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On the Lagrangian Theory of Anti-Self-Dual Fields in Four-Dimensional Euclidean Space

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Abstract. We show that a certain four-dimensional field theory has powerful structures in common with the two-dimensional $0(1,3)$ non-linear σ-model.

I. Introduction

By now, many non-equivalent two-dimensional relativistic and non-relativistic *integrable* field theories have been identified. Both their classical aspects, e.g. soliton solutions, action-angle variables, phase shifts and their quantum theoretical aspects, e.g. conservation laws, spectrum, scattering matrix, vacuum expectation values are being intensively studied. However, a complete and explicit classification of these theories is still lacking.

To the author's knowledge, not a single four-dimensional integrable field theoretical model – different from the free field – has been identified be it relativistic or not. Leaving aside the completeness requirement for the set of conserved charges which enters into the definition of integrable systems, actually, the very existence of a non-free four-dimensional continuum field theory with an infinite number of conserved charges has not been established, yet. On account of theorems due to Aks [1] on the one hand and to Coleman and Mandula [2] on the other hand it seems very unlikely that reasonable four-dimensional non-trivial *relativistic* theories exist which possess an infinite number of conserved *local* charges.

In this communication we show that a certain classical four-dimensional local Euclidean field theory admits an infinite number of continuity equations involving non-local expressions of the field variables and an infinite number of corresponding non-local symmetries. The theory in question was introduced into the literature by Yang [3]. It is a local translation-invariant though not manifestly SO(4)-invariant Lagrangian theory. Its extremal classical configurations are closely related to the anti-self-dual SU(2) Yang-Mills gauge fields, whence the name: Lagrangian theory of anti-self-dual fields.

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II. The Model

Let us consider the sourceless SU(2) Yang-Mills theory in Euclidean four-dimensional real space and let us impose the anti-self-duality condition on the real gauge potentials

$$A^a_\mu \qquad a=1,2,3\,; \qquad \mu=1,\ldots,4\,.$$

We set

$$B_\mu=\frac{1}{2i}A^a_\mu\sigma_a \quad \text{and} \quad B_{\mu,\nu}=\frac{\partial}{\partial x^\nu}B_\mu$$

where σ_a denote the Pauli matrices and where the summation convention is implied. Correspondingly, we pass from the real-valued field strengths $F^a_{\mu\nu}$ to the Lie-algebra-valued field strengths

$$F_{\mu\nu}=\frac{1}{2i}F^a_{\mu\nu}\sigma_a\,.$$

$$B^\dagger_\mu=-B_\mu\,, \qquad F^\dagger_{\mu\nu}=-F_{\mu\nu}\,.$$

Here the symbol † stands for the Hermitean adjoint. Then

$$F_{\mu\nu}=B_{\mu,\nu}-B_{\nu,\mu}-[B_\mu,B_\nu]\,.$$

Anti-self-duality means exactly

$$F_{\mu\nu}=-2^{-1}\varepsilon^{\mu\nu\alpha\beta}F_{\alpha\beta} \tag{0}$$

where $\varepsilon^{\mu\nu\alpha\beta}$ is the Levi-Civita symbol, $\varepsilon^{1234}=+1$. This is a set of three independent Lie-algebra-valued equations.

We introduce the complex variables (the bar denotes complex conjugation)

$$y=\tfrac{1}{2}(x^1+ix^2)\,, \qquad \bar{y}=\tfrac{1}{2}(x^1-ix^2)$$

$$z=\tfrac{1}{2}(x^3+ix^4)\,, \qquad \bar{z}=\tfrac{1}{2}(x^3-ix^4)\,,$$

the corresponding derivatives and the corresponding linear combinations of the gauge potentials:

$$\partial_y=\frac{\partial}{\partial x^1}-i\frac{\partial}{\partial x^2}\,, \ \ldots$$

$$\partial_z=\frac{\partial}{\partial x^3}-i\frac{\partial}{\partial x^4}\,, \ \ldots$$

$$B_y=B_1-iB_2\,, \ \ldots$$

$$B_z=B_3-iB_4\,, \ \ldots.$$

The anti-self-duality equations now read

$$F_{yz}=0=F_{\bar{y}\bar{z}}\,, \qquad F_{y\bar{y}}+F_{z\bar{z}}=0\,.$$

The first two equations can be integrated immediately taking the reality of the gauge potentials A_μ^a into account.

$$B_y = D^{-1}D_{,y}, \qquad B_z = D^{-1}D_{,z}$$

$$B_{\bar{y}} = -D^\dagger_{,\bar{y}}D^{\dagger-1}, \qquad B_{\bar{z}} = -D^\dagger_{,\bar{z}}D^{\dagger-1}$$

where $D \in \mathrm{SL}(2, \mathbb{C})$.

A gauge transformation is the replacement

$$D \to D \cdot g, \qquad g = g(y, \bar{y}, z, \bar{z}) \in \mathrm{SU}(2).$$

All this is familiar from Yang's discussion [3].

Less familiar are perhaps the following observations. We note that

$$\chi = D \cdot D^\dagger$$

is a gauge invariant quantity. It is a positive 2×2 matrix whose determinant is equal to 1. The third of the anti-self-duality equations takes its simplest form when written as an equation for χ

$$\{\chi^{-1}\chi_{,y}\}_{,\bar{y}} + \{\chi^{-1}\chi_{,z}\}_{,\bar{z}} = 0, \qquad \chi \in \mathrm{SL}(2, \mathbb{C}), > 0 \tag{1}$$

or

$$\chi_{,y\bar{y}} + \chi_{,z\bar{z}} + \{\chi_{,\bar{y}}(\chi^{-1})_{,y} + \chi_{,\bar{z}}(\chi^{-1})_{,z}\}\chi = 0, \qquad \chi \in \mathrm{SL}(2, \mathbb{C}), > 0. \tag{1'}$$

Thus, to any real anti-self-dual gauge configuration there corresponds a positive 2×2 matrix χ of determinant 1 satisfying Eq. (1) or equivalently Eq. (1′). Conversely, let χ be a positive 2×2 matrix of determinant 1 satisfying Eq. (1) [or (1′)]. Then by taking the hermitean square root

$$K = \chi^{1/2}$$

and forming

$$B_y = K^{-1}K_{,y}, \qquad B_z = K^{-1}K_{,z}$$

$$B_{\bar{y}} = -K_{,\bar{y}}K^{-1}, \qquad B_{\bar{z}} = -K_{,\bar{z}}K^{-1}.$$

we obtain an anti-self-dual gauge field (in the so-called Hermitean gauge).

If we parametrize χ with the help of Poincaré coordinates for the forward unit mass hyperboloid:

$$\chi = \frac{1}{\phi}\begin{pmatrix} 1, & \bar{\varrho} \\ \varrho, & \phi^2 + |\varrho|^2 \end{pmatrix}, \qquad \phi \text{ real}, \quad \varrho \text{ complex},$$

equation (1) reads

$$\phi[\phi_{,y\bar{y}} + \phi_{,z\bar{z}}] - \phi_{,y}\phi_{,\bar{y}} - \phi_{,z}\phi_{,\bar{z}} + \varrho_{,y}\bar{\varrho}_{,\bar{y}} + \varrho_{,z}\bar{\varrho}_{,\bar{z}} = 0$$

$$\phi[\varrho_{,y\bar{y}} + \varrho_{,z\bar{z}}] - 2\varrho_{,y}\phi_{,\bar{y}} - 2\varrho_{,z}\phi_{,\bar{z}} = 0$$

$$\phi[\bar{\varrho}_{,y\bar{y}} + \bar{\varrho}_{,z\bar{z}}] - 2\bar{\varrho}_{,\bar{y}}\phi_{,y} - 2\bar{\varrho}_{,\bar{z}}\phi_{,z} = 0.$$

These are Yang's equations "in the R-gauge". As we have seen, it is the Poincaré parameter formulation of the gauge invariant Eq. (1′).

Next, it is important to note that Yang's equations – and hence Eq. (1′) – are the Euler-Lagrange equations for a variational problem with the Lagrangian density

$$\mathscr{L}=\frac{1}{2}\,\frac{\phi_{,y}\phi_{,\bar{y}}+\phi_{,z}\phi_{,\bar{z}}+\varrho_{,y}\bar{\varrho}_{,\bar{y}}+\varrho_{,z}\bar{\varrho}_{,\bar{z}}}{\phi^2}.$$

This Lagrangian density defines the model.

We may avoid the employment of special parametrizations of the positive 2×2 matrices with determinant 1. In order to make contact with other models we introduce instead a real four vector q by setting

$$\chi=q^0\mathbb{1}_2+\sum_1^3 q^i\sigma_i.$$

Det $\chi=1$ implies

$$q^a q_a=1$$

where

$$q_a=g_{ab}q^b$$

with

$$g_{ab}=\mathrm{diag}(+1,-1,-1,-1).$$

Finally, the positivity of the matrix χ implies

$$q\in V_1^{\dagger}$$

i.e. q takes its values on the forward unit hyperboloid in a Minkowskian colour space.

The differential Eq. (1′) for χ becomes

$$q_{,y\bar{y}}+q_{,z\bar{z}}+(q_{,y}\cdot q_{,\bar{y}}+q_{,z}\cdot q_{,\bar{z}})q+i[q\,;q_{,\bar{y}}\,;q_{,y}]+i[q\,;q_{,\bar{z}}\,;q_{,z}]=0. \tag{1''}$$

Here the dots denote Minkowskian scalar products and the brackets mean vector products e.g.

$$[q\,;q_{,\bar{y}}\,;q_{,y}]^a=\varepsilon^{abcd}q_b q_{c,\bar{y}}q_{d,y}.$$

The Lagrangian and, consequently, the field equation and the constraint are invariant under internal Lorentztransformations

$$\chi\to A\chi A^{\dagger}$$

where A is an SL$(2,C)$-matrix-valued function of $\bar{y}$ and $\bar{z}$. They are translation-invariant and "invariant" under the following group of (coordinate) transformations

$$\begin{aligned}
&y\to y'=\cos\delta[\cos\alpha e^{i\beta}\lambda y+\sin\alpha e^{i\gamma}\lambda z]-i\sin\delta[-\sin\alpha e^{i\beta}\lambda y+\cos\alpha e^{i\gamma}\lambda z]\\
&z\to z'=-i\sin\delta[\cos\alpha e^{i\beta}\lambda y+\sin\alpha e^{i\gamma}\lambda z]+\cos\delta[-\sin\alpha e^{i\beta}\lambda y+\cos\alpha e^{i\gamma}\lambda z] \qquad (3_1)\\
&\chi(y,\bar{y},z,\bar{z})\to\chi'(y',\bar{y}',z',\bar{z}')=\chi(y,\bar{y},z,\bar{z})
\end{aligned}$$

where α, β, γ, δ, and λ are arbitrary real parameters.

They are obviously not invariant under arbitrary rotations in the four-dimensional Euclidean coordinate space, let alone inversion of the coordinates.

However, on the set of solutions of the field equation the action of the full rotation group SO(4) can be defined, such that solutions are mapped into solutions and that the action for the solutions does not change. Actually, it suffices to describe the transformation law for rotations around an angle $2\cdot\theta$ in the $(1,3)$-plane

$$\begin{aligned} & y \to y' = y\cos^2\theta - \bar{y}\sin^2\theta - (z+\bar{z})\sin\theta\cos\theta\,, \qquad \bar{y}\to\bar{y}' = \dots \\ R_{e^{2i\theta}}\colon\ & z \to z' = (y+\bar{y})\sin\theta\cos\theta + z\cos^2\theta - \bar{z}\sin^2\theta\,, \qquad \bar{z}\to\bar{z}' = \dots \\ & \chi(y,\bar{y},z,\bar{z}) \to \chi^{(\theta)}(y',\bar{y}',z',\bar{z}') = \mathscr{A}^{(\theta)}\chi(y,\bar{y},z,\bar{z})\mathscr{A}^{(\theta)\dagger} \end{aligned} \tag{3_2}$$

where

$$\mathscr{A}^{(\theta)} = \mathscr{A}^{(\theta)}_{[\chi]}(y,\bar{y},z,\bar{z}) \in \mathrm{SL}(2,\mathbb{C})$$

satisfies the following system of linear differential equations

$$\begin{aligned} & \mathscr{A}^{(\theta)}_{,y} - \mathrm{tg}\theta\,\mathscr{A}^{(\theta)}_{,\bar{z}} = \mathrm{tg}\theta\,\mathscr{A}^{(\theta)}\chi_{,\bar{z}}\chi^{-1} \\ & \mathscr{A}^{(\theta)}_{,z} + \mathrm{tg}\theta\,\mathscr{A}^{(\theta)}_{,\bar{y}} = -\mathrm{tg}\theta\,\mathscr{A}^{(\theta)}\chi_{,\bar{y}}\chi^{-1}\,. \end{aligned}$$

For $\chi \in \mathrm{SL}(2,\mathbb{C}), >0$ this system is compatible if and only if χ satisfies (1). It is equivalent to the linear system advertised by Belavin and Zakharov [4][1].

The composition law for two successive rotations in the $(1,3)$-plane

$$(y,\bar{y},z,\bar{z}) \xrightarrow{\ 2\theta\ } (y',\bar{y}',z',\bar{z}') \xrightarrow{\ 2\theta'\ } (y'',\bar{y}'',z'',\bar{z}'')$$

is

$$\mathscr{A}^{(\theta+\theta')}_{[\chi]}(y,\bar{y},z,\bar{z}) = \mathscr{A}^{(\theta')}_{[\mathscr{A}^{(\theta)}\chi\mathscr{A}^{(\theta)\dagger}]}(y',\bar{y}',z',\bar{z}')\,\mathscr{A}^{(\theta)}_{[\chi]}(y,\bar{y},z,\bar{z})\,.$$

The field equation and the constraint remind us of the two-dimensional $O(1,3)$ non-linear σ-model. In fact, solutions of Eq. $(1'')$ depending only on the coordinates x^1 and x^3, or x^1 and x^4, or x^2 and x^3, or x^2 and x^4 solve the field equation of the classical Euclidean two-dimensional $O(1,3)$ non-linear σ-model

$$q_{,w\bar{w}} + (q_{,w}\cdot q_{,\bar{w}})q = 0\,, \qquad q\cdot q = 1$$

e.g.

$$w = \frac{x^1 + ix^3}{2}\,, \qquad \bar{w} = \frac{x^1 - ix^3}{2}\,.$$

On the other hand, solutions of $(1'')$ depending only on the coordinates x^1 and x^2, or x^3 and x^4 satisfy the explicitly soluble equation [cf. (1)]

$$q_{,y\bar{y}} + (q_{,y}\cdot q_{,\bar{y}})q + i[q\,;q_{,\bar{y}}\,;q_{,y}] = 0\,, \qquad q\cdot q = 1\,.$$

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1 In terms of θ, $\mathscr{A}^{(\theta)}$ and χ, their parameter λ and their fundamental matrix Ψ are given by

$$\lambda = \mathrm{tg}\theta\,, \qquad \Psi = (\mathscr{A}^{(\theta)}D)^\dagger$$

where for instance in the Hermitean gauge

$$D = K = \chi^{1/2} = \text{Hermitean}$$

Thus we might think of the "theory of anti-self-dual gauge fields" as being composed of the free theory in some planes and the non-linear $O(1,3)$ σ-model in some other planes of the four-dimensional Euclidean space.

III. Non-Local Continuity Equations and Non-Local Infinitesimal Symmetries

We go back to the system of linear differential equations defining $\mathscr{A}^{(\theta)}$. Setting

$$\mathrm{tg}\,\theta=\zeta$$

and taking the Hermitean adjoint, it reads

$$\begin{aligned}&\mathscr{A}^{(\theta)\dagger}_{,\bar{y}}-\zeta\mathscr{A}^{(\theta)\dagger}_{,z}-\zeta\chi^{-1}\chi_{,z}\mathscr{A}^{(\theta)\dagger}=0\\&\mathscr{A}^{(\theta)\dagger}_{,\bar{z}}+\zeta\mathscr{A}^{(\theta)\dagger}_{,y}+\zeta\chi^{-1}\chi_{,y}\mathscr{A}^{(\theta)\dagger}=0\,.\end{aligned}\tag{4}$$

We expand $\mathscr{A}^{(\theta)\dagger}$ in powers of the parameter ζ:

$$\mathscr{A}^{(\theta)\dagger}=A_0+A_1\zeta+A_2\zeta^2+\dots,$$

insert this expansion into the left hand sides of the system (4), collect all terms of the same order in ζ and set the resulting coefficients separately equal to zero. In this way we obtain the following set of equations

$$\begin{aligned}&A_{0,\bar{y}}=0\,;\quad A_{0,\bar{z}}=0\\&A_{n+1,\bar{y}}-A_{n,z}-\chi^{-1}\chi_{,z}A_n=0\,;\quad A_{n+1,\bar{z}}+A_{n,y}+\chi^{-1}\chi_{,y}A_n=0\,,\\&\qquad n=0,1,2,\dots\end{aligned}$$

from which we derive the following infinite set of non-local continuity equations.

$$\begin{aligned}&\{\chi^{-1}\chi_{,z}A_n+A_{n,z}\}_{,\bar{z}}+\{\chi^{-1}\chi_{,y}A_n+A_{n,y}\}_{,\bar{y}}=0\,,\\&\qquad n=0,1,2,\dots.\end{aligned}\tag{5}$$

Without loss of generality we may assume

$$A_0\equiv\mathbb{1}\,,\qquad \mathrm{Tr}A_1\equiv 0\,.\tag{6}$$

This set is the direct generalization of the infinitely many non-local conservation laws corresponding to the non-local charges Q^a_{n+1} of the Minkowskian two-dimensional $O(4)$ non-linear σ-model [6].

Along with the non-local continuity equations go non-local symmetries. In their infinitesimal form they are given by

$$\begin{aligned}&\delta\chi=\chi(\boldsymbol{\varepsilon}\cdot\boldsymbol{\sigma})+\text{h.c.}\\&\delta\chi=\chi[A_1,(\boldsymbol{\varepsilon}\cdot\boldsymbol{\sigma})]+\text{h.c.}\\&\delta\chi=\chi([A_2,(\boldsymbol{\varepsilon}\cdot\boldsymbol{\sigma})]+\tfrac{1}{2}[A_1,[A_1,(\boldsymbol{\varepsilon}\cdot\boldsymbol{\sigma})]])+\text{h.c.},\dots\end{aligned}\tag{7}$$

where $\boldsymbol{\varepsilon}\cdot\bar{\boldsymbol{\varepsilon}}\ll 1$.

IV. Backlund Transformations

Both lower dimensional theories i.e., the Euclidean two-dimensional $O(1,3)$ non-linear σ-model and the free theory possess Backlund transformations mapping solutions of

$$\check{q}_{,w\bar{w}}-(\check{q}_{,w}\cdot\check{q}_{,\bar{w}})\check{q}=0, \qquad \check{q}\cdot\check{q}=-1$$

and

$$\check{q}_{,y\bar{y}}-(\check{q}_{,y}\cdot\check{q}_{,\bar{y}})\check{q}-i[\check{q};\check{q}_{,\bar{y}};\check{q}_{,y}]=0, \qquad \check{q}\cdot\check{q}=-1$$

respectively into solutions of

$$q_{,w\bar{w}}+(q_{,w}\cdot q_{,\bar{w}})q=0, \qquad q\cdot q=+1$$

and

$$q_{,y\bar{y}}+(q_{,y}\cdot q_{,\bar{y}})q+i[q;q_{,\bar{y}};q_{,y}]=0, \qquad q\cdot q=+1$$

respectively. In both cases the image vector is orthogonal to the original vector.

It is a remarkable fact that these Backlund transformations can be wed to a single one in four-dimensional Euclidean space mapping solutions of

$$\check{q}_{,y\bar{y}}+\check{q}_{,z\bar{z}}-(\check{q}_{,y}\cdot\check{q}_{,\bar{y}}+\check{q}_{,z}\cdot\check{q}_{,\bar{z}})\check{q}-i[\check{q};\check{q}_{,\bar{y}};\check{q}_{,y}]-i[\check{q};\check{q}_{,\bar{z}};\check{q}_{,z}]=0, \quad \check{q}\cdot\check{q}=-1 \tag{2''}$$

[2]

into solutions of (1″)

$$q_{,y\bar{y}}+q_{,z\bar{z}}+(q_{,y}\cdot q_{,\bar{y}}+q_{,z}\cdot q_{,\bar{z}})q+i[q;q_{,\bar{y}};q_{,y}]+i[q;q_{,\bar{z}};q_{,z}]=0, \quad q\cdot q=+1.$$

The Backlund transformation is

$$q\cdot q=+1, \qquad \check{q}\cdot\check{q}=-1, \qquad q\cdot\check{q}=0 \tag{8_1}$$

B_+:

$$q_{,y}-\check{q}_{,\bar{z}}=-(q\cdot\check{q}_{,\bar{z}})q+(q\cdot\check{q}_{,y})\check{q}-i[\check{q};q_{,\bar{z}}+\check{q}_{,y};q]. \tag{8_2}$$

Note that this Backlund transformation consists of only *one* complex vectorial differential equation.

Compatibility of the defining equations of B_+ – including the reality requirements for q and $\check{q}$ – implies

$$\check{q}_{,y\bar{y}}+\check{q}_{,z\bar{z}}-(\check{q}_{,y}\cdot\check{q}_{,\bar{y}}+\check{q}_{,z}\cdot\check{q}_{,\bar{z}})\check{q}-i[\check{q};\check{q}_{,\bar{y}};\check{q}_{,y}]-i[\check{q};\check{q}_{,\bar{z}};\check{q}_{,z}]=\kappa q \tag{2^+}$$

$$q_{,y\bar{y}}+q_{,z\bar{z}}+(q_{,y}\cdot q_{,\bar{y}}+q_{,z}\cdot q_{,\bar{z}})q+i[q;q_{,\bar{y}};q_{,y}]+i[q;q_{,\bar{z}};q_{,z}]=\kappa\check{q}. \tag{1^+}$$

Here the proportionality factor κ is arbitrary. This situation, namely that the compatibility of the defining equations of a Backlund transformation does not enforce exactly the field equations, is familiar from the Euclidean two-dimensional $O(1,3)$ non-linear σ-model.

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2 To $\check{q}$: $\check{q}\cdot\check{q}=-1$ correspond real SU(1, 1) gauge potentials

If, however, $\check{q}$ satisfies the differential (2″), then $\kappa \equiv 0$ and q solves (1″). Or, if q satisfies (1″), then $\kappa \equiv 0$ and $\check{q}$ solves (2″).

If we multiply (8_2) vectorially by q and $\check{q}$ and pass to the complex conjugate of the resulting equation, we obtain

$$q_{,z} + \check{q}_{,\bar{y}} = (q \cdot \check{q}_{,\bar{y}})q + (q \cdot \check{q}_{,z})\check{q} + i[\check{q}; q_{,\bar{y}} - \check{q}_{,z}; q]. \qquad (8'_2)$$

In terms of the hermitean 2×2 matrices $\check{\chi}$ and χ, the Backlund transformation B_+ reads

$$\chi \cdot \tilde{\chi} = \mathbb{1}, \quad \check{\chi} \cdot \tilde{\check{\chi}} = -\mathbb{1}, \quad \chi \cdot \tilde{\check{\chi}} + \check{\chi}\tilde{\chi} = 0$$

$$\{\chi \cdot \tilde{\check{\chi}}\}_{,y} \check{\chi} = -\chi\{\tilde{\chi}\check{\chi}\}_{,\bar{z}}$$

$$(\{\chi \cdot \tilde{\check{\chi}}\}_{,z} \check{\chi} = \chi\{\tilde{\chi}\check{\chi}\}_{,\bar{y}})$$

where

$$\tilde{\chi} = q^0 \mathbb{1} - \sum_{i=1}^{3} q^i \sigma_i, \quad \tilde{\check{\chi}} = \check{q}^0 \mathbb{1} - \sum_{i=1}^{3} \check{q}^i \sigma_i.$$

The Backlund transformation B_+ does not commute with the transformations (3_2). Hence, by forming

$$T_{\gamma+} = R_\gamma^{-1} B_+ R_\gamma, \qquad \gamma = e^{2i\theta}$$

we obtain a one parameter family of Backlund transformations (*c.f.* [5], *Eq.* (*VI*.6))

$$\chi \xrightarrow{T_{\gamma+}} \check{\chi}(\cdot\,; \gamma+) = (\chi^{(\theta)})^{\vee(-\theta)}.$$

The corresponding vectors q and $\check{q}\,(\cdot\,; \gamma+)$ are no longer orthogonal to each other whereas extensions of B_+ obtained by analogous combinations with the transformations (3_1) do not affect the orthogonality of the original and the image vector.

V. The Analogs of the Two-Dimensional Local Conservation Laws

In order to derive yet another infinite set of continuity equations – corresponding to the set of local conservation laws of the Minkowskian two-dimensional $O(4)$ non-linear σ-model – we need two continuity equations involving both the original and the Backlund transformed solutions q and $\check{q}$ in combinations which are invariant under internal Lorentz transformations. They should express the information that

$$Q \doteq \{q_{,y\bar{y}} + q_{,z\bar{z}} + i[q; q_{,\bar{y}}; q_{,y}] + i[q; q_{,\bar{z}}; q_{,z}]\}$$

and

$$\check{Q} \doteq \{\check{q}_{,y\bar{y}} + \check{q}_{,z\bar{z}} - i[\check{q}; \check{q}_{,\bar{y}}; \check{q}_{,y}] - i[\check{q}; \check{q}_{,\bar{z}}; \check{q}_{,z}]\}$$

are orthogonal to $\check{q}$ and to q respectively. To formulate such continuity equations, it seems mandatory to parametrize q and $\check{q}$ such that the algebraic equations (8_1) are satisfied identically. The complex equations (8_2) and ($8'_2$) take their simplest form if for the parametrization of q and $\check{q}$ Poincaré coordinates are used:

$$(q^a) = \left(\frac{1 + (|\varrho|^2 + \phi^2)}{2\phi}, \frac{\varrho^1}{\phi}, \frac{\varrho^2}{\phi}, \frac{1 - (|\varrho|^2 + \phi^2)}{2\phi} \right), \quad \varrho = \varrho^1 + i\varrho^2$$

$$\check{q} = \operatorname{tgh}\alpha(\cos\beta r + \sin\beta s) + \cosh^{-1}\alpha\, t$$

with

$$(r^a)=\left(\frac{1+(|\varrho|^2-\phi^2)}{2\phi},\frac{\varrho^1}{\phi},\frac{\varrho^2}{\phi},\frac{1-(|\varrho|^2-\phi^2)}{2\phi}\right),$$

$$(s^a)=(\varrho^1,1,0,-\varrho^1),$$

$$(t^a)=(\varrho^2,0,1,-\varrho^2).$$

We combine the internal coordinates α and β into a complex coordinate

$$\check{\omega}=-\operatorname{tgh}\frac{\alpha}{2}e^{i\beta}.$$

Now equations (8_2) and $(8'_2)$ read

$$B_+:\quad\begin{aligned}\check{\omega}_{,u}&=\left(\frac{\varrho^1_{,\bar{v}}-\varrho^2_{,u}+i\phi_{,\bar{v}}}{2\phi}\right)-i\check{\omega}\left(\frac{\varrho^1_{,u}+\varrho^2_{,\bar{v}}}{\phi}\right)\\&\quad+\check{\omega}^2\left(\frac{\varrho^1_{,\bar{v}}-\varrho^2_{,u}-i\phi_{,\bar{v}}}{2\phi}\right),\\\check{\omega}_{,v}&=-\left(\frac{\varrho^1_{,\bar{u}}+\varrho^2_{,v}+i\phi_{,\bar{u}}}{2\phi}\right)-i\check{\omega}\left(\frac{\varrho^1_{,v}-\varrho^2_{,\bar{u}}}{\phi}\right)\\&\quad-\check{\omega}^2\left(\frac{\varrho^1_{,\bar{u}}+\varrho^2_{,v}-i\phi_{,\bar{u}}}{2\phi}\right)\end{aligned}\tag{9}$$

where

$$u=(x^1+ix^4)/2\quad\text{and}\quad v=(x^2+ix^3)/2.$$

For any parametrization of q and $\check{q}$ it is not difficult to show that the relations

$$0=(Q\cdot\check{q})\pm(q\cdot\check{Q})$$

can be cast into the form of continuity equations. These simplify considerably if Poincaré coordinates and the relations B_+ are used. The result is the following complex continuity equation

$$\begin{aligned}&\left\{\left(\frac{\varrho^1_{,v}-\varrho^2_{,\bar{u}}}{\phi}\right)-\left(\frac{\phi_{,\bar{u}}+i\{\varrho^1_{,\bar{u}}+\varrho^2_{,v}\}}{\phi}\right)\check{\omega}\right\}_{,u}\\&-\left\{\left(\frac{\varrho^1_{,u}+\varrho^2_{,\bar{v}}}{\phi}\right)+\left(\frac{\phi_{,\bar{v}}+i\{\varrho^1_{,\bar{v}}-\varrho^2_{,u}\}}{\phi}\right)\check{\omega}\right\}_{,v}=0.\end{aligned}\tag{10}$$

If we start from the pair $q^{(\theta)}(y',\bar{y}',z',\bar{z}')$, $(q^{(\theta)}(y',\bar{y}',z',\bar{z}'))\check{}$ instead of the pair $q(y,\bar{y},z,\bar{z})$, $(q(y,\bar{y},z,\bar{z}))\check{}$, we arrive at the one-parameter family of continuity equations

$$\begin{aligned}&\left\{\left(\frac{\varrho^1_{,v'}-\varrho^2_{,\bar{u}'}}{\phi}\right)-\left(\frac{\phi_{,\bar{u}'}+i\{\varrho^1_{,\bar{u}'}+\varrho^2_{,v'}\}}{\phi}\right)\check{\omega}^{(\theta)}\right\}_{,u'}\\&-\left\{\left(\frac{\varrho^1_{,u'}+\varrho^2_{,\bar{v}'}}{\phi}\right)+\left(\frac{\phi_{,\bar{v}'}+i\{\varrho^1_{,\bar{v}'}-\varrho^2_{,u'}\}}{\phi}\right)\check{\omega}^{(\theta)}\right\}_{,v'}=0\end{aligned}\tag{11}$$

with

$\partial_{u'} = \cos^2\theta\,\partial_u - \sin^2\theta\,\partial_{\bar{u}} - i\sin\theta\cos\theta(\partial_v - \partial_{\bar{v}})$,

$\partial_{v^\cdot} =$ dito up to the replacement $u \leftrightarrow v$,

$\partial_{v'} = \cos^2\theta\,\partial_v + \sin^2\theta\,\partial_{\bar{v}} - i\sin\theta\cos\theta(\partial_u + \partial_{\bar{u}})$,

$\partial_{u^\cdot} =$ dito up to the replacement $u \leftrightarrow v$,

and where, up to an integration "constant", $\check{\omega}^{(\theta)}$ is defined by two compatible Riccati equations

$$T_{\gamma+}: \quad \begin{aligned} \check{\omega}^{(\theta)}_{,u'} &= \left(\frac{\varrho^1_{,\bar{v}^\cdot} - \varrho^2_{,u'} + i\phi_{,\bar{v}^\cdot}}{2\phi}\right) - i\check{\omega}^{(\theta)}\left(\frac{\varrho^1_{,u'} + \varrho^2_{,\bar{v}^\cdot}}{\phi}\right) + \check{\omega}^{(\theta)2}\left(\frac{\varrho^1_{,\bar{v}^\cdot} - \varrho^2_{,u'} - i\phi_{,\bar{v}^\cdot}}{2\phi}\right) \\ \check{\omega}^{(\theta)}_{,v'} &= -\left(\frac{\varrho^1_{,\bar{u}^\cdot} + \varrho^2_{,v'} + i\phi_{,\bar{u}^\cdot}}{2\phi}\right) - i\check{\omega}^{(\theta)}\left(\frac{\varrho^1_{,v'} - \varrho^2_{,\bar{u}^\cdot}}{\phi}\right) - \check{\omega}^{(\theta)2}\left(\frac{\varrho^1_{,\bar{u}^\cdot} + \varrho^2_{,v'} - i\phi_{,\bar{u}^\cdot}}{2\phi}\right). \end{aligned} \tag{12}$$

Just as in the two-dimensional case [5], we may expand the l.h.s. of Eq. (11) in powers of $\gamma = e^{2i\theta}$ (or $\gamma^{-1} = e^{-2i\theta}$) around $\gamma = 0$ (or $\gamma = \infty$) after having inserted the appropriate expansion of $\check{\omega}^{(\theta)}$ obtained from $T_{\gamma+}$. We then may collect the terms accompanied by equal powers of γ (or γ^{-1}) and set them separately equal to zero. In this way we would get an infinite series of complex continuity equations the densities of which are invariant under internal Lorentz transformations. In contrast to the two-dimensional case they involve non-local expressions of the field q. Even worse, these nonlocal expressions have not been obtained explicitly. They require the iterative solution of two compatible linear differential equations involving the parameter γ for some two by two matrix i.e. of the linear systems associated with the Riccati equations (12). Unfortunately, only the formal expression for the zero order term is at our disposal.

VI. Conclusions

In this note we have formulated a local Lagrangian field theory in four-dimensional Euclidean space. The extrema of its action are related to the anti-self-dual SU(2) gauge potentials. The model in question is a composition of the free theory in some planes and the $O(1,3)$ non-linear σ-model in some other planes of the four-dimensional Euclidean space.

Using the known structures of the two-dimensional $O(1,3)$ non-linear σ-model as a guiding principle and scale for the analysis of the model, we gave a Backlund transformation and a one-parameter extension of it. In addition, it was shown that the model admits an infinite number of non-local continuity equations and an infinite number of corresponding non-local symmetries. A generating functional for the non-local current densities both for the variant ones and invariant ones under internal Lorentz-transformations is essentially provided by a fundamental matrix introduced in a somewhat different context by Belavin and Zakharov and by the Backlund transformation $T_{\gamma+}$ and eq. (10) respectively.

The Euclidean theory discussed in this article does not correspond to a possible dynamics in one time and three space dimensions. However its "projections" to certain three dimensional Euclidean subspaces presumably correspond to dynamical theories in one time and two space dimensions. It might be worthwhile to study the associated quantum theories.

The present note recapitulates and extends results already contained in an earlier preprint [7]. There is some overlap of our results with the conclusions of two recent papers by Prasad, Sinha and Wang [8].

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