Commun. Math. Phys. 75, 207–227 (1980)

Communications in
**Mathematical
Physics**

On the Equivalence of the First and Second Order Equations for Gauge Theories*

Clifford Henry Taubes†
Lyman Laboratory of Physics, Harvard University, Cambridge, MA 02138 USA

Abstract. We prove that every solution to the SU(2) Yang–Mills equations, invariant under the lifting to the principle bundle of the action of the group, O(3), of rotations about a fixed line in $\mathbb{R}^4$, with locally bounded and globally square integrable curvature is either self-dual or anti-self dual. In other words we prove, under the above assumptions, that every critical point of the Yang–Mills functional is a global minimum.

We prove also that every finite extremal of the Ginzburg–Landau action functional on $\mathbb{R}^2$, with the coupling constant equal to one, is a solution to the first order Ginzburg–Landau equations. The relationship between the Ginzburg–Landau equations and the O(3) symmetric, SU(2) Yang–Mills equations on $\mathbb{R}^2 \times S^2$ is established.

I. Introduction

On Euclidean four space the value of the Yang–Mills action evaluated on a connection is bounded below by the topological invariant. Any connection whose action achieves this minimum is a solution to the Yang–Mills equations with self (or anti-self) dual curvature; in fact, self duality is equivalent to a set of first order differential equations. Atiyah, Drinfeld, Hitchin, Manin [1] demonstrated a construction for any self or anti-self dual finite action connection. An important open question in the classical theory is whether there exist finite action solutions to the second order equations which are not solutions to the first order equations [2]. Insight may be obtained by answering this question in simpler models. In particular, we shall study the O(3) symmetric, SU(2) Yang–Mills equations on $\mathbb{R}^4$ and the Ginzburg–Landau equations [3] with critical coupling constant in two dimensions. The Ginzburg–Landau equations are related to the four dimensional Yang–Mills equations because any O(3) symmetric solution to the SU(2) Yang–Mills equations on the space $\mathbb{R}^2 \times S^2$ with the natural Riemannian metric determines a solution to the Ginzburg–Landau equations and vice versa, c.f. Sect. V.

* This work supported in part through funds provided by the National Science Foundation under Grant PHY 79-16812.
† Harvard University GSAS Fellow.

0010-3616/80/0075/0207/$04.20

We examine first the Ginzburg–Landau equations. In this case the topological invariant is the integral over $\mathbb{R}^2$ of the curvature form of a U(1) connection on the principle bundle $\mathbb{R}^2 \times \mathrm{U}(1)$. The space of C^0 connections with finite Ginzburg–Landau action separates into disjoint path components labelled by the first Chern number (vortex number) [4]. Bogomol'nyi [5] demonstrated that the action on each path component is bounded below by a multiple of the topological invariant, and that any field configuration which achieves this minimum satisfies a set of first order equations. In a recent paper [6], the author proved that the solution manifold of the first order equations with vortex number N is naturally isomorphic to $\mathbb{R}^{2|N|}$; the isomorphism given by specifying the zeroes on $\mathbb{R}^2$ of the complex scalar field. In this paper it is proved that there are no solutions to the second order Ginzburg–Landau equations which are not solutions to the first order equations.

A precise statement of this result is made in Sect. II; this is the content of Theorems I and II. Theorem I is proved in Sect. III and Theorem II is proved in Sect. IV. In Sect. V we prove the equivalence of the O(3) symmetric Yang–Mills equations on $\mathbb{R}^2 \times S^2$ and the Ginzburg–Landau equations on $\mathbb{R}^2$ with the critical value of the coupling constant.

Our second model is the SU(2) Yang–Mills equations restricted to the space, $\mathscr{C}_{\mathrm{O}(3)}$, of connections which are invariant under the lifting to the principal bundle of the action of the group of rotations about a fixed line in $\mathbb{R}^4$, the group O(3) [7,8]. The SU(2) Yang–Mills equations when restricted to $\mathscr{C}_{\mathrm{O}(3)}$ reduce to the variational equations of the Ginzburg–Landau functional on the hyperbolic plane, and any solution to these variational equations is a solution to the Yang–Mills equations on $\mathbb{R}^4$, cf. Sect. VI. The O(3) symmetric instantons found by Witten are the solutions to the first order equations. We prove in Section VII that there are no finite action strong solutions to the SU(2) Yang–Mills equations restricted to $\mathscr{C}_{\mathrm{O}(3)}$ (henceforth called the O(3) symmetric Yang–Mills equations) which are not either self dual or anti-self dual, cf. Theorem III. In the final Section we prove that any weak solution in $\mathscr{C}_{\mathrm{O}(3)}$ is gauge equivalent to a strong solution in $\mathscr{C}_{\mathrm{O}(3)}$, and hence Theorem III holds for weak solutions also.

II. The Ginzburg–Landau Equations

Let E denote the vector bundle $\pi : \mathbb{R}^2 \times \mathbb{C} \to \mathbb{R}^2$. The Ginzburg–Landau action is a functional on the set $\mathscr{C}(E) \oplus C^\infty(E)$; $\mathscr{C}(E)$ is the set of C^∞, U(1), connections on E and $C^\infty(E)$ is the set of C^∞ cross sections of E. Because E is trivial, the set of connections, $\mathscr{C}(E)$, can be identified with $\Lambda^1(\mathbb{R}^2)$, the set of C^∞ sections of the cotangent bundle. For the same reason, $C^\infty(E)$ can be identified with the set of C^∞ complex valued functions on $\mathbb{R}^2$. These identifications will be made implicitly in this paper.

A connection in $\mathscr{C}(E)$ is given by $-i\pi^*a$ with $a \in \Lambda^1(\mathbb{R}^2)$ and $i = \sqrt{-1}$. The curvature form of the connection will be denoted $-i\pi^* F_a$ with $F_a = da \,\varepsilon\, \Lambda^2(\mathbb{R}^2)$. If $\phi \,\varepsilon\, C^\infty(E)$ is any section, $\phi^*(\pi^* F_a) = F_a$. The connection defines a map from $C^\infty(E)$ to $\Lambda^1(\mathbb{R}^2) \oplus C^\infty(E)$ via the covariant derivative; for $\phi \,\varepsilon\, C^\infty(E)$

$$D_a\phi = d\phi - ia\phi \tag{2.1}$$

With $*$ denoting the duality isomorphism $*: \Lambda^k(\mathbb{R}^2) \to \Lambda^{2-k}(\mathbb{R}^2)$ as defined in the usual way by the (flat) Riemannian structure on $\mathbb{R}^2$, we define the Ginzburg–Landau action as a functional on $\mathscr{C}(E) \oplus C^\infty(E)$ by

$$\mathscr{A}(a, \phi) = \frac{1}{2} \int_{\mathbb{R}^2} \left\{ F_a \wedge *F_a + D_a\phi \wedge *\overline{D_a\phi} + \frac{\lambda}{4} *(\phi\bar{\phi} - 1)^2 \right\}. \tag{2.2}$$

The coupling constant λ will be taken to equal one in the remainder of this paper. The variational equations of the Ginzburg–Landau action are

$$d*F_a - \frac{i}{2} *(\phi\overline{D_a\phi} - \bar{\phi}D_a\phi) = 0, \tag{2.3a}$$

$$- D_a * D_a\phi + *\tfrac{1}{2}(\phi\bar{\phi} - 1)\phi = 0. \tag{2.3b}$$

The boundary conditions for Eqns. (2.3a, b) are specified by the Chern number

$$\frac{1}{2\pi} \int_{\mathbb{R}^2} F_a = N. \tag{2.4}$$

In the physics literature, the usual statement of the problem demands the stronger pointwise conditions

$$\lim_{|x| \to \infty} a = N d\theta; \quad \lim_{|x| \to \infty} \phi = e^{iN\theta}. \tag{2.5}$$

where θ is the polar angle in the plane.

As Bogomol'nyi [5] pointed out, a lower bound on the action results from integrating by parts,

$$\begin{aligned}\mathscr{A}(a, \phi) = \tfrac{1}{2} \int_{\mathbb{R}^2} \{ &\tfrac{1}{2}(D_a\phi \pm i*D_a\phi) \wedge *(\overline{D_a\phi} \mp i*\overline{D_a\phi}) \\ &+ (*F \mp \tfrac{1}{2}(\phi\bar{\phi} - 1)) \wedge *(*F \mp \tfrac{1}{2}(\phi\bar{\phi} - 1))\} \mp \tfrac{1}{2} \int_{\mathbb{R}^2} F\end{aligned} \tag{2.6}$$

so

$$\mathscr{A} \geqq |N|\pi. \tag{2.7}$$

This lower bound is realized if and if (a, ϕ) satisfy

$$\begin{aligned} &D_a\phi - i* D_a\phi = 0 \\ &*F + \tfrac{1}{2}(\phi\bar{\phi} - 1) = 0 \quad \text{for } N \geqq 0; \end{aligned} \tag{2.8}$$

$$\begin{aligned} &D_a\phi + i* D_a\phi = 0 \\ &*F - \tfrac{1}{2}(\phi\bar{\phi} - 1) = 0 \quad \text{for } N \leqq 0. \end{aligned} \tag{2.9}$$

In [6] the solution manifold of (2.8) and (2.9) on $\mathscr{C}(E) \oplus C^\infty(E)$ for fixed N (defined by (2.4)) and modulo gauge transformations $(a, \phi) \to (a + df, \phi e^{if})$ for $f \in C^\infty(\mathbb{R}^2)$ was proven to be isomorphic to $2N$ dimensional Euclidean space. The main result of this paper is the proof that any solution to the second order equations (2.3a, b) with N defined by (2.4) must be either a solution to (2.8) if $N \geqq 0$ or a solution to (2.9) if $N \leqq 0$. To make this precise some preliminary definitions are necessary.

Let $\Omega \subseteq \mathbb{R}^2$ be an open set with compact closure. Define [9] the space

$H^{1,2}(\Omega; T^*)$ as the completion of the set of C^∞ sections of T^* over Ω in the norm

$$\|a\|_{1,2}^2 = \int_\Omega \{da \wedge *da + d*a \wedge *d*a + a \wedge *a\}. \tag{2.10}$$

Let $H_0^{1,2}(\Omega; T^*)$ be the space of sections of T^* compactly supported in Ω. In a like manner define the spaces $H^{1,2}(\Omega; E)\,(H_0^{1,2}(\Omega; E))$ as the completion of the set of C^∞ sections of E over Ω (resp. C^∞ sections of E with compact support in Ω) in the norm

$$\|\phi\|_{1,2}^2 = \int_\Omega \{d\phi \wedge *d\bar{\phi} + *\phi\bar{\phi}\}. \tag{2.11}$$

By a weak solution of equations (2.3a) and (2.3b) we will mean a section (a, ϕ) of $T^* \oplus E$ with the following properties

1. $(a, \phi) \in H^{1,2}(\Omega, T^*) \oplus H^{1,2}(\Omega, E)$ for all $\Omega \subset \mathbb{R}^2$ with compact closure,
2. $\alpha(a, \phi) \leqq \infty$,
3. For all $(b, \eta) \in H_0^{1,2}(\mathbb{R}^2; T^*) \oplus H_0^{1,2}(\mathbb{R}^2; \mathbb{C})$,

$$\int_{\mathbb{R}^2} \Big\{ db \wedge *F_a - \frac{i}{2} b \wedge *(\phi \overline{D_a\phi} - \bar{\phi} D_a \phi)$$
$$+ D_a\eta \wedge *\overline{D_a\phi} + D_a\phi \wedge *\overline{D_a\eta} + *\tfrac{1}{2}(\phi\bar{\phi} - 1)(\phi\bar{\eta} + \eta\bar{\phi}) \Big\} = 0. \tag{2.12}$$

The precise statement of our result is

Theorem I. *Let (a, ϕ) be a weak solution of equations (2.3a) and (2.3b) such that a is a C^3 section of $T^*(\mathbb{R}^2)$ and ϕ is a C^2 section of E. If the number N defined by equation (2.4) is nonnegative then (a, ϕ) is a solution to equations (2.8). If N is nonpositive then (a, ϕ) is a solution to equations (2.9).*

Secondly we show if (a, ϕ) is a weak solution to equations (2.3a) and (2.3b), it is related by a gauge transformation to a C^∞ solution:

Theorem II. *Let (a, ϕ) be a weak solution of equations (2.3a) and (2.3b). Then there exists a pair $(\hat{a}, \hat{\phi}) \in \Lambda^1(\mathbb{R}^2) \oplus C^\infty(E)$ related to (a, ϕ) by $(\hat{a}, \hat{\phi}) = (a + d\psi, \phi e^{i\psi})$ with the function $\psi \in H^{2,2}(\Omega)$ for all open sets $\Omega \subset \mathbb{R}^2$ with compact closure.*

III. Equivalence of First and Second Order Ginzburg–Landau Equations

In proving Theorem I it is convenient to introduce functions f and w in $C^2(\mathbb{R}^2)$ defined by

$$f = *F_a,$$
$$w = \tfrac{1}{2}(1 - \bar{\phi}\phi). \tag{3.1}$$

Assume that (a, ϕ) satisfy (2.3a) and (2.3b) with $a \in C^3(T^*(\mathbb{R}^2))$ and $\phi \in C^2(E)$. Applying the operator $*d*$ to both sides of equation (2.3a) and using (3.1) gives

$$\Delta f = -i*(D_a\phi \wedge \overline{D_a\phi}) + (1 - 2w) f \tag{3.2}$$

where $\Delta = *d*d$ is the Laplace operator on $\mathbb{R}^2$. The definition of w and equation

(2.3b) imply that w must satisfy

$$\Delta w = -*(D_a\phi \wedge *\overline{D_a\phi}) + (1-2w)w. \tag{3.3}$$

We now use mean value theorems associated with the Laplace operator to prove two Lemmas.

Lemma 3.1. *Let (a, ϕ) be a weak solution of equations (2.3a, b). Then $\|\phi\|_\infty \leqq 1$. Further, if $w \in C^2(\mathbb{R}^2)$ then either $w = 0$ or $w > 0$ on $\mathbb{R}^2$.*

Lemma 3.2. *Let f, $w \in C^2(\mathbb{R}^2)$ satisfy (3.2) and (3.3) respectively. Suppose further that the action is finite. Then*

$$|f| \leqq w \quad on \quad \mathbb{R}^2.$$

The behaviour of the action functional under scaling transformations will be used to prove

Lemma 3.3. *Under the conditions of Theorem I the following equality must hold for any solution (a, ϕ):*

$$\int_{\mathbb{R}^2} *f^2 = \int_{\mathbb{R}^2} *w^2. \tag{3.4}$$

Together, Lemmas 3.1, 3.2 and 3.3 imply that $|f| = w$. If the number $N = 0$ then $f = w = 0$ and the solution is trivial. If $N \neq 0$ then

$$\begin{aligned} f &= w \qquad \text{for } N > 0, \\ f &= -w \qquad N < 0. \end{aligned} \tag{3.5a}$$

Using equations (3.5a) and (3.5b) in equations (3.2) and (3.3) gives

$$D_a\phi \wedge *\overline{D_a\phi} - iD_a\phi \wedge \overline{D_a\phi} = 0 \quad \text{for} \quad N > 0, \tag{3.6a}$$

$$D_a\phi \wedge *\overline{D_a\phi} + iD_a\phi \wedge \overline{D_a\phi} = 0 \quad \text{for} \quad N < 0. \tag{3.6b}$$

Equations (3.6a) and (3.6b) may be rewritten as

$$(D_a\phi - i*D_a\phi) \wedge *(\overline{D_a\phi} + i*\overline{D_a\phi}) = 0 \quad N > 0, \tag{3.7a}$$

$$(D_a\phi + i*D_a\phi) \wedge *(\overline{D_a\phi} - i*\overline{D_a\phi}) = 0 \quad N < 0. \tag{3.7b}$$

Together the pairs of equations (3.5a), (3.7a) and (3.5b), (3.7b) imply the theorem.

We now prove Lemmas 3.1, 3.2 and 3.3.

Let $g(x) \in C_0^\infty(\mathbb{R}^2)$ have the properties

$$g(x) = \begin{cases} 1 & |x| \leqq 1 \\ 0 & |x| \geqq 2, \quad 0 \leqq g(x) \leqq 1. \end{cases} \tag{3.8}$$

For $R > 0$ define

$$g_R(x) = g(x/R). \tag{3.9}$$

For the function $g_R(x)$,

$$dg_R(x) = \frac{1}{R}(dg)\left(\frac{x}{R}\right),$$

$$\Delta g_R(x) = \frac{1}{R^2}(\Delta g)\left(\frac{x}{R}\right). \tag{3.10}$$

Let $\eta \in H_0^{1,2}(\Omega; E)$ for some bounded open set $\Omega \in \mathbb{R}^2$. The fact that (a, ϕ) is a weak solution implies

$$\int_{\mathbb{R}^2} \{D_a\eta \wedge *\overline{D_a\phi} + D_a\phi \wedge *\overline{D_a\eta} - *\tfrac{1}{2}(1-\phi\bar{\phi})(\phi\bar{\eta} + \bar{\eta}\phi)\} = 0. \tag{3.11}$$

Proof of Lemma 3.1. We prove that for any weak solution, (a, ϕ), $w = 1/2(1-\phi\bar{\phi}) \geqq 0$. Assume this inequality. It follows from equation (3.3), that for $w \in C^2(\mathbb{R}^2)$

$$(\Delta - 1)w \leqq 0. \tag{3.12}$$

pointwise. By the maximum principle, equation (3.12) implies that w cannot have a nonpositive minimum unless $w \equiv 0$ (see, e.g. Gilbarg and Trudinger, [11], Theorem 3.5). We now prove that $w \geqq 0$.

Suppose that (a, ϕ) is a weak solution of equations (2.3a) and (2.3b). For $R > 0$ define the section $\eta_R \in H_0^{1,2}(D_{2R}(0); E)$ by

$$\eta_R = \begin{cases} g_R(|\phi| - 1)\dfrac{\phi}{|\phi|} & \text{if } |\phi| > 1 \\ 0 & \text{if } |\phi| \leqq 1 \end{cases} \tag{3.13}$$

where g_R is defined in equation (3.9). Let $e = \phi|\phi|^{-1}$. From the definition, $e\bar{e} = 1$.

$$D_a\eta_R = (d|\phi|e + (|\phi| - 1)D_a e)g_R + (|\phi| - 1)\mathrm{ed}g_R. \tag{3.14}$$

Equation (3.11) reads

$$\int_{\Omega_{2R}} \{2g_R[d|\phi| \wedge *d|\phi| + (|\phi| - 1)|\phi| D_a e \wedge *\overline{D_a e}$$
$$+ \tfrac{1}{2}*(|\phi|^2 - 1)(|\phi| - 1)|\phi|] + (|\phi| - 1)2dg_R \wedge *d|\phi|\} = 0. \tag{3.15}$$

Where $\Omega_{2R} = \{x \in \mathbb{R}^2 \,|\, |\phi|(x) > 1\} \cap D_{2R}(0)$.

We have used

$$\bar{e}D_a e + e\overline{D_a e} = d(\bar{e}e) = 0,$$
$$\bar{\phi}D_a\phi + \phi\overline{D_a\phi} = 2|\phi|d|\phi|, \tag{3.16}$$

valid on Ω_R, in deriving (3.15). Equation (3.15) implies:

$$\int_{\Omega_R} \{d|\phi| \wedge *d|\phi| + |\phi|(|\phi| - 1)D_a e \wedge *\overline{D_a e} + *\tfrac{1}{2}(|\phi| + 1)|\phi|(|\phi| - 1)^2\}$$
$$\leqq \frac{1}{2}\left[\int_{\Omega_{2R}} *(|\phi| - 1)^4\right]^{1/2}\left[\int_{\mathbb{R}^2} *(g_{2R}\Delta g_R)^2\right]^{1/2}. \tag{3.17}$$

The last step we use integration by parts and Holder's inequality. The function $\sigma(x)$ defined by

$$\sigma(x)=\begin{cases}(|\phi|-1)(x) & \text{if} \quad |\phi|>1\\ 0 & \text{if} \quad |\phi|<1\end{cases}$$

is in $L(\mathbb{R}^2)$. Since $|\sigma(x)|^2 \leqq |w(x)| \in L_2(\mathbb{R}^2)$ as the action is finite. Combining this result with equation (3.17) gives the inequality

$$\int_{\Omega_R} \{d|\phi| \wedge *d|\phi| + |\phi|(|\phi|-1)D_a e \wedge *\overline{D_a e} + *\tfrac{1}{2}(|\phi|+1)|\phi|(|\phi|-1)^2\} \leqq \frac{4k}{R} a$$

$$\text{where} \qquad \kappa = \left[\int_{\mathbb{R}^2} *(g_2 \Delta g)^2\right]^{1/2}. \tag{3.18}$$

Since $\Omega_R \subseteq \Omega_{R'}$ for $R' \geqq R$ we conclude that the set Ω_∞ has zero measure. Hence $\|\phi\|_\infty \leqq 1$. Since $w \in C^2(\mathbb{R}^2)$ we infer $w \geqq 0$ to complete the proof.

Proof of Lemma 3.2. The Schwarz inequality asserts that

$$|-i*(D_a\phi \wedge \overline{D_a\phi})| \leqq *(D_a\phi \wedge *\overline{D_a\phi}). \tag{3.19}$$

Inequality (3.19) with equations (3.2) and (3.3) imply the two inequalities

$$\Delta\,(w+f) \leqq (1-2w)(w+f), \tag{3.20a}$$

$$\Delta\,(w-f) \leqq (1-2w)(w-f). \tag{3.20b}$$

Since $w \leqq 1/2$, neither $(w+f)$ or $(w-f)$ can achieve a nonpositive minimum on $\mathbb{R}^2$ (see, e.g. [11] Theorem 3.5). If $\lim\limits_{|x|\to\infty} w(x), f(x) \to 0$ pointwise then there is nothing left to prove. However, we need to prove the Lemma under the weaker hypothesis that $a < \infty$. Since F_a satisfies equation (2.3a), and $|\phi| \leqq 1$ (Lemma 3.1),

$$\int_{\mathbb{R}^2} d*F_a \wedge *d*F_a \leqq \int_{\mathbb{R}^2} D_a\phi \wedge *\overline{D_a\phi} \tag{3.21}$$

and $*d*F_a \in L_2(\mathbb{R}^2)$.

It follows from the definition of w and the fact that $|\phi| \leqq 1$ that also

$$\int_{\mathbb{R}^2} dw \wedge *dw \leqq \int_{\mathbb{R}^2} D_a\phi \wedge *\overline{D_a\phi}. \tag{3.22}$$

Let $u = w + f$. From equations (3.21) and (3.22) $u \varepsilon H^{1,2}(\mathbb{R}^2) \cap C^2(\mathbb{R}^2)$.

$$\text{Let } u_- = \begin{cases}-u & \text{if } u<0\\ 0 & \text{otherwise}\end{cases} \tag{3.23}$$

and $u_R = g_R u_-$. The function u_R is in $H_0^{1,2}(D_{2R}(0)) \cap C^0(\mathbb{R}^2)$ and $u_R \geqq 0$. Equation (3.29a) implies that

$$\int_\Omega *\{-g_R u \Delta\, u\} \leqq 0 \tag{3.24}$$

where $\Omega = \{x \in \mathbb{R}^2 | u(x) < 0\}$. Integrating the left side of equation (3.24) by parts gives

$$\int_\Omega g_R(du \wedge *du) + \tfrac{1}{2}\int_\Omega dg_R \wedge *d(u^2) \leqq 0\,. \tag{3.25}$$

An integration by parts again and the Schwarz inequality give

$$\int_\Omega g_R(du \wedge *du) \leqq \frac{1}{2}\left[\int_\Omega *(g_{2R}u^4)\right]^{1/2}\frac{\kappa}{R}; \quad \kappa = \left[\int_{\mathbb{R}^2} *(g_2\, \Delta\, g)^2\right]^{1/2}. \tag{3.26}$$

The Sobolev Embedding Theorem asserts that $H^{1,2}(\mathbb{R}^2) \subset L^4(\mathbb{R}^2)^{10}$. Let $\|u\|_{1,2}$ denote the finite $H^{1,2}$ norm of u. Then there exists a constant $c > 0$ such that

$$\int_{\Omega \cap D_R(0)} (du \wedge *du) \leqq \frac{c}{R}\|u\|_{1,2}^2. \tag{3.27}$$

Since $\Omega \cap D_R(0) \subset \Omega \cap D_{R'}(0)$ for $R' > R$, taking lim inf over R proves that the set Ω has zero measure. Since $u \in C^2(\mathbb{R}^2)$, $u \geqq 0$. The proof for $u = w - f$ is the same. Q.E.D.

Proof of Lemma 3.3. To prove Lemma 3.3 it is convenient to define complex coordinates z, $\bar{z}$ on $\mathbb{R}^2$ by

$$z = x_1 + ix_2,$$
$$\bar{z} = x_1 - ix_2.$$

The one forms dz and $d\bar{z}$ form a basis for $T^*_{\mathbb{C}}(\mathbb{R}^2)$ and we expand the one-form a in this basis;

$$a = \alpha dz + \bar{\alpha} d\bar{z}. \tag{2.28}$$

Equation (3.28) defines $\alpha \in C^3(\mathbb{R}^2; \mathbb{C})$. We define functions u, $v \in C^2(\mathbb{R}^2)$ and h_+, $h_- \in C^1(\mathbb{R}^2; \mathbb{C})$ by

$$\begin{aligned} u &= w + f, \\ v &= w - f, \\ h_\pm &= (D_a\phi)_1 \mp i(D_a\phi)_2. \end{aligned} \tag{3.29}$$

The functions $h_\pm$ have an equivalent definition as

$$\begin{aligned} D_a\phi + i*D_a\phi &= h_+ dz, \\ D_a\phi - i*D_a\phi &= h_- d\bar{z}. \end{aligned} \tag{3.30}$$

Using the definitions (3.1) for f and w and equations (2, 3a, b), we derive a coupled system of equations for $h_\pm$, u and v:

$$\partial u = -\tfrac{1}{2}\bar{\phi} h_+, \tag{3.31a}$$
$$\partial v = -\tfrac{1}{2}\phi \bar{h}_-, \tag{3.31b}$$
$$(\bar{\partial} - i\bar{\alpha})h_+ = -\tfrac{1}{2}\phi u, \tag{3.31c}$$
$$(\bar{\partial} + i\bar{\alpha})\bar{h}_- = -\tfrac{1}{2}\bar{\phi} v, \tag{3.31d}$$

where

$$\partial = \frac{\partial}{\partial z} \quad \text{and} \quad \bar{\partial} = \frac{\partial}{\partial \bar{z}}.$$

The equations (3.31a-d) imply that

$$\partial(uv) = \bar{\partial}(\bar{h}_- h_+). \tag{3.32}$$

To see this, multiply both sides of (3.31a) by v; both sides of (3.31b) by u and add the resulting equations to get

$$\partial(uv) = -\tfrac{1}{2}\bar{\phi} v h_+ - \tfrac{1}{2}\phi u \bar{h}_-. \tag{3.33}$$

Similarly, multiply both sides of (3.31c) by $\bar{h}_-$ and both sides of (3.31d) by h_+. Adding the resulting two equation gives

$$\bar{\partial}(\bar{h}_- h_+) = -\tfrac{1}{2}\bar{\phi} v h_+ + \tfrac{1}{2}\phi u \bar{h}_-. \tag{3.34}$$

Equation (3.32) follows immediately. To derive (3.4), multiply both sides of (3.32) by $z g_R(x)$ and integrate over $\mathbb{R}^2$.

$$\int_{\mathbb{R}^2} * g_R z \partial(uv) = \int_{\mathbb{R}^2} * g_R z \bar{\partial}(\bar{h}_- h_+). \tag{3.35}$$

An integration by parts in (3.35) and using the facts that $\partial z = 1$, $\bar{\partial} z = 0$ gives

$$\int_{\mathbb{R}^2} * g_R uv = -\int_{\mathbb{R}^2} *(uvz\partial g_R - \bar{h}_- h_+ z \bar{\partial} g_R). \tag{3.36}$$

To estimate the right side of (3.36) we make use of (3.10), and the fact that $\partial g_R = (1 - g_{R/2})\partial g_R$. These facts and equation (3.36) imply that

$$\begin{aligned} \int_{\mathbb{R}^2} * g_R uv &\leqq 4\|\partial g\|_\infty \left\{ \left| \int_{\mathbb{R}^2} *(1-g_{R/2}) uv \right| + \left| \int_{\mathbb{R}^2} *(1-g_{R/2}) \bar{h}_- h_+ \right| \right\}, \\ &\leqq 4\|\partial g\|_\infty \int_{\mathbb{R}^2} *(1-g_{R/2})(f^2 + w^2 + h_+\bar{h}_+ + h_-\bar{h}_-) \end{aligned} \tag{3.37}$$

The right hand side of (3.37) is bounded by a constant times $\mathscr{A}$. From Lemma (3.2), $uv \geqq 0$. Therefore

$$\int_{\mathbb{R}^2} g_R uv \leqq \int_{\mathbb{R}^2} g_{R'} uv, \quad \text{for} \quad R \leqq R'. \tag{3.38}$$

We now take lim inf over R as $R \to \infty$ on the right hand side of (3.37). The lim inf is zero, proving the lemma.

IV. Every Weak Solution is Gauge Equivalent to a Strong Solution

In this section we prove Theorem II. Let Ω be an open subset of $\mathbb{R}^2$; if V is any vector bundle over Ω with C^∞ Hermitian metric $\langle \cdot, \cdot \rangle$ defined on the fibres, define the spaces $H^{p,q}(\Omega; V)$ [9], $p, q \geqq 1$ as the completion of $C^\infty(\Omega; V)$ in the norm

$$\|w\|^q_{p,q:\Omega} = \sum_{|\beta| \leqq p} \int_\Omega * \langle \partial_1^{\beta_1} \partial_2^{\beta_2} w, \partial_1^{\beta_1} \partial_2^{\beta_2} w \rangle^q$$

$$\text{with} \quad \partial_i = \frac{\partial}{\partial x^i}, i = 1, 2. \tag{4.1}$$

In particular this defines the Banach spaces $H^{p,q}(\Omega; T^*)$ and $H^{p,q}(\Omega; E)$ for $p, q \geqq 1$.
Let D be the disc of radius one about the origin.

Proposition 4.1. *Let (a, ϕ) be a weak solution of equations (2.3a) and (2.3b). Then there exists a pair $(a_1, \phi_1) \in \Lambda^1(D) \oplus C^\infty(D; E)$ related to (a, ϕ) by $(a_1, \phi_1) = (a + d\psi_1, \phi e^{i\psi_1})$ with the function $\psi_1 \in H^{2,2}(D)$.*

Proof of Proposition 4.1. It is a standard [12] result that there exists a unique $\psi_1 \in H^{2,2}(D_2(0))$ such that in D

$$d * d\psi_1 = -d * a,$$
$$\psi_1|_{\partial D_2(0)} = 0 \tag{4.2}$$

where $D_2(0)$ is the disc of radius 2 about the origin. Define a 1-form a_1 and section ϕ_1 of $T^* \oplus E|_D$ by

$$a_1 = a + d\psi_1, \tag{4.3a}$$
$$\phi_1 = \phi e^{i\psi_1}. \tag{4.3b}$$

Because (a, ϕ) is a weak solution of equations (2.3a) and (2.3b) and $d * a_1 = 0$, for any $(b, \eta) \in H_0^{1,2}(D; T^*) \oplus H_0^{1,2}(D; E)$

$$\int_D \left\{ db \wedge * da_1 + d * b \wedge * d * a_1 - \frac{i}{2} b \wedge * (\phi_1 \overline{D_{a_1}\phi_1} - \bar{\phi}_1 D_{a_1}\phi_1) \right\} = 0, \tag{4.4}$$

$$\int_D \left\{ D_{a_1}\eta \wedge * \overline{D_{a_1}\phi_1} + D_{a_1}\phi_1 \wedge * \overline{D_{a_1}\eta} + \tfrac{1}{2}(\phi_1\bar{\phi}_1 - 1)(\phi_1\bar{\eta} + \eta\bar{\phi}_1) \right\} = 0. \tag{4.5}$$

Define a 1-form J by

$$J = -\frac{i}{2}(\phi_1 \overline{D_{a_1}\phi_1} - \bar{\phi}_1 D_{a_1}\phi_1). \tag{4.6}$$

It follows from Lemma 3.1 that if (a, ϕ) is a weak solution, $\|\phi\|_\infty \leqq 1$; therefore $J \in H^{0,2}(\mathbb{R}^2; T^*) = L^2(\mathbb{R}^2, T^*)$ and

$$\|J\|^2_{0,2:D} \leqq \int_D D_{a_1}\phi_1 \wedge * \overline{D_{a_1}\phi_1} \leqq \mathcal{A}((a, \phi)). \tag{4.7}$$

Equations (4.4), (4.7) and standard regularity estimates (see e.g. Morrey [12], Theorem 6.4.3) assert that $a_1 \in H^{2,2}(D; T^*)$. Using the Sobolev Imbedding Theorem 9, 10, $a_1 \in C^0(D; T^*)$. The fact that a_1 is continuous in D, equation (4.5), and the regularity theorem, Theorem 6.4.3 of Morrey [12] imply $\phi_1 \in H^{2,2}(D; E)$. The

Banach space $H^{2,2}(D; E)$ imbeds in $H^{1,4}(D; E)$;[9] therefore $J \in H^{1,2}(D; T^*)$ and $\phi_1\bar{\phi}_1 \in H^{2,2}(D)$. With these results, Theorem 6.4.3 of [12] asserts that $a_1 \in H^{3,2}(D; T^*)$. Continuing this iteration proves that $a_1 \in \Lambda^1(D)$ and $\phi_1 \in C^\infty(D; E)$. We leave out the details.

Because $da = da_1$ in D, the function $f = * da$ is in $C^\infty(D)$. Similarly the function $w \in C^\infty(D)$ as are the forms $J, D_a\phi \wedge *\overline{D_a\phi}, iD_a\phi \wedge \overline{D_a\phi}$. The choice of the origin was completely arbitrary so in fact f, w, J, $D_a\phi \wedge *\overline{D_a\phi}$, $id_a\phi \wedge \overline{D_a\phi}$ are C^∞ on $\mathbb{R}^2$.

The two-form $F = da$ is infinitely differentiable on $\mathbb{R}^2$; therefore there exists a 1-form $\hat{a} \in \Lambda^1(\mathbb{R}^2)$ such that $d\hat{a} = F$. Let D_R and $D_{R'}$ be two discs of radii $R > R'$ about the origin. The identical analysis as before proves the existence of a pair $(a_R, \phi_R) \in \Lambda^1(D_R; T^*) \oplus C^\infty(D_R; E)$ and a function $\psi_R \in H^{2,2}(D_R)$ such that $(a_R, \phi_R) = (a + d\psi_R, \phi e^{i\psi_R})$ in D_R. Similarly there exists a pair $(a_{R'}, \phi_{R'})$ in $\Lambda^1(D_{R'}; T^*) \oplus C^\infty(D_{R'}; E)$ and a function $\psi_{R'} \in H^{2,2}(D_{R'})$ such that $(a_{R'}, \phi_{R'}) = (a + d\psi_{R'}, \phi e^{i\psi_{R'}})$. In D_R, there exists a C^∞ function σ_R such that

$$\hat{a} = a_R + d\sigma_R = a + d(\sigma_R + \psi_R). \tag{4.9}$$

In the intersection $D_R \cap D_{R'}$ the pair $\sigma_{R'} + \psi_{R'}$ may be chosen (it is unique up to constant) so that

$$\sigma_{R'} + \psi_{R'} = \sigma_R + \psi_R \quad \text{in} \quad D_R \cap D_{R'}. \tag{4.10}$$

The set of functions $\{\sigma_R + \psi_R\}_{R>1}$ have the property that on $D_R \cap D_{R'}$ equation (4.10) holds. The set $\{\sigma_R + \psi_R\}_{R>1}$ defines a single function ψ such that

$$\psi|_{D_R} = \psi_R + \sigma_R. \tag{4.11}$$

By construction, $\psi \in H^{2,2}(\Omega)$ for any bounded set $\Omega \subset \mathbb{R}^2$. The same kind of argument can be used to show that the section $\hat{\phi}$ of E defined on each disc D_R by

$$\hat{\phi}|_{D_R} = \phi e^{i(\sigma_R + \psi_R)} = \phi e^{i\psi}|_{D_R} \tag{4.12}$$

is in $C^\infty(E)$. This completes the proof of Theorem II.

V. Yang–Mill's Equations on $\mathbb{R}^2 \times S^2$

In this section we will show that the SU(2) Yang–Mill's equations on the four dimensional manifold $\mathbb{R}^2 \times S^2$ with Riemannian line element $ds^2 = dx_1^2 + dx_2^2 + 2(d\theta^2 + \sin^2\theta d\chi^2)$ reduce to equations (2.3a) and (2.3b) when the connection is required to be invariant under the lifting (unique up to conjunction), of the group of rotations, O(3), acting on S^2 to the principle bundle. Let $\mathscr{C}_{O(3)}$ be the set of such invariant C^∞ connections.

Let $\{\sigma^j\}_{j=1}^3$ be the 2×2 Pauli matrices. Define an element Q in the Lie algebra of $SU(2)$ by

$$Q = i\{\cos\theta\sigma^3 + \sin\theta(\cos\chi\sigma^1 + \sin\chi\sigma^2)\}. \tag{5.1}$$

The matrix Q satisfies the relation $Q^2 = -1$. A connection in $\mathscr{C}_{O(3)}$ has the form, [7, 8]

$$A = \tfrac{1}{2}a_\mu dx^\mu Q + \tfrac{1}{2}(\phi_1 - 1)QdQ + \tfrac{1}{2}\phi_2 dQ \tag{5.2}$$

with $a_\mu = a_\mu(x_1, x_2)$; $\mu = 1, 2$ and $\phi_j = \phi_j(x_1, x_2)$; $j = 1, 2$. Let T^α, $\alpha = 1, 2, 3$ denote the generators of the action of SU(2) in the adjoint representation; let L^α, $\alpha = 1, 2, 3$ be the generators of the action of the group O(3) on $T^*(S^2)$. Then the curvature of the above connection transforms trivially under the action of $T + L$.

To make the relation to the Ginzburg–Landau equations on $\mathbb{R}^2$ manifest, define a two-form F and a complex scalar ϕ by

$$F = da = \partial_\nu a_\mu dx^\nu \wedge dx^\mu, \quad a = a_\mu dx^\mu$$
$$\phi = \phi_1 + i\phi_2. \tag{5.3}$$

The curvature two-form of the connection defined by equation (5.2) is

$$\mathscr{F} = \tfrac{1}{2} F Q + \tfrac{1}{2} \operatorname{Re} D_a\phi \wedge Q dQ + \tfrac{1}{2} \operatorname{Im} D_a\phi \wedge dQ + \tfrac{1}{4}(\phi\bar{\phi} - 1) dQ \wedge dQ. \tag{5.4}$$

The duality relations on $\mathbb{R}^2 \times S^2$ are

$$\begin{aligned} &*(dx^\mu \wedge dx^\nu) = 2\sin\theta d\theta \wedge d\chi \varepsilon^{\mu\nu}, \\ &*(dx^\mu \wedge d\theta) = -\varepsilon^{\mu\nu} dx^\nu \wedge \sin\theta d\chi, \\ &*(dx^\mu \wedge \sin\theta d\chi) = \varepsilon^{\mu\nu} dx^\nu \wedge d\theta, \\ &*(\sin\theta d\theta \wedge d\chi) = \tfrac{1}{4}\varepsilon^{\mu\nu} dx^\mu \wedge dx^\nu. \end{aligned} \tag{5.5}$$

Using these relations and the fact that

$$dQ \wedge dQ = -2Q \sin\theta d\theta \wedge d\chi \tag{5.6}$$

one can compute the Yang–Mills equations.

The four dimensional equations

$$d * \mathscr{F} + A \wedge * \mathscr{F} - * \mathscr{F} \wedge A = 0 \tag{5.7}$$

are

$$\begin{aligned} &\left[d * F - \frac{i}{2} * (\phi \overline{D_a \phi} - \bar{\phi} D_a \phi) \right] \wedge \sin\theta d\theta \wedge d\chi Q \\ &\quad - \tfrac{1}{2} \operatorname{Re}(-D_a * D_a\phi + * \tfrac{1}{2}(\phi\bar{\phi} - 1)\phi) \wedge dQ \\ &\quad + \tfrac{1}{2} \operatorname{Im}(-D_a * D_a\phi + * \tfrac{1}{2}(\phi\bar{\phi} - 1)\phi) \wedge Q dQ = 0. \end{aligned} \tag{5.8}$$

Where in equation (5.8) the symbol $*$ denotes the Hodge duality operator on the two dimensional subspaces spanned by (x_1, x_2). The SU(2) action for the invariant connection defined by (5.2) is a constant times the Ginzburg–Landau action defined by equation (2.2). The second Chern number of $\mathscr{F}$ is exactly the first Chern number for F_a defined in (2.4). In fact the integration over the variables on the two spheres is trivial.

The equations for the O(3) symmetric instanton solutions are equivalent to equations (2.8) and (2.9). These facts, along with Theorems I and II of this paper, imply that there are no O(3) invariant solutions to the SU(2) Yang–Mills equations on $\mathbb{R}^2 \times S^2$ which are not either self dual or anti-self dual. Combined with our previous results [6] there is a $2|N|$ parameter family of O(3) symmetric instanton solutions on $\mathbb{R}^2 \times S^2$ with topological charge N. The $2|N|$ parameters describing a solution are the positions of the zeroes of the field ϕ in the (x_1, x_2) plane.

VI. The O(3) Symmetric, SU(2) Yang–Mills Equations

If we fix a line in $\mathbb{R}^4$, then the group of rotations about that line, O(3), lifts to an action on the bundle $\mathbb{R}^4 \times \mathrm{SU}(2)$ in a unique way (up to conjugation) [8]. Let $\mathscr{C}_{O(3)}$ denote the set of C^∞ connections invariant under this action. We prove that all solutions to the SU(2) Yang–Mills equations in $\mathscr{C}_{O(3)}$ are either self dual or anti-self dual, cf. Theorem III. Before stating the theorem, some preliminary definitions are necessary.

We take coordinates (t, r, θ, χ) on $\mathbb{R}^4$ with line element $ds^2 = dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta d\chi^2)$. The fixed line in $\mathbb{R}^4$ we take to be the line $r = 0$. A connection in $\mathscr{C}_{O(3)}$ can be put in the following canonical form:

$$A = \tfrac{1}{2}a_\mu dx^\mu Q + \tfrac{1}{2}(\phi_1 - 1)QdQ + \tfrac{1}{2}\phi_2 dQ \tag{6.1}$$

with $dx^0 = dt, dx^1 = dr; a_\mu = a_\mu(r, t), \mu = 0, 1; \phi_j = \phi_j(r, t), j = 1, 2$; and Q defined in (5.1). Define a two-form F_a and a complex scalar ϕ by

$$\begin{aligned} F_a &= da = (\partial_t a_r - \partial_r a_t)dt \wedge dr \\ \phi &= \phi_1 + i\phi_2 \end{aligned} \tag{6.2}$$

The curvature two-form of the connection defined by (6.1) is

$$\mathscr{F} = \tfrac{1}{2}F_a Q + \tfrac{1}{2}(d\phi_1 + a\phi_2) \wedge QdQ + \tfrac{1}{2}(d\phi_2 - a\phi_1) \wedge dQ + \tfrac{1}{4}(\phi\bar{\phi} - 1)dQ \wedge dQ. \tag{6.3}$$

The Yang–Mills action density is the scalar invariant $-*tr(\mathscr{F} \wedge *\mathscr{F})$. For the connection in (6.1)

$$-*tr(\mathscr{F} \wedge *\mathscr{F}) = \frac{1}{2}\left\{(\partial_\mu a_\nu \varepsilon^{\mu\nu})^2 + \frac{2}{r^2}(D_a\phi)_\mu \overline{(D_a\phi)}_\mu + \frac{1}{r^4}(\phi\bar{\phi} - 1)^2\right\} \tag{6.4}$$

where the index μ runs from 0 to 1 and $\partial_0 = \partial/\partial t$, $\partial_1 = \partial/\partial r$.

The action is the integral of (6.4) over $\mathbb{R}^4$; the angular integration is trivial, leaving

$$\mathscr{A} = -tr \int_{\mathbb{R}^4} \mathscr{F} \wedge *\mathscr{F} = \mathscr{A}(a, \phi)$$

and

$$\mathscr{A}(a, \phi) = 2\pi \iint_{\{(r,t) \in \mathbb{R}^2 \mid r \geqq 0\}} \left\{r^2 F_a \wedge *F_a + 2D_a\phi \wedge *\overline{D_a\phi} + \frac{1}{r^2}*(\phi\bar{\phi} - 1)^2\right\}. \tag{6.5}$$

The $*$ operation in (6.5) is defined by $*dt = dr$, $*dr = -dt$, $*1 = dt \wedge dr$. The Yang–Mills equations are the variational equations of the reduced action, $\mathscr{A}$, defined in (6.5);

$$d*r^2F - i*(\phi\overline{D_a\phi} - \bar{\phi}D_a\phi) = 0, \tag{6.6a}$$

$$-D_a*D_a\phi + *\frac{1}{r^2}(\phi\bar{\phi} - 1)\phi = 0. \tag{6.6b}$$

The second Chern number of the SU(2) connection A in (6.1) can be expressed in terms of the first Chern number of the U(1) connection on $\mathbb{R}^2_+$ as

$$\frac{1}{2\pi}\int_{\mathbb{R}^2_+} F_a = N = -\frac{1}{8\pi^2} tr \int_{\mathbb{R}^4} \mathscr{F} \wedge \mathscr{F}. \tag{6.7}$$

The SU(2) action, $\mathscr{F}(a, \phi)$, is bounded below by a multiple of $|N|$ and achieves its minimum for fixed N if and only if

$$\left.\begin{aligned} &D_a\phi + i*D_a\phi = 0 \\ &r*F + \frac{1}{r}(1-\phi\bar{\phi}) = 0 \end{aligned}\right\} \text{ If } \quad N \leqq 0, \tag{6.8a}$$

$$\left.\begin{aligned} &D_a\phi - i*D_a\phi = 0 \\ &r*F - \frac{1}{r}(1-\phi\bar{\phi}) = 0 \end{aligned}\right\} \text{ if } \quad N \geqq 0. \tag{6.8b}$$

Equations (6.8a) and (6.8b) are the O(3) symmetric instanton equations for which Witten [7] found a $2|N|$ parameter family of solutions.

Define the (open) subset $\mathbb{R}^2_+ \subset \mathbb{R}^2$ as

$$\mathbb{R}^2_+ = \{(r, t) \in \mathbb{R}^2 | r > 0\}. \tag{6.9}$$

We now state

Theorem III. *Let the* SU(*2*) *connection* $A \in \mathscr{C}_{O(3)}$; *A is given by* (*6.1*). *Assume that the one-form* $a \in C^3(\mathbb{R}^2_+, T^*)$ *and* $\phi \in C^2(\mathbb{R}^2_+; \mathbb{C})$ *with* (a, ϕ) *a strong solution to* (*6.6a, b*) *in* $\mathbb{R}^2_+$. *Suppose further that* $\mathscr{A}(a, \phi) < \infty$ *and that there exists* $\varepsilon > 0$ *such that*

$$\operatorname{tr} \int_{\mathbb{R}^4} \left(\frac{1}{r^{1+\varepsilon}} \mathscr{F} \wedge *\mathscr{F}\right) < \infty. \tag{6.10}$$

Then A satisfies (*6.8*).

Two Remarks: First K. Uhlenbeck [13], proved that if $\mathscr{A} < \infty$ and A is a strong solution to the Yang–Mills equations, then A defines a connection on a principal SU(2) bundle on S^4 and hence has integer second Chern number. Secondly, if the curvature is $L^4(\mathbb{R}^4)$ then condition (6.10) is superfluous. If $\mathscr{F}$ is locally bounded, then we may also dispense with (6.10). This is the case for any strong solution.

We strengthen Theorem III by proving that the hypothesis on the differentiability of (a, ϕ) may be relaxed to allow for weak solutions.

Define a weak solution to the O(3) symmetric Yang–Mills equations as a connection $A \in \mathscr{C}_{O(3)}$ given by (6.1) such that

1. The one-form $a \in H^{1,2}(\Omega; T^*)$ for all open sets $\Omega \in \mathbb{R}^2_+$ with compact closure.
2. The function $\phi \in H^{1,2}(\Omega; \mathbb{C})$ for all open sets $\Omega \in \mathbb{R}^2_+$ with compact closure.
3. $\mathscr{A}(a, \phi) < \infty$.
4. For all pairs $(b, \eta) \in H^{1,2}_0(\mathbb{R}^2_+; T^*) \oplus H^{1,2}_0(\mathbb{R}^2_+; \mathbb{C})$

$$\int_{\mathbb{R}^2_+} \left\{ r^2 db \wedge *F_a - ib \wedge *(\phi\overline{D_a\phi} - \bar{\phi} D_a\phi) + D_a\eta \wedge *\overline{D_a\phi} \right.$$
$$\left. + D_a\phi \wedge *\overline{D_a\eta} + \frac{1}{r^2}*(\phi\bar{\phi} - 1)(\phi\bar{\eta} + \eta\bar{\phi}) \right\} = 0. \tag{6.11}$$

Theorem IV states that every weak solution is gauge equivalent to a strong solution.

Theorem IV. Let the connection A in (6.1) be a weak solution to the O(3) symmetric SU(2) Yang–Mills equations. Then there exists a pair $(\hat{a}, \hat{\phi}) \in C^\infty(\mathbb{R}^2_+; T^*) \oplus C^\infty(\mathbb{R}^2_+; \mathbb{C})$ related to (a, ϕ) by $(\hat{a}, \hat{\phi}) = (a + d\psi; \phi e^{i\psi})$. The function $\psi \in H^{2,2}(\Omega)$ for all open $\Omega \in \mathbb{R}^2_+$ with compact closure.

VII. The Equivalence of the First and Second Order O(3) Symmetric SU(2) Yang–Mills Equations

The proof of Theorem III is conceptually similar to the proof of Theorem I. Certain estimates will differ. Define functions f, $w \in C^2(\mathbb{R}^2_+)$ and $h_+, h_- \in C^1(\mathbb{R}^2_+; \mathbb{C})$ by

$$\begin{aligned} f &= r^2 * F_a, \\ w &= (1 - \phi\bar{\phi}), \\ h_+ &= D_t\phi - iD_r\phi, \\ h_- &= D_t\phi + iD_r\phi, \end{aligned}$$

where

$$D_\mu\phi = \partial_\mu\phi - ia_\mu\phi_\mu \tag{7.1}$$

In $\mathbb{R}^2_+$, the function w pointwise satisfies

$$\Delta w = -2*(D_a\phi \wedge *\overline{D_a\phi}) + \frac{2}{r^2}(1-w)w,$$

where

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{\partial^2}{\partial t^2}. \tag{7.2}$$

Equation (7.2) follows from (6.6b) and the definition of w. From equation (6.6a) we derive an equation for f,

$$\Delta f = -2i*(D_a\phi \wedge \overline{D_a\phi}) + \frac{2}{r^2}(1-w)f. \tag{7.3}$$

Equation (6.10) is equivalent to the statement that

$$\int_{\mathbb{R}^2_+} * \frac{1}{r^{1+\varepsilon}} \left\{ \frac{f^2}{r^2} + \frac{w^2}{r^2} + h_+\bar{h}_+ + h_-\bar{h}_- \right\} < \infty. \tag{7.4}$$

In the proofs that follow, $g_R(r, t) \in C_0^\infty(\mathbb{R}^2)$ (for $R > 0$) is defined in (3.8). For $\lambda > 0$, we define a function $\beta_\lambda \in C^\infty(\mathbb{R}^2_+)$ by

$$\beta_\lambda(r, t) = \begin{cases} 0 & \text{if } r \leqq \lambda \\ 0 \leqq \beta_\lambda \leqq 1 & \text{if } \lambda < r < 2\lambda, \\ 1 & \text{if } 2\lambda \leqq r. \end{cases} \tag{7.5}$$

Lemma 7.1. The function w is either identically zero on $\mathbb{R}^2_+$ or $w > 0$.

Proof of Lemma 7.1. Define a function $w_- \in H^{1,2}_{LOC}(\mathbb{R}^2_+)$ by

$$w_- = \begin{cases} -w & \text{if } w < 0, \\ 0 & \text{if } w \geqq 0. \end{cases} \tag{7.6}$$

Define the set $\Omega \subset \mathbb{R}^2_+$ by

$$\Omega = \{x \in \mathbb{R}^2_+ \,|\, w(x) < 0\}. \tag{7.7}$$

From (7.2) we have

$$\int_\Omega \beta_\lambda g_R \frac{1}{r} w_- d * dw = -\int_\Omega 2\beta_\lambda g_R \frac{w_-}{r} D_a\phi \wedge * \overline{D_a\phi} + 2\int_\Omega * \beta_\lambda g_R \frac{w_- w}{r^3}(1 - w). \tag{7.8}$$

Using the definition of w_- in (7.8),

$$-\int_\Omega \beta_\lambda g_R \frac{1}{r} w d * dw = -2\int_\Omega \beta_\lambda g_R \left\{ \frac{|w|}{r} D_a\phi \wedge * \overline{D_a\phi} + * \frac{w^2}{r^3}(1 + |w|) \right\}. \tag{7.9}$$

The left side of (7.9) can be integrated by parts to give

$$\begin{aligned} -\int_\Omega \beta_\lambda g_R \frac{1}{r} w d * dw &= \int_\Omega d\left(\frac{g_R \beta_\lambda w}{r}\right) \wedge * dw = \int_\Omega g_R \beta_\lambda \frac{1}{r} dw \wedge * dw \\ &\quad + \int_\Omega d\frac{w^2}{2} \wedge * \left\{ \frac{1}{r}\beta_\lambda dg_R + \frac{g_R}{r} d\beta_\lambda + g_R \beta_\lambda d\frac{1}{r} \right\}. \end{aligned} \tag{7.10}$$

Integration by parts on the last term on the right side of (7.10) yields

$$\begin{aligned} -\int_\Omega \beta_\lambda g_R \frac{1}{r} w d * dw &= \int_\Omega g_R \beta_\lambda \frac{1}{r} dw \wedge * dw - \int_\Omega \beta_\lambda \frac{w^2}{2r^2}(r d * dg_R - 2 dg_R \wedge * dr) \\ &- \int_\Omega * \frac{w^2}{r^3} g_R \beta_\lambda - \int_\Omega \frac{w^2}{r}(d\beta_\lambda \wedge * dg_R) - \int_\Omega g_R \frac{w^2}{2r^2}(r d * d\beta_\lambda - 2 d\beta_\lambda \wedge * dr). \end{aligned} \tag{7.11}$$

Together, equations (7.11) and (7.9) imply

$$\begin{aligned} &\int_\Omega g_R \beta_\lambda \left\{ \frac{1}{r} dw \wedge * dw + 2\frac{|w|}{r} D_a\phi \wedge * \overline{D_a\phi} + * 2\frac{w^2}{r^3}(1/2 + |w|) \right\} = \\ &\quad + \int_\Omega \beta_\lambda \frac{w^2}{2r^2}(r d * dg_R - 2 dg_R \wedge * dr) + \int_\Omega g_R \frac{w^2}{2r^2}(r d * d\beta_\lambda - 2 d\beta_\lambda \wedge * dr) \\ &\quad + \int_\Omega \frac{w^2}{r}(d\beta_\lambda \wedge * dg_R). \end{aligned} \tag{7.12}$$

The three terms on the right side of (7.12) are effectively boundary terms since the integrands are nonzero only near the boundary of the supports of β_λ and g_R. To estimate them, define a constant K_1 by

$$K_1 = \max(2\|d * dg\|_\infty, 2\|dg\|_\infty), \tag{7.13}$$

with $g = g_{R=1}$. Using the definition of K_1 and equation (3.10),

$$\left| \int_{\Omega} \beta_\lambda \frac{w^2}{2r^2}(rd*dg_R - 2dg_R \wedge *dr) \right| \leqq \frac{K_1}{R} \int_{\mathbb{R}^2_+} * \frac{w^2}{r^2}. \tag{7.14}$$

Define a constant K_2 by

$$K_2 = \max(2\|d*d\beta_1\|_\infty, 2\|d\beta_1\|_\infty). \tag{7.15}$$

Using the definition of K_2 and equations (7.4) and (7.5),

$$\left| \int_{\Omega} g_R \frac{w^2}{2r^2}(rd*d\beta_\lambda - 2d\beta_\lambda \wedge *dr) \right| \leqq 2^{1+\varepsilon}\lambda^\varepsilon K_2 \int_{\mathbb{R}^2_+} * \frac{w^2}{r^{3+\varepsilon}}. \tag{7.16}$$

For the final term on the right side of (7.12), we have

$$\left| \int_{\Omega} \frac{w^2}{r}(d\beta_\lambda \wedge *dg_R) \right| \leqq \frac{2}{R} K_1 K_2 \int_{\mathbb{R}^2_+} * \frac{w^2}{r^2}. \tag{7.17}$$

Equations (7.12), (7.14), (7.16) and (7.17) imply the inequality

$$\int_{\Omega} g_R \beta_\lambda \left\{ \frac{1}{r} dw \wedge *dw + 2\frac{|w|}{r} D_a\phi \wedge *\overline{D_a\phi} + *\frac{2}{r^3} w^2(1/2 + |w|) \right\} \leqq$$
$$+ \frac{K_1}{R}(1 + 2K_2) \int_{\mathbb{R}^2_+} * \frac{w^2}{r^2} + 2^{1+\varepsilon}\lambda^\varepsilon K_2 \int_{\mathbb{R}^2_+} * \frac{w^2}{r^{3+\varepsilon}}. \tag{7.18}$$

The finite action assumption implies that

$$\int_{\mathbb{R}^2_+} * \frac{w^2}{r^2} < \infty.$$

From (7.4),

$$\int_{\mathbb{R}^2_+} * \frac{w^2}{r^{3+\varepsilon}} < \infty.$$

Taking lim inf over (R, λ) as $R \to \infty$ and as $\lambda \to 0$ on the right side of (7.18) proves that the set Ω has zero measure. As $w \in C^2(\mathbb{R}^2_+)$ we conclude that $w \geqq 0$. The maximum principle (see, e.g. [11], Theorem 3.5) asserts that w cannot have a nonpositive minimum on $\mathbb{R}^2_+$, thus proving Lemma 7.2.

It is convenient to introduce complex coordinates on $\mathbb{R}^2_+$ by

$$z = t + ir, \quad \bar{z} = t - ir, \tag{7.19}$$

The one-forms dz and $d\bar{z}$ form a basis for $T^*_{\mathbb{C}}(\mathbb{R}^2_+)$ and we can expand the one-form a in this basis as

$$a = \alpha dz + \bar{\alpha} d\bar{z}. \tag{7.20}$$

Equation (7.20) defines $\alpha \in C^3(\mathbb{R}^2_+; \mathbb{C})$. The functions h_+, h_- defined in (7.1) have an equivalent definition,

$$h_\pm dz = D_a\phi \pm i * D_a\phi. \tag{7.21}$$

Let $u = w + f$ and $v = w - f$. From equations (6.6a, b), (7.2) and (7.3),

$$\bar{\partial}\partial u = \frac{1}{2r^2}u(1-w) - \frac{1}{2}h_+\bar{h}_+, \tag{7.22a}$$

$$\bar{\partial}\partial v = \frac{1}{2r^2}v(1-w) - \frac{1}{2}h_-\bar{h}_-, \tag{7.22b}$$

and

$$\partial u = -h_+\bar{\phi}, \tag{7.23a}$$

$$\partial v = -\bar{h}_-\phi, \tag{7.23b}$$

$$(\bar{\partial} - i\bar{\alpha})h_+ = -\frac{1}{2r^2}u\phi, \tag{7.23c}$$

$$(\bar{\partial} + i\bar{\alpha})\bar{h}_- = -\frac{1}{2r^2}v\bar{\phi}. \tag{7.23d}$$

Lemma 7.2. The functions u, v are either identically zero on $\mathbb{R}^2_+$ or $u, v > 0$.

Proof of Lemma 7.2. The proof of Lemma 7.2 is essentially the same as that of Lemma 7.1, except that instead of Eq. (7.2), equation (7.22a) is used for u, (7.22b) is used for v.

Equations (7.23) can be used to derive the equality

$$2\bar{\partial}(\bar{h}_-h_+) = \frac{1}{r^2}\partial(uv). \tag{7.24}$$

To derive (7.24), multiply (7.23a) by v and (7.23b) by u and add the resulting equations. Multiply (7.23c) by $\bar{h}_-$ and (7.23d) by h_+ and add these two equations. The results are:

$$\partial(uv) = -vh_+\bar{\phi} - u\bar{h}_-\phi,$$

$$\bar{\partial}(\bar{h}_-h_+) = -\frac{1}{2r^2}(vh_+\bar{\phi} + u\bar{h}_-\phi). \tag{7.25}$$

Equation (7.24) follows directly.

Lemma 7.3. With u, v defined above,

$$\int_{\mathbb{R}^2_+} * \frac{uv}{r^3} = 0. \tag{7.26}$$

Proof of Lemma 7.3. Multiply both sides of (7.26) by $q_R\beta_\lambda$ and integrate. We have

$$2\int_{\mathbb{R}^2_+} * \beta_\lambda g_R\bar{\partial}(\bar{h}_-h_+) = \int_{\mathbb{R}^2_+} * \beta_\lambda g_R\frac{1}{r^2}\partial(uv). \tag{7.27}$$

Integrating both sides of (7.27) by parts and multiplying by i gives

$$-2i\int_{\mathbb{R}^2_+} *\bar{h}_- h_+ \bar{\partial}(g_R\beta_\lambda) = +\int_{\mathbb{R}^2_+} *\beta_\lambda g_R \frac{uv}{r^3} - i\int_{\mathbb{R}^2_+} *\frac{uv}{r^2}\partial(g_R\beta_\lambda). \tag{7.28}$$

To estimate the derivatives of β_λ and g_R in (7.28), we note that

$$|\partial(g_R\beta_\lambda)| \leqq \frac{1}{R}\|\partial g_1\|_\infty + \frac{\lambda^\varepsilon}{r^{1+\varepsilon}}\|\partial\beta_1\|_\infty 2^{1+\varepsilon}. \tag{7.29}$$

Inequality (7.29) and (7.28) imply the inequality

$$\begin{aligned}\int_{\mathbb{R}^2_+} *\beta_\lambda g_R\frac{uv}{r^3} \leqq{}& \frac{1}{R}\|\partial g_1\|_\infty\left(\int_{\mathbb{R}^2_+} *\left\{h_+\bar{h}_+ + h_-\bar{h}_- + \frac{w^2}{r^2} + \frac{f^2}{r^2}\right\}\right)\\ &+\lambda^\varepsilon 2^{1+\varepsilon}\|\partial\beta_1\|_\infty\left(\int_{\mathbb{R}^2_+} *\frac{1}{r^{1+\varepsilon}}\left\{h_+\bar{h}_+ + h_-\bar{h}_- + \frac{w^2}{r^2} + \frac{f^2}{r^2}\right\}\right)\end{aligned} \tag{7.30}$$

The right hand side of (7.30) exists, this from the fact that $\mathscr{A}(a, \phi) < \infty$ and (7.4). Taking lim inf over (R, λ) as $R \to \infty$ and $\lambda \to 0$ on the left side of (7.30) gives the result.

Theorem III is essentially proved. Lemmas 7.2 and 7.3 imply the either $u = 0$ on $\mathbb{R}^2_+$ or $v = 0$ on $\mathbb{R}^2_+$. If $u = 0$ then $f = -w$ is negative from Lemma 7.1 and the second Chern class is negative. From (7.23a), $h_+ = 0$. Therefore (a, ϕ) satisfies equation (6.8a). If $v = 0$ then likewise (a, ϕ) satisfies (6.8b).

VIII. Regularity of Weak Solutions

The proof of Theorem IV is only slightly more difficult than the proof of Theorem II. The techniques are the same. Define $D_\rho \subset \mathbb{R}^2_+$ by

$$D_\rho = \{(r, t) \in \mathbb{R}^2_+ \,|\, (r^2 + t^2)^{1/2} \leqq \rho, \quad r \geqq 1/\rho\}. \tag{8.1}$$

Note that $D_\rho \subset D_{\rho+1}$ and $\cup_{\rho=1}^\infty D_\rho = \mathbb{R}^2_+$.

Lemma 8.1. Let A be a weak solution to the O(3) symmetric SU(2) Yang–Mills equations. With (a, ϕ) defined by (6.1), and (6.2)

$$\|\phi\|_\infty \leqq 1. \tag{8.2}$$

We shall postpone proving Lemma 8.1. Assume for the moment that it is true. We use it to prove a local regularity result

Proposition 8.2. Let A be a weak solution to the O(3) symmetric SU(2) Yang–Mills equations. Then there exists a pair $(a_\rho, \phi_\rho) \in C^\infty(D_\rho; T^*) \oplus C^\infty(D_\rho; \mathbb{C})$ related to (a, ϕ) by $(a_\rho, \phi_\rho) = (a + d\psi_\rho, e^{i\psi_\rho})$. The function $\psi_\rho \in H^{2,2}(D_\rho)$.

Proof of Proposition 8.2. The proof is essentially the same as the proof of Proposition 4.1. The function ψ_ρ is the unique solution [12] to the equation

$$\begin{aligned} d*d\psi_\rho &= -d*a,\\ \psi_\rho|_{\partial D_{2\rho}} &= 0. \end{aligned} \tag{8.3}$$

The argument used in proving Proposition 4.1 is valid because $1/\rho \leqq r \leqq \rho$ in D_ρ.

The construction used in the proof of the global existence of functions (a, ϕ) in Theorem II, on $\mathbb{R}^2$, works word for word on $\mathbb{R}^2_+$. Only the simply connected topology of $\mathbb{R}^2$ was necessary in the proof. $\mathbb{R}^2_+$ is simply connected.

Therefore, we have reduced the proof of Theorem IV to that of Lemma 8.1

Proof of Lemma 8.1. The assumption that A be a weak solution means that for all $\eta \in H_0^{1,2}(\mathbb{R}^2_+; \mathbb{C})$

$$\int_{\mathbb{R}^2_+} \left\{ D_a\eta \wedge *\overline{D_a\phi} + D_a\phi \wedge *\overline{D_a\eta} + *\frac{1}{r^2}(\phi\bar{\phi} - 1)(\phi\bar{\eta} + \eta\bar{\phi}) \right\} = 0. \tag{8.4}$$

Substitute in (8.4) the function η defined by

$$\eta = \begin{cases} \frac{1}{r} g_R \beta_\lambda (|\phi| - 1)\phi/|\phi| & \text{if } |\phi| > 1, \\ 0 & \text{if } |\phi| < 1. \end{cases} \tag{8.5}$$

The function g_R, $R > 0$ was defined in (3.9) and the function β_λ, $\lambda > 0$ was defined in (7.5). Let $e = \phi/|\phi|$ and

$$\Omega = \{(r, t) \in \mathbb{R}^2_+ \,|\, |\phi| > 1\}. \tag{8.6}$$

The function $\beta_\lambda g_R e \in H_0^{1,2}(\mathbb{R}^2_+; \mathbb{C})$. We compute

$$D_a\eta = \frac{1}{r}\beta_\lambda g_R d|\phi| e + \frac{1}{r} g_R \beta_\lambda (|\phi| - 1) D_a e + (|\phi| - 1) e d\left(\frac{g_R \beta_\lambda}{r}\right). \tag{8.7}$$

The substitution of (8.7) into (8.4) gives

$$\begin{aligned} &2\int_\Omega \frac{\beta_\lambda g_R}{r} \left\{ d|\phi| \wedge *d|\phi| + (|\phi| - 1)|\phi| D_a e \wedge *\overline{D_a e} + *\frac{1}{r^2}|\phi|(|\phi| - 1)^2(|\phi| + 1) \right\} \\ &\quad + 2\int_\Omega \left\{ (|\phi| - 1) d|\phi| \wedge *d\left(\frac{g_R\beta_\lambda}{r}\right) \right\} = 0. \end{aligned} \tag{8.8}$$

To evaluate the second term in (8.8), expand $d((g_R\beta_\lambda)/r)$ and integrate by parts. The result is

$$\begin{aligned} &2\int_\Omega \left\{ (|\phi| - 1) d|\phi| \wedge *d\left(\frac{g_R\beta_\lambda}{r}\right) \right\} \\ &= -2\int_\Omega * \frac{(|\phi| - 1)^2}{r^3} \beta_\lambda g_R - \int_\Omega \frac{(|\phi| - 1)^2}{r^2} \left\{ \beta_\lambda (r d * dg_R - 2 dg_R \wedge * dr) \right. \\ &\quad \left. + g_R(r d * d\beta_\lambda - 2 d\beta_\lambda \wedge * dr) + 2r dg_R \wedge * d\beta_\lambda \right\}. \end{aligned} \tag{8.9}$$

We note that in Ω, $|\phi|(|\phi| + 1) \geqq 2$ and $(|\phi| - 1)^2 \leqq (\phi\bar{\phi} - 1)^2$. Using these facts, (8.9) and (8.8) give the inequality

$$2\int\limits_{\Omega}\frac{\beta_\lambda g_R}{r}\left\{d|\phi|\wedge *d|\phi|+(|\phi|-1)|\phi|D_a e\wedge *\overline{D_a e}+*\frac{1}{r^2}(|\phi|-1)^2\right\}\leqq$$
$$+\left|\int\limits_{\Omega}\frac{(\phi\bar{\phi}-1)^2}{r^2}\{rd*dg_R-2dg_R\wedge *dr\}\right|+\left|\int\limits_{\Omega}\frac{(\phi\bar{\phi}-1)^2}{r^{3+\varepsilon}}\{r^{2+\varepsilon}d*d\beta_\lambda\right.$$
$$\left.-2r^{1+\varepsilon}d\beta_\lambda\wedge *dr\}\right|+\left|\int\limits_{\Omega}\frac{(\phi\bar{\phi}-1)^2}{r^2}2rdg_R\wedge *d\beta_\lambda\right|. \tag{8.10}$$

The terms on the right side of (8.10) contain derivatives of g_R and β_λ; they were evaluated in equations (7.14), (7.16) and (7.17). With the constants K_1 defined in (7.13) and K_2 defined in (7.15) we have

$$2\int\limits_{\Omega}\frac{\beta_\lambda g_R}{r}\left\{d|\phi|\wedge *d|\phi|+(|\phi|-1)|\phi|D_a e\wedge *\overline{D_a e}+*\frac{1}{r^2}(|\phi|-1)^2\right\}$$
$$\leqq 2\frac{K_1}{R}(1+2K_2)\int\limits_{\mathbb{R}^2_+}*\frac{w^2}{r^2}+2^{2+\varepsilon}\lambda^\varepsilon K_2\int\limits_{\mathbb{R}^2_+}*\frac{w^2}{r^{3+\varepsilon}}. \tag{8.11}$$

By assumption, the integrals on the right hand side of (8.11) exist. Taking lim inf over (R, λ) as $R\to\infty$ and $\lambda\to 0$ on the right hand side of (8.11) proves that the set Ω has zero measure, which proves Lemma 8.1.

Acknowledgements. The author wishes to thank Professor A. Jaffe for stressing the importance of the analogies with four dimensional Yang–Mills equations and for his critical comments.

References

1. Atiyah, M. F., Drinfeld, V. A., Hitchin, N. J., Mannin, Yu. I.,: Phys. Lett. **65A**, 185 (1978) Drinfeld, V. A., Manin, Yu. I.: Commun. Math. Phys. **63**, 177 (1978)
2. A proof that local minima of the Yang–Mills action are either self or anti-self dual is given in: Bourguignon, J. P., Lawson, H. B., Simons, J.: Proc. Nat. Acad. Sci., USA, **76**, 1550 (1979). A proof that there are no non-self dual solutions in an open neighborhood of a self-dual solution is given above and in: Flume: Phys. Lett. **76B**, 593 (1978)
3. Ginzburg, V. L., Landau, L. D. : Zh. Eksp. Teor. Fiz. **20**, 1064 (1950)
4. Weinberg, E. : Phys. Rev. **D19**, 10, 3008 (1979)
5. Bogomol'nyi, E. B.: Sov. J. Nucl. Phys. **24**, 449 (1976)
6. Taubes, C. H.: Arbitrary N-vortex solutions to the first order Ginzburg–Landau equations. Commun. Math. Phys. to appear
7. Witten, E. : Phys. Rev. Lett. **38**, 121 (1977)
8. Forgacs, P., Manton, N.: Space time symmetries in gauge theories. Commun. Math. Phys., to appear
9. Palais, R.: Foundations of global nonlinear analysis. New York: Benjamin 1968
10. Adams, R.: Sobolev spaces. New York: Academic Press 1975
11. Gilbarg, D., Trudinger, S.: Elliptic partial differential equations of second order. Berlin, Heidelberg, New York: Springer 1977
12. Morrey, C.: Multiple integrals in the calculus of variations. Berlin, Heidelberg, New York: Springer 1966
13. Uhlenbeck, K.: Removable singularities in Yang-Mills fields, Preprint,

Communicated by A. Jaffe

Received October 11, 1979; in revised from November 2, 1979