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On the Cancellation of Hard Anomalies in Gauge Field Models: a Regularization Independent Proof

G. Bandelloni, C. Becchi, A. Blasi, and R. Collina
Istituto di Scienze Fisiche dell 'Università di Genova, Istituto Nazionale di Fisica Nucleare, Sezione di Genova, Italy

Abstract. In this paper we prove Bardeen's conjecture that the anomaly of the Adler–Bardeen–Bell–Jackiw–Schwinger type in gauge models are definitely absent if they are cancelled at the first order of the $\hbar$ perturbation expansion. Our analysis develops within the regularization independent B.P.H.Z. renormalization scheme. We discuss the possible appearance of anomalies in an enlarged class of gauge models admitting soft violations of the Slavnov–Taylor identities which prescribe the gauge transformation properties of the Green functions. By a repeated use of the Callan–Symanzik equation we conclude that the lowest non vanishing contributions to the anomalies must necessarily correspond to the first order in the $\hbar$ perturbation expansion, hence if they are cancelled at this order the theory will be definitely anomaly free.

I. Introduction

One of the main reasons of the general interest in the Adler–Bardeen–Bell–Jackiw–Schwinger [1, 2] anomaly to the Ward Takahashi identities for the axial vector current (A.B.A. for short) is its peculiar property of being free from radiative corrections or, in other words, that the anomaly is completely determined by its one loop contributions. The phenomenological implications of this fact are widely discussed in the literature [3].

It is also well known that a necessary requisite for the renormalizability of the gauge models is that the corresponding Slavnov–Taylor Identities (S.I.) [4], prescribing the wanted gauge transformation properties of the Green functions, be free of anomalies. This, at the one loop level, sets a precise constraint on the fermion fields content of these models. The conjecture that this constraint is sufficient to kill also the higher order contributions to the anomaly would be a direct consequence of the absence of radiative corrections to the anomaly itself.

The validity of this property for the P.C.A.C. anomaly was first suggested in a paper by Adler and Bardeen [5] and later on confirmed, by regularization independent analyses, in some models of phenomenological interest [6, 7] along the lines first proposed by Zee [8]. The method is based on the Callan–Symanzik

[9] equation which for the coefficient of the anomaly assumes, by the power counting constraints, the form of a linear homogeneous partial differential equation of the first order. One then proves that the only solution of this equation, analytic in a neighbourhood of the origin in coupling constant space, is a homogeneous polynomial whose degree corresponds to the one loop contributions.

The anomalies arising in gauge models need a separate discussion. The goal here is to show that a gauge theory is free from anomalies to all orders of perturbation theory provided it is so at the one loop level.

This possibility has been first discussed by Bardeen [10] in the framework of a special regularization procedure. Bardeen's argument is that, in a dimensionally regularized scheme, the only sources of anomalies appear to be the fermion loops; hence if the contributions of such loops cancel, the anomaly is definitely absent.

More recently Costa et al. [11] have proposed to approach the question via the Callan–Symanzik equation. Even if this strategy is of general applicability, many steps of their proof are explicitely based on the particular regularization employed and the analysis itself is developed completely only for some special models.

A general and regularization independent proof that the condition for a gauge model to be anomaly free reduces to the one loop constraint, to our knowledge, is still lacking; our task in this paper is to try to fill this gap.

The method we employ is based on the Callan–Symanzik equation and follows in some aspects the approach chosen in [11]. The general strategy is to study the properties of the coefficients of the A.B.A. at their lowest non vanishing order in perturbation theory. Our analysis will develop within the B.P.H.Z. [12] scheme and will not need any particular regularization procedure. We shall use the known results on the renormalizability of gauge theories and extend them to a larger class of models whose Slavnov identity admits soft violations, i.e. breakings which can be neglected in the region of large Euclidean momenta.

We shall analyse the anomalous softly broken Slavnov Identity (S.B.S.I.) and show that the A.B.A. at its lowest non-trivial $\hbar$ order does not depend upon any dimensional parameter including the ones which characterize the soft breakings to the Slavnov Identity. This agrees with the expected result that the anomaly is exclusively related to the short distance behaviour of the models. We shall also prove that the lowest order contributions to the anomaly are independent from the gauge parameters of the theory. Furthermore by selecting among the models with anomalous S.B.S.I. the special ones without any superrenormalizable coupling we will be able to prove that the coefficients of the A.B.A. are polynomials in the coupling constants associated with the gauge invariant vertices and obey a "natural" factorization property in the charges defining the gauge field couplings. Owing to the independence of these coefficients from the dimensional parameters this result will also hold in the whole class of models under consideration.

The functional dependence of the anomaly on the gauge invariant coupling constants will be further discussed by means of the Callan–Symanzik equation of the theory along the lines suggested in [11]. This analysis will enable us to show that if the A.B.A. appears in the Slavnov Identity of a given model, this must happen at the first order of the $\hbar$ perturbation development.

Therefore we may conclude that if the anomaly is suitably cancelled at this order, it will never appear again.

The paper is so organized:

In Sect. II we analyze at the classical limit the models with S.B.S.I., whose renormalizability is proved in Appendix A;

In Sect. III we describe a suitable parametrization of the renormalized theories and derive their Callan–Symanzik equations. We also prove that the coefficients of the anomaly are independent from any dimensional parameter;

In Sect. IV we prove the gauge invariance of the lowest order anomalous terms;

In Sect. V we show the polynomial character of the anomalies which may appear in the models with S.B.S.I.; some technical aspects of this proof are in appendix B;

The final analysis involving the Callan–Symanzik equation, some steps of which are given in Appendix C, is developed and concluded in Sect. VI.

II. The Softly Broken Slavnov Identity at the Classical Level

The renormalizability of a general class of gauge models has been widely discussed in the literature [13, 14, 15]. The focal point of all these analyses is the validity to all orders of perturbation theory of the Slavnov Identity.

The general framework is as follows: it is given a compact group G (gauge group); a set of gauge vector fields $\mathscr{A}_\mu^\alpha(x)$ and anticommuting scalar fields $c^\alpha(x)$ and $\bar{c}^\alpha(x)$ (Faddeev–Popov ghosts) in one to one correspondence with the generators of the Lie algebra $\mathscr{G}$ of G; a set of spinless and spin $\frac{1}{2}$ matter fields which carry a fully reducible representation of $\mathscr{G}$. The further assignment of a vacuum expectation value for the scalar fields φ_i and the choice of gauge functions of 't Hooft type $\partial_\mu \mathscr{A}_\mu^\alpha + \rho_i^\alpha \varphi_i$ with suitable conditions on the ρ_i^α's uniquely identifies the Slavnov transformations [4, 14]. With respect to these transformations the classical Lagrangian is then defined as the most general invariant polynomial compatible with the power counting constraints.

The possibility of extending the Slavnov invariance of the theory to the quantum level is analyzed by introducing in the Lagrangian a set of external fields coupled to the Slavnov variations of the quantized ones and by discussing the stability under radiative corrections of the ensuing S.I.

The results thus far obtained insure that in models involving only massive quantized fields the only obstruction to the S.I. is the A.B.A. [14].

In the presence of massless particles the Feynman amplitude are ill defined if the involved couplings do not satisfy a suitable set of I.R. dimensional constraints. Assigning an I.R. dimension to all quantized fields by giving it the value two for the massive fields, the naive dimension for the massless ones and dimension one to each space time derivative, such a constraint is that the global infrared dimension of any coupling be greater than or equal to four. Now I.R. pathologies may arise [16] in gauge models with massless fields if the S.I. develops a breaking whose compensation requires the introduction of counterterms violating the I.R. constraint, in much the same way as the A.B.A. is not compensable without violating the ultraviolet (U.V.) power counting rules.

Following the lines given in the introduction, the first step of our analysis is to enlarge the class of gauge models under study by allowing the S.I. to be softly broken (S.B.S.I.) by terms with naive dimension lower than five. At the classical level the Lagrangian of these models are identified by the most general hard, four dimensional, terms invariant under Slavnov transformations and by all possible soft terms compatible with the I.R. constraints which may arise from the presence of massless particles. For example in the completely massive case one can add to the hard Lagrangian any term with naive dimension smaller than or equal to three.

The field content is specified by the quantized gauge vectors $\mathscr{A}^{\alpha}_{\mu}(x)$, the scalars $\varphi_i(x)$, the left fermions $\psi_L(x)$ and right ones $\psi_R(x)$, the Faddeev–Popov $c^{\alpha}(x)$, $\bar{c}^{\alpha}(x)$ and the external fields $\gamma^{\alpha_s}_{\mu}(x), \gamma_i(x), \eta_L(x), \eta_R(x), \zeta^{\alpha_s}(x)$. The components of the fields $\mathscr{A}^{\alpha}_{\mu}(x), c^{\alpha}(x), \bar{c}^{\alpha}(x)$ are identified by the index α labelling a basis of $\mathscr{G}$; the fields $\gamma^{\alpha_s}_{\mu}(x), \zeta^{\alpha_s}(x)$ are restricted to the semisimple factor $\mathscr{G}_s$ of $\mathscr{G}$ while $\varphi_i(x)$, $\gamma_i(x); \psi_L(x), \eta_L(x); \psi_R(x), \eta_R(x)$ define three different representation spaces for $\mathscr{G}$. The fields are also characterized by a conserved Faddeev–Popov charge which is: 0 for $\mathscr{A}^{\alpha}_{\mu}(x), \varphi_i(x), \psi_L(x), \psi_R(x)$; $+1$ for $c^{\alpha}(x), \gamma^{\alpha_s}_{\mu}(x), \gamma_i(x), \eta_L(x), \eta_R(x)$, $+2$ for $\zeta^{\alpha_s}(x)$ and -1 for $\bar{c}^{\alpha}(x)$.

The hard classical action Γ^{cl}_h obeys the following hard S.I.:

$$
\begin{aligned}
(\mathscr{S}_h \Gamma^{\mathrm{cl}}_h) = \int dx \Bigg[& \frac{\delta}{\delta \gamma^{\alpha_s}_{\mu}(x)} \Gamma^{\mathrm{cl}}_h \frac{\delta}{\delta \mathscr{A}^{\alpha_s}_{\mu}(x)} \Gamma^{\mathrm{cl}}_h + \frac{\delta}{\delta \mathscr{A}^{\alpha_A}_{\mu}(x)} \Gamma^{\mathrm{cl}}_h \partial_{\mu} \bar{c}^{\alpha_A}(x) \\
& + \frac{\delta}{\delta \gamma_i(x)} \Gamma^{\mathrm{cl}}_h \frac{\delta}{\delta \varphi_i(x)} \Gamma^{\mathrm{cl}}_h + \frac{\vec{\delta}}{\delta \eta^{\dagger}_R(x)} \Gamma^{\mathrm{cl}}_h \Gamma^{\mathrm{cl}}_h \frac{\overleftarrow{\delta}}{\delta \psi_L(x)} \\
& + \frac{\vec{\delta}}{\delta \eta^{\dagger}_L(x)} \Gamma^{\mathrm{cl}}_h \Gamma^{\mathrm{cl}}_h \frac{\overleftarrow{\delta}}{\delta \psi_R(x)} + \frac{\vec{\delta}}{\delta \psi^{\dagger}_L(x)} \Gamma^{\mathrm{cl}}_h \Gamma^{\mathrm{cl}}_h \frac{\overleftarrow{\delta}}{\delta \eta_R(x)} \\
& + \frac{\vec{\delta}}{\delta \psi^{\dagger}_R(x)} \Gamma^{\mathrm{cl}}_h \Gamma^{\mathrm{cl}}_h \frac{\overleftarrow{\delta}}{\delta \eta_L(x)} + \frac{\delta}{\delta c^{\alpha}(x)} \Gamma^{\mathrm{cl}}_h \partial_{\mu} \mathscr{A}^{\alpha}_{\mu}(x) \\
& + \frac{\delta}{\delta \bar{c}^{\alpha_s}(x)} \Gamma^{\mathrm{cl}}_h \frac{\delta}{\delta \zeta^{\alpha_s}(x)} \Gamma^{\mathrm{cl}}_h \Bigg]
\end{aligned}
\tag{1}
$$

and the supplementary condition

$$
\frac{\delta}{\delta c^{\alpha}(x)} \Gamma^{\mathrm{cl}}_h = \Lambda^{\alpha\beta} \left\{ \delta^{\beta\beta_s} \frac{\delta}{\delta \gamma^{\beta_s}_{\mu}(x)} \Gamma^{\mathrm{cl}}_h + \delta^{\beta\beta_A} \Box \bar{c}^{\beta_A}(x) \right\}
\tag{2}
$$

where the gauge parameters $\Lambda^{\alpha\beta}$ are given as an arbitrary real, symmetric, positive definite matrix.

In order to describe the general solution of Eq. (1) it is useful to introduce the charge matrices involved in the couplings of the gauge fields $\mathscr{A}^{\alpha}_{\mu}(x)$. First of all we specify the symmetric, invariant, positive definite charge tensor $e^{\alpha\beta}$ on the algebra $\mathscr{G}$. Clearly $e^{\alpha\beta}$ has no elements connecting the semisimple factor $\mathscr{G}_s$ of $\mathscr{G}$ to the abelian one $\mathscr{G}_A$. Furthermore the restriction of $e^{\alpha\beta}$ to each simple component $\mathscr{s}$ is proportional to the Killing form. Choosing a basis in $\mathscr{G}_S$ such that the Killing form becomes the identity matrix we have $e^{\alpha\beta}|_{\mathscr{s}} = e_{\mathscr{s}}\delta^{\alpha_s\beta_s}$, thus identifying the simple charges $e_{\mathscr{s}}$. Concerning the restriction of $e^{\alpha\beta}$ to the abelian factor $\mathscr{G}_A$, the only requirement is the symmetry and positive definiteness.

Secondly we introduce the infinitesimal generators $t^\alpha, T_L^\alpha, T_R^\alpha$ of the gauge group in the scalar, left and right handed spinor fields representations respectively. They obey

$$(t^\alpha)^T = -t^\alpha, \quad (T_L^\alpha)^\dagger = T_L^\alpha, \quad (T_R^\alpha)^\dagger = -T_R^\alpha, \tag{3a}$$

$$[t^\alpha, t^\beta] = f^{\alpha\beta\gamma}t^\gamma, \quad [T_L^\alpha, T_L^\beta] = f^{\alpha\beta\gamma}T_L^\gamma,$$
$$[T_R^\alpha, T_R^\beta] = f^{\alpha\beta\gamma}T_R^\gamma \tag{3b}$$

where the matrices t^α are real and $f^{\alpha\beta\gamma}$ are the structure constants of $\mathscr{G}$. Moreover the basis in the abelian factor $\mathscr{G}_A$ is chosen so that

$$\tfrac{8}{3} T_r[(T_R^{\alpha_A})^\dagger T_R^{\beta_A} + (T_L^{\alpha_A})^\dagger T_L^{\beta_A}] + \tfrac{1}{3} T_r[(t^{\alpha_A})^T t^{\beta_A}] = \delta^{\alpha_A\beta_A} \tag{4}$$

This can be made without any loss of generality since, for any choice of the basis in $\mathscr{G}_A$, the matrix in the l.h.s. of Eq. (4) is real, symmetric and positive definite except in the trivial case where free abelian photons are present.

The couplings of the gauge fields $\mathscr{A}_\mu^\alpha(x)$ can now be expressed in terms of the tensors

$$(ef)^{\alpha_s\beta_s\gamma_s} = e_{\mathscr{s}} f^{\alpha_s\beta_s\gamma_s} \tag{5a}$$

for any simple factor $\mathscr{s}$ of $\mathscr{G}_S$, and of the matrices

$$(et)^\alpha = e^{\alpha\beta}t^\beta, \quad (eT_L)^\alpha = e^{\alpha\beta}T_L^\beta, \quad (eT_R)^\alpha = e^{\alpha\beta}T_R^\beta \tag{5b}$$

and the general solution of Eq. (1) can be given up to the following [17] field transformations

$$\mathscr{A}_\mu^\alpha \to Z^{\alpha\beta}\mathscr{A}_\mu^\beta \text{ with } Z^{\alpha_s\beta_A} = Z^{\alpha_A\beta_s} = 0, \tag{6a}$$

$$\varphi_i \to \sigma_{ij}\varphi_j, \tag{6b}$$

$$\psi_L \to Z_L^F\psi_L, \quad \psi_R \to Z_R^F\psi_R \tag{6c}$$

$$\bar{c}^\alpha \to \bar{z}^{\alpha\beta}\bar{c}^\beta, \text{ with } \bar{z}^{\alpha_A\beta_A} = Z^{\alpha_A\beta_A}, \tag{6d}$$

$$c^\alpha \to Z^{\alpha\beta}c^\beta, \tag{6e}$$

$$\gamma_\mu^{\alpha_s} \to (Z^{-1})^{T\alpha_s\beta_s}\gamma_\mu^{\beta_s}, \tag{6f}$$

$$\gamma_i \to (\sigma^{-1})_{ij}^T\gamma_j, \tag{6g}$$

$$\zeta^{\alpha_s} \to (\bar{z}^{-1})^{T\alpha_s\beta_s}\zeta^{\beta_s}, \tag{6h}$$

$$\eta_L^\dagger \to (Z_R^{F-1})^T\eta_L^\dagger, \quad \eta_R^\dagger \to (Z_L^{F-1})^T\eta_R^\dagger,$$
$$\eta_L \to (Z_R^{F-1})^T\eta_L, \quad \eta_R \to (Z_L^{F-1})^T\eta_R \tag{6i}$$

which leave Eq. (1) invariant, as

$$\begin{aligned}
\mathscr{L}^h = & -\tfrac{1}{4}\mathscr{G}^\alpha_{\mu\nu}\mathscr{G}^{\mu\nu}_\alpha - \tfrac{1}{2}(D_\mu\varphi)_i(D^\mu\varphi)_i - i\psi^\dagger_L \not{D}_L\psi_L \\
& - i\psi^\dagger_R\not{D}_R\psi_R - (\psi^\dagger_L g^i\psi_R + \psi^\dagger_R g^{i\dagger}\psi_L)\varphi_i - \frac{h_{ijkl}}{4!}\varphi_i\varphi_j\varphi_k\varphi_l \\
& - \Lambda^{\alpha\beta}\left\{\frac{\partial_\mu\mathscr{A}^\mu_\alpha\partial_\nu\mathscr{A}^\nu_\beta}{2} - c^\alpha(\Box\bar{c}^\beta + (ef)^{\beta\delta\gamma}\partial_\mu(\mathscr{A}^\mu_\gamma\bar{c}_\delta))\right\} \\
& + \gamma^{\alpha_s}_\mu[\partial^\mu\bar{c}^{\alpha_s} + (ef)^{\alpha_s\rho\sigma}\mathscr{A}^\mu_\sigma\bar{c}_\rho] + \gamma_i(et)^\alpha_{ij}\varphi_j\bar{c}^\alpha + \eta^\dagger_R(eT_L)^\alpha\psi_L\bar{c}^\alpha \\
& + \psi^\dagger_L(eT^\dagger_L)^\alpha\eta_R\bar{c}^\alpha + \psi^\dagger_R(eT^\dagger_R)^\alpha\eta_L\bar{c}^\alpha + \eta^\dagger_L(eT_R)^\alpha\psi_R\bar{c}^\alpha + \tfrac{1}{2}\zeta^{\alpha_s}(ef)^{\alpha_s\gamma\delta}\bar{c}^\gamma\bar{c}^\delta,
\end{aligned} \tag{7}$$

where

$$\mathscr{G}^\alpha_{\mu\nu} = \partial_\mu\mathscr{A}^\alpha_\nu - \partial_\nu\mathscr{A}^\alpha_\mu - (ef)^{\alpha\beta\gamma}\mathscr{A}^\beta_\mu\mathscr{A}^\gamma_\nu, \tag{8a}$$

$$D_\mu\varphi_i = \partial_\mu\varphi_i - \mathscr{A}^\alpha_\mu(et)^\alpha_{ij}\varphi_j, \tag{8b}$$

$$\not{D}_L\psi_L = (\not{\partial} - \not{\mathscr{A}}^\alpha(eT_L)^\alpha)\psi_L, \tag{8c}$$

$$\not{D}_R\psi_R = (\not{\partial} - \not{\mathscr{A}}^\alpha(eT_R)^\alpha)\psi_R. \tag{8d}$$

The couplings between the scalar fields φ_i and the spinors in Eq. (7) are described by the matrices g^i which due to invariance, satisfy

$$g^i T^\alpha_R - T^\alpha_L g^i = t^\alpha_{ij} g^j \tag{9}$$

and the quartic couplings among the φ_i fields are given by the real symmetric tensor h_{ijkl} whose invariance condition is

$$h_{mjkl}t^\alpha_{mi} + h_{mikl}t^\alpha_{mj} + h_{mijl}t^\alpha_{mk} + h_{mijk}t^\alpha_{ml} = 0. \tag{10}$$

Finally we spend a few words on the parameter content of the hard Lagrangian in Eq. (7); let us recall that the matrix $e^{\alpha\beta}$ is assigned by the simple charges $e_\mathscr{S}$ and the abelian matrix elements $e^{\alpha_A\beta_A}$ with $\alpha_A \geqq \beta_A$ (according to an ordering of the basis of $\mathscr{G}$). The coupling matrices g^i and the tensor h_{ijkl} will be parametrized by means of the coefficients of their development on suitable bases, which will be explicitely given when necessary. All these charge and coupling parameters will be denoted by the collective symbol $\{\lambda_a\}$. No explicit parametrization of the gauge matrix $\Lambda^{\alpha\beta}$ will be needed in the following.

III. The Renormalized Models with S.B.S.I. and their Callan–Symanzik Equations

In the previous section we have characterized at the classical limit the hard part of the Lagrangian of the models with S.B.S.I. The analysis of the renormalizability of these theories is compared, in Appendix A, with those of models with exact S.I. [14, 15]. The result is that, as in the exact S.I. case, the S.B.S.I. is implementable to all orders of perturbation theory ($\hbar$ expansion) except when the A.B.A. appears. Furthermore, while the exact S.I. may be affected with the I.R. pathologies mentioned in Sect. II, this is not the case for the S.B.S.I. since the breakings which are not compensable without violating the I.R. constraint are indeed soft.

We shall explicitly write the above results by means of the functional language which we now briefly recall. Let $Z(J, \tau)$ be the functional generating the Feynman graphs of the model, where $J(x)$ stands collectively for the sources $J^\alpha_\mu(x)$, $J_i(x)$, $\xi^\alpha(x)$, $\bar{\xi}^\alpha(x)$, $F_L(x)$, $F_R(x)$ of the quantized fields $\mathscr{A}^\alpha_\mu(x)$, $\varphi_i(x)$, $\bar{c}^\alpha(x)$, $c^\alpha(x)$, $\psi_L(x)$, $\psi_R(x)$ respectively and $\tau(x)$ denotes the external fields $\gamma^{\alpha s}_\mu(x)$, $\gamma_i(x)$, $\zeta^{\alpha s}(x)$, $\eta_L(x)$, $\eta_R(x)$. We shall indicate by $Z_c(J, \tau)$ the connected graphs generator and by $\Gamma(\Phi, \tau)$ its Legendre transform which gives the proper vertices. The variable $\Phi(x)$ is conjugate to $J(x)$ and stands collectively for all quantized fields.

We shall also use the following

Definition. The vertex functionals $\Gamma(\Phi, \tau)$ and $\Gamma'(\Phi, \tau)$ are "hard equivalent", in symbols

$$\Gamma(\Phi, \tau) \sim \Gamma'(\Phi, \tau), \tag{11}$$

if the corresponding proper graphs have the same leading behaviour in the region of large Euclidean momenta, non exceptional in the sense of Symanzik [18].

We can now express the S.B.S.I. as $(\mathscr{S}_h\Gamma)(\Phi, \tau) \sim 0$, where $(\mathscr{S}_h\Gamma)(\Phi, \tau)$ is given in Eq. (1). The situation which we shall be concerned with in this paper is when the A.B.A. appears and the S.B.S.I. modifies into the anomalous form

$$(\mathscr{S}_h\Gamma)(\Phi, \tau) \sim A(\Phi) + O(\hbar A) \tag{12}$$

where the symbol A denotes the local functional

$$\begin{aligned} A = \hbar^n \sum_i r_i \varepsilon^{\mu\nu\rho\sigma} \int dx \Big[& - D_i^{\alpha\beta\gamma} \bar{c}_\alpha(x) \partial_\rho \mathscr{A}^\beta_\mu(x) \partial_\sigma \mathscr{A}^\gamma_\nu(x) \\ & + \frac{F_i^{\alpha\beta\gamma\delta}}{12} (\partial_\mu \bar{c}_\alpha(x)) \mathscr{A}^\beta_\nu(x) \mathscr{A}^\gamma_\rho(x) \mathscr{A}^\delta_\sigma(x) \Big] \equiv \sum_i r_i A_i \end{aligned} \tag{13}$$

with

$$F_i^{\alpha\beta\gamma\delta} = D_i^{\alpha\beta\lambda}(ef)^{\lambda\gamma\delta} + D_i^{\alpha\gamma\lambda}(ef)^{\lambda\delta\beta} + D_i^{\alpha\delta\lambda}(ef)^{\lambda\beta\gamma} \tag{14}$$

and $D_i^{\alpha\beta\gamma}$ are rank three invariant symmetric tensors on the algebra $\mathscr{G}$. This expression of A is the well known form of A.B.A. [2, 19] at the lowest non vanishing order $\hbar^n (n \geqq 1)$ and $O(\hbar A)$ in the r.h.s. of Eq. (12) denotes all the higher order breakings induced by A.

Our program is to discuss the possible appearance of the A.B.A. analyzing, by means of the Callan–Symanzik equation, the functional dependence of the coefficients r_i in Eq. (13) on the parameters of the theory; in this line we choose now a definite parametrization for our renormalized models.

A theory to be interpreted as an operator theory in Fock space requires that the physical and unphysical parameters of the model be identified by the mass, wave function, coupling constant normalization conditions; in our case we adopt an intermediate renormalization scheme where, these normalization conditions being not yet imposed, we are free to choose our parameters in the way best suited for later developments.

For this purpose, let us recall shortly that our analysis is carried out within the regularization independent B.P.H.Z. scheme as extended by Zimmermann [21], Clark and Lowenstein [12, 20] to include massless particles. In this scheme the Feynman and subtraction rules are assigned by means of an effective Lagrangian which is a Normal Product Operator (N.P.O.) in the sense of Zimmermann [21]. The subtraction rules are defined by assigning to each field an U.V. dimension, which in our case coincides with the naive one, and an I.R. dimension according to the prescriptions given in Sect. II and equipping each N.P.O. with appropriate U.V. (δ) and I.R. (ρ) indices (explicitely indicated in the Zimmermann's symbol N^{ρ}_{δ}) which for the effective Lagrangian will be chosen equal to four.

In this formalism the parameters of the renormalized model are all contained in $\mathscr{L}_{\text{eff}}$ which is a polynomial in the fields and their derivatives whose coefficients may be considered as formal power series in $\hbar$.

The effective Lagrangian of our model is so identified: its hard part (with U.V. dimension equal to four) is obtained from the Slavnov invariant expression in Eq. (7) parametrized with λ_a, $\Lambda^{\alpha\beta}$, by performing the field transformations $\phi \to Z\phi$, $\tau \to (Z^{-1})^T\tau$ given in Eq. (6) and adding the non-invariant hard counter-terms needed to satisfy the S.B.S.I. The soft part of the Lagrangian consists of the most general polynomial of U.V. dimension $\leqq 3$ compatible with the I.R. constraints and parametrized with the dimensional coefficients μ_i of its single terms.

In the renormalized models thus defined, the Callan–Symanzik equations can be derived by means of the Lowenstein's Quantum Action Principle (Q.A.P.) [22] and the Zimmermann's reduction formulae connecting N.P.O.'s with different subtraction indices [20, 21].

Lowenstein's Q.A.P. states that, in any given renormalizable model, the partial derivative of a Green function with respect to any parameter is equivalent to the insertion into the Green functions of an internal vertex (integrated N.P.O.) with subtraction indices equal to four. Now the very structure of the B.P.H.Z. method makes it clear that the correspondence between parameters and insertions is one to one if the parameters are completed to describe all the models with the same field content and the same U.V. and I.R. rules, thus neglecting any symmetry.

Referring to the case at hand, we shall denote collectively by the vector $\vec{\sigma}$ the parameters λ_a, $\Lambda^{\alpha\beta}$, μ_i, Z of the theories with S.B.S.I. and by the vector $\vec{\vartheta}$ the ones needed to accomplish the above mentioned completion. In this parametrization we can write

$$\Xi_4\Gamma(\vec{\sigma},\vec{\vartheta})=[\vec{X}^{\Xi}_{\sigma}(\vec{\sigma},\vec{\vartheta})\vec{\nabla}_{\sigma}+\vec{X}^{\Xi}_{\vartheta}(\vec{\sigma},\vec{\vartheta})\vec{\nabla}_{\vartheta}]\Gamma(\vec{\sigma},\vec{\vartheta}) \tag{15}$$

where the l.h.s. means the insertion of the vertex Ξ_4 with subtraction indices equal to 4 in the functional $\Gamma(\vec{\sigma},\vec{\vartheta})$ and $\vec{\nabla}_{\sigma}$, $\vec{\nabla}_{\vartheta}$ are gradient operators.

The Zimmermann's reduction formulae enable us to reduce to a soft (N^3_3) N.P.O. the hard insertion corresponding to a derivative with respect to any dimensional parameter of the theory. This reduction yields both soft and hard radiative correction terms: the last ones can be rewritten as derivatives with respect to the parameters using Eq. (15). In our case we have:

$$\{\mu_i\partial_{\mu_i} + \hbar(\vec{X}^{(i)}_\sigma(\vec{\sigma},\vec{\vartheta})\vec{\nabla}_\sigma + \vec{X}^{(i)}_\vartheta(\vec{\sigma},\vec{\vartheta})\vec{\nabla}_\vartheta)\}\Gamma(\vec{\sigma},\vec{\vartheta}) = Q^{(i)}_3\Gamma(\vec{\sigma},\vec{\vartheta}) \tag{16}$$

where $Q^{(i)}_3$ is an integrated N.P.O. with subtraction indices equal to 3.

The softness of the $Q^{(i)}_3$ insertions, allows us to restrict Eq. (16) to the models satisfying the S.B.S.I. as in the following:

Proposition I. *In models with S.B.S.I. to any dimensional parameter μ_i there corresponds an integrated N.P.O. $Q^{(i)}_3$ whose insertion in the vertex functional generates an infinitesimal transformation of the parameters $\vec{\sigma}$ according to:*

$$[\mu_i\partial_{\mu_i} + \hbar\vec{X}^{(i)}_\sigma(\vec{\sigma},0)\vec{\nabla}_\sigma]\Gamma(\vec{\sigma},0) = Q^{(i)}_3\Gamma(\vec{\sigma},0). \tag{17}$$

Furthermore this translation does not affect the A.B.A. at its lowest non vanishing order since we have

$$[\mu_i\partial_{\mu_i} + \hbar\vec{X}^{(i)}_\sigma(\vec{\sigma},0)\vec{\nabla}_\sigma](\mathscr{S}_h\Gamma)(\vec{\sigma},0) \sim 0 \tag{18}$$

and therefore, if the S.B.S.I. is anomalous:

$$[\mu_i\partial_{\mu_i} + \hbar\vec{X}^{(i)}_\sigma(\vec{\sigma},0)\vec{\nabla}_\sigma]A \sim 0(\hbar A). \tag{19}$$

To prove Proposition I, let us consider the functional $\Gamma(\vec{\sigma} + \varepsilon\vec{X}^{(i)}_\sigma(\vec{\sigma},0), \varepsilon\vec{X}^{(i)}_\vartheta(\vec{\sigma},0))$ obtained from $\Gamma(\vec{\sigma},0)$ by an ε translation along the vectors $\vec{X}^{(i)}$ as in Eq. (16). By a straightforward substitution from Eq. (16) we have

$$\Gamma(\vec{\sigma} + \varepsilon\vec{X}^{(i)}_\sigma(\vec{\sigma},0), \varepsilon\vec{X}^{(i)}_\vartheta(\vec{\sigma},0)) = \Gamma(\vec{\sigma},0) + \varepsilon Q^{(i)}_3\Gamma(\vec{\sigma},0) + O(\varepsilon^2) \sim \Gamma(\vec{\sigma},0) + O(\varepsilon^2). \tag{20}$$

The last line in Eq. (20) follows from Weinberg's power counting theorem [23] and the softness of the $Q^{(i)}_3$ insertion. Likewise from Eq. (12) we have

$$\begin{aligned}
&\mathscr{S}_h\Gamma(\vec{\sigma} + \varepsilon\vec{X}^{(i)}_\sigma(\vec{\sigma},0), \varepsilon\vec{X}^{(i)}_\vartheta(\vec{\sigma},0))\\
&\quad= \mathscr{S}_h[\Gamma(\vec{\sigma},0)] + \varepsilon\int dx\left[\frac{\delta}{\delta\tau_a(x)}(Q^{(i)}_3\Gamma)(\vec{\sigma},0)\frac{\delta}{\delta\Phi_a(x)}\Gamma(\vec{\sigma},0)\right.\\
&\quad+\frac{\delta}{\delta\tau_a(x)}\Gamma(\vec{\sigma},0)\frac{\delta}{\delta\Phi_a(x)}(Q^{(i)}_3\Gamma)(\vec{\sigma},0) + \partial_\mu\bar{c}^{\alpha_A}(x)\frac{\delta}{\delta\mathscr{A}^{\alpha_A}_\mu(x)}(Q^{(i)}_3\Gamma)(\vec{\sigma},0)\\
&\quad\left.+\partial_\mu\mathscr{A}^\alpha_\mu(x)\frac{\delta}{\delta c^\alpha(x)}(Q^{(i)}_3\Gamma)(\vec{\sigma},0)\right] + O(\varepsilon^2)\\
&\quad\sim \mathscr{S}_h\Gamma(\vec{\sigma},0) + O(\varepsilon^2) \sim A + O(\hbar A) + O(\varepsilon^2)
\end{aligned} \tag{21}$$

where we have used a short hand form of the r.h.s. of Eq. (1) with the collective variables τ and Φ.

The last line in Eq. (21) shows that the translation along the vector $\varepsilon\vec{X}^{(i)}$ preserves to first order in ε the S.B.S.I., together with the possible anomalies at their lowest order. From the renormalizability itself of the models with S.B.S.I. it follows that the $\vec{X}^{(i)}$ vector has vanishing components in the ϑ space, thus proving Eq. (17). Now Eq. (18) and Eq. (19) follow directly from Eq. (21) by selecting the first order terms in ε. Proposition I is thus proved.

To complete the derivation of the Callan–Symanzik equations we have to express the derivatives with respect to the Z parameters appearing in the hard part of $\mathscr{L}_{\mathrm{eff}}$, in terms of the generators of multiplicative renormalizations of the fields.

Let us denote by ρ^a an element of a basis for the Lie algebra of the group of transformations given in Eq. (6); a derivative with respect to Z can thus be written as a linear combination of the Lie derivatives L^a's corresponding to the generators ρ^a's.

Conversely we denote by $\mathscr{L}^a$ a generator of the infinitesimal transformations $\Gamma(\Phi,\tau)\to\Gamma(Z\Phi,(Z^{-1})^T\tau)$ induced by the substitutions $\Phi\to Z\Phi, \tau\to(Z^{-1})^T\tau$ shown in Eq. (6). For example the operators

$$\int d^4x\left[\mathscr{A}^\mu_{\alpha_s}(x)\frac{\delta}{\delta\mathscr{A}^\mu_{\beta_s}(x)}+c^{\alpha_s}(x)\frac{\delta}{\delta c^{\beta_s}(x)}-\gamma^\mu_{\alpha_s}(x)\frac{\delta}{\delta\gamma^\mu_{\beta_s}(x)}\right], \tag{22a}$$

$$\int d^4x\left[\mathscr{A}^\mu_{\alpha_A}(x)\frac{\delta}{\delta\mathscr{A}^\mu_{\beta_A}(x)}+c^{\alpha_A}(x)\frac{\delta}{\delta c^{\beta_A}(x)}+\bar{c}^{\alpha_A}(x)\frac{\delta}{\delta\bar{c}^{\beta_A}(x)}\right], \tag{22b}$$

$$\int d^4x\left[\bar{c}^{\alpha_s}(x)\frac{\delta}{\delta\bar{c}^{\beta_s}(x)}-\zeta^{\alpha_s}(x)\frac{\delta}{\delta\zeta^{\beta_s}(x)}\right] \tag{22c}$$

are elements of the set $\{\mathscr{L}^a\}$.

The $\mathscr{L}^a$ and L^a operators, which by their definitions act in distinct ways on the functional $\Gamma(\Phi,\tau)$, can be connected by the Q.A.P. and Zimmermann's reduction formulae. Indeed, according to the Q.A.P., the action of an $\mathscr{L}^a$ operator on $\Gamma(\Phi,\tau)$ is equivalent to the insertion of an integrated N.P.O. as

$$\mathscr{L}^a\Gamma(\Phi,\tau)=\Omega^a_4\Gamma(\Phi,\tau) \tag{23}$$

In theories without massless particles the r.h.s. of this equation can be written as a derivative with respect to the $\vec{\sigma}$ parameters only, since the transformations generated by the $\mathscr{L}^a$'s leave the $\mathscr{S}_h$ operator invariant. However, if massless particles are present, the Ω^a_4 insertion corresponds to a N.P.O. with I.R. index equal to $3(N^3_4)$ which cannot be directly reduced to the hard (N^4_4) N.P.O.'s considered in Eq. (15). In this case the wanted reduction involves the addition of soft (N^3_3) terms yielding

$$\mathscr{L}^a\Gamma(\Phi,\tau)=\vec{\omega}^a\cdot\vec{\nabla}_\sigma\Gamma(\Phi,\tau)+Q^a_3\Gamma(\Phi,\tau). \tag{24}$$

In the tree approximation the $\mathscr{L}^a$ and L^a operators have the same action on the hard part $\Gamma^{\mathrm{cl}}_h(\Phi,\tau)$ of $\Gamma^{\mathrm{cl}}(\Phi,\tau)$, so that

$$\mathscr{L}^a\Gamma^{\mathrm{cl}}(\Phi,\tau)=L^a\Gamma^{\mathrm{cl}}(\Phi,\tau)+Q^{a,\,\mathrm{cl}}_3(\Phi,\tau) \tag{25}$$

where $Q^{a,\,\mathrm{cl}}_3(\Phi,\tau)$ is an integrated local functional of naive dimension $\leqq 3$.

Taking into account the above relation, the r.h.s. of Eq. (24) can be rewritten as

$$\begin{aligned}\mathscr{L}^a\Gamma(\Phi,\tau)=&\,L^a\Gamma(\Phi,\tau)\\&+\hbar\vec{X}^a\vec{\nabla}_\sigma\Gamma(\Phi,\tau)+(Q^a_3\Gamma)(\Phi,\tau).\end{aligned} \tag{26}$$

Now, solving this equation for $L^a\Gamma(\Phi, \tau)$ and Eq. (17) for $\mu_i\partial_{\mu_i}\Gamma(\Phi, \tau)$ we get

$$\begin{aligned}&\{\mu_i\partial_{\mu_i} + \hbar(\vec{\beta}^i_\lambda\vec{\nabla}_\lambda + \vec{\alpha}^i_\Lambda\vec{\nabla}_\Lambda + \gamma^i_a\mathscr{L}^a)\}\Gamma(\Phi, \tau)\\ &\quad = Q'^i_3\Gamma(\Phi, \tau) \sim 0\end{aligned} \tag{27}$$

where we have distinguished the contributions of the parameters λ_a, $\Lambda^{\alpha\beta}$ and Z explicitely.

Defining the scaling operator:

$$\mu\partial_\mu = \sum_i \mu_i\partial_{\mu_i} \tag{28a}$$

and setting

$$\vec{\beta}_\lambda = \sum_i \vec{\beta}^i_\lambda, \tag{28b}$$

$$\vec{\alpha}_\Lambda = \sum_i \vec{\alpha}^i_\Lambda, \tag{28c}$$

$$\gamma_a = \sum_i \gamma^i_a, \tag{28d}$$

$$Q_3 = \sum_i Q'^i_3 \tag{28e}$$

we finally obtain the Callan–Symanzik equation

$$\begin{aligned}&\{\mu\partial_\mu + \hbar(\vec{\beta}_\lambda\vec{\nabla}_\lambda + \vec{\alpha}_\Lambda\vec{\nabla}_\Lambda + \gamma_a\mathscr{L}^a)\}\Gamma(\Phi, \tau)\\ &\quad = Q_3\Gamma(\Phi, \tau) \sim 0.\end{aligned} \tag{29}$$

The last step is the translation of the parametric equations thus far obtained into differential equations for the r_j coefficients in Eq. (13).

First of all we select the n-th order terms in $\hbar$ on both sides of Eq. (19) getting

$$\mu_i\partial_{\mu_i}\left(\sum_j r_j A_j\right) = 0 \tag{30}$$

from which we have

Proposition II. *The coefficients r_j of the* A.B.A. *at their lowest non vanishing order are independent from any dimensional parameter of the theory.*

This Proposition formalizes the expected result that the A.B.A. only depends upon the short distance properties of the models and it will be a basic ingredient of the analysis to follow.

To exploit Eq. (29) let us remark that if the A.B.A. appears at the order $\hbar^n$, as shown explicitely in Eq. (13), the anomalous S.B.S.I. can be extended to order $\hbar^{n+1}$ as

$$(\mathscr{S}_h\Gamma)(\Phi, \tau) \sim \hbar^n(A\Gamma)(\Phi, \tau) + \hbar^{n+1}B(\Phi, \tau) + O(\hbar^{n+2}) \tag{31}$$

where the first term on the r.h.s. is the generator of diagrams containing the vertex corresponding to the A.B.A. and B is a local functional of U.V. dimension 5 collecting only the hard breaking terms appearing at the order $\hbar^{n+1}$. At this order any contribution of lower dimension can be transferred into the soft breaking of the

S.B.S.I. Now, writing explicitely Eq. (18) we have

$$\{\mu\partial_\mu + \hbar(\vec{\beta}_\lambda \vec{\nabla}_\lambda + \vec{\alpha}_A \cdot \vec{\nabla}_A + \gamma_a \mathscr{L}^a)\}\,(\mathscr{S}_h \Gamma)\,(\Phi, \tau) \sim 0 \tag{32}$$

so that the substitution of Eq. (31), selecting the order $\hbar^{n+1}$ terms, yields:

$$\frac{1}{\hbar}\mu\partial_\mu(A\Gamma)(\Phi,\tau)\Big|_{1\text{ loop}} + \mu\partial_\mu B(\Phi,\tau) + (\vec{\beta}_\lambda^{(1)}\vec{\nabla}_\lambda + \vec{\alpha}_A^{(1)}\vec{\nabla}_A + \gamma_a^{(1)}\mathscr{L}^a)A(\Phi,\tau) \sim 0 \tag{33}$$

where the first term on the l.h.s. stands for the generator of the one loop diagrams containing one A.B.A. vertex and $\vec{\beta}_\lambda^{(1)}, \vec{\alpha}_A^{(1)}, \gamma_a^{(1)}$ are the well known one loop contributions to the coefficients $\vec{\beta}_\lambda, \vec{\alpha}_A, \gamma_a$ of the Callan–Symanzik equation. Taking now into account that $\mu\partial_\mu B = 0$, since the local functional B has U.V. dimension equal to five and the coefficients of its terms are dimensionless functions of the parameters, and employing the gauge invariance of the A.B.A., which will be proved in the next Section, we obtain

$$\frac{1}{\hbar}\mu\partial_\mu(A\Gamma)\,(\Phi,\tau)\Big|_{1\text{ loop}} + (\vec{\beta}_\lambda^{(1)}\vec{\nabla}_\lambda + \gamma_a \mathscr{L}^a)A(\Phi,\tau) \sim 0 \tag{34}$$

Recalling that the functional $\sum_j r_j A_j$ depends only on the fields $\mathscr{A}_\mu^\alpha, \bar{c}^\beta$, (compare with Eq. (15)) we can write the last term on the l.h.s. as

$$\gamma_a^{(1)}\mathscr{L}^a\Big(\sum_j r_j A_j(Z\Phi,(Z^{-1})^T\tau)\Big) = \Big(\gamma_2^{(1)\alpha\beta}\int dy\,\mathscr{A}_\mu^\alpha(y)\frac{\delta}{\delta\mathscr{A}_\mu^\beta(y)} + \gamma_1^{(1)\alpha\beta}\int dy\,\bar{c}^\alpha(y)\frac{\delta}{\delta\bar{c}^\beta(y)}\Big)\sum_j r_j A_j(Z\Phi,(Z^{-1})^T\tau), \tag{35}$$

so that Eq. (34) becomes

$$\frac{1}{\hbar}\mu\partial_\mu\sum_j r_j(\vec{\lambda})\,(A_j\Gamma)\,(\vec{\sigma};\,\Phi,\tau)\Big|_{1\text{ loop}} + \Big(\vec{\beta}_\lambda^{(1)}\vec{\nabla}_\lambda \; \gamma_2^{(1)\alpha\beta}\int dy\,\mathscr{A}_\mu^\alpha(y)\frac{\delta}{\delta\mathscr{A}_\mu^\beta(y)} + \gamma_1^{(1)\alpha\beta}\int dy\,\bar{c}_\alpha(y)\frac{\delta}{\delta\bar{c}_\beta(y)}\Big)\sum_j r_j(\vec{\lambda})A_j(\Phi,\tau) \sim 0 \tag{36}$$

Finally introducing the test operators

$$X^{\alpha\beta\gamma}(\eta p,\eta q) = -\frac{1}{2\cdot 4!}\frac{\varepsilon^{\mu\gamma\rho\sigma}}{\eta^2}\frac{\partial}{\partial p_\rho}\frac{\partial}{\partial q_\sigma}\frac{\delta}{\delta\bar{c}^\alpha(o)}\frac{\delta}{\delta\tilde{\mathscr{A}}_\mu^\beta(\eta p)}\frac{\delta}{\delta\tilde{\mathscr{A}}_\nu^\gamma(\eta q)}, \tag{37}$$

where $\dfrac{\delta}{\delta\tilde{\mathscr{A}}_\mu^\rho(\eta p)}$ denotes the functional derivative with respect to the Euclidean Fourier transform of the field $\mathscr{A}_\mu^\beta(x)$, and observing that from Eq. (13) it follows

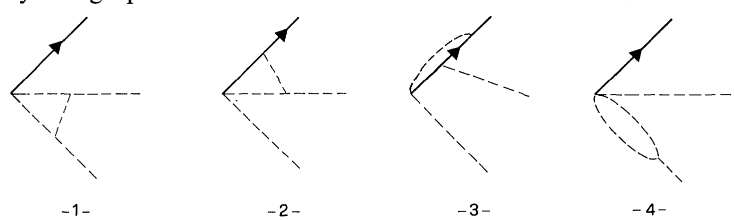
$$X^{\alpha\beta\gamma}(\eta p,\eta q)\sum_j r_j A_j(\Phi,\tau)\Big|_{\phi=\tau=0} = \sum_j r_j D_j^{\alpha\beta\gamma} \tag{38}$$

we can derive the wanted system of partial first order differential equations for the coefficients r_j. Indeed by acting on Eq. (36) with the operators $X^{\alpha\beta\gamma}(\eta p,\eta q)$ in the limit $\eta \to \infty$ and noticing that, as denoted by the hard equivalence symbol, the r.h.s. of Eq. (36) vanishes up to terms which can be neglected in the deep Euclidean region and are therefore killed by these test operators we get:

$$\frac{1}{\hbar}\sum_j r_j(\vec{\lambda}) \lim_{\eta\to\infty} X^{\alpha\beta\gamma}(\eta p, \eta q)\mu\partial_\mu(A_j\Gamma)(\vec{\sigma};\Phi,\tau)\Big|_{\substack{1\ \text{loop}\\ \Phi=\tau=0}} + \Bigg(\beta^{(1)}_{e_{\alpha'_A\alpha''_A}}\frac{\partial}{\partial e_{\alpha'_A\alpha''_A}}$$
$$+ \beta^{(1)}_e \frac{\partial}{\partial e_{\mathscr{s}}} + \vec{\beta}^{(1)}_{\mathscr{g}}\vec{\nabla}_{\mathscr{g}} + \vec{\beta}^{(1)}_{\mathscr{h}}\vec{\nabla}_{\mathscr{h}}\Bigg)\sum_j r_j(\vec{\lambda})D_j^{\alpha\beta\gamma} + \gamma_1^{(1)\alpha\alpha'}\sum_j r_j(\vec{\lambda})D_j^{\alpha'\beta\gamma}$$
$$+ \gamma_2^{(1)\beta\beta'}\sum_j r_j(\vec{\lambda})D_j^{\alpha\beta'\gamma} + \gamma_2^{(1)\gamma\gamma'}\sum_j r_j(\vec{\lambda})D_j^{\alpha\beta\gamma'} = 0 \tag{39}$$

where we have distinguished the contributions of the parameters $e^{\alpha_A\beta_A}$, $e_{\mathscr{s}}$, $\mathscr{g}^i$, $\mathscr{h}$.

The first term on the l.h.s. of Eq. (39) can be directly computed in terms of the following Feyman graphs



where the dashed lines and the solid ones denote respectively the $\mathscr{A}^\alpha_\mu$ and $\bar{c}^\beta$ fields.

IV. Independence of the Anomaly from the Gauge Parameters

We analyze here the dependence of the coefficients r_j in the expression of A.B.A. from the gauge parameters $\Lambda^{\alpha\beta}$. The method is based on the use of the Quantum Action Principle of Lowenstein, already employed in Sec. III, according to which the functional $Z(J, \tau)$ satisfies

$$-i\hbar\frac{\partial}{\partial\Lambda^{\alpha\beta}}Z(J,\tau) = \Xi_4^{\alpha\beta}\, Z(J,\tau) \tag{40}$$

where the r.h.s. denotes the insertion of an internal vertex (integrated N.P.O.) with U.V. subtraction index equal to four. The vertex $\Xi_4^{\alpha\beta}$ can be described as a field dependent local functional with coefficients formal power series in $\hbar$, whose zeroth order contribution is the term proportional to $\Lambda^{\alpha\beta}$ in the classical action in Eq. (7).

The structure of the N.P.O. $\Xi_4^{\alpha\beta}$ in its fully quantized form can be analyzed along the lines indicated by Lowenstein and Schroer in ref. [7]. To this end we introduce the bilocal operator:

$$P_\varepsilon^{\alpha\beta}(x) = \frac{1}{4}\Bigg\{\partial_\mu\frac{\delta}{\delta J^\alpha_\mu(x+\varepsilon)}\partial_\nu\frac{\delta}{\delta J^\beta_\nu(x-\varepsilon)} - \frac{\delta}{\delta\bar{\xi}^\alpha(x+\varepsilon)}\frac{\delta}{\delta\Sigma^\beta(x-\varepsilon)}$$
$$-\frac{\delta}{\delta\bar{\xi}^\beta(x-\varepsilon)}\frac{\delta}{\delta\Sigma^\alpha(x+\varepsilon)} + (\varepsilon \leftrightarrow -\varepsilon)\Bigg\} \tag{41}$$

where we have set

$$\frac{\delta}{\delta\Sigma^\alpha(x)} = \delta^{\alpha\alpha_s}\partial_\mu \frac{\delta}{\delta\gamma_\mu^{\alpha_s}(x)} + \delta^{\alpha\alpha_A}\Box \frac{\delta}{\delta\xi^{\alpha_A}(x)} \tag{42}$$

We further define the modified functional

$$Z^\varepsilon(J,\tau,\rho) = (1 + i\hbar^2 \int dx \rho^{\alpha\beta}(x) P_\varepsilon^{\alpha\beta}(x)) Z(J,\tau) + O(\rho^2) \tag{43}$$

where $\rho^{\alpha\beta}(x)$ is a set of external commuting fields coupled to the bilocal vertex generated by the $P_\varepsilon^{\alpha\beta}(x)$ operator. The connected graphs generator $Z_c^\varepsilon(J,\tau,\rho)$ corresponding to $Z^\varepsilon(J,\tau,\rho)$ is given by

$$\begin{aligned} Z_c^\varepsilon(J,\tau,\rho) = {} & Z_c(J,\tau) + i\hbar \int dx \rho^{\alpha\beta}(x) P_\varepsilon^{\alpha\beta}(x) Z_c(J,\tau) \\ & + \frac{1}{4}\int dx \rho^{\alpha\beta}(x) \Bigg[\frac{\delta}{\delta\bar{\xi}^\alpha(x+\varepsilon)} Z_c(J,\tau) \frac{\delta}{\delta\Sigma^\beta(x-\varepsilon)} Z_c(J,\tau) \\ & + \frac{\delta}{\delta\bar{\xi}^\beta(x-\varepsilon)} Z_c(J,\tau) \frac{\delta}{\delta\Sigma^\alpha(x+\varepsilon)} Z_c(J,\tau) \\ & - \partial_\mu \frac{\delta}{\delta J_\mu^\alpha(x+\varepsilon)} Z_c(J,\tau) \partial_\nu \frac{\delta}{\delta J_\nu^\beta(x-\varepsilon)} Z_c(J,\tau) \\ & + (\varepsilon \leftrightarrow -\varepsilon) \Bigg] + O(\rho^2). \end{aligned} \tag{44}$$

It is now straightforward to verify that in the tree approximation, the $P_\varepsilon^{\alpha\beta}(x)$ operator at $\varepsilon = 0$ introduces into the Green functions the vertex corresponding to the term proportional to $\Lambda^{\alpha\beta}$ in the classical Lagrangian.

One can also verify by direct computation that the introduction of the $\rho_{(x)}^{\alpha\beta}$ external fields does not alter the anomalous S.B.S.I., which for the connected functional $Z_c(J,\tau)$ writes as

$$\begin{aligned} \mathscr{S}_h Z_c(J,\tau) \equiv \int dx \Bigg\{ \Bigg[& J_i(x) \frac{\delta}{\delta\gamma_i(x)} + J_\mu^{\alpha_s}(x) \frac{\delta}{\delta\gamma_\mu^{\alpha_s}(x)} \\ & + J_\mu^{\alpha_A}(x)\partial^\mu \frac{\delta}{\delta\xi^{\alpha_A}(x)} - \zeta^{\alpha_s}(x) \frac{\delta}{\delta\zeta^{\alpha_s}(x)} + \bar{\xi}^\alpha(x)\partial^\mu \frac{\delta}{\delta J_\alpha^\mu(x)} + F_R^{(x)} \frac{\vec{\delta}}{\delta\eta_R^\dagger(x)} \\ & + F_L(x) \frac{\vec{\delta}}{\delta\eta_L^\dagger(x)} \Bigg] Z_c(J,\tau) + Z_c(J,\tau) \Bigg[\frac{\overleftarrow{\delta}}{\delta\eta_R(x)} F_R^\dagger(x) + \frac{\overleftarrow{\delta}}{\delta\eta_L(x)} F_L^\dagger(x) \Bigg] \Bigg\} \\ & \sim -(AZ_c)(J,\tau) + O(\hbar A). \end{aligned} \tag{45}$$

Indeed, taking into account only the leading terms in the deep Euclidean region and recalling (from Eq. (2)) that the dominant contribution to $\frac{\delta}{\delta\Sigma^\alpha(x)} Z_c(J,\tau)$ is

$(\Lambda^{-1})^{\alpha\gamma}\xi^{\gamma}(x)$ we find,

$$\mathscr{S}_h Z_c^{\varepsilon}(J,\tau,\rho) \sim -(AZ_c)(J,\tau) + \tfrac{1}{4}\int dx \rho^{\alpha\beta}(x)$$
$$\cdot\Bigg[+ \partial_\mu \frac{\delta}{\delta J_\mu^\alpha(x+\varepsilon)}(AZ_c)(J,\tau)\partial_\nu \frac{\delta}{\delta J_\nu^\beta(x-\varepsilon)} Z_c(J,\tau)$$
$$+ \partial_\mu \frac{\delta}{\delta J_\mu^\alpha(x+\varepsilon)} Z_c(J,\tau)\partial_\nu \frac{\delta}{\delta J_\nu^\beta(x-\varepsilon)}(AZ_c)(J,\tau) +$$
$$+ \frac{\delta}{\delta \bar{\xi}^\alpha(x+\varepsilon)}(AZ_c)(J,\tau)\Lambda^{-1\alpha\gamma}\xi^\gamma(x+\varepsilon) + (\varepsilon \leftrightarrow -\varepsilon)\Bigg] + O(\hbar A, \rho^2). \tag{46}$$

where we have used the fact that $P_\varepsilon^{\alpha\beta}(x)$ commutes with the hard Slavnov operator $\mathscr{S}_h$. To show that $Z_c^\varepsilon(J,\tau,\rho)$ satisfies Eq. (45) we observe that the first two terms in the r.h.s. of Eq. (46) describe, up to contributions of order $\hbar A$ or $(\rho^2)^{\alpha\beta}$, the insertion of the vertex A into $Z_c^\varepsilon(J,\tau,\rho)$.

We can now define, up to first order in $\rho^{\alpha\beta}$ the functional $Z_c(J,\tau,\rho)$ as the $\varepsilon \to 0$ limit of $Z_c^\varepsilon(J,\tau,\rho)$ by deleting the singular contribution in the Wilson expansion of the bilocal operator $P_\varepsilon^{\alpha\beta}(x)$ in the second term on the r.h.s. of Eq. (44), all others being clearly convergent. The singular terms which have been so dropped are annihilated by the $\mathscr{S}_h$ operator at least up to the order $\hbar A$; indeed all the terms in the r.h.s. of Eq. (46) are regular at $\varepsilon = 0$. It follows that the anomalous S.B.S.I. in Eq. (45) is maintained for the functional $Z_c(J,\tau,\rho)$ and the A.B.A. at its lowest non vanishing order is left unaltered by the introduction of the $\rho^{\alpha\beta}(x)$ fields.

We can now put together the various pieces of information so far collected. First of all we can write

$$\partial_{\Lambda^{\alpha\beta}} Z_c(J,\tau) - \int dx \frac{\delta}{\delta\rho^{\alpha\beta}(x)} Z_c(J,\tau,\rho)\big|_{\rho=0} \equiv D^{\alpha\beta} Z_c(J,\tau,\rho)\big|_{\rho=0} = O(\hbar). \tag{47}$$

Furthermore, since both $\Lambda^{\alpha\beta}$ and $\int dx \dfrac{\delta}{\delta\rho^{\alpha\beta}(x)}$ commute with the Slavnov operator, we have, from Eq. (45):

$$D^{\alpha\beta}(AZ_c)(J,\tau,\rho)\big|_{\rho=0} + O(\hbar A) \sim 0. \tag{48}$$

Rewriting the l.h.s. of Eq. (48) in terms of the vertex generator $\Gamma(\Phi,\tau,\rho)$, which is the Legendre transformation of $Z_c(J,\tau,\rho)$ we find

$$D^{\alpha\beta}(A(\Phi,\tau,\rho))\big|_{\rho=0} + \int dx \frac{\delta}{\delta J_a(x)} D^{\alpha\beta} Z_c(J,\tau,\rho) \frac{\delta}{\delta\Phi_a(x)}(A(\Phi,\tau,\rho))\Big|_{\substack{\rho=0\\ \Phi=\frac{\delta Z}{\delta J}}}$$
$$+ O(\hbar A) \sim 0 \tag{49}$$

and hence, substituting from Eq. (47)

$$D^{\alpha\beta}(A(\Phi,\tau,\rho))\big|_{\rho=0} + O(\hbar A) \sim 0. \tag{50}$$

Finally recalling that the A.B.A. is, at its lowest non vanishing order, independent from the fields $\rho^{\alpha\beta}$, we conclude that the coefficients r_j satisfy

$$\partial_{A^{\alpha\beta}} r_j = 0 \tag{51}$$

which is the desired gauge independence property.

V. The Polynomial Character of the Adler Bardeen Anomaly

Summarizing the results thus far obtained, the coefficients r_j in the expression of A.B.A. (Eq. (13)) depend only on the parameters λ_a defining the hard gauge invariant part of the Lagrangian, and therefore they do not change if we consider models built with an exact S.I. or else models where we allow soft breakings to the S.I. itself.

From a naive point of view one could conjecture that the Feynman amplitudes of these theories, and in particular the coefficients r_j which to Feynman amplitudes are related, are polynomials in the λ_a's. The ansatz that the coefficients r_j are polynomials in the λ_a's has been fully exploited in the analysis of the radiative corrections to the A.B.A. given in [11]. This point is also central to our study and will be proved in this Section.

The proof begins analyzing the special class of the models in which all the couplings with naive dimension less than four are absent, so that the soft part of the Lagrangian is reduced to the mass terms, and the gauge parameters $A^{\alpha\beta}$ are in diagonal form

$$A^{\alpha\beta} = \frac{1}{K}\delta^{\alpha\beta}. \tag{52}$$

Within this special class we can prove the following sequence of properties:

Property 0. The fields $c^{\alpha_A}(x)$, $\bar{c}^{\alpha_A}(x)$ corresponding to the abelian factor $\mathscr{G}_A$ of the gauge algebra $\mathscr{G}$ are free, their couplings being restricted only to the ones with the external fields $\gamma_i, \eta_L, \eta_R, \eta_L^\dagger, \eta_R^\dagger$.

This property is of course true at the tree approximation and can be straightforwardly implemented to all orders of perturbation theory. Consequently the supplementary renormalization conditions corresponding to Eqs. (A.1) (A.2) in Appendix A, become

$$\begin{aligned}\frac{\delta}{\delta\bar{c}^{\alpha_A}(x)}\Gamma(\Phi,\tau) = &-K^{-1}(\Box c^{\alpha_A}(x) + \mu^{\alpha_A\beta}c^\beta(x))\\ &+\gamma_i(et)^\alpha_{ij}\varphi_j(x) + [\eta_R^+(x)(eT_L)^{\alpha_A}\psi_L(x) + \psi_R^+(x)(eT_R^+)^{\alpha_A}\eta_L(x)\\ &+\text{h.c.}] + Z_i^{\alpha_A}\gamma_i(x) + y^{\alpha_A\beta_s}\partial_\mu\gamma_\mu^{\beta_s}(x),\end{aligned} \tag{53}$$

$$K\frac{\delta}{\delta c^{\alpha_A}(x)}\Gamma(\Phi,\tau) = \Box\bar{c}^{\alpha_A}(x) + \mu^{\alpha_A\beta}\bar{c}^\beta(x), \tag{54a}$$

$$\left[\frac{\delta}{\delta c^{\alpha_s}(x)} - K^{-1}\partial_\mu\frac{\delta}{\delta\gamma_\mu^{\alpha_s}(x)} - \frac{\delta}{\delta\mathscr{C}^{\alpha_s}(x)}\right]\Gamma(\Phi,\tau) = 0\,. \tag{54b}$$

Let us remark that, as shown in ref. [15], Eqs. (53), (54) uniquely fix the couplings of the Faddeev–Popov fields with the quantized ones.

We now introduce the following

Definition. A proper amplitude $\Gamma_{\mathscr{A}^{\alpha_1}_{\mu_1}\dots\mathscr{A}^{\alpha_i}_{\mu_i},\bar{c}^{\beta_1}\dots\bar{c}^{\beta_j},c^{\gamma_1}\dots c^{\gamma_\alpha},\gamma^{\delta_1}_{\nu_1}\dots\gamma^{\delta_l}_{\nu_l},\zeta^{\rho_1}\dots\zeta^{\rho_n},\dots}$ where with the subscripts $\mathscr{A}^{\alpha_1}_{\mu_1}\dots\zeta^{\rho_n}$ we have explicitely labelled the amputated external legs of the diagram corresponding to the fields $\mathscr{A}^{\alpha_1}_{\mu_1}\dots\zeta^{\rho_n}$ and the dots indicate possible amputated external legs corresponding to scalar or fermion matter fields, is said to be charge factorized if:

$$\begin{aligned}&\Gamma_{\mathscr{A}^{\alpha_1}_{\mu_1}\dots\mathscr{A}^{\alpha_i}_{\mu_i},\bar{c}^{\beta_1}\dots\bar{c}^{\beta_j},c^{\gamma_1}\dots c^{\gamma_k},\gamma^{\delta_1}_{\nu_1}\dots\gamma^{\delta_l}_{\nu_l},\zeta^{\rho_1}\dots\zeta^{\rho_n},\dots}\\&\quad=e^{\alpha_1\alpha'_1}\dots e^{\rho_n\rho'_n}\bar{\Gamma}_{\mathscr{A}^{\alpha'_1}_{\mu_1}\dots\mathscr{A}^{\alpha'_i}_{\mu_i},\bar{c}^{\beta'_1}\dots\bar{c}^{\beta'_j},c^{\gamma'_1}\dots c^{\gamma'_k},\gamma^{\delta'_1}_{\nu_1}\dots\gamma^{\delta'_l}_{\nu_l},\zeta^{\rho'_1}\dots\zeta^{\rho'_n},\dots}\end{aligned}\tag{55}$$

The coefficient $\bar{\Gamma}_{\mathscr{A}^{\alpha'_1}_{\mu_1}\dots\zeta^{\rho'_n}\dots}$ being a polynomial in the parameters λ_a which vanishes with λ_a if matter fields external legs are present.
Using this definition we have also:

Property 1. If the Lagrangian contains only invariant couplings
(a) the Feynman amplitudes are polynomials in the parameters λ_a;
(b) among these amplitudes the proper non trivial ones, i.e. those corresponding to diagrams with at least one loop, are charge factorized;
(c) the same factorization property also holds for the diagrams which give leading contributions in the deep Euclidean region to the functional $(\mathscr{S}_h\Gamma)(\Phi,\tau)$ defined in Eq. (1).

Properties 1a and b follow directly from the restriction to the invariant couplings and from the results given in Appendix B. In particular the validity of Eq. (55) depends strictly on the fact that $\Gamma_{\mathscr{A}^{\alpha_1}_{\mu_1}\dots\zeta^{\rho_n},\dots}$ is a proper non trivial diagram. For instance, each $\bar{c}^{\alpha_s}(x)$, $c^{\alpha_s}(x)$, $\gamma^{\alpha_s}_\mu(x)$, $\zeta^{\alpha_s}(x)$ external line emerges from a trilinear vertex proportional to the parameter e_σ and whose remaining two lines are necessary internal to the diagram.

Concerning the diagrams contributing to $(\mathscr{S}_h\Gamma)(\Phi,\tau)$ one sees, making use of Eq. (55), that the factorization property can be violated only by $\int dx\frac{\delta}{\delta\gamma^{\alpha_s}_\mu(x)}\Gamma(\Phi,\tau)\frac{\delta}{\delta\mathscr{A}^{\alpha_s}_\mu(x)}\Gamma(\Phi,\tau)$ and $\int dx\frac{\delta}{\delta\zeta^{\alpha_s}(x)}\Gamma(\Phi,\tau)\frac{\delta}{\delta\bar{c}^{\alpha_s}(x)}\Gamma(\Phi,\tau)$ if the contributions to $\frac{\delta}{\delta\mathscr{A}^{\alpha_s}_\mu(x)}\Gamma(\Phi,\tau)$ and $\frac{\delta}{\delta\bar{c}^{\alpha_s}(x)}\Gamma(\Phi,\tau)$ arise from mass terms, i.e. $\frac{\delta}{\delta\mathscr{A}^{\alpha_s}_\mu(x)}\Gamma(\Phi,\tau)=M^{\alpha_s\beta}\mathscr{A}^\beta_\mu(x)$ and $\frac{\delta}{\delta\bar{c}^{\alpha_s}(x)}\Gamma(\Phi,\tau)=\mu^{\alpha_s\beta}c^\beta(x)$. It is however clear that these terms are non leading in the region of large Euclidean momenta.

Property 2. In the hypothesis that, up to a given order in $\hbar$ the counterterms in $\mathscr{L}_{\text{eff}}$ are charge factorized, the same holds true for
(a) any proper nontrivial diagram;
(b) the leading contributions to $(\mathscr{S}_h\Gamma)(\Phi,\tau)$ in the deep Euclidean region;
(c) the counterterms needed to compensate the hard breakings both to the sup-

plementary renormalization conditions in Eq.s (53) (54) and to the S.B.S.I. at the same order.

Statements 2a and b are obvious extensions of Property 1.

In order to prove 2c, let us remark that, according to the analysis developed in Appendix A, the coefficients of the hard breakings are computed as infinite momentum limits of suitable terms in $(\mathscr{S}_h\Gamma)(\Phi,\tau)$ and in the l.h.s. of Eq. (54b). It hence follows from property 2b and the analysis in Appendix B, that the coefficients of the independent breakings do satisfy Eq. (55). Now the factorization property of the counterterms necessary to implement the supplementary condition (Eq. (54b)) can be easily deduced looking at the analysis in ref. [14].

Finally we must unfold the relation between the compensable hard breakings to the S.B.S.I. and the corresponding non invariant counterterms. From the dedailed discussion in refs. [14–15], it turns out that the counterterms $\hat{\Delta}_i(x)$ are obtained from the breakings $\Delta_i(x)$ by solving a chain of equations of the type

$$\text{a)}\quad \int dy\,\partial_\mu \bar{c}^\alpha(y)\frac{\delta}{\delta\mathscr{A}^\alpha_\mu(y)}\hat{\Delta}_i(x)=\Delta_i(x) \tag{56a}$$

or

$$\text{b)}\quad d\hat{\Delta}_i(x)=\Delta_i(x) \tag{56b}$$

where $\hat{\Delta}_i(x)$ and $\Delta_i(x)$ are local polynomials in the fields and their derivatives. The operator d in Eq. (56b) is given by

$$d=R^\alpha e^{\alpha\beta}\bar{c}^\beta(x)+\frac{1}{2}(ef)^{\alpha_s\beta_s\gamma_s}\bar{c}^{\alpha_s}(x)\bar{c}^{\beta_s}(x)\frac{\partial}{\partial\bar{c}^{\alpha_s}(x)} \tag{57}$$

where R^α is the infinitesimal generator of a global gauge transformation on the fields $\mathscr{A}^\alpha_\mu(x)$, $\gamma^{\alpha_s}_\mu(x)$, $\zeta^{\alpha_s}(x)$, $\partial_\mu\bar{c}^\alpha(x)$, $\varphi_i(x)$, $\psi_L(x)$, $\psi_R(x)$, $\eta_L(x)$, $\eta_R(x)$, $\eta^\dagger_L(x)$, $\eta^\dagger_R(x)$, $\gamma_i(x)$ and their derivatives considered as independent variables. Recall that the fields $\mathscr{A}^\alpha_\mu(x)$, $\gamma^{\alpha_s}_\mu(x)$, $\zeta^{\alpha_s}(x)$, $\partial_\mu\bar{c}^\alpha(x)$ transform according to the adjoint representation, thence for example the contribution to R^α from the gauge vector fields and their first derivative is

$$f^{\beta\alpha\gamma}\left[\mathscr{A}^\beta_\mu(x)\frac{\delta}{\partial\mathscr{A}^\gamma_\mu(x)}+\partial_\gamma\mathscr{A}^\beta_\mu(x)\frac{\delta}{\partial(\partial_\gamma\mathscr{A}^\gamma_\mu(x))}\right] \tag{58a}$$

while the scalar fields contribute with:

$$t^\alpha_{ij}\varphi_i(x)\frac{\partial}{\partial\varphi_j(x)} \tag{58b}$$

and so on. Now any dependence from the charges $e^{\alpha\beta}$, and consequently from the whole set of parameters λ_a in the operator d, can be reabsorbed by the field transformation

$$\begin{pmatrix}\mathscr{A}'^\alpha_\mu\\ \gamma'^\alpha_\mu\\ \zeta'^\alpha\\ \bar{c}'^\alpha\end{pmatrix}=e^{\alpha\beta}\begin{pmatrix}\mathscr{A}^\beta_\mu\\ \gamma^\beta_\mu\\ \zeta^\beta\\ \bar{c}^\beta\end{pmatrix} \tag{59}$$

After this transformation the coefficients of the breakings $\Delta_i(x)$ remain polynomials in the parameters λ_a due to charge factorization (Eq. (55)). Furthermore the solutions of Eqs. (56) obtained as in ref. [14] by deleting the invariant contributions, yield counterterms, in the primed fields, whose coefficients are λ_a independent linear combinations of those of the breakings. The back transformation to unprimed fields restores the charge factorization property of these counterterms.

Making use iteratively of Properties 1 and 2 it is straightforward to verify the following

Proposition III. *Within the special class of models whose effective Lagrangians do not contain any coupling with dimension less than four, the coefficients of the hard non invariant counterterms possess the charge factorization property given in Eq. (55).*

This result is readily extended to the coefficients of A.B.A. since the anomaly is a hard breaking to the S.B.S.I. and, on the basis of Proposition II in Sect. III, we have the stronger formulation:

Proposition IV. *In any model with anomalous* S.B.S.I. *the coefficients of the Adler Bardeen Anomaly at their lowest order are charge factorized according to Eq. (55).*

VI. Analysis of the Anomaly by the Callan–Symanzik Equation

On the basis of the results obtained in the previous sections we can conclude the analysis of the anomalous S.B.S.I. exploiting the Callan–Symanzik equation for the A.B.A. (Eq. (39)).

In order to discuss Eq. (39) it is convenient to consider separately the following possibilities:

a) The indices α, β, γ all belong to the abelian invariant factor $\mathscr{G}_A$ of $\mathscr{G}$.
b) The index α belongs to $\mathscr{G}_A$, while β and γ are in the same simple component $\mathscr{s}$ of the semisimple factor $\mathscr{G}_S$ of $\mathscr{G}$.
c) The indices α, β, γ all belong to a simple factor $\mathscr{s}$ of $\mathscr{G}_S$.

By the invariance and symmetry under permutation of the indices of the tensor $D^{\alpha\beta\gamma}$ we can reduce any other non trivial possibility to one of the mentioned cases.

Starting with case a), we notice that there are as many rank 3 invariant symmetric tensors on $\mathscr{G}_A$ as the choices of the indices α, β, γ with $\alpha \leqq \beta \leqq \gamma$. We thus parametrize the tensor

$$\sum_j r_j D_j^{\alpha\beta\gamma} = r^{\alpha\beta\gamma} \tag{60}$$

with its components for $\alpha \leqq \beta \leqq \gamma$. Furthermore the contribution of the graphs in Fig. 1 to the first term in the l.h.s. of Eq. (36) vanishes for α, β, γ in $\mathscr{G}_A$, hence, from Eq. (60), we obtain

$$\left(\beta^{(1)}_{e_{\alpha'_A\alpha''_A}} \frac{\partial}{\partial e_{\alpha'_A\alpha''_A}} + \beta^{(1)}_{e_s} \frac{\partial}{\partial e_s} + \vec{\beta}^{(1)}_{g} \vec{\nabla}_{g} + \vec{\beta}^{(1)}_{h} \vec{\nabla}_{h}\right) r^{\alpha\beta\gamma}$$

$$+ \gamma_1^{(1)\alpha\tau} r^{\tau\beta\gamma} + \gamma_2^{(1)\beta\tau} r^{\alpha\tau\gamma} + \gamma_2^{(1)\gamma\tau} r^{\alpha\beta\tau} = 0. \tag{61}$$

Observe now that in the abelian factor $\mathscr{G}_A$ we have the following relations [7]:

$$(\beta^{(1)}e^{-1})^{\alpha_A\beta_A} = -\gamma_1^{(1)\alpha_A\beta_A}, \tag{62a}$$

$$\gamma_1^{(1)\alpha_A\beta_A} = \gamma_2^{(1)\alpha_A\beta_A} \tag{62b}$$

so that the substitution

$$r^{\alpha\beta\gamma} = e^{\alpha\alpha'} e^{\beta\beta'} e^{\gamma\gamma'} \bar{r}^{\alpha'\beta'\gamma'} \tag{63}$$

reduces Eq. (61) to the form

$$\left(\beta^{(1)}_{e_{\alpha'_A\alpha''_A}} \frac{\partial}{\partial e_{\alpha'_A\alpha''_A}} + \beta^{(1)}_{e_{\mathscr{s}}} \frac{\partial}{\partial e_{\mathscr{s}}} + \vec{\beta}^{(1)}_{\mathscr{g}} \vec{\nabla}_{\mathscr{g}} + \vec{\beta}^{(1)}_{\mathscr{h}} \vec{\nabla}_{\mathscr{h}}\right) \bar{r}^{\alpha\beta\gamma} = 0. \tag{64}$$

We emphasize that, according to Proposition IV the coefficients $r^{\alpha\beta\gamma}$ are charge factorized (Eq. (55)) so that $\bar{r}^{\alpha\beta\gamma}$ is a polynomial in the parameters.

Concerning case b) we note that there are as many rank 3 invariant tensors with one index (α) in $\mathscr{G}_A$ and the remaining (β, γ) in $\mathscr{G}_S$ as the couples made with an abelian generator and a simple factor $\mathscr{s}$ of $\mathscr{G}_S$. This follows from the fact that the only rank 2 invariant tensor on $\mathscr{G}_S$ is its Killing form. We can thus parametrize $\sum_j r_j D_j^{\alpha\beta\gamma}$ for this choice of the indices as

$$\sum_j r_j D_j^{\alpha\beta\gamma} = r_{\alpha\mathscr{s}} \delta_{\mathscr{s}}^{\beta\gamma} \tag{65}$$

where $\delta_{\mathscr{s}}^{\beta\gamma}$ is the restriction of the Kronecker symbol (the Killing form) to the simple factor $\mathscr{s}$. Now Eq. (39) becomes

$$\begin{aligned} &\frac{1}{\hbar}\sum_j r_j(\vec{\lambda}) \lim_{\eta\to\infty} X^{\alpha\beta\gamma}(\eta p, \eta q)\mu\partial_\mu(A_j\Gamma)(\vec{\sigma};\Phi,\tau)\Big|_{1\text{ loop}} \\ &+\left(\beta^{(1)}_{e_{\alpha'_A\alpha''_A}} \frac{\partial}{\partial e_{\alpha'_A\alpha''_A}} + \beta^{(1)}_{e} \frac{\partial}{\partial e_{\mathscr{s}}} + \vec{\beta}^{(1)}_{\mathscr{g}} \vec{\nabla}_{\mathscr{g}} + \vec{\beta}^{(1)}_{\mathscr{h}} \vec{\nabla}_{\mathscr{h}}\right) r_{\alpha\mathscr{s}} \delta_{\mathscr{s}}^{\beta\gamma} \\ &+\gamma_1^{(1)\alpha\alpha'} r_{\alpha'\mathscr{s}} \delta_{\mathscr{s}}^{\beta\gamma} + (\gamma_2^{(1)\beta\gamma} + \gamma_2^{(1)\beta\gamma}) r_{\alpha\mathscr{s}} = 0 \end{aligned} \tag{66}$$

where $\gamma_1^{(1)\alpha\alpha'}$ obeys Eq. (62a) and $\gamma_2^{(1)\beta\gamma}$ is given explicitly by

$$\begin{aligned} \gamma_2^{(1)\beta\gamma} = \frac{e_{\mathscr{s}}^2}{(8\pi)^2}\Big\{ &-(3+K)c_2(\mathscr{s}) - \Big[\frac{22}{3}c_2(\mathscr{s}) \\ &+\frac{8}{3}(T_L^2(\mathscr{s}) + T_R^2(\mathscr{s})) + \frac{1}{3}t_B^2(\mathscr{s})\Big]\Big\} \end{aligned} \tag{67}$$

with

$$c_2(\mathscr{s})\delta_{\mathscr{s}}^{\beta\gamma} = \sum_{\sigma,\rho} f^{\beta\sigma\rho} f^{\gamma\sigma\rho}, \tag{68a}$$

$$T_L^2(\mathscr{s})\delta_{\mathscr{s}}^{\beta\gamma} = T_r((T_L^\beta)^\dagger T_L^\gamma)\,;\, T_R^2(\mathscr{s})\delta_{\mathscr{s}}^{\beta\gamma} = T_r((T_R^\beta)^\dagger T_R^\gamma), \tag{68b}$$

$$t_B^2(\mathscr{s})\delta_{\mathscr{s}}^{\beta\gamma} = T_r((t^\beta)^T t^\gamma) \tag{68c}$$

and the gauge parameter K chosen as in Eq. (52). Furthermore in the l.h.s. of

Eq. (66) the function $\beta^{(1)}_{e_\mathscr{s}}$ is (see ref. [24]):

$$\beta^{(1)}_{e} = \frac{e^3_\mathscr{s}}{(8\pi)^2}\left[-\frac{22}{3}c_2(\mathscr{s}) + \frac{8}{3}(T^2_L(\mathscr{s}) + T^2_R(\mathscr{s})) + \frac{1}{3}t^2_B(\mathscr{s})\right]^1 \tag{69}$$

and the first term, which receives contributions only from diagrams 1 and 4 in Fig. 1, is computed to be

$$\begin{aligned}&\frac{1}{\hbar}\sum_j r_j(\vec{\lambda}) \lim_{\eta\to\infty} X^{\alpha\beta\gamma}(\eta p, \eta q)\mu\partial_\mu(A_j\Gamma)(\vec{\sigma};\Phi,\tau)\big|_{1\ \text{loop}}\\ &\quad = \frac{e^2_\mathscr{s}}{(8\pi)^2}2(3+K)c_2(\mathscr{s})r_{\alpha\mathscr{s}}\delta^{\beta\gamma}_\mathscr{s}.\end{aligned} \tag{70}$$

Now substituting Eqs. (70), (69), (67), (62a) into Eq. (66) and setting

$$r_{\alpha\mathscr{s}} = e_{\alpha\alpha'}e^2_\mathscr{s}\bar{r}_{\alpha'\mathscr{s}} \tag{71}$$

we find that $\bar{r}_{\alpha'\mathscr{s}}$ satisfies Eq. (64). Remember that by the charge factorization property $\bar{r}_{\alpha'\mathscr{s}}$ is a polynomial in the parameters.

Concerning case c) we recall that, up to a normalization constant, at most one, invariant, symmetric rank 3 tensor can be built on any simple Lie algebra. Therefore the restriction to a simple factor $\mathscr{s}$ of the tensor $\sum_j r_j D^{\alpha\beta\gamma}_j$ is written $r_\mathscr{s} D^{\alpha\beta\gamma}_\mathscr{s}$ and parametrized by the coefficient $r_\mathscr{s}$. Then in case c) Eq. (39) becomes

$$\begin{aligned}&\frac{1}{\hbar}\sum_j r_j(\vec{\lambda}) \lim_{\eta\to\infty} X^{\alpha\beta\gamma}(\eta p, \eta q)\mu\partial_\mu(A_j\Gamma)(\vec{\sigma};\Phi,\tau)\big|_{1\ \text{loop}}\\ &\quad + \left(\beta^{(1)}_{e_{\alpha'_A\alpha''_A}}\frac{\partial}{\partial e_{\alpha'_A\alpha''_A}} + \beta^{(1)}_{e_\mathscr{s}}\frac{\partial}{\partial e_\mathscr{s}} + \vec{\beta}^{(1)}_{g}\vec{\nabla}_{g} + \vec{\beta}^{(1)}_{h}\vec{\nabla}_{h}\right) r_\mathscr{s}(\vec{\lambda})D^{\alpha\beta\gamma}_\mathscr{s}\\ &\quad + \gamma^{(1)\alpha\alpha'}_1 r_\mathscr{s}(\vec{\lambda})D^{\alpha'\beta\gamma}_\mathscr{s} + \gamma^{(1)\beta\beta'}_2 r_\mathscr{s}(\vec{\lambda})D^{\alpha\beta'\gamma}_\mathscr{s}\\ &\quad + \gamma^{(1)\gamma\gamma'}_2 r_\mathscr{s}(\vec{\lambda})D^{\alpha\beta\gamma'}_\mathscr{s} = 0\end{aligned} \tag{72}$$

where $\gamma^{(1)\beta\beta'}_2$ is given in Eq. (67), and the restriction of $\gamma^{(1)\alpha\alpha'}_1$ to the simple factor $\mathscr{s}$ is

$$\begin{aligned}\gamma^{(1)\alpha\alpha'}_1 = \delta^{\alpha\alpha'}\frac{e^2_\mathscr{s}}{(8\pi)^2}\bigg[&-2Kc_2(\mathscr{s}) - \frac{22}{3}c_2(\mathscr{s})\\ &+\frac{8}{3}(T^2_R(\mathscr{s}) + T^2_L(\mathscr{s})) + \frac{1}{3}t^2_B(\mathscr{s})\bigg].\end{aligned} \tag{73}$$

The first term on the l.h.s. of Eq. (72), which now receives contributions from all four graphs in Fig. 1, is computed to be

$$\begin{aligned}&\frac{1}{\hbar}\sum_j r_j(\vec{\lambda}) \lim_{\eta\to\infty} X^{\alpha\beta\gamma}(\eta p, \eta q)\mu\partial_\mu(A_j\Gamma)(\vec{\sigma},\Phi,\tau)\big|_{1\ \text{loop}}\\ &\quad = \frac{e^2_\mathscr{s}}{(8\pi)^2}r_\mathscr{s}(\vec{\lambda})D^{\alpha\beta\gamma}_\mathscr{s}(6+4K)c_2(\mathscr{s}).\end{aligned} \tag{74}$$

1 We shall exclude the pathological cases where the representation content of the model is such that some of the functions $\beta^{(1)}_{e_\mathscr{s}}$ do vanish for $e_\mathscr{s} \neq 0$

Substituting Eqs. (73), (74) and $r_{\mathscr{d}} = e_{\mathscr{d}}^3 \bar{r}_{\mathscr{d}}$ into Eq. (72) we find that $\bar{r}_{\mathscr{d}}$ obeys Eq. (64). Also in this case one should remember that, due to the charge factorization property proved in Section V, $\bar{r}_{\mathscr{d}}$ is a polynomial in the parameters.

We have thus arrived at the final step of our analysis, that of finding the general polynomial solution $f(e, \mathscr{g}, \mathscr{h})$ of the equation

$$\left(\beta^{(1)}_{e_{\alpha'_A\alpha''_A}} \frac{\partial}{\partial e_{\alpha'_A\alpha''_A}} + \beta^{(1)}_{e_{\mathscr{d}}} \frac{\partial}{\partial e_{\mathscr{d}}} + \vec{\beta}^{(1)}_{\mathscr{g}} \vec{\nabla}_{\mathscr{g}} + \vec{\beta}^{(1)}_{\mathscr{h}} \vec{\nabla}_{\mathscr{h}}\right) f(e, \mathscr{g}, \mathscr{h}) = 0. \tag{75}$$

The method relies on the coefficients of the differential operator being themselves polynomials in the parameters; in particular $\beta^{(1)}_{e_{\mathscr{d}}}$ and $\beta^{(1)}_{e_{\alpha_A\beta_A}}$ are cubic homogeneous in the charges $e_{\mathscr{d}}, e_{\alpha_A\beta_A}$ and do not depend upon the remaining parameters. The vector coefficient $\vec{\beta}^{(1)}_{\mathscr{g}}$ depends only on the parameters e, $\mathscr{g}$ and decomposes two terms

$$\vec{\beta}^{(1)}_{\mathscr{g}} = \vec{b}_{\mathscr{g}}(\mathscr{g}) + \vec{c}_{\mathscr{g}}(\mathscr{g}, e) \tag{76}$$

where $\vec{b}_{\mathscr{g}}(\mathscr{g})$ and $\vec{c}_{\mathscr{g}}(\mathscr{g}, e)$ are homogeneous of degree 3 and 1 respectively in $\mathscr{g}$. Likewise the components of $\vec{\beta}_{\mathscr{h}}$ decompose into two terms

$$\vec{\beta}^{(1)}_{\mathscr{h}} = \vec{b}_{\mathscr{h}}(\mathscr{h}) + \vec{c}_{\mathscr{h}}(\mathscr{h}, e, \mathscr{g}) \tag{77}$$

where $\vec{b}_{\mathscr{h}}(\mathscr{h})$ is homogeneous of second degree and $\vec{c}_{\mathscr{h}}(\mathscr{h}, e, \mathscr{g})$ is at most linear in $\mathscr{h}$.

The analysis proceeds along the lines suggested in ref. [11]. Selecting the highest degree contributions in the $\mathscr{h}$ variables to the l.h.s. of Eq. (75) yields

$$(\vec{b}_{\mathscr{h}}(\mathscr{h}) \vec{\nabla}_{\mathscr{h}}) f_{\mathscr{h}} = 0 \tag{78}$$

where $f_{\mathscr{h}}$ is the highest order part of f in the same variables.

We shall show in Appendix C, by a detailed study of the coefficient $\vec{b}_{\mathscr{h}}$, that Eq. (78) admits no polynomial solutions except a constant; from which we immediately conclude that f is independent from the variables $\mathscr{h}$.

Applying now the same procedure to the $\mathscr{g}$ variables one gets

$$(\vec{b}_{\mathscr{g}}(\mathscr{g}) \vec{\nabla}_{\mathscr{g}}) f_{\mathscr{g}} = 0 \tag{79}$$

with $f_{\mathscr{g}}$ the highest order contribution to f in these variables. Here again the dis- discussion of Eq. (79) in Appendix C shows that $f_{\mathscr{g}}$ can only be a constant and hence f is also $\mathscr{g}$ independent.

At this point Eq. (75) reduces to the following partial differential equation in the $e_{\mathscr{d}}, e_{\alpha_A\beta_A}$ variables only:

$$\left(\sum_{\mathscr{d}} e_{\mathscr{d}}^3 \left[-\frac{22}{3} c_2(\mathscr{d}) + \frac{8}{3}(T_L^2(\mathscr{d}) + T_R^2(\mathscr{d})) + \frac{1}{3} t_B^2(\mathscr{d}) \right] \frac{\partial}{\partial e_{\mathscr{d}}} \right.$$
$$\left. + \sum_{\alpha_A\beta_A} (e^3)_{\alpha_A\beta_A} \frac{\partial}{\partial e_{\alpha_A\beta_A}} \right) f(e) = 0 \tag{80}$$

which is obtained by substituting into Eq. (75) the explicit form of the coefficient

$\beta^{(1)}_{e_\sigma}$ given in Eq. (69) and $\beta^{(1)}_{e_{\alpha_A\beta_A}}$ as computed from

$$(e\beta^{(1)}+\beta^{(1)}e)_{\alpha_A\beta_A}=\frac{1}{2(8\pi)^2}\Bigg\{\sum_{\rho_A\sigma_A}(e^2)_{\alpha_A\rho_A}T_r\Bigg[\frac{8}{3}(T_R^{\rho_A'}T_R^{\sigma_A}+T_L^{\rho_A'}T_L^{\sigma_A})+\frac{1}{3}t^{T\rho_A}t^{\sigma_A}\Bigg](e^2)_{\alpha_A\beta_A}\Bigg\} \tag{81}$$

taking into account the condition in Eq. (4).

Equation (80) can be analyzed by the same method illustrated above; indeed, by selecting the highest order terms f_{e_σ} and $f_{e_{\alpha_A\beta_A}}$ in the variables e_σ and $e_{\alpha_A\beta_A}$ respectively we get

$$\sum_\sigma e_\sigma^3\left[-\frac{22}{3}c_2(\sigma)+\frac{8}{3}(T_L^2(\sigma)+T_R^2(\sigma))+\frac{1}{3}t_B^2(\sigma)\right]\frac{\partial}{\partial e_\sigma}f_{e_\sigma}=0 \tag{82a}$$

$$\sum_{\alpha_A\beta_A}(e^3)_{\alpha_A\beta_A}\frac{\partial}{\partial e_{\alpha_A\beta_A}}f_{e_{\rho_A\sigma_A}}=0. \tag{82b}$$

From equation (82a) we easily derive that, provided $\beta^{(1)}_{e_\sigma}\neq 0$ (see Note (1)), f_{e_σ} is a constant and identical conclusion is reached for $f_{e_{\rho_A\sigma_A}}$ in Eq. (82b) by employing a procedure analogous to the one used for Eq. (79).

Thus the result of our analysis of the Callan–Symanzik equation is that the coefficients r_j are cubic homogeneous polynomials in the charges e_σ, $e_{\alpha_A\beta_A}$.

Our final task is to connect this result with the order of the Feynman diagrams contributing to these coefficients. This is achieved by referring to the models without soft couplings introduced in Sect. III and recalling that in these models all radiative corrections to the couplings satisfy the charge factorization property (Proposition III). As a consequence the radiative corrections to the tree approximation vertices have an higher order in the λ_a parameters.

Now one can easily deduce[2] that the only Feynman diagrams whose contributions to r_j are compatible with the behaviour fixed by the Callan–Symanzik equation are the single loop ones built with the vertices of the classical Lagrangian: such diagrams have order one in $\hbar$. On the basis of the results in Sect. III (Proposition II), the above conclusion is seen to hold in any gauge model.

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Appendix A

Our task in this Appendix is to show that the renormalization program for models with S.B.S.I. leads to the same results obtained for an exact S.I.; in particular we shall prove that the only non compensable breaking term to the S.B.S.I. is the Adler–Bardeen anomaly (A.B.A.).

We shall illustrate the renormalization procedure in the framework of the

2 The analysis is obviously simplified by the admissible choice $\hbar=g=0$

B.P.H.Z. scheme [12] only up to the point where it completely overlaps with the one for the exact S.I. given in ref. [15] thus yielding the same results.

As a simplifying hypothesis, after refs. [14, 15], we shall assume that all the quantized fields are massive; the extension of our discussion to models involving massless fields can be straightforwardly obtained by employing the formalism and techniques developed by Zimmermann, Lowenstein and Clark [12, 20, 21]. We recall that, as already mentioned in Sect. III the theories with S.B.S.I. are free from infrared (I.R.) pathologies.

Our models are specified by assigning an effective Lagrangian as a normal product operator (N.P.O.) [21] with ultraviolet (U.V.) subtraction index equal to four and requiring that the corresponding Green's functions satisfy to all orders of perturbation theory ($\hbar$ formal power series) the S.B.S.I. together with two systems of supplementary renormalization conditions. The first system fixes the equations of motion of the $\bar{c}^{\alpha}(x)$ fields and the second one expresses the super-renormalizability of the couplings of the $c^{\alpha_A}(x)$ fields (i.e. the ones corresponding to the abelian factor of $\mathscr{G}$) with the quantized ones.

The classical limit of the effective Lagrangian of these models is described in Sect. II. Denoting the quantized fields collectively by $\Phi(x)$ and the external ones by $\tau(x)$, we can write the equation of motion of the $\bar{c}^{\alpha}(x)$ fields in terms of the vertex functional $\Gamma(\Phi, \tau)$ as

$$\frac{\delta}{\delta c^{\alpha}(x)}\Gamma(\Phi,\tau) - \Lambda^{\alpha\beta}\left(\delta^{\beta\beta_s}\frac{\delta}{\delta\gamma_{\mu}^{\beta_s}(x)}\Gamma(\Phi,\tau) - \delta^{\beta\beta_A}\Box\bar{c}^{\beta_A}(x)\right)$$
$$= (\Delta_2^{\alpha}(x)\Gamma)(\Phi,\tau) \tag{A.1}$$

where the r.h.s. means the insertion in $\Gamma(\Phi,\tau)$ of the N.P.O. $N_2[\Delta^{\alpha}(x)]$ which accounts for the effect of the super renormalizable couplings corresponding to the soft breakings of the S.B.S.I. Likewise the super renormalizability requirement for the $c^{\alpha_A}(x)$ couplings is written as

$$\frac{\delta}{\delta\bar{c}^{\alpha_A}(x)}\Gamma(\Phi,\tau) + y^{\alpha_A\beta}(\Lambda^{\beta\beta'}\Box c_{\beta'}(x) + \delta^{\beta\beta_s}\partial_{\mu}\gamma^{\mu}_{\beta_s}(x)) + \gamma_i(x)(et)^{\alpha_A}_{ij}\varphi_j(x)$$
$$+(\eta_R^{\dagger}(x)(eT_L)^{\alpha_A}\psi_L(x) + \eta_L^{\dagger}(eT_R)^{\alpha_A}\psi_R(x) + \text{h.c.}) = (\Delta_2^{\prime\alpha_A}(x)\Gamma)(\Phi,\tau) \tag{A.2}$$

where the matrix $y^{\alpha_A\beta}$, which is $\delta^{\alpha_A\beta}$ at the classical level, is kept free for adjustments at the higher orders and the r.h.s. means the insertion in $\Gamma(\Phi,\tau)$ of the N.P.O. $N_2[\Delta^{\prime\alpha_A}(x)]$.

To discuss the implementability to all orders of perturbation theory of Eqs. (A.1) (A.2), which obviously hold at the tree approximation, we first need their complete functional translation. This is accomplished by coupling the soft breakings $\Delta_2^{\alpha}(x)$ and $\Delta_2^{\prime\alpha_A}(x)$ to external fields $\mathscr{C}^{\alpha}(x)$ and $\bar{\mathscr{C}}^{\alpha_A}(x)$ which are assigned U.V. dimension equal to two and $+1$ and -1 Faddeev–Popov charge respectively. By introducing these new vertices into the Lagrangian, thereby defining a new generator $\Gamma(\Phi, \tau, \mathscr{C}, \bar{\mathscr{C}})$, we rewrite Eqs. (A.1) (A.2) as

$$D^{\alpha}(x)\Gamma(\Phi, \tau, \mathscr{C}, \bar{\mathscr{C}}) \equiv \left[\frac{\delta}{\delta c^{\alpha}(x)} - \Lambda^{\alpha\beta}\left(\delta^{\beta\beta_s}\partial_{\mu}\frac{\delta}{\delta\gamma_{\mu}^{\beta_s}(x)}\right) - \frac{\delta}{\delta\mathscr{C}^{\alpha}(x)}\right]\Gamma(\Phi, \tau, \mathscr{C}, \bar{\mathscr{C}}) = -\Lambda^{\alpha\beta}\delta^{\beta\beta_A}\Box\,\bar{c}^{\beta_A}(x) + \Theta^{\alpha\alpha_A}\bar{\mathscr{C}}^{\alpha_A}(x), \tag{A.3}$$

$$\begin{aligned}\mathscr{E}^{\alpha_A}(x)\Gamma(\Phi, \tau, \mathscr{C}\,\bar{\mathscr{C}}) &\equiv \left[\frac{\delta}{\delta\bar{c}^{\alpha_A}(x)} - \frac{\delta}{\delta\bar{\mathscr{C}}^{\alpha_A}(x)}\right]\Gamma(\Phi, \tau, \mathscr{C}\,\bar{\mathscr{C}})\\ &= -y^{\alpha_A\beta}[\Lambda^{\beta\beta'}\Box\,c^{\beta'}(x) + \delta^{\beta\beta_s}\partial_{\mu}\gamma^{\mu}_{\beta_s}(x)] - \gamma_i(x)(e\,t)^{\alpha_A}_{ij}\,\varphi_j(x)\\ &\quad - (\eta^{\dagger}_R(x)(e\,T_L)^{\alpha_A}\psi_L(x) + \eta^{\dagger}_L(x)(e\,T_R)^{\alpha_A}\psi_R(x) + \text{h.c.}) + \bar{\Theta}^{\alpha_A\beta}\mathscr{C}^{\beta}(x)\end{aligned} \tag{A.4}$$

where the last term on the r.h.s. of both equations accounts for the corrections induced by the $\mathscr{C}^{\alpha}(x)$, $\bar{\mathscr{C}}^{\alpha_A}(x)$ couplings and satisfy $\bar{\Theta}^{\alpha_A\beta} + \Theta^{\beta\alpha_A} = 0$.

The proof of the renormalizability of Eqs. (A.3), (A.4) can now be performed by following verbatim the procedure described in details in ref. [15].

From now on we shall consider only those effective Lagrangians which satisfy the supplementary conditions in Eqs. (A.1), (A.2).

The second step of the analysis is to discuss, within this restricted class of $\mathscr{L}_{\text{eff}}$ the renormalizability of the S.B.S.I. which we shall write for the connected generator $Z_c(J, \tau)$ as

$$\begin{aligned}\int dx\Bigg\{&\left[J_i(x)\frac{\delta}{\delta\gamma_i(x)} + J^{\alpha_s}_{\mu}(x)\frac{\delta}{\delta\gamma^{\alpha_s}_{\mu}(x)} + J^{\alpha_A}_{\mu}(x)\partial_{\mu}\frac{\delta}{\delta\xi^{\alpha_A}(x)} + F_L(x)\frac{\vec{\delta}}{\delta\eta^{\dagger}_L(x)}\right] + \left[F_R(x)\frac{\vec{\delta}}{\delta\eta^{\dagger}_R(x)}\right.\\ &\left. - \zeta^{\alpha_s}(x)\frac{\delta}{\delta\zeta^{\alpha_s}(x)} - \xi^{\alpha}(x)\partial_{\mu}\frac{\delta}{\delta J^{\alpha}_{\mu}(x)}\right]Z_c(J, \tau) + Z_c(J, \tau)\left[\frac{\overleftarrow{\delta}}{\delta\eta_L(x)}F^{\dagger}_L(x)\right.\\ &\left.\left. + \frac{\overleftarrow{\delta}}{\delta\eta_R(x)}F^{\dagger}_R(x)\right]\right\} \equiv (\mathscr{S}_h Z_c)\,(J, \tau) = \int dx(\Delta_4(x)Z_c)\,(J, \tau)\end{aligned} \tag{A.5}$$

Here the r.h.s. denotes the insertion in $Z_c(J, \tau)$ of the integrated N.P.O. $\int dx\,\Delta_4(x)$ with U.V. subtraction index equal to four. We introduce an external field $\sigma(x)$ with U.V. dimension equal to zero and $+1$ Faddeev–Popov charge, coupled to the soft breaking $\Delta_4(x)$. The addition of this new vertex to the Lagrangian allows us to define the connected generator $Z_c(J, \tau, \sigma)$ so that Eq. (A.5) becomes

$$\mathscr{S}^{\sigma}_h Z_c(J, \tau, \sigma)\Big|_{\sigma=0} \equiv \left(\mathscr{S}_h - \int dx\frac{\delta}{\delta\sigma(x)}\right)Z_c(J, \tau, \sigma)\Big|_{\sigma=0} = 0. \tag{A.6}$$

The analysis of this equation is based on the Quantum Action Principle (Q.A.P.) [22] according to which for a generic choice of $\mathscr{L}_{\text{eff}}$, which we will however choose in agreement with the supplementary conditions, Eq. (A.6) is affected by breaking terms with U.V. dimension uniformly bounded by five. Hence we have in general

$$\mathscr{S}^{\sigma}_h Z_c(J, \tau, \sigma) = \int dx\,\sigma(x)\Sigma_5(x)Z_c(J, \tau, \sigma) + \int dx\,\Delta_5(x)Z_c(J, \tau, \sigma) \tag{A.7}$$

Here the $\sigma(x)$ independent breakings of naive (U.V.) dimensions equal to five have been isolated in $\Delta_5(x)$ and possible lower dimensionality ones are reabsorbed by a suitable redefinition of the $\sigma(x)$ field coupling. The insertion $\Sigma_5(x)$ summarizes the breakings deriving from the $\sigma(x)$ coupling itself.

Now our aim is to prove, order by order in $\hbar$, that the breaking $\Delta_5(x)$ may be compensated by a suitable counterterm to be introduced in the effective Lagrangian without violating the supplementary conditions in Eqs. (A.1) (A.2).

At the classical limit (i.e. the $\hbar$ independent contribution) $\Delta_5(x)$ vanishes and the proof proceeds inductively by considering the first non vanishing contribution $\Delta_5^{(n)}(x)$ of order $\hbar^n (n \geqq 1)$. The compensability of $\Delta_5^{(n)}(x)$ by an appropriate choice of $\mathscr{L}_{\text{eff}}^{(n)}$ is deduced from the consistency conditions which $\Delta_5^{(n)}(x)$ must satisfy in order to meet the supplementary renormalization requirements in Eqs. (A.1), (A.2).

The explicit form of these consistency conditions is obtained by the same functional method employed in ref. [14]. We introduce an external field $\beta(x)$ with U.V. dimension equal to -1 and $+1$ Faddeev–Popov charge, coupled to the breaking $\Delta_5^{(n)}$ and the modified effective Lagrangian

$$\mathscr{L}_{\text{eff}}(\Phi, \tau, \sigma, \beta) = \mathscr{L}_{\text{eff}}(\Phi, \tau, \sigma) + \beta(x)\Delta_5^{(n)}(x). \tag{A.8}$$

The application of the Q.A.P. yields, for the corresponding connected functional $Z_c(J, \tau, \sigma, \beta)$,

$$\begin{aligned}\mathscr{S}_h^\sigma Z_c(J, \tau, \sigma, \beta) = {} & \int dx \Sigma_5(x) Z_c(J, \tau, \sigma, \beta) \\ & + \int dx \frac{\delta}{\delta\beta(x)} Z_c(J, \tau, \sigma, \beta) + \int dx \beta(x) \Delta_6'(x) Z_c(J, \tau, \sigma, \beta) + O(\beta^2)\end{aligned} \tag{A.9}$$

where the $\beta(x)$ proportional term on the r.h.s. summarizes the corrections (to first order in $\beta(x)$) arising from the coupling of the $\beta(x)$ field. By direct computation, via the identity $(\mathscr{S}_h^\sigma)^2 = \mathscr{S}_h^2$, we find, employing Eqs. (A.6), (A.7), (A.9),

$$\begin{aligned}(\mathscr{S}_h^\sigma)^2 Z_c(J, \tau, \sigma, \beta)\big|_{\sigma=\beta=0} = \mathscr{S}_h^2 Z_c(J, \tau) = {} & -\int dx \Sigma_5(x) Z_c(J, \tau, \sigma, \beta)\big|_{\sigma=\beta=0} \\ & - \int dx \frac{\delta}{\delta\beta(x)} \mathscr{S}_h^\sigma Z_c(J, \tau, \sigma, \beta)\big|_{\sigma=\beta=0}\end{aligned} \tag{A.10}$$

and the further substitution of Eq. (A.9) into the last term of the r.h.s. yields $\left(\text{by } \int dx dy \frac{\delta}{\delta\beta(x)} \frac{\delta}{\delta\beta(y)} = 0\right)$ at $\sigma = \beta = 0$

$$\mathscr{S}_h^2 Z_c(J, \tau) = -\int dx \Sigma_5(x) Z_c(J, \tau) - \int dx \Delta_6'(x) Z_c(J, \tau) \tag{A.11}$$

At this point we are ready to impose the constraints derived from the supplementary condition in Eq. (A.1). By direct computation we find:

$$\mathscr{S}_h^2 Z_c(J, \tau) = -\int dx \xi^\alpha(x) \left(\delta^{\alpha\beta_s} \partial_\mu \frac{\delta}{\delta\gamma_\mu^{\beta_s}(x)} + \delta^{\alpha\beta_A} \Box \frac{\delta}{\delta\xi^{\beta_A}(x)} \right) Z_c(J, \tau) \tag{A.12}$$

so that comparing Eq. (A.11) with Eq. (A.12) and translating the result in terms of the vertex functional $\Gamma(\Phi, \tau)$ we obtain

$$\begin{aligned}&\int dx \frac{\delta}{\delta c^{\alpha}(x)} \Gamma(\Phi, \tau)(\Lambda^{-1})^{\alpha\beta}(\Delta_2^{\beta}(x)\Gamma)(\Phi, \tau)\\&\quad = \int dx[(\Sigma_5(x)\Gamma)(\Phi, \tau) + (\Delta_6'(x)\Gamma)(\Phi, \tau)]\end{aligned} \tag{A.13}$$

where the l.h.s. has been reduced by means of Eq. (A.1) and the property $\int dxdy \frac{\delta}{\delta c^{\alpha}(x)} \frac{\delta}{\delta c^{\beta}(y)} (\Lambda^{-1})^{\alpha\beta} = 0.$

Now an analysis of Eq. (A.13) in the deep Euclidean region leads to a constraint for $\int dx \Delta_6'^{(n)}(x)$, i.e. the contribution of order $\hbar^n$ to $\int dx \Delta_6'(x)$.

Let the symbol $X_{a_i}(x)$ stand collectively for all fields quantized or external ones, and $\tilde{X}_{a_i}(p)$ for the corresponding Euclidean Fourier transform; we define the test operators

$$X(a_0, p_1 a_1, \ldots, p_n a_n) \equiv X(p)_{a_0 a_1 \ldots a_n} = \prod_{i=1}^{n} \frac{\delta}{\delta \tilde{X}_{a_i}(p_i)} \frac{\delta}{\delta X_{a_0}(0)} \tag{A.14}$$

where the momenta $p_1 \ldots p_n$ are non exceptional in the sense of Symanzik. Let Furthermore

$$D_{a_0 a_1 \ldots a_n} = \sum_{i=1}^{n} \bar{d}_{a_i} + \bar{d}_{a_0} \tag{A.15}$$

with $\bar{d}_{a_i}$ the U.V. dimension of the field $X_{a_i}(x)$. With these definitions, on the basis of Weinberg power counting rules, we obtain for all choices $D_{a_0 a_1 \ldots a_n} \leqq 6$

$$\lim_{\rho \to \infty} \rho^{D_{a_0 a_1 \ldots a_n} - 6} X_{a_0 a_1 \ldots a_n}(\rho p) \int dx \frac{\delta}{\delta c^{\alpha}(x)} \Gamma(\Phi, \tau)(\Delta_2^{\alpha}(x)\Gamma)(\Phi, \tau) = 0 \tag{A.16a}$$

$$\lim_{\rho \to \infty} \rho^{D_{a_0 a_1 \ldots a_n} - 6} X_{a_0 a_1 \ldots a_n}(\rho p) \int dx (\Sigma_5(x)\Gamma)(\Phi, \tau) = 0 \tag{A.16b}$$

$$\begin{aligned}&\lim_{\rho \to \infty} \rho^{D_{a_0 a_1 \ldots a_n} - 6} X_{a_0 a_1 \ldots a_n}(\rho p) \int dx (\Delta_6'(x)\Gamma)(\Phi, \tau)\\&\quad = X_{a_0 a_1 \ldots a_n}(p) \int dx \Delta_6'^{(n)}(x) + O(\hbar^{n+1})\end{aligned} \tag{A.16c}$$

where on the r.h.s. of Eq. (A.16c) we have isolated the contribution $\int dx \Delta_6'^{(n)}(x)$. This result can now be substituted into Eq. (A.13) to yield at the order $\hbar^n$:

$$X_{a_0 a_1 \ldots a_n}(p) \int dx \Delta_6'^{(n)}(x) = 0 \tag{A.17}$$

for all test operators with $D_{a_0 a_1 \ldots a_n} \leqq 6$ and hence

$$\int dx \Delta_6'^{(n)}(x) = 0 \tag{A.18}$$

which contains all the algebraic constraints on $\Delta_5^{(n)}(x)$ implied by the supplementary condition in Eq. (A.1).

The link between $\Delta_6'^{(n)}(x)$ and $\Delta_5^{(n)}(x)$ can be obtained by decomposing $\Delta_5^{(n)}(x)$ as

$$\Delta_5^{(n)}(x) = \Delta^{a}(x)\tau_a(x) + \Delta_5^{0}(x) \tag{A.19}$$

where $\Delta_5^{0}(x)$ is independent from the external fields, and then proceeding along the lines given in ref. [14] to get

$$\int dx \Delta_6'^{(n)}(x) = \int dx [\mathscr{s}^{\Delta(n)} \mathscr{L}^{(0)}_{\text{eff}}(\Phi, \tau) + \mathscr{s}^{P(0)} \Delta_5^{(n)}(x)]. \tag{A.20}$$

The substitution operators in Eq. (A.20) are defined by

$$\mathscr{s}^{\Delta(n)} = \int dx (-1)^{n_a} \Delta^a(x) \frac{\delta}{\delta \Phi_a(x)} \tag{A.21a}$$

$$\mathscr{s}^{P(0)} = \int dx P_a^{(0)}(x) \frac{\delta}{\delta \Phi_a(x)} \tag{A.22a}$$

where n_a is the Faddeev–Popov charge of the external field τ_a and $P_a^{(0)}(x)$ is the tree approximation Slavnov variation of the field $\Phi_a(x)$.

We conclude by noting that the algebraic consistency conditions obtained substituting Eq. (A. 20) into Eq. (A. 18) are exactly the same as the ones given in refs. [14, 15]; these coupled with the super renormalizability requirement in Eq. (A. 2) allow us to repeat verbatim the algebraic and power counting analysis of $\Delta_5^{(n)}(x)$ performed in ref. [15] and therefore to reach the result that only the presence of the Adler Bardeen anomaly may spoil the renormalizability to all orders of the S.B.S.I.

Appendix B

The purpose of this Appendix is to prove that in models with S.B.S.I., renormalized according to the B.P.H.Z. scheme, if all the coupling constants are polynomials in the parameters λ_a, the following properties hold:

a) any Feynman amplitude is a polynomial in the parameters λ_a

b) if in an open neighborhood of a given choice of the parameters λ_a, a Feynman amplitude has a finite limit at infinite Euclidean momenta, then this limit is a polynomial in the parameters.

We first notice that, within the B.P.H.Z. renormalization scheme, any Feynman amplitude D is given by an absolutely convergent integral [12] of a rational function of the type:

$$I_D = \frac{P_D(p, k)}{\prod\limits_i (l_i^2(p, k) + m_i^2)^{n_i}} \tag{B.1}$$

where k and p denote the internal and external momenta respectively, $l_i(p, k)$ are linear combinations of p and k and $m_i^2 \geqq 0$.

Secondly we remark that in models with S.B.S.I. the masses m_i do not depend on the dimensionless parameters λ_a (the situation being quite different in a model with exact S.I. where these masses may be generated by the Higgs–Kibble mechanism) so that the only dependence of I_D from λ_a is contained in its numerator P_D which is a polynomial in the coupling constants.

We can decompose the numerator P_D and make explicit its dependence from the parameters λ_a as

$$P_D(p, k) = \sum_\alpha M_\alpha(\lambda_a) X_D^\alpha(p, k) \tag{B.2}$$

where $M_\alpha(\lambda_a)$ is a set of independent monomials. The corresponding Feynman amplitude factorizes according to

$$\int d^{4L}K \frac{P_D(p,k)}{\prod_i (l_i^2(p,k)+m_i^2)^{n_i}}$$
$$= \sum_\alpha M_\alpha(\lambda_a) \int d^{4L}k \frac{X_D^\alpha(p,k)}{\prod_i (l_i^2(p,k)+m_i^2)^{n_i}} \tag{B.3}$$

where the single integrals in the r.h.s. are absolutely convergent as shown in ref. [25]. This proves property a).

It is also straightforward to verify that, due to the independence of the monomials $M_\alpha(\lambda_a)$, the existence of the limit

$$f_D(\lambda_a) \equiv \lim_{\eta\to\infty} \int d^{4L}k \frac{P_D(\eta p,k)}{\prod_i (l_i^2(\eta p,k)+m_i^2)^{n_i}} \tag{B.4}$$

is an open neighborhood around λ_a, insures that of

$$X_D^\alpha \equiv \lim_{\eta\to\infty} \int d^{4L}k \frac{X_D^\alpha(\eta p,k)}{\prod_i (l_i^2(\eta p,k)+m_i^2)^{n_i}} \tag{B.5}$$

from which we obtain

$$f_D(\lambda_a) = \sum_\alpha M_\alpha(\lambda_a) X_D^\alpha \tag{B.6}$$

thereby proving property b).

Appendix C

In this Appendix we shall first show that the only polynomial solution of Eq. (78) is a constant, and then reach the same conclusion concerning Eq. (79).

The first step in the analysis is to choose a suitable basis in the real linear space $\mathscr{H}$ of the symmetric rank four tensors h describing the quadrilinear coupling of the scalar fields (with components $\hbar_{ijkl}$). These tensors are invariant on the real completely reducible representation of the gauge group carried by the fields.

In the space $\mathscr{H}$ we introduce the scalar product:

$$(\hbar, \hbar') = \sum_{ijkl} \hbar_{ijkl} \hbar'_{ijkl} \tag{C.1}$$

and the tensor product:

$$\{\hbar, \hbar'\}_{ijkl} = \sum_{mn} (\hbar_{ijmn}\hbar'_{mnkl} + \hbar_{ikmn}\hbar'_{mnjl}$$
$$+ \hbar_{ilmn}\hbar'_{mnjk}) + (\hbar \leftrightarrow \hbar') \tag{C.2}$$

We now choose a basis $\{\varphi_i\}$ for the scalar fields completely reducing the gauge group representation into a sequence of irreducible components labelled with capital latin letters A, B, C ... and whose dimensionalities are $n_A, n_B, n_C \ldots$. The decomposition of the representation on the basis $\{\varphi_i\}$ automatically induces a reduction of $\mathscr{H}$ into subspaces $\mathscr{H}^{(ABCD)}$ which are contained in the symmetrized tensor product of the components A, B, C, D. It is clear from the very definition that these subspaces, some of which may be empty due to the invariance conditions in Eq. (10), are mutually orthogonal; hence an orthonormal basis in $\mathscr{H}$ may be given by the union of those for the subspaces $\mathscr{H}^{(ABCD)}$.

We now select some special elements of these bases. For each component A and subspaces $\mathscr{H}^{(AAAA)}$ we can choose the tensor

$$X^A_{ijkl} = \frac{1}{\sqrt{3n_A(n_A+2)}}(\delta^A_{ij}\delta^A_{kl} + \delta^A_{ik}\delta^A_{jl} + \delta^A_{il}\delta^A_{jk}) \tag{C.3}$$

where the δ^A_{ij} is the usual Kronecker symbol if both indices i and j belong to the component A and vanishes otherwise. Similarly for each couple $A, B\,(A < B)$ we identify as an element of the basis in $\mathscr{H}^{(AABB)}$ the tensor

$$X^{AB}_{ijkl} = \frac{1}{\sqrt{6n_A n_B}}(\delta^A_{ij}\delta^B_{kl} + \delta^A_{ik}\delta^B_{jl} + \delta^A_{il}\delta^B_{jk} + (A \leftrightarrow B)) \tag{C.4}$$

and for each triple ABB' with the representation B equivalent to B', we introduce in $\mathscr{H}^{(AABB')}$ the tensor

$$\begin{aligned} X^{ABB'}_{ijkl} = & \frac{1}{\sqrt{6n_A n_B}}(\delta^A_{ij}U^{BB'}_{kl} + \delta^A_{ik}U^{BB'}_{jl} + \delta^A_{il}U^{BB'}_{jk} \\ & + \delta^A_{ij}U^{BB'}_{lk} + \delta^A_{ik}U^{BB'}_{lj} + \delta^A_{il}U^{BB'}_{kj}) \end{aligned} \tag{C.5}$$

where $U^{BB'}_{kl}$ is the matrix intertwining the representation B and B' normalized to $\sum_{kl}(U^{BB'}_{kl})^2 = n_B$.

The remaining basis elements in $\mathscr{H}$, obtained by completion of the orthonormal sub-bases in each subspace, will collectively be denoted by $Z^\chi, \chi = 1, 2, \ldots, N$. For notation convenience we shall assume that the tensors Z^χ in $\mathscr{H}^{(ABCD)}$ with $A < B < C < D$ correspond to values of χ in the range $\bar{N} \leqq \chi \leqq N$.

We now give some relations among the elements of this basis which will be useful in the following. From Eq. (C.2), (C.3), (C.4), (C.5) we get

$$\{X^A, X^B\}_{ijkl} = \delta^{AB}\frac{2(n_A+8)}{\sqrt{3n_A(n_A+2)}}X^A_{ijkl} \tag{C.6a}$$

$$\{X^A, X^{CD}\}_{ijkl} = \frac{n_A+2}{\sqrt{3n_A(n_A+2)}}(\delta^{AC}X^{AD}_{ijkl} + \delta^{AD}X^{AC}_{ijkl}) \tag{C.6b}$$

$$\{X^A, X^{BCC'}\}_{ijkl} = \delta^{AB}\frac{(n_A+2)}{\sqrt{3n_A(n_A+2)}}X^{ACC'}_{ijkl} \tag{C.6c}$$

Moreover

$$\{X^A, Z^\chi\}_{ijkl} = c^{A\chi}Z^\chi_{ijkl} \tag{C.7}$$

with $c^{A\chi} \geqq 0$. To derive this last relation we observe that its l.h.s. vanishes for any A unless Z^χ is of the type $h^{(AABC)}$ (i.e. belongs to $\mathcal{H}^{(AABC)}$) for any choices of B and C; in this case we compute from Eqs. (C.2), (C.3)

$$\{X^A, h^{(AABC)}\}_{ijkl} = \frac{1}{\sqrt{3n_A(n_A+2)}}\left(\sum_{\binom{\text{all permutations}}{\text{of } ijkl}} \delta^A_{ij}\delta^A_{mn} h^{(AABC)}_{mnkl} + \sum_{\binom{\text{all permutations}}{\text{of } ijkl}} 2\delta^A_{im}\delta^A_{jn} h^{(AABC)}_{mnkl} \right) \tag{C.8}$$

and remark that Eq. (C.7) is immediately recovered provided the first term on the r.h.s. vanishes. Now the invariant rank two tensor $\delta^A_{mn} h^{(AABC)}_{mnkl}$ can be nonzero only when the representation B and C are equivalent; and in this instance the orthogonality of Z^χ with X^A, X^{AB}, X^{ABC} also ensures its vanishings.

In complete analogy with Eq. (C.7) one can derive

$$\{X^{AB}, Z^\chi\}_{ijkl} = c^{\chi AB} Z^\chi_{ijkl} \tag{C.9}$$

where

$$c^{\chi AB} = \frac{2}{\sqrt{6n_A n_B}}$$

if Z^χ is of the type $h^{(ABCD)}$ for some C and D, and $c^{\chi AB} \equiv 0$ otherwise.

Having given an orthonormal basis in $\mathcal{H}$, the tensor h decomposes as

$$h_{ijkl} = \sum_A x_A X^A_{ijkl} + \sum_{A<B} x_{AB} X^{AB}_{ijkl} + \sum_{A<B<B'} x_{ABB'} X^{ABB'}_{ijkl} + \sum_\chi z_\chi Z^\chi_{ijkl} \tag{C.10}$$

where the coefficients $x_A, x_{AB}, x_{ABB'}, z_\chi$ give the wanted parametrization for the quadrilinear scalar field couplings.

We now go back to Eq. (78); by direct computation one can show that, for any choice of an orthonormal basis in $\mathcal{H}$, the vector coefficients $\vec{b}_h$ in Eq. (77) is given by

$$\vec{b}_h = \vec{\nabla}_h B \tag{C.11}$$

with

$$B = \frac{1}{3(8\pi)^2}(h, \{h, h\}) \tag{C.12}$$

so that in our case Eq. (78) becomes

$$\left[\sum_A \frac{\partial B}{\partial x_A}\frac{\partial}{\partial x_A} + \sum_{A<B} \frac{\partial B}{\partial x_{AB}}\frac{\partial}{\partial x_{AB}} + \sum_{A<B<B'} \frac{\partial B}{\partial x_{ABB'}}\frac{\partial}{\partial x_{ABB'}} + \sum_\chi \frac{\partial B}{\partial z_\chi}\frac{\partial}{\partial z_\chi}\right] f_h(x_A, x_{AB}, x_{ABB'}, z_\chi) = 0. \tag{C.13}$$

In order to discuss the general polynomial solutions of this equation we first isolate the terms of B depending on x_A. Substituting into Eq. (C.12) the expression in Eq. (C.10), taking into account Eqs. (C.6) (C.7) and the orthonormality of the basis, yields:

$$B(x_A, x_{AB}, x_{ABB'}, z_\chi) = \frac{1}{3(8\pi)^2}\left\{\sum_A \frac{2(n_A+2)}{\sqrt{3n_A(n_A+2)}} x_A^3 + \sum_{A,B\neq A} \frac{(n_A+2)}{\sqrt{2n_A(n_A+2)}} x_A x_{AB}^2 + \sum_{\substack{A,C\neq A\\ C'\neq A}} \frac{(n_A+2)}{\sqrt{3n_A(n_A+2)}} x_A x_{ACC'}^2 + \sum_{A,\chi} c^{A\chi} x_A z_\chi^2\right\} + \bar{B}(x_{AB}, x_{ABB'}, z_\chi) \tag{C.14}$$

where $\bar{B}(x_{AB}, x_{ABB'}, z_\chi)$ is x_A independent.

Now substituting Eq. (C.14) into Eq. (C.13) and selecting the highest order contributions in x_A yields:

$$x_A\left[\frac{2(n_A+2)}{\sqrt{3n_A(n_A+2)}}\left(3x_A\frac{\partial}{\partial X_A} + x_{AB}\frac{\partial}{\partial x_{AB}} + x_{ACC'}\frac{\partial}{\partial x_{ACC'}}\right) + 2c^{A\chi} z_\chi \frac{\partial}{\partial z_\chi}\right] f_{\hbar}^A(x_A, x_{AB}, x_{ACC'}, z_\chi) = 0 \tag{C.15}$$

where $f_{\hbar}^A$ denotes the highest degree terms in x_A of $f_{\hbar}$. From this equations, recalling that the coefficients $c^{A\chi}$ are nonnegative, we immediately conclude that $f_{\hbar}^A$ is of degree zero in x_A and $f_{\hbar}$ is independent from $x_A, x_{AB}, x_{ACC'}$ for any A, B and CC' and from the z_χ parameters for which the coefficients $c^{A\chi}$ in Eq. (C.7) do not vanish (i.e. $\chi \leqq \bar{N}$).

After these specifications we are left with the equation

$$\sum_{N<\chi\leqq N} \frac{\partial \bar{B}}{\partial z_\chi}\frac{\partial}{\partial z_\chi} f_{\hbar} = 0. \tag{C.16}$$

The substitution of Eq. (C.9) into Eq. (C.12) yields the following decomposition for the function $\bar{B}$ defined in Eq. (C.14),

$$\bar{B} = \sum_{\substack{A,B,\chi\\ \chi>N}} c^{\chi AB} x^{AB} z_\chi^2 + \bar{\bar{B}} \tag{C.17}$$

where $\bar{\bar{B}}$ is independent from the parameters x_{AB}. Inserting Eq. (C.17) into Eq. (C.16) and selecting the terms linear in x_{AB}, gives

$$\sum_{\chi>\bar{N}} c^{\chi AB} z_\chi \frac{\partial}{\partial z_\chi} f_{\hbar} = 0 \tag{C.18}$$

which implies that $f_{\hbar}$ is also independent from the parameters z_χ, $\chi > \bar{N}$ since for any such χ there is at least a choice of A and B with $c^{\chi AB} > 0$.

This shows that the only polynomial solution of Eq. (78) is a constant.

We turn our attention to Eq. (79) employing a technique quite similar to the one already used. Introducing the spinor representation space $\psi \equiv \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$ the Yukawa type couplings are specified by matrices

$$Y^i = \left(\begin{array}{c|c} 0 & \mathscr{G}^i \\ \hline \mathscr{G}^i & 0 \end{array}\right) \tag{C.19}$$

satisfying the invariance condition given in Eq. (9) which is written here as:

$$[Y^i, \mathscr{Q}^\alpha] = t^\alpha_{ij} Y^j \tag{C.20}$$

with

$$\mathscr{Q}^\alpha = \left(\begin{array}{c|c} T^\alpha_L & 0 \\ \hline 0 & T^\alpha_R \end{array}\right) \tag{C.21}$$

Let us consider the real linear space of the solutions $\hat{Y}$ (with components Y^i) of Eq. (C.20) equipped with the scalar product

$$(\hat{Y}, \hat{Y}') = \sum_i \mathrm{Tr}(Y^i Y'^i) \tag{C.22}$$

let us further choose in this space a basis $\{\hat{Y}_\lambda\}$ orthonormal with respect to this scalar product, and parametrize the Yukawa coupling matrices by the coefficients y_λ of their expansion on the basis $\{\hat{Y}_\lambda\}$

$$Y^i = \sum_\lambda y_\lambda Y^i_\lambda. \tag{C.23}$$

With this choice of the parameters the components b_{y_λ} of the vector coefficient $\vec{b}_g$ in Eq. (79) are given by

$$b_{y_\lambda} = \frac{\partial}{\partial y_\lambda} F(y_\gamma) \tag{C.24}$$

where the function F, as shown in ref. [26] is the following quartic homogeneous positive definite polynomial:

$$F = \frac{1}{2(8\pi)^2} \sum_{ij} \{[\mathrm{Tr}(Y^i Y^j)]^2 + \mathrm{Tr}(\{Y^i, Y^j\}^2) - \mathrm{Tr}((Y^i)^2 (Y^j)^2)\}. \tag{C.25}$$

We then perform the polar-like change of variables:

$$\rho = F^{1/4} \tag{C.26a}$$

$$\theta_\lambda = \theta_\lambda(y_\eta) \tag{C.26b}$$

for a choice of the variables θ_λ satisfying

$$\sum_\eta \frac{\partial \rho}{\partial y_\eta} \frac{\partial \theta_\lambda}{\partial y_\eta} = 0 \tag{C.27}$$

After this change of variables and setting $\tilde{f}(\rho, \theta) = f_y(y(\rho, \theta))$ we rewrite Eq. (80) as

$$\sum_\lambda \left(\frac{\partial \rho}{\partial y_\lambda}\right)^2 \frac{\partial F}{\partial \rho} \frac{\partial \tilde{f}}{\partial \rho}(\rho, \theta) = 4 \sum_\lambda \left(\frac{\partial \rho}{\partial y_\lambda}\right)^2 \rho^3 \frac{\partial \tilde{f}}{\partial \rho}(\rho, \theta) = 0 \tag{C.28}$$

from which it follows that $\tilde{f}(\rho, \theta)$ is independent from the parameter ρ.

Since $\tilde{f}$ is a homogeneous polynomial in ρ whose degree is equal to that of $f_{\mathscr{g}}$ we conclude that $f_{\mathscr{g}}$ is necessarily a constant.

References

1. Adler, S. L.: Phys. Rev. **117**, 2426 (1969)
   Bell, J. S. Jackiw, R.: Nuovo Cimento **60**, 47 (1969)
   Schwinger, J.: Phys. Rev. **82**, 664 (1951)
2. Bardeen, W. A.: Phys. Rev. **184**, 1848 (1969)
3. See for example Adler, S. L.: Anomalies in perturbation theory. In: Lectures on elementary particles and fields (eds. S. Deser, M. Grisaru, H. Pendleton). Cambridge, Mass.: M.I.T. Press 1970
4. Slavnov, A.: Theor. Math. Phys. (UDSSR) **13**, 174 (1972)
   Taylor, J. C.: Nucl. Phys. **33B**, 436 (1971)
5. Adler, S. L., Bardeen, W. A.: Phys. Rev. **182**, 1517 (1969)
6. Becchi, C.: Commun. Math. Phys. **33**, 97 (1973)
   Pi, S., Shei S. S.: Phys. Rev. **D11**, 2946 (1975)
7. Lowenstein, J. H., Schroer, B.: Phys. Rev. **D7**, 1929 (1975)
8. Zee, A.: Phys. Rev. Lett. **29**, 1198 (1972)
9. Callan, C. G.: Phys. Rev. **D2**, 1541 (1970)
   Symanzik K.: Commun. Math. Phys. **18**, 227 (1970)
10. Bardeen, W. A.: In: Renormalization of Yang-Mills fields and application to particle physics. C. N. R. S. (Marseille) report 72/P 470, 29 (1972), and: Proceedings of the Sixteenth International Conference on High Energy Physics, Fermilab (1972) Vol. 2, 295
11. Costa, G., Julve, J., Marinucci, T., Tonin M.: Nuovo Cimento **38A**, 373 (1977)
12. Lowenstein J. H.: B. P. H. Z. Renormalization. In: Renormalization theory (eds., G. Velo, A. S. Wightman) Dordrecht, Stuttgart: Reidel 1976, and references quoted therein
13. Lee, B. W.: Gauge theories. In: Methods in field theories, (eds. R. Balian, J. Zinn-Justinn) p. 79, Amsterdam: North Holland 1976, and references quoted therein
14. Becchi, C., Rouet, A., Stora, R.: Ann. Phys. (N. Y.) **98**, 287 (1976)
15. Bandelloni, G., Blasi, A., Becchi, C., Collina, R.: Ann. Inst. H. Poincaré **XXVIII**, 255 (1978)
16. Clark, T. E. Piguet, O., Sibold K.: Nucl. Phys. **13B**, 292 (1977)
    Bandelloni, G., Blasi, A. Becchi, C., Collina R.: Commun. Math. Phys. 67, 142 (1979)
17. Bandelloni, G., Blasi, A., Becchi, C., Collina, R.: Ann. Inst. H. Poincaré **XXVIII**, 522 (1978)
18. Symanzik K.: Commun. Math. Phys. **23**, 49 (1971)
19. Becchi, C., Rouet, A., Stora, R.: Renormalizable theories with symmetry breaking. Marseille preprint 75/P 734 (1975)
20. Clark, T., Lowenstein, J. H.: Nucl. Phys. **113B**, 109 (1976)
21. Zimmermann W.: Local operator products and renormalization. In: Lectures on elementary particles and field (eds. S. Deser, M. Grisaru, H. Pendleton). Cambridge, Mass: M. I. T. Press 1970
22. Lam, Y. M. P.: Phys. Rev. **D6**, 2145 (1972); **D7**, 2943 (1973)
    Lowenstein J. H.: Commun. Math. Phys. **24**, 1 (1971)
23. Weinberg, S.: Phys. Rev. **118**, 838 (1960)
    Zimmermann, W.: Commun. Math. Phys. **11**, 1 (1968)
24. Gross, D. J., Wilzek, F.: Phys. Rev. **D8**, 3633 (1973)
25. Yan, Y., Zimmermann Wj: Commun. Math. Phys. **10**, 330 (1968)
26. Coleman, S., Gross, D. J.: Phys. Rev. Lett. **31**, 851 (1973)

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