_\Sigma^{(\cdot)}e^{-Q_{\omega_n}(Y)}\|_{L^p}\\&\leqq e^{\sum_m l^2(E_c^q-E_c^m)|Y|_m}e^{-3\tau_2\lambda^{-2}(|\Sigma|+|\Sigma'|)}\prod_{\Delta^1}n(\Delta^1)!\prod_{\Delta^1\subseteqq Y_d}(1+O(\lambda))\,,\end{aligned} \tag{5.3.30}$$

and the proof is complete. □

5.4. *Lower Bounds for Z_{Δ^m} and Z_Σ*

In this section we prove Proposition 2.5.2 and a lower bound on $Z_\Sigma(\mathbb{V})$ which will be important in the proof of Proposition 2.5.6, Sect. 5.7.

Proof of Proposition 2.5.2. By (2.4.16) we need to show that for an l-lattice square $\Delta\subseteqq\Lambda$

$$\left(\int\chi_{\Sigma\equiv m}e^{-\int_\Delta\left(:\mathscr{P}(\phi(x)):-\frac{m_m^2}{2}:\psi_m(x)^2:-E_c^m\right)dx}d\mu_{m_m^2,\partial\Delta}(\psi_m)\right)^{\pm1}\leqq e^{a(\eta,C)\lambda l^2}. \tag{5.4.1}$$

Write the integral as

$$\begin{aligned}&\int \chi_{\Sigma\equiv m} e^{-\tilde{U}(\Delta)} d\mu_{m_{\tilde{m}}^2,\partial\Delta}(\psi_m)\\&\quad =1+\int \chi_{\Sigma\equiv m}(e^{-\tilde{U}(\Delta)}-1)d\mu_{m_{\tilde{m}}^2,\partial\Delta}(\psi_m)+\int(\chi_{\Sigma\equiv m}-1)d\mu_{m_{\tilde{m}}^2,\partial\Delta}(\psi_m). \qquad (5.4.2)\end{aligned}$$

As in (5.3.29)–(5.3.30), the second term is bounded by

$$\prod_{\Delta^1\subseteq\Delta}(1+O(\lambda))-1\leqq O(\lambda l^2). \qquad (5.4.3)$$

Write the third term as $-\sum_{\Sigma\neq m}\int \chi_\Sigma d\mu_{m_{\tilde{m}}^2,\partial\Delta}(\psi_m)$ and observe that for $n\neq m$

$$\begin{aligned}&\zeta:\delta\phi_\kappa^2(x):-\log\chi_n(\bar{\phi}(\Delta))+\tfrac{1}{2}(m_m^2-\eta):\psi_m(x)^2:\\&\geqq -b\log\kappa+\frac{\eta}{2}\begin{cases}(\phi_\kappa-\xi_m)^2+(\phi_\kappa-\bar{\phi})^2+(\bar{\phi}-\frac{1}{2}(\xi_m+\xi_n))^2, & |\bar{\phi}-\xi_m|\leqq\frac{1}{2}C^{-1}\lambda^{-1}\\(\phi_\kappa-\xi_m)^2+(\phi_\kappa-\bar{\phi})^2, & |\bar{\phi}-\xi_m|\geqq\frac{1}{2}C^{-1}\lambda^{-1}\end{cases}\\&\geqq -b\log\kappa+O(\lambda^{-2}). \qquad (5.4.4)\end{aligned}$$

Thus (5.3.25) holds for $t(\Delta)\equiv 0$, $\chi_m^{(\prime)}$ replaced by χ_n, and with a term $a\lambda^{-2}$. With $Y_t=\Delta$, $F_i=0$, the remainder of the proof of Proposition 5.3.7 can be applied to yield

$$\int \chi_\Sigma d\mu_{m_{\tilde{m}}^2,\partial\Delta}(\psi_m)\leqq \prod_{\Delta^1:\Sigma(\Delta^1)\neq m} e^{-O(\lambda^{-2})}. \qquad (5.4.5)$$

Thus

$$\sum_{\Sigma\neq m}\int \chi_\Sigma d\mu_{m_{\tilde{m}}^2,\partial\Delta}(\psi_m)\leqq(1+e^{-O(\lambda^{-2})})^{l^2}-1\leqq e^{-O(\lambda^{-2})}. \qquad (5.4.6)$$

This completes the proof. □

Proposition 5.4.1. *Under the conditions of Proposition 2.5.1,*

$$Z_\Sigma(\mathbb{V})e^{E_{\mathbb{W}}^{p(\mathbb{V})}l^2|\mathbb{V}|}\geqq e^{-a(\eta,C)\lambda^{-2}l^2|\mathbb{V}|}. \qquad (5.4.7)$$

Here a is independent of Σ for Σ compatible with $\mathbb{V}$.

Proof. This lower bound is very weak because of possible phase boundaries. Nevertheless it is essential in the proof of Proposition 2.5.6.

Write $p(\mathbb{V})=p$, $d\mu_{m_{\tilde{p}}^2,\partial\mathbb{V}}(\psi_p)=d\psi_p$. By (2.4.16), we need a lower bound on $\int\chi_\Sigma e^{-V_{p}(\mathbb{V})}d\psi_p$. Recall that

$$\chi_\sigma(\xi+\xi_p)=\pi^{-1/2}\int_{(\xi_{\sigma-1}+\xi_\sigma)/2-\xi_p}^{(\xi_\sigma+\xi_{\sigma+1})/2-\xi_p} e^{-(\xi-z)^2}dz. \qquad (5.4.8)$$

Choose an interval $[z_0, z_0+1]$, $z_0=O(\lambda^{-1})$ in the range of z-integration. Since $e^{-(\xi-z)^2}\geqq e^{-2\xi^2}e^{-O(\lambda^{-2})}$ for z in the interval, we have a lower bound on χ_σ:

$$\chi_\sigma(\xi+\xi_p)\geqq e^{-2\xi^2}e^{-O(\lambda^{-2})}. \qquad (5.4.9)$$

Thus

$$\int\chi_\Sigma e^{-V_p(\mathbb{V})}d\psi_p\geqq e^{-O(\lambda^{-2})l^2|\mathbb{V}|}\int\prod_{\Delta^1\subseteq\mathbb{V}} e^{-2\psi(\Delta^1)^2}e^{-V_p(\mathbb{V})}d\psi_p. \qquad (5.4.10)$$

By Jensen's inequality, the integral on the right is bounded below by

$$\exp\left(-\int\left(\sum_{\Delta^1\subseteq \mathbb{V}} 2\psi(\Delta^1)^2+V_p(\mathbb{V})\right)d\psi_p\right)\geqq e^{-O(1)l^2|\mathbb{V}|}, \tag{5.4.11}$$

completing the proof. □

5.5. Estimates on Mass Shift Normalization Factors

We need estimates on the factors $Z_{\omega_k\omega_{k+1}}(s)$ arising in the expansion of Sect. 2.4. It is important to get the correct volume dependence and to estimate the deviation as a surface effect.

Recall that

$$Z_{\omega_1\omega_2}(s)=\int e^{\int(\omega_1(x)-\omega_2(x)):\psi(x)^2:dx}d\mu_{\omega_1,s}(\psi), \tag{5.5.1}$$

where the Wick order is with respect to the free covariance with mass $m_{\bar{q}}\equiv\sqrt{\omega_0}$. In this section only, we shall use ordinary units (not l- or l^2-units) to measure lengths and areas.

Proposition 5.5.1. *Let $\omega_1(x)$, $\omega_2(x)$ be constant on unit lattice squares and lie in the range $[\eta/2,\bar{m}^2]$ for all x. Let D be a finite union of unit lattice squares, and suppose $\omega_1(x)=\omega_2(x)$, $x\notin D$ and $\omega_i(x)=\bar{\omega}_i$, $x\in D$. Let $s=\{s_b\}$ be an arbitrary set of decoupling parameters for the bonds of the l-lattice, subject to the requirement that $s_b=1$ for b intersecting the interior of D. Finally, suppose $|\bar{\omega}_2-\bar{\omega}_1|\leqq\hat{\omega}_1-\eta/4$, where $\hat{\omega}_1=\inf_x\omega_1(x)$. Then*

$$\left|\log Z_{\omega_1\omega_2}(s)-\left(\frac{\bar{\omega}_2}{8\pi}\log\frac{\bar{\omega}_2}{\omega_0}-\frac{\bar{\omega}_1}{8\pi}\log\frac{\bar{\omega}_1}{\omega_0}-\frac{\bar{\omega}_2-\bar{\omega}_1}{8\pi}\right)|D|\right|\leqq O(1)|\partial D|, \tag{5.5.2}$$

where $O(1)$ depends only on η and $\bar{m}$. Here $|D|$ is the volume of D and $|\partial D|$ is the length of the boundary of D.

Proof. We have the formula

$$\begin{aligned}\log Z_{\omega_1\omega_2}(s)&=-\tfrac{1}{2}\mathrm{tr}\log(1-(\omega_1-\omega_2)C_{\omega_1}(s))-\tfrac{1}{2}\mathrm{tr}(\omega_1-\omega_2)C_{\omega_0}\\&=\sum_{n=2}^{\infty}\frac{1}{2n}\mathrm{tr}((\omega_1-\omega_2)C_{\omega_1}(s))^n+\tfrac{1}{2}\mathrm{tr}(\omega_1-\omega_2)(C_{\omega_1}(s)-C_{\omega_0}),\end{aligned} \tag{5.5.3}$$

where C_{ω_0} is the free covariance. Convergence follows from our assumed bound on $|\omega_1-\omega_2|$ and $C_{\omega_1}(s)\leqq(-\Delta+\hat{\omega}_1)^{-1}$. We begin by comparing each term of this sum with the corresponding term after replacing $C_{\omega_1}(s)$ with the free covariance $C_{\bar{\omega}_1}$. Using $C_{\omega_1}(s)\leqq C_{\hat{\omega}_1}$, $C_{\bar{\omega}_1}\leqq C_{\hat{\omega}_1}$, we have

$$\begin{aligned}&\left|\frac{1}{n}\mathrm{tr}[(\omega_1-\omega_2)C_{\omega_1}(s)]^n-\frac{1}{n}\mathrm{tr}[(\omega_1-\omega_2)C_{\bar{\omega}_1}]^n\right|\\&\leqq\frac{1}{n}\sum_{j=0}^{n-1}\mathrm{tr}(\bar{\omega}_1-\bar{\omega}_2)^nC_{\bar{\omega}_1}^j|C_{\omega_1}(s)-C_{\bar{\omega}_1}|\chi_D C_{\omega_1}(s)^{n-j-1}\\&\leqq|\bar{\omega}_1-\bar{\omega}_2|^n\,\mathrm{tr}\,C_{\hat{\omega}_1}^{n-1}|C_{\omega_1}(s)-C_{\bar{\omega}_1}|\chi_D\\&\leqq|\bar{\omega}_1-\bar{\omega}_2|^n(\mathrm{tr}\,C_{\hat{\omega}_1}^{n-1}|C_{\bar{\omega}_1}(0)-C_{\bar{\omega}_1}|+\mathrm{tr}\,C_{\hat{\omega}_1}^{n-1}|C_{\omega_1}(s)-C_{\bar{\omega}_1}(s)|\chi_D).\end{aligned} \tag{5.5.4}$$

Here $C_{\omega_1}(0)$ has Dirichlet boundary conditions on ∂D, C_{ω_1} is the free covariance, and χ_D is the operator of multiplication by the characteristic function of D.

Standard estimates on the Wiener integral representation for $(C_{\bar{\omega}_1}(0)-C_{\bar{\omega}_1})(x,y)$ [17, 27] yield a decay $\exp(-c(|x-y|+\mathrm{dist}(y,\partial D)))$ for some $c>0$. Thus

$$\|C_{\bar{\omega}_1}(0)(\cdot,y)-C_{\bar{\omega}_1}(\cdot,y)\|_{L^2(dx)} \leqq K e^{-c\,\mathrm{dist}(y,\partial D)}. \tag{5.5.5}$$

The covariances $C_{\hat{\omega}_1}$ map L^2 to $L^\infty \cap L^2$, so

$$C_{\hat{\omega}_1}^{n-1}|C_{\bar{\omega}_1}(0)-C_{\bar{\omega}_1}|(x,y) \leqq K\hat{\omega}_1^{-n+2} e^{-c\,\mathrm{dist}(y,\partial D)} \tag{5.5.6}$$

and hence

$$\sum_{n=2}^{\infty} |\bar{\omega}_1-\bar{\omega}_2|^n \operatorname{tr} C_{\hat{\omega}_1}^{n-1}|C_{\bar{\omega}_1}(0)-C_{\bar{\omega}_1}| \leqq K|\partial D|. \tag{5.5.7}$$

The second term in (5.5.4) is handled by putting $\omega_t = t\omega_1+(1-t)\bar{\omega}$ and observing that

$$|C_{\omega_1}(s)-C_{\bar{\omega}_1}(s)| = \left|\int_0^1 -C_{\omega_t}(s)(\omega_1-\bar{\omega}_1)C_{\omega_t}(s)dt\right| \leqq K\int_0^1 C_{\omega_t}(s)\chi_{\sim D}C_{\omega_t}(s)dt. \tag{5.5.8}$$

Since

$$\|\chi_{\sim D}C_{\omega_t}(s)\chi_D(\cdot,y)\|_{L^2(dx)} \leqq K e^{-c\,\mathrm{dist}(y,\partial D)}, \tag{5.5.9}$$

the second term is also bounded by $K|\partial D|$.

In a similar fashion we can prove

$$\operatorname{tr}(\omega_1-\omega_2)|C_{\omega_1}(s)-C_{\bar{\omega}_1}| \leqq K|\partial D|. \tag{5.5.10}$$

We have expressed $\log Z_{\omega_1\omega_2}(s)$ as

$$\sum_{n=2}^{\infty} \frac{(\bar{\omega}_1-\bar{\omega}_2)^n}{2n} \operatorname{tr}(\chi_D C_{\bar{\omega}_1})^n + \tfrac{1}{2}(\bar{\omega}_1-\bar{\omega}_2)\operatorname{tr}\chi_D(C_{\bar{\omega}_1}-C_{\omega_0}) \tag{5.5.11}$$

up to an error $K|\partial D|$. We next control the substitution $(\chi_D C_{\bar{\omega}_1})^{n-1} \to C_{\bar{\omega}_1}^{n-1}$. Since $C_{\bar{\omega}_1}-\chi_D C_{\bar{\omega}_1} = \chi_{\sim D}C_{\bar{\omega}_1}$, we obtain a sum of terms

$$\frac{(\bar{\omega}_1-\bar{\omega}_2)^n}{2n} \operatorname{tr}(\chi_D C_{\bar{\omega}_1})^j \chi_{\sim D} C_{\bar{\omega}_1}^{n-j}, \qquad j=1,\dots,n-1 \tag{5.5.12}$$

as in (5.5.4). By (5.5.9), the sum of these terms is bounded by $K|\partial D|$.

With just one χ_D in each trace, we can use translation invariance to divide by $|D|$ simply:

$$\frac{1}{|D|}(\log Z_{\omega_1\omega_2}(s)+O(|\partial D|)) = \sum_{n=2}^{\infty} \frac{(\bar{\omega}_1-\bar{\omega}_2)^n}{2n} C_{\bar{\omega}_1}^n(0,0) + \tfrac{1}{2}(\bar{\omega}_1-\bar{\omega}_2)(C_{\omega_1}-C_{\omega_0})(0,0). \tag{5.5.13}$$

We calculate the right-hand side in momentum space:

$$\sum_{n=2}^{\infty} \frac{(\bar{\omega}_1-\bar{\omega}_2)^n}{2n} \int \frac{d^2p}{(2\pi)^2} \frac{1}{(p^2+\bar{\omega}_1)^n} + \tfrac{1}{2}(\bar{\omega}_1-\bar{\omega}_2) \int \frac{d^2p}{(2\pi)^2} \left(\frac{1}{p^2+\omega_1} - \frac{1}{p^2+\omega_0} \right)$$
$$= \frac{\bar{\omega}_1}{8\pi} \sum_{n=2}^{\infty} \frac{(\bar{\omega}_1-\bar{\omega}_2)^n}{n(n-1)\bar{\omega}_1^n} + \frac{\bar{\omega}_1-\bar{\omega}_2}{8\pi} \log\frac{\omega_0}{\omega_1}. \tag{5.5.14}$$

Since

$$\sum_{n=2}^{\infty} \frac{x^n}{n(n-1)} = (1-x)\log(1-x)+x \quad \text{for} \quad |x|<1, \tag{5.5.15}$$

this is equal to

$$\frac{\bar{\omega}_2}{8\pi}\log\frac{\bar{\omega}_2}{\omega_0} - \frac{\bar{\omega}_1}{8\pi}\log\frac{\bar{\omega}_1}{\omega_0} - \frac{\bar{\omega}_2-\bar{\omega}_1}{8\pi}, \tag{5.5.16}$$

which completes the proof. □

Corollary 5.5.2. *The bound* (5.5.2) *of Proposition 5.5.1 holds without the restriction on* $|\bar{\omega}_2-\bar{\omega}_1|$.

Proof. The elementary identity

$$Z_{\omega_1\omega_3}(s) = Z_{\omega_1\omega_2}(s) Z_{\omega_2\omega_3}(s) \tag{5.5.17}$$

allows us to write $Z_{\omega_1\omega_2}(s)$ as a product of Z's for which Proposition 5.5.1 is applicable. The terms

$$\frac{\omega}{8\pi}\log\frac{\omega}{\omega_0} - \frac{\omega}{8\pi}$$

cancel for the intermediate ω's. □

This trick also proves the bound

$$Z_{\omega_n\omega}(s) \leqq e^{K(\eta,\bar{m})|Y_{ct}|} \tag{5.5.18}$$

used in Eqs. (5.3.21)–(5.3.22).

Proposition 5.5.3. *Under the conditions of Proposition 5.5.1, let* $\omega_1(x)$, $\omega_2(x)$, *and* ω_0 *depend on a parameter* μ *in such a way that* $\sup_x \frac{\partial}{\partial \mu}\omega_i(x) \leqq C$. *Then*

$$\left| \frac{\partial}{\partial \mu} \left[\log Z_{\omega_1\omega_2}(s) - \left(\frac{\bar{\omega}_2}{8\pi}\log\frac{\bar{\omega}_2}{\omega_0} - \frac{\bar{\omega}_1}{8\pi}\log\frac{\bar{\omega}_1}{\omega_0} - \frac{\bar{\omega}_2-\bar{\omega}_1}{8\pi} \right)|D| \right] \right| \leqq O(1)|\partial D|. \tag{5.5.19}$$

Proof. Expand the above difference as in the proof of Proposition 5.5.1, and differentiate each term. Derivatives of $(\bar{\omega}_1-\bar{\omega}_2)$ factors do not affect the estimates. For derivatives of covariances, we apply

$$\frac{\partial}{\partial \mu} C_\omega(s) = -C_\omega(s) \frac{\partial \omega}{\partial \mu} C_\omega(s). \tag{5.5.20}$$

(This formula depends on the fact that $\bar{m}$ is independent of μ.) When $C_\omega(s)$ is an isolated covariance (one that has not been finite-differenced) we still have a bounded operator from L^2 to $L^2 \cap L^\infty$ and the surface estimate works as before. (There are $n-1$ terms, but since the sum on n converges geometrically this causes no problem.)

It remains for us to consider the terms involving

$$\frac{\partial}{\partial\mu}(C_{\bar\omega_1}(s)-C_{\bar\omega_1}), \quad \frac{\partial}{\partial\mu}\chi_D(C_{\omega_1}(s)-C_{\bar\omega_1}(s))\chi_D, \quad \text{or} \quad \frac{\partial}{\partial\mu}(\chi_D C_{\bar\omega_1}-C_{\bar\omega_1}).$$

The first term is equal to

$$-(C_{\bar\omega_1}(s)-C_{\bar\omega_1})\frac{\partial\bar\omega_1}{\partial\mu}C_{\bar\omega_1}(s)-C_{\bar\omega_1}\frac{\partial\bar\omega_1}{\partial\mu}(C_{\bar\omega_1}(s)-C_{\bar\omega_1}).$$

In each term the factor $|C_{\bar\omega_1}(s)-C_{\bar\omega_1}| \leqq |C_{\bar\omega_1}(0)-C_{\bar\omega_1}|$ provides the necessary localization at the boundary. The second case above may be written as

$$\begin{aligned}&-\chi_D(C_{\omega_1}(s)-C_{\bar\omega_1}(s))\frac{\partial\omega_1}{\partial\mu}C_{\omega_1}(s)\chi_D-\chi_D C_{\bar\omega_1}(s)\frac{\partial(\omega_1-\bar\omega_1)}{\partial\mu}\\ &\cdot C_{\omega_1}(s)\chi_D-\chi_D C_{\bar\omega_1}(s)\frac{\partial\bar\omega_1}{\partial\mu}(C_{\omega_1}(s)-C_{\bar\omega_1}(s))\chi_D.\end{aligned}$$

The first and third terms are localized at the boundary as in (5.5.8)–(5.5.9). Since $\frac{\partial}{\partial\mu}(\omega_1-\bar\omega_1) \leqq K\chi_{\sim D}$, the second term can be handled similarly. The third case above is equal to $-\chi_{\sim D}C_{\bar\omega_1}(\partial\bar\omega_1/\partial\mu)C_{\bar\omega_1}$, which forces the preceding covariance to stretch between D and $\sim D$ as in (5.5.12). This completes the proof. □

Corollary 5.5.4. *Proposition 5.5.3 holds without the restriction on* $|\bar\omega_2-\bar\omega_1|$.

Proof. This follows from (5.5.17) as in the previous corollary. □

5.6. Decoupling Expansion Estimates

This section is devoted to proving Propositions 2.5.3 and 2.5.4. We require the bound

$$\begin{aligned}&\Bigg\| \int ds_\Gamma \sum_{\pi\in\mathscr{P}(\Gamma)} \int \prod_{j=1}^{n} \Bigg[e^{Q_{\omega_j}(\mathbb{Z})-Q_{\omega_{j-1}}(\mathbb{Z})} \prod_{\substack{\alpha\in\pi\\ k(\alpha)=j}} [\tfrac{1}{2}\partial_s^\alpha C_{\omega_j}(s_\Gamma)\cdot\Delta_\psi] \Bigg] \\ &\quad \cdot R_Z\chi_\Sigma e^{-Q_{\omega_1}(Z)} d\mu_{\omega_n,s_\Gamma}(\psi) \prod_{k=1}^{n-1} Z_{\omega_k\omega_{k+1}^\kappa}(s_{\Gamma_k}) \Bigg\|_{L^p} \\ &\leqq (\lambda^{1/2})(\lambda^{-\deg R}) \prod_{\Delta} (N(\Delta)!)^{1/2} e^{Kl\deg R} e^{-3\tau_1 l|Z_\kappa|} e^{-2\tau_2\lambda^{-2}|\Sigma|} \\ &\quad \cdot e^{\sum_m (E_c^q - E_w^m - E_c^{p(\mathbb{Z})}+E_w^{p(\mathbb{Z})})l^2|\mathbb{Z}|_m} e^{\sum_m (E_c^m - E_w^m - E_c^{p(\mathbb{Z})}+E_w^{p(\mathbb{Z})})l^2|\mathrm{Int}_m\mathbb{Z}|}. \end{aligned} \tag{5.6.1}$$

Here $Z=Z_\kappa$, $\Gamma=\Gamma\cap Z$, $\Sigma=\Sigma\cap Z$, and $\mathbb{Z}=(Z,\Sigma,\Gamma)$ is a cluster. The factor $(\lambda^{1/2})$ is conditional; it is present only if $\deg R=0$. Likewise $(\lambda^{-\deg R})$ is present only if $\Sigma \not\equiv m$ (R is a monomial in ψ_m). We assume $|\mathbb{Z}|>1$ if $\deg R=0$. The L^p norm is with respect to the variables in R, and $p\in[1,\infty)$. See Sects. 2.4 and 2.5.

We expand $\partial_s^\alpha C$ into its l-lattice localizations and apply the derivatives Δ_ψ. Using Hölder's inequality, split the integral into a Gaussian part times an interaction part. Apply Proposition 2.5.1 to the interaction integral. This produces factors

$$e^{\sum_m (E_c^q - E_c^m) l^2 |\mathbb{Z}|_m} e^{-3\tau_2 \lambda^{-2}(|\Sigma| + |\Sigma'|)} e^{a\lambda l^2 |Z \cap \Lambda|} \prod_{\Delta^1} n(\Delta^1)!\,.$$

The first factor accounts for some of the energy factors in (5.6.1); what remains to be accounted for is

$$\exp\left(\sum_m (E_c^m - E_w^m - E_c^{p(\mathbb{Z})} + E_w^{p(\mathbb{Z})}) l^2 |\mathbb{Z}^{\mathrm{int}}|_m\right). \tag{5.6.2}$$

(Recall that $\mathbb{Z}^{\mathrm{int}}$ extends into $\mathrm{IInt}\,\mathbb{Z}$.) We use

$$E_c^m - E_w^m - E_c^p + E_w^p = \frac{m_m^2}{8\pi} \log \frac{m_m^2}{m_{\bar{q}}^2} - \frac{m_p^2}{8\pi} \log \frac{m_p^2}{m_{\bar{q}}^2} - \frac{m_m^2 - m_p^2}{8\pi} \tag{5.6.3}$$

and Corollary 5.5.2 to obtain this factor from the $Z_{\omega_k \omega_{k+1}^\kappa}(s_{\Gamma_k})$ factors. The mass prevailing over a particular square in $\mathbb{Z}^{\mathrm{int}}$ is in general shifted many times. However, the energy factors associated with intermediate values of the mass cancel, by (5.5.2). Thus up to an error $\exp\left(\sum_k O(1)|\partial \hat{D}_k|\right)$, we obtain a factor

$$\exp\left(\int_{\mathbb{Z} \cup \mathrm{IInt}\,\mathbb{Z}} \frac{\omega_n(x)}{8\pi} \log \frac{\omega_n(x)}{m_{\bar{q}}^2} - \frac{m_p^2}{8\pi} \log \frac{m_p^2}{m_{\bar{q}}^2} - \frac{\omega_n(x) - m_p^2}{8\pi} dx\right), \tag{5.6.4}$$

where we have put $\omega_n(x) = m_m^2$ for $x \in \mathrm{IInt}_m \mathbb{Z}$. Recall that $\omega_n(x)$ differed from $m_{\sigma(x)}^2$ only when $\mathrm{dist}(x, \Sigma) \leqq L/2$. Thus (5.6.4) agrees with (5.6.2) up to an error $e^{O(1)L^2|\Sigma|}$. Since $\sum_k |\partial \hat{D}_k| \leqq |\Sigma|$, both errors can be absorbed into $e^{-\delta \tau_2 \lambda^{-2} |\Sigma|}$.

We next estimate the Gaussian integrals that were split off with Hölder's inequality above. We use the estimate

$$\|\partial_s^\alpha C\|_{L^q(\Delta_{j_1} \times \Delta_{j_2})} \leqq e^{Kl} e^{-cld((j_1, j_2), \alpha)} e^{-cl|\alpha|} \sum_{o \in L(\alpha)} e^{-cl|o|}, \tag{5.6.5}$$

proven at the end of this section. Here $q < \infty$,

$$d((j_1, j_2), \alpha) = \sup_{b \in \alpha} (\mathrm{dist}(\Delta_{j_1}, b) + \mathrm{dist}(\Delta_{j_2}, b))/l\,,$$

and o is a linear ordering of the bonds in α. We define $|o|$ as follows. If $o = (b_1, \ldots, b_n)$, let $I = (i_1, \ldots, i_k)$ be any subset of $\{1, \ldots, n\}$ with $i_\alpha < i_{\alpha+1}$. Then $|o| = \sup_I \sum_{\alpha=2}^k \mathrm{dist}(b_{i_{\alpha-1}}, b_{i_\alpha})/l$. This is not quite the same definition as in [17]. There is an analogous bound on the single-variable kernel $\partial_s^\alpha C(x, x)$.

Assuming (5.6.5), the remainder of the proof of (5.6.1) is fairly standard [27, 29]. We indicate only the main points. The $e^{-cl|o|}$ factors control the sum over partitions of π, up to an effect $e^{O(1)|\mathbb{Z}|}$. The $e^{-cld((j_1, j_2), \alpha)}$ factor controls localization sums. Derivatives in $\partial_s^\alpha C_{\omega_{k(\alpha)}} \cdot \Delta_\psi$ can contract to $Q_{\omega_{k(\alpha)}}$. There must be a bond in α contained in $\mathscr{B}_{k(\alpha)}(\Sigma)$, and $\omega_{k(\alpha)}$ is the correct mass-squared within $L/2$ of such bonds. Furthermore there are no phase boundaries within L of such bonds. Thus the coefficients in $Q_{\omega_{k(\alpha)}}$ are $O(\lambda)$ within $L/2$ of the bond, and if the contraction is to Δ with $\mathrm{dist}(b,\Delta)\leqq L/2$, then a factor $O(\lambda)$ is brought down. Otherwise factors $O(\lambda^{-1})$ can be introduced, but this is compensated by $e^{-c'L/2}\leqq\lambda^2$ arising from $d((j_1,j_2),\alpha)>L/2l$.

Derivatives of χ_Σ can be pinned to the factor $e^{-\tau_2\lambda^{-2}}$ arising from $|\Sigma'|$ and hence they also yield $O(\lambda)$ factors. If $\deg R=0$, $|\mathbb{Z}|>1$, there must be at least one derivative or phase boundary, hence the factor $(\lambda^{1/2})$ in (5.6.1).

Translation from ψ_m to ψ in R will produce factors $O(\lambda^{-\deg R})$ if $\Sigma\not\equiv m$. This has been taken into account in (5.6.1). Some derivatives may act on R; the associated e^{Kl} goes into the $e^{Kl\deg R}$ factor in (5.6.1).

At least a certain fraction of the derivatives not contracted to R will contract to Q or to χ_Σ and yield factors $O(\lambda)$. Altogether we have at least a factor e^{-cl} for every derivative bond and e^{-Kl} for every vertex. Since derivative bonds are "dense" in regions away from phase boundaries, we obtain the overall volume convergence $e^{-cl|\mathbb{Z}|}$.

Finally there are factorials to control. With $d=\deg\mathscr{P}$, $N(\Delta)=\deg R_\Delta$, $M(\Delta)$ contractions in Δ, and $n(\Delta^1)$ contractions to $\chi_\sigma(\Delta^1)$, we have factors $\prod_\Delta(N(\Delta)+dM(\Delta))!^{1/2}$ from the Gaussian integration, $\prod_{\Delta^1}n(\Delta^1)!$ from the vacuum energy bound, and $\prod_\Delta(e^{O(1)(N(\Delta)+M(\Delta))}M(\Delta)^{M(\Delta)})$ from summing over different ways of applying the derivatives. After extracting $e^{O(1)N(\Delta)}(N(\Delta)!)^{1/2}$ for (5.6.1), we must bound

$$(O(1)M(\Delta))^{M(\Delta)}\prod_{\gamma\text{ contracted to }\Delta}e^{-c\,\mathrm{dist}(\Delta,\gamma)}$$

by $O(1)$ to get a controllable volume effect. This is accomplished as in [17] by taking account of the rate α is forced away from Δ as $M(\Delta)$ becomes large. This completes the proof of Propositions 2.5.3 and 2.5.4.

We now prove (5.6.5). Using the Neumann series (5.3.17) for $C_\omega(s)$ we have

$$\begin{aligned}\partial_s^\alpha C_\omega(s)&=\partial_s^\alpha\sum_{n=1}^\infty C_{\bar m^2}(s)[(\bar m^2-\omega)C_{\bar m^2}(s)]^{n-1}\\&=\sum_{n=1}^\infty\sum_{\substack{(\alpha_1,\ldots,\alpha_n)\\ \bigcup_i\alpha_i=\alpha}}\partial_s^{\alpha_1}C_{\bar m^2}(s)\prod_{i=2}^n[(\bar m^2-\omega)\partial_s^{\alpha_i}C_{\bar m^2}(s)]\,.\end{aligned}\tag{5.6.6}$$

The α_i's are disjoint, and possibly empty. When $\alpha_i=\emptyset$ or $i\geqq n-1$, we use a standard type of estimate [17, 27]:

$$\begin{aligned}&\|\partial_s^{\alpha_i}C_{\bar m^2}(s)(\cdot,x)\|_{L^q(\Delta^1)}\\&\quad\leqq c^{|\alpha_i|+1}\sum_{o\in L(\alpha_i)}e^{-cl|o|}e^{-c\delta(\Delta^1,\alpha_i)}e^{-c\delta(x,\alpha_i)}e^{-c\,\mathrm{dist}(x,\Delta^1)}\,.\end{aligned}\tag{5.6.7}$$

Here $\delta(\Delta^1, \alpha_i) = \sup_{b\in\alpha_i} \operatorname{dist}(\Delta^1, b)$, $\delta(x, \alpha_i) = \sup_{b\in\alpha_i} \operatorname{dist}(x, b)$, and $q\in[1, \infty)$. If α_i is empty, these distances are defined to be zero. Summing over unit squares $\Delta^1 \subseteqq \mathbb{R}^2$, we obtain

$$\begin{aligned}&\int \partial_s^{\alpha_i} C_{\bar{m}^2}(s)(x, y) e^{c\delta(x,\alpha_i)} e^{c\delta(y,\alpha_i)} e^{c|x-y|} dy \\ &\quad \leqq c^{|\alpha_i|+1} \sum_{o\in L(\alpha_i)} e^{-cl|o|} . \end{aligned} \tag{5.6.8}$$

When $\alpha_i = \emptyset$ we may use

$$\begin{aligned}&\int C_{\bar{m}^2}(s)(x, y) e^{\eta^{1/2}|x-y|/2} dx \\ &\quad \leqq \int C_{\bar{m}^2}(1)(x, 0) e^{\eta^{1/2}(|x_0|+|x_1|)/2} dx \\ &\quad = \int \frac{d^2p}{(2\pi)^2} \frac{e^{ipx}}{p^2+\bar{m}^2} e^{\eta^{1/2}(|x_0|+|x_1|)/2} dx = \frac{1}{\bar{m}^2 - \eta/2} . \end{aligned} \tag{5.6.9}$$

We have from (5.6.7) that if $n \geqq 2$,

$$\begin{aligned}&\int e^{c\delta(x,\alpha_{n-1})} e^{c\delta(y,\alpha_{n-1})} e^{c|x-y|} \partial_s^{\alpha_{n-1}} C_{\bar{m}^2}(s)(x, y) \\ &\quad \cdot e^{c\delta(y,\alpha_n)} e^{c\delta(z,\alpha_n)} e^{c|y-z|} \partial_s^{\alpha_n} C_{\bar{m}^2}(s)(y, z) dy \\ &\quad \leqq c^{|\alpha_{n-1}|+|\alpha_n|+1} \sum_{o\in L(\alpha_{n-1})} e^{-cl|o|} \sum_{o'\in L(\alpha_n)} e^{-cl|o'|}\end{aligned}$$

for each x and z. Equations (5.6.8) and (5.6.9) bound the norms of the integral operators $\partial_s^{\alpha_i} C_{\bar{m}^2}(s)$ after extracting some decay factors. Putting these estimates together yields

$$\begin{aligned}&|(\partial_s^{\alpha} C_{\omega}(s) - \partial_s^{\alpha} C_{\bar{m}^2}(s))(x_0, x_n)| \\ &\quad \leqq \sum_{n=2}^{\infty} \sum_{(\alpha_1,\ldots,\alpha_n)} (c\bar{m}^2)^{|\alpha|+2} \left[\frac{\bar{m}^2 - 2\eta}{\bar{m}^2 - \eta/2}\right]^{n-|\alpha|-2} \\ &\quad \cdot \sup_{(x_1,\ldots,x_{n-1})} \prod_{i=1}^{n} \left[\sum_{o_i\in L(\alpha_i)} e^{-cl|o_i|} e^{-c\delta(x_{i-1},\alpha_i)} e^{-c\delta(x_i,\alpha_i)} e^{-c|x_{i-1}-x_i|}\right] . \end{aligned} \tag{5.6.10}$$

We have used $\omega(x) \geqq 2\eta$. Put $(\bar{m}^2 - 2\eta)/(\bar{m}^2 - \eta/2) = 1 - 2\varepsilon$. With a factor $(1-\varepsilon)^{-n}\varepsilon^{-|\alpha_i|}$ we can choose $|\alpha_1|, \ldots, |\alpha_n|$, because $\prod_{i=1}^{n}\left[\sum_{|\alpha_i|=0}^{\infty} \varepsilon^{|\alpha_i|}\right] = (1-\varepsilon)^{-n}$. Having made this choice, a set of α_i's with their linear orderings uniquely determines a linear ordering o of α. Furthermore,

$$|o| \leqq \sum_{i=1}^{n} |o_i| + \sum_{i=1}^{n-1} (\delta(x_i, \alpha_i) + \delta(x_i, \alpha_{i+1}) + |x_i - x_{i+1}|) , \tag{5.6.11}$$

and if $x_0 \in \Delta_{j_1}$, $x_n \in \Delta_{j_2}$, then

$$ld((j_1, j_2), \alpha) \leqq \sum_{i=1}^{n} (\delta(x_{i-1}, \alpha_i) + \delta(x_i, \alpha_i) + |x_{i-1} - x_i|) . \tag{5.6.12}$$

Altogether we have bounded the right-hand side of (5.6.10) by

$$\sum_{n=2}^{\infty} c^{|\alpha|+1} \left[\frac{1-2\varepsilon}{1-\varepsilon}\right]^n \sum_{o\in L(\alpha)} e^{-cl|o|} e^{-cld((j_1,j_2),\alpha)} .$$

Thus

$$\|\partial_s^\alpha C_\omega(s) - \partial_s^\alpha C_{\bar{m}^2}(s)\|_{L^q(\Delta_{j_1}\times\Delta_{j_2})} \leqq l^{4/q} c^{|\alpha|+1} \sum_{o\in L(\alpha)} e^{-cl|o|} e^{-cld((j_1,j_2),\alpha)}, \tag{5.6.13}$$

and since

$$l^4 c^{|\alpha|+1} e^{-cl|o|/2} \leqq e^{Kl} e^{-c'l|\alpha|}, \tag{5.6.14}$$

we obtain (5.6.5) for the difference $\partial_s^\alpha C_\omega(s) - \partial_s^\alpha C_{\bar{m}^2}(s)$. The bound (5.6.5) is well known for $\partial_s^\alpha C_{m^2}(s)$, so this completes the proof for $\partial_s^\alpha C_\omega(s)$. Equation (5.6.13) holds for the single-variable kernels as well if we replace $L^q(\Delta_{j_1}\times\Delta_{j_2})$ with $L^q(\Delta_{j_1})$. Again, comparison with $\partial_s^\alpha C_{\bar{m}^2}$ proves the bound analogous to (5.6.5) for single-variable kernels.

5.7. The Bounded Spin Approximation

In this section we prove the bound

$$\left|\frac{\partial}{\partial\mu^i}\log(Z_\Sigma(\mathbb{V})e^{E^{p(\mathbb{V})}_{\mathbb{V}}l^2|\mathbb{V}|})\right| \leqq K\lambda^{-2}l^2|\mathbb{V}| \tag{5.7.1}$$

of Proposition 2.5.6. This derivative will turn out to be a sum of expectations of quantities like $:\psi_p^j(\mathbb{V}): \equiv \int_{\mathbb{V}} :\psi_p(x)^j: dx$. If ψ_p were a bounded variable, such expectations would be automatically bounded. (Having bounded spins simplified the Pirogov-Sinai work at this point [24].) Since ψ_p is in fact unbounded, we must show that the error incurred in treating it as bounded is small enough to dominate the vacuum energy volume divergences.

Write p for $p(\mathbb{V})$, $d\psi_p$ for $d\mu_{m_p^2,\partial\mathbb{V}}(\psi_{p(\mathbb{V})})$, μ for μ^i, and C for the covariance of $d\psi_p$. Using (2.4.16) we compute

$$\begin{aligned}\frac{\partial}{\partial\mu}(Z_\Sigma(\mathbb{V})e^{E^p_{\mathbb{V}}l^2|\mathbb{V}|}) &= \frac{\partial}{\partial\mu}\int \chi_\Sigma e^{-V_p(\mathbb{V})}d\psi_p \\ &= \int\left(\sum_{\Delta^1\subseteq\mathbb{V}}\frac{\partial}{\partial\mu}\log\chi_{\sigma(\Delta)}\right)\chi_\Sigma e^{-V_p(\mathbb{V})}d\psi_p \\ &\quad -\int\frac{\partial V_p(\mathbb{V})}{\partial\mu}\chi_\Sigma e^{-V_p(\mathbb{V})}d\psi_p + \int\frac{1}{2}\frac{\partial C}{\partial\mu}\cdot\Delta_{\psi_p}\chi_\Sigma e^{-V_p(\mathbb{V})}d\psi_p\,.\end{aligned} \tag{5.7.2}$$

From the formula (5.4.8) for $\chi_\sigma(\phi(\Delta)) = \chi_\sigma(\psi_p(\Delta)+\xi_p)$, we have

$$\frac{\partial}{\partial\mu}\chi_\sigma = \pi^{-1/2}\frac{\partial B}{\partial\mu}e^{-(\psi_p(\Delta)-B)^2} - \pi^{-1/2}\frac{\partial A}{\partial\mu}e^{-(\psi_p(\Delta)-A)^2}$$

$$\chi_\sigma = \tfrac{1}{2}\operatorname{erfc}(\psi_p(\Delta)-B) - \tfrac{1}{2}\operatorname{erfc}(\psi_p(\Delta)-A)\,,$$

where

$$A = \tfrac{1}{2}(\xi_{\sigma-1}+\xi_\sigma)-\xi_p\,, \qquad B = \tfrac{1}{2}(\xi_\sigma+\xi_{\sigma+1})-\xi_p\,.$$

Note that $\partial A/\partial\mu$ and $\partial B/\partial\mu$ are $O(\lambda^{-1})$.

We use the asymptotic expansion

$$\operatorname{erfc}(z)=\pi^{-1/2}z^{-1}e^{-z^2}(1+O(z^{-2}))\,, \qquad z\gg 1 \tag{5.7.3}$$

to bound $\left|\frac{\partial}{\partial\mu}\log\chi_\sigma\right|$ by $O(\lambda^{-1})(\psi_p(\Delta)+1)$ for $\psi_p(\Delta)\geqq B$. There is an analogous bound for $\psi_p(\Delta)\leqq A$, and for $\psi_p(\Delta)\in[A,B]$ we have $\chi_\sigma^{-1}\leqq O(1)$. Thus

$$\left|\frac{\partial}{\partial\mu}\log\chi_\sigma\right|\leqq O(\lambda^{-1})|\psi_p(\Delta)|+O(\lambda^{-1})\,, \tag{5.7.4}$$

and the first term in (5.7.2) is bounded by

$$O(\lambda^{-1})\int\Big(\sum_{\Delta\subseteqq\mathbb{V}}|\psi_p(\Delta)|\Big)\chi_\Sigma e^{-V_p(\mathbb{V})}d\psi_p+O(\lambda^{-1})\,.$$

The second term in (5.7.2) is a sum of terms

$$O(\lambda^{j-2})\int:\psi_p^j(\mathbb{V}):\chi_\Sigma e^{-V_p(\mathbb{V})}d\psi_p\,,\qquad 0\leqq j\leqq d=\deg\mathscr{P}\,. \tag{5.7.5}$$

This includes terms arising from differentiating the mass in the Wick ordering. (The free covariance is always used in the Wick ordering.)

Using integration by parts, we have

$$\begin{aligned}\frac{\partial C}{\partial\mu}\cdot\Delta_{\psi_p}&=\int\frac{\delta}{\delta\psi_p(x)}C(x,y)\frac{\partial m_p^2}{\partial\mu}C(y,z)\frac{\delta}{\delta\psi_p(z)}dxdydz\\ &=\frac{\partial m_p^2}{\partial\mu}\int:\psi_p(y)^2:_C dy\,.\end{aligned} \tag{5.7.6}$$

The difference $:\psi_p^2(\mathbb{V}):-:\psi_p^2(\mathbb{V}):_C$ is a constant $O(l^2|\mathbb{V}|)$, so the third term in (5.7.2) is also of the form (5.7.5).

Altogether we have

$$\left|\frac{\partial}{\partial\mu}(Z_\Sigma(\mathbb{V})e^{E_\infty^{\mathscr{P}}l^2|\mathbb{V}|})\right|\leqq O(\lambda^{-1})\Big\langle\sum_{\Delta\subseteqq\mathbb{V}}|\psi_p(\Delta)|\Big\rangle_\Sigma+\sum_{j=0}^{d}O(\lambda^{j-2})|\langle:\psi_p^j(\mathbb{V}):\rangle_\Sigma|\,, \tag{5.7.7}$$

where

$$\langle\cdot\rangle_\Sigma=\frac{\int\cdot\chi_\Sigma e^{-V_p(\mathbb{V})}d\psi_p}{\int\chi_\Sigma e^{-V_p(\mathbb{V})}d\psi_p}\,.$$

Lemma 5.7.1. *There exists a constant $K_0(C)>0$ such that for all $K\geqq K_0$ the following is true. With $j\leqq\deg\mathscr{P}$ let $\chi_+(\chi_-)$ be the characteristic function of*

$$:\psi_p^j(\mathbb{V}):\geqq\lambda^{-j}Kl^2|\mathbb{V}|\qquad(:\psi_p^j(\mathbb{V}):\leqq-\lambda^{-j}Kl^2|\mathbb{V}|)\,,$$

respectively. Then

$$\left|\int\chi_\pm:\psi_p^j(\mathbb{V}):\chi_\Sigma e^{-V_p(\mathbb{V})}d\psi_p\right|\leqq e^{-K\lambda^{-2}l^2|\mathbb{V}|/4C}\,. \tag{5.7.8}$$

Proof. Let $\bar{V}=V_p(\mathbb{V})-\lambda^{j-2}:\psi_p^j(\mathbb{V}):/2C$. Then

$$\begin{aligned}\int\chi_+:\psi_p^j(\mathbb{V}):\chi_\Sigma e^{-V_p(\mathbb{V})}d\psi_p&=\int\chi_+:\psi_p^j(\mathbb{V}):e^{-\lambda^{j-2}:\psi_p^j(\mathbb{V}):/2C}\chi_\Sigma e^{-\bar{V}}d\psi_p\\ &\leqq\lambda^{-j}Kl^2|\mathbb{V}|e^{-\lambda^{-2}Kl^2|\mathbb{V}|/2C}\int e^{-\bar{V}}d\psi_p\,,\end{aligned} \tag{5.7.9}$$

because $\chi_\Sigma \leqq 1$, and because $Xe^{-\lambda^{j-2}X/2C}$ attains its maximum at the minimum value of $X = \lambda^{-j}Kl^2|\mathbb{V}|$, for K large enough. In fact,

$$\lambda^{-j}Kl^2|\mathbb{V}|e^{-\lambda^{-2}Kl^2|\mathbb{V}|/6C} \leqq 1$$

for K large enough. Notice that $\bar{V}$ is bounded below even if $j = d = \deg \mathscr{P}$ because the coefficient of ψ_p^d in V_q is at least $C^{-1}\lambda^{d-2}$. Furthermore, $\bar{V}$ is of the form $\lambda^{-2}\bar{V}_1(\lambda\psi_p)$, so the lower bound is $O(\lambda^{-2})$.

Standard linear lower bound estimates [15] will yield

$$\int e^{-\bar{V}} d\psi_p \leqq e^{O(\lambda^{-2})l^2|\mathbb{V}|}. \tag{5.7.10}$$

This can be absorbed into $e^{-\lambda^{-2}Kl^2|\mathbb{V}|/12C}$, for K large enough.

The proof for χ_- is similar, using $\bar{V} = V_p(\mathbb{V}) + \lambda^{j-2} :\psi_p^j(\mathbb{V}):/2C$. □

Lemma 5.7.2. *Let χ_+ be the characteristic function of $\sum_{\Delta \subsetneqq \mathbb{V}} |\psi_p(\Delta)| \geqq \lambda^{-1}Kl^2|\mathbb{V}|$, and suppose K is sufficiently large. Then*

$$\int \chi_+ \sum_{\Delta \subsetneqq \mathbb{V}} |\psi_p(\Delta)| \chi_\Sigma e^{-V_p(\mathbb{V})} d\psi_p \leqq e^{-K\lambda^{-2}l^2|\mathbb{V}|/4C}. \tag{5.7.11}$$

Proof. We modify the proof of Lemma 5.7.1. Write the integral as a sum of $2^{l^2|\mathbb{V}|}$ terms according to whether $\psi_p(\Delta) < 0$ or $\psi_p(\Delta) \geqq 0$ for each $\Delta^1 \subsetneqq \mathbb{V}$. (That is, insert the corresponding partition of unity into the measure.) Each term can be bounded as before if we take

$$\bar{V} = V_p(\mathbb{V}) - \lambda^{-1}/2C \sum_{\Delta \subsetneqq \mathbb{V}} \varepsilon(\Delta)\psi_p(\Delta).$$

Here $\varepsilon(\Delta) = 1$ for the $\psi_p(\Delta) \geqq 0$ term, $\varepsilon(\Delta) = -1$ for the $\psi_p(\Delta) < 0$ term. The combinatoric factor $2^{l^2|\mathbb{V}|}$ is controlled by $e^{-\lambda^{-2}Kl^2|\mathbb{V}|/12C}$. □

Proof of Proposition 2.5.6. With χ_+, χ_- as in Lemma 5.7.1 and $\chi_0 = 1 - \chi_+ - \chi_-$, we have

$$\begin{aligned} \langle :\psi_p^j(\mathbb{V}):\rangle_\Sigma &= \frac{\int \chi_0 :\psi_p^j(\mathbb{V}): \chi_\Sigma e^{-V_p(\mathbb{V})} d\psi_p}{\int \chi_0 \chi_\Sigma e^{-V_p(\mathbb{V})} d\psi_p} + \frac{\int (\chi_+ + \chi_-) :\psi_p^j(\mathbb{V}): \chi_\Sigma e^{-V_p(\mathbb{V})} d\psi_p}{\int \chi_0 \chi_\Sigma e^{-V_p(\mathbb{V})} d\psi_p} \\ &\quad + \int_0^1 dt \frac{d}{dt} \frac{\int :\psi_p^j(\mathbb{V}): \chi_\Sigma e^{-V_p(\mathbb{V})} d\psi_p}{\int (t(\chi_+ + \chi_-) + \chi_0)\chi_\Sigma e^{-V_p(\mathbb{V})} d\psi_p}. \end{aligned} \tag{5.7.12}$$

The first term is an expectation of a variable bounded by $\lambda^{-j}Kl^2|\mathbb{V}|$ and so it is also bounded by $\lambda^{-j}Kl^2|\mathbb{V}|$. By Lemma 5.7.1 we have

$$\int t(\chi_+ + \chi_-)\chi_\Sigma e^{-V_p(\mathbb{V})} d\psi_p \leqq 2e^{-K\lambda^{-2}|\mathbb{V}|/4C}, \tag{5.7.13}$$

since $t\chi_+ \leqq \chi_+ :\psi_p^j(\mathbb{V}):$, $t\chi_- \leqq -\chi_- :\psi_p^j(\mathbb{V}):$. Thus by Proposition 5.4.1,

$$\begin{aligned} \int (t(\chi_+ + \chi_-) + \chi_0)\chi_\Sigma e^{-V_p(\mathbb{V})} d\psi_p &\geqq e^{-a\lambda^{-2}l^2|\mathbb{V}|} - 2e^{-K\lambda^{-2}l^2|\mathbb{V}|/4C} \\ &\geqq e^{-2a\lambda^{-2}l^2|\mathbb{V}|}. \end{aligned} \tag{5.7.14}$$

Thus the second term is bounded by $2\exp[(-K/4C+2a)\lambda^{-2}l^2|\mathbb{V}|]\leqq 1$ for K large enough. The third term is bounded by

$$\frac{|\int :\psi_p^j(\mathbb{V}):\chi_\Sigma e^{-V_p(\mathbb{V})}d\psi_p|\int(\chi_+ +\chi_-)\chi_\Sigma e^{-V_p(\mathbb{V})}d\psi_p}{(\int(t(\chi_+ +\chi_-)+\chi_0)\chi_\Sigma e^{-V_p(\mathbb{V})}d\psi_p)^2}$$
$$\leqq 2\exp[(O(1)-K/4C+4a)\lambda^{-2}l^2|\mathbb{V}|]\leqq 1\,. \tag{5.7.15}$$

The $:\psi_p^j(\mathbb{V}):$ integral has been split with Hölder's inequality and bounded as usual using Proposition 2.5.1. Putting these bounds together, we obtain

$$\lambda^{j-2}|\langle :\psi_p^j(\mathbb{V}):\rangle_\Sigma|\leqq K\lambda^{-2}l^2|\mathbb{V}|+2\lambda^{j-2}\leqq 2K\lambda^{-2}l^2|\mathbb{V}|\,. \tag{5.7.16}$$

The bound $\lambda^{-1}\left\langle \sum_{\Delta\subseteq\mathbb{V}}|\psi_p(\Delta)|\right\rangle_\Sigma \leqq 2K\lambda^{-2}l^2|\mathbb{V}|$ can be proven in the same way. By (5.7.7), this completes the proof. $\square$

5.8. *Smoothness in μ*

In this section we prove the bound

$$\left|\frac{\partial}{\partial\mu^i}\left(\varrho_{\Lambda,q}(\mathbb{Y})e^{\sum_m(-E_c^q+E_w^m+E_c^{p(\mathbb{Y})}-E_w^{p(\mathbb{Y})})l^2|\mathbb{Y}|_m}e^{\sum_m(-E_c^m+E_w^m+E_c^{p(\mathbb{Y})}-E_w^{p(\mathbb{Y})})l^2|\mathrm{Int}_m\mathbb{Y}|}\right)\right|$$
$$\leqq \lambda^{1/2}e^{-\frac{5}{2}\tau_1 l|\mathbb{Y}|}e^{-\tau_2\lambda^{-2}|\Sigma_{\mathbb{Y}}|} \tag{5.8.1}$$

of Proposition 2.5.5. An expression for $\varrho_{\Lambda,q}(\mathbb{Y})$ may be found in (5.6.1). We take $R=1$, $|\mathbb{Y}|\geqq 1$, $p=p(\mathbb{Y})$. Almost everything in (5.6.1) depends on $\mu=\mu^i$; we will show that after deriving each element of (5.6.1) the structure of the estimates in Sect. 5.6 need be modified only slightly. Without the derivative with respect to μ, (5.8.1) would reduce to Proposition 2.5.3; the energy factors have merely been moved to the other side of the inequality.

Consider first the dependence of $Z_{\omega_k\omega_{k+1}^\kappa}(s_{\Gamma_k})$ on μ. Divide some of the energy factors amongst the Z's in accordance with Proposition 5.5.3. By Corollary 5.5.4, differentiating

$$Z_{\omega_k\omega_{k+1}^\kappa}(s_{\Gamma_k})$$
$$\cdot\exp\left(\int_{\mathbb{Z}\cup\mathrm{Int}\,\mathbb{Z}}\frac{\omega_k(x)}{8\pi}\log\frac{\omega_k(x)}{m_q^2}-\frac{\omega_{k+1}^\kappa(x)}{8\pi}\log\frac{\omega_{k+1}^\kappa(x)}{m_q^2}+\frac{\omega_{k+1}^\kappa(x)-\omega_k(x)}{8\pi}dx\right)$$

brings down a factor $O(1)|\partial\hat{D}_{k+1}|$. Thus differentiating all the $Z\exp(\ldots)$ factors introduces a factor no worse than $O(1)|\Sigma|$, which can be absorbed into $e^{-2\tau_2\lambda^{-2}|\Sigma|}$ in (5.6.1). A factor λ can also be extracted, since $|\Sigma|\geqq 1$ whenever there are Z-factors to differentiate.

In differentiating the remaining energy factors, consider two cases. If $|\Sigma|\geqq 1$, the differentiation introduces a factor $O(\lambda^{-2})l^2|\mathbb{Y}|$. When multiplied by $e^{-\delta\tau_2\lambda^{-2}|\Sigma|}e^{-\delta\tau_1 l|\mathbb{Y}|}$, taken out of (5.6.1), we are left with $O(\lambda)$. This is sufficient to stand as a contribution to the bound in (5.8.1). If $|\Sigma|=0$, then there are no Z-factors and the energy factors degenerate to $e^{(-E_c^q+E_c^p)l^2|\mathbb{Y}|}$. We combine this with Q before differentiating. The product over j in (5.6.1) degenerates to one term, with all functional derivatives acting on $\chi_\Sigma e^{-Q_{m_p^2}(Z)}$. The coefficients in $Q_{m_p^2}(Z)$

$-(E_c^q - E_c^p)l^2|\mathbb{Y}|$ are $O(\lambda)$, so that differentiation with respect to μ will bring down terms $O(\lambda):\psi_p^j(\mathbb{Y}):$. Such terms are bounded by Proposition 2.5.4; they correspond to a sum of $|\mathbb{Y}|l^2$ terms with $R = :\psi_p^j(\Delta):$. The factor $|\mathbb{Y}|l^2$ is absorbed into $e^{-3\tau_1 l|\mathbb{Y}|}$, leaving an overall $O(\lambda)e^{Klj} \leqq \lambda^{3/4}$ which is small enough for (5.8.1).

Returning to the case $|\Sigma| \geqq 1$, we consider the effect of differentiating $Q_{\omega_1}(Z)$ or the mass-shifts $Q_{\omega_j}(Z) - Q_{\omega_{j+1}}(Z)$. Coefficients of $:\psi_p^j:$ will be $O(\lambda^{j-2})$, but since $|\Sigma| \geqq 1$ we must include a factor λ^{-j} from translation, as in Proposition 2.5.4. Having some R-factors precede some of the functional derivatives does not affect the estimates in Sect. 5.6. Thus we have terms

$$O(\lambda^{-2})e^{Klj}l^2|\mathbb{Y}|e^{-\delta\tau_1 l|\mathbb{Y}|}e^{-\delta\tau_2\lambda^{-2}|\Sigma|} \leqq \lambda,$$

which is small enough.

We next consider μ-derivatives of χ_Σ. As in (5.7.2), we have

$$\begin{aligned}\frac{\partial}{\partial\mu}\chi_{\sigma(\Delta)}(\psi(\Delta)+g(\Delta)) &= \pi^{-1/2}\frac{\partial B}{\partial\mu}e^{-(\phi(\Delta)-(\xi_\sigma+\xi_{\sigma+1})/2)^2} \\ &\quad - \pi^{-1/2}\frac{\partial A}{\partial\mu}e^{-(\phi(\Delta)-(\xi_{\sigma-1}+\xi_\sigma)/2)^2},\end{aligned} \tag{5.8.2}$$

where $A = (\xi_{\sigma-1}+\xi_\sigma)/2 - g(\Delta)$, $B = (\xi_\sigma + \xi_{\sigma+1})/2 - g(\Delta)$. We see that $\partial\chi_\sigma/\partial\mu$ satisfies the same bound in (5.2.5) as for $\partial\chi_\sigma/\partial\phi(\Delta)$, except for a factor $|\partial A/\partial\mu| + |\partial B/\partial\mu| = O(\lambda^{-1})$. It is easy to see that derivatives $\frac{\partial^n}{\partial\phi(\Delta)^n}\frac{\partial}{\partial\mu}\chi_\sigma$ also satisfy (5.2.5), up to a factor $O(\lambda^{-1})$. Thus the vacuum energy bound will hold with $|\Sigma'| \geqq 1$ whenever some χ_σ is differentiated with respect to μ. This introduces a factor $O(\lambda^{-1})e^{-\delta\tau_2\lambda^{-2}} \leqq \lambda$, which survives the decoupling expansion estimates. There are also $|\mathbb{Y}|l^2\chi_\sigma$'s to differentiate, but this is controlled by $e^{-\delta\tau_1 l|\mathbb{Y}|}$.

The masses in the covariance $C = C_{\omega_n}(s_\Gamma)$ depend on μ, so we must differentiate the measure $d\mu_{\omega_n, s_\Gamma}(\psi)$. As in (5.7.6), this corresponds to inserting a factor $\int \frac{\partial\omega_n(x)}{\partial\mu}:\psi(x)^2:_C$ before everything else in the functional integral in (5.6.1). Since $\partial\omega_n(x)/\partial\mu$ is $O(1)$, this produces a factor $O(\lambda^{-2})l^2|\mathbb{Y}|$ if $|\Sigma| \geqq 1$ or $O(1)l^2|\mathbb{Y}|$ if $|\Sigma| = 0$. These factors are dominated by $e^{-\delta\tau_2\lambda^{-2}|\Sigma|}e^{-\delta\tau_1 l|\mathbb{Y}|}$ as before. As long as $|\mathbb{Y}| \geqq 2$, the overall factor $\lambda^{1/2}$ can still be obtained as in (5.6.1) because at least one functional derivative must be applied to $\chi_\Sigma e^{-Q}$ if $|\Sigma| = 0$. When $|\mathbb{Y}| = 1$, we use the fact that $:\psi(x)^2:_C$ has no self-lines when integrated in $d\mu_C$. Integrating one power of ψ by parts then gives us the missing derivative on $\chi_\Sigma e^{-Q}$. This yields a factor $O(\lambda) \leqq e^{-Kl}e^{-a\lambda l^2}\lambda^{3/4}e^{-\frac{5}{2}\tau_1 l}$, supplying the missing factors in (5.8.1).

The last type of term to consider involves differentiation of $\partial_s^\alpha C_{\omega_j}(s_\Gamma)$. We require the estimate

$$\left\|\frac{\partial}{\partial\mu}\partial_s^\alpha C\right\|_{L^q(\Delta_{j_1}\times\Delta_{j_2})} \leqq e^{Kl}e^{-cld((j_1,j_2),\alpha)}e^{-cl|\alpha|}\sum_{o\in L(\alpha)} e^{-cl|o|}, \tag{5.8.3}$$

and the analogous one for the single-variable kernel. This is the same as the estimate we used for $\partial_s^\alpha C$, Eq. (5.6.5). Since there are no more than $|\pi| \leqq |\Gamma| \leqq 2|\mathbb{Y}|$

covariances to differentiate, (5.8.3) will suffice to control all terms involving a $\frac{\partial}{\partial\mu}\partial_s^\alpha C$. [The factor $\lambda^{1/2}$ is already present in (5.6.1) because $|\mathbb{Y}|\geqq 2$.]

We compute

$$\frac{\partial}{\partial\mu}\partial_s^\alpha C(x,y)=\sum_{\beta\cup\gamma=\alpha}-\int\partial_s^\beta C(x,z)\frac{\partial\omega_n(z)}{\partial\mu}\partial_s^\gamma C(z,y)dz\,. \tag{5.8.4}$$

Summing over z-localizations Δ_{j_3}, we can apply (5.6.5) to obtain

$$\begin{aligned}\left\|\frac{\partial}{\partial\mu}\partial_s^\alpha C\right\|_{L^q}\leqq O(1)\sum_{\beta\cup\gamma=\alpha}\sum_{\Delta_{j_3}\subseteqq\mathbb{Y}}e^{2kl}e^{-cld((j_1,j_3),\beta)}\left(\sum_{o_\beta\in L(\beta)}e^{-cl|o_\beta|}\right)\\ \cdot e^{-cld((j_3,j_2),\gamma)}e^{-cl(|\beta|+|\gamma|)}\left(\sum_{o_\gamma\in L(\gamma)}e^{-cl|o_\gamma|}\right).\end{aligned} \tag{5.8.5}$$

If $\bar{o}$ is the linear ordering of α defined by (o_β,o_γ), then

$$\begin{gathered}|\bar{o}|\leqq|o_\beta|+d((j_1,j_3),\beta)+d((j_3,j_2),\gamma)+|o_\gamma|+2\,,\\ d((j_1,j_2),\alpha)\leqq d((j_1,j_3),\beta)+d((j_3,j_2),\gamma)\,,\\ |\alpha|=|\beta|+|\gamma|\,,\\ \sum_{\Delta_{j_3}\subseteqq\mathbb{Y}}e^{-cld((j_1,j_3),\beta)/2}\leqq O(1)\,.\end{gathered} \tag{5.8.6}$$

Note that there are $|\alpha|-1$ pairs (o_β,o_γ) that could correspond to any $\bar{o}\in L(\alpha)$. Hence with a factor $|\alpha|$ we can replace $\sum_{\beta\cup\gamma=\alpha}\sum_{o_\beta\in L(\beta)}\sum_{o_\gamma\in L(\gamma)}$ with $\sum_{\bar{o}\in L(\alpha)}$. Altogether we obtain

$$\left\|\frac{\partial}{\partial\mu}\partial_s^\alpha C\right\|_{L^q}\leqq\sum_{\bar{o}\in L(\alpha)}e^{K'l}e^{-cld((j_1,j_2),\alpha)/3}e^{-cl|\alpha|}|\alpha|e^{-cl|\bar{o}|/3}. \tag{5.8.7}$$

Estimate (5.8.3) now follows with a change in c. This completes the proof of Proposition 2.5.5.

References

See Part I, this volume p. 261–304