Commun. Math. Phys. 68, 129–140 (1979)

Communications in
Mathematical
Physics

On the Hénon Transformation

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Abstract. In [4] Hénon studied a transformation which maps the plane into itself and appears to have an attractor with locally the structure of a Cantor set cross an interval. By making use of the characteristic exponent, frequency spectrum, and a theorem of Smale, our numerical experiments provide evidence for the existence of two distinct strange attractors for some parameter values, an exponential rate of mixing for the parameter values studied by Hénon, and an argument that there is a Cantor set in the trapping region of Hénon.

1. Introduction

In [4] Hénon, motivated by computer studies of the Lorenz system performed by Pomeau, studied a transformation which maps the plane into itself. Hénon was able to prove, among other things, that the transformation which he considered was the most general quadratic map which carries the plane into itself and has constant Jacobian determinant. Then in a remarkable sequence of computer graphics he gave strong numerical evidence that the transformation he studied has a strange attractor whose local structure is the product of a one-dimensional manifold by a Cantor set, at least in the neighborhood of one of the stationary solutions.

Further, Hénon was able to show for the specific parameter values which he considered that there exist a compact set M, called a "trapping region", which is carried into itself by the action of the transformation. Subsequently, Feit in [2] has generalized the above result by giving a characterization of the compact set of nondivergent points for Hénon's transformation – a point in [2] is called nondivergent provided its forward orbit under the action of the transformation is bounded.

In [2] characteristic exponents were also computed for a substantial set of parameter values for the Hénon map. If the characteristic exponent is less than

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** The National Center for Atmospheric Research is sponsored by the National Science Foundation

0010-3616/79/0068/0129/$02.40

zero then neighboring trajectories approach each other at an exponential rate and if it is positive then we have exponential separation of nearby trajectories; hence, the characteristic exponent provides a measure of sensitivity to initial conditions. Feit found, for the parameter values studied by Hénon, that the associated characteristic exponent was positive. For a review of the properties of characteristic exponents we refer the reader to [6] and the bibliography cited there.

Another quantity which is an indicator of sensitivity to initial conditions is the decay of time correlations; the time correlation function is a normalized time covariance function and the decay of this function to zero provides some evidence that a given dynamical system is mixing [6]. By computing the Fourier transform of the covariance, i.e., the frequency spectrum, it is possible to determine which frequencies contribute most to the variance of a process. If the frequency spectrum consists of solitary narrow spikes then the underlying process is (multiply) periodic, while if there is a broad band of frequencies present then the process has continuous spectrum and is not periodic. Gollub and Swinney [3] have measured frequency spectra for the velocity field in a rotating fluid and have found a broad band of frequencies present.

Despite the graphics of Hénon and the work of Feit, some doubt has been expressed that there is a Cantor set in the trapping region. Indeed, Newhouse [5] has suggested that Hénon may have found a very long periodic orbit. If this is the case then it is certain that computational error will prevent one from establishing the period of the closed orbit.

In order to demonstrate the effects of computational error on the iteration of Hénon's transformation consider the following simple experiments: Given an initial condition, compute the sixtieth iterate of the transformation using two different machines (a CDC 7600 and a CRAY-1) – both machines carry fourteen significant digits in single precision. We found that there was no agreement in the output of the machines by the sixtieth iterate of the transformation!

Since the computational error is definitely significant after sixty iterations of Hénon's map, it is apparent that it will be difficult, if not impossible, to establish numerically that what we see in [4] is only a very long periodic orbit. Further, although the above experiment illustrates that the rounding error makes it impossible to predict the coordinates of a high iterate of Hénon's map, the numerical experiments whose result we report in this article indicate that rounding error does not affect the gross statistical properties of the transformation.

In the following sections we shall make use of the characteristic exponent, frequency spectrum and a theorem of Smale [7] to further study the Hénon transformation. In Sect. 2 we present the model of Hénon and describe a few of its properties. Section 3 is devoted to reporting the results of our numerical studies. In Sect. 4 we end with a discussion of our findings. In the appendix we estimate the round-off in repeated iterations of the Hénon transformation.

2. Preliminary Result

The mapping of Hénon is defined as follows:

$$T(x,y)=(1+y-ax^2, bx)\,. \tag{1}$$

T is invertible with inverse

$$T^{-1}(x,y)=(b^{-1}y, x-1+ab^{-2}y^2)\,. \tag{2}$$

We shall assume throughout that $a>0$ and $b=0.3$.

It is immediate that T has two fixed points whose coordinates are given by

$$x=\frac{(b-1)\pm\sqrt{(1-b)^2+4a}}{2a}\,,\ y=bx\,. \tag{3}$$

One fixed point is in the left half plane while the other is in the right. We are particularly interested in those parameter values for which the stationary solution in the first quadrant is unstable. This is the case provided $a>3(1-b)^2/4$ ($a>0.3675$ when $b=0.3$).

Two of the main tools which we will be using in our analysis of T are the characteristic exponent and frequency spectrum. The characteristic exponent is given by the following expression

$$\lim_{n\to\infty}\frac{1}{n}\log|DT^n(\mathbf{x})\mathbf{v}|\,, \tag{4}$$

where by T^n we mean T composed with itself n times and D denotes differentiation with respect to the two-vector $\mathbf{x}$, $\mathbf{v}$ is a two-vector chosen at random. The frequency spectrum is the Fourier cosine transform of the lag covariances where the k^{th} lag is given by

$$\Gamma_k=\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}x(i+k)x(i) \tag{5}$$

and $x(i)$ is some quantity of interest (e.g., position) which is assumed to have mean zero.

3. Numerical Results

We find it convenient to present our numerical results in three separate subsections, devoted to characteristic exponents, frequency spectra, and unstable manifolds.

a) Characteristic Exponents

The characteristic exponents were computed following the procedure described in [2]. However, we generally took one initial point and two random vectors. We then iterated the initial point 60,000 times. This led to values of the characteristic exponent which agreed to at least four decimal places for the two different random vectors.

As we indicated in the introduction, Feit has computed the characteristic exponent for a large set of values of a and b. The result of one of those computations was two distinct values of the characteristic exponent, having opposite sign, when $a=1.0752$. The corresponding limit sets were a stable orbit

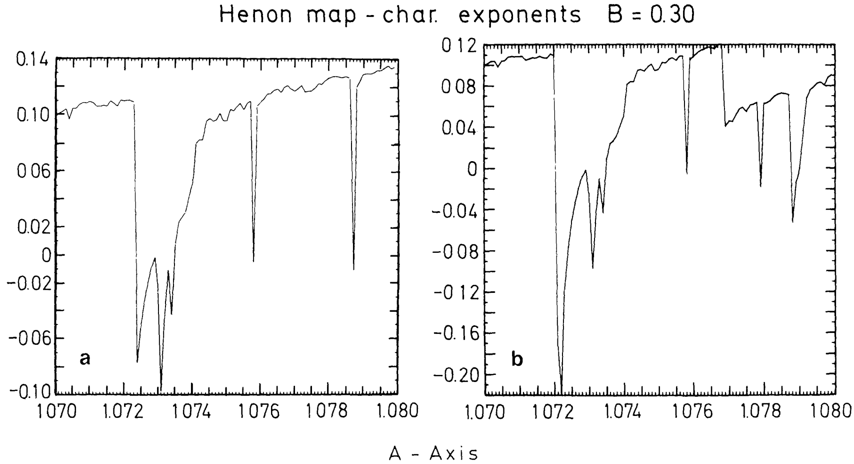

Fig. 1. **a** Graph of the characteristic exponent for $a \varepsilon$ [1.07, 1.08] and initial condition (0, 0). **b** Same as **a** only starting from a different initial condition

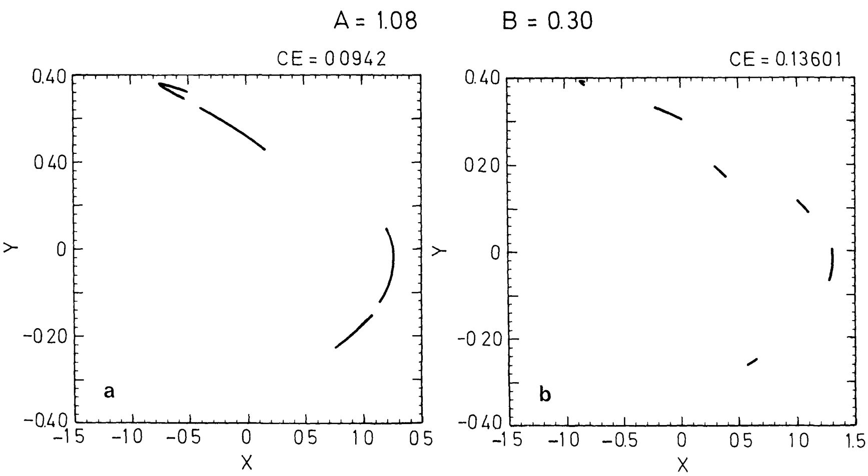

Fig. 2. **a** Graph of a strange attractor when $a = 1.08$, initial condition the origin. **b** Graph of a strange attractor when $a = 1.08$, initial condition on a previous attractor

having period 24 and a set consisting of four arcs which are permuted under the action of T. The phenomenon of several different attractors being present for the same parameter value is known as hysteresis.

In Figs. 1a, b we see the graph of characteristic exponent as a function of a, $a \varepsilon$ [1.07, 1.08]. The variable parameter was incremented by a fixed amount (0.0001) until the right end-point of the interval was reached. The graphs are the result of computing the characteristic exponent for two different initial conditions, the origin and the last point from the previous experiment (when $a = 1.07$, the initial conditions in both computations were equal).

A comparative examination of the phase portraits associated with the above graphs reveal that for $a = 1.0721$ we have two attractors present, a stable orbit having period 12 and four stable arcs.

For $a = 1.0724$ both phase plots are again in agreement and the agreement seems to persist until around $a = 1.0768$. Recall, however, that Feit found two distinct attractors when $a = 1.0752$. The fact that we did not find the stable cycle having period 24 can be understood by realizing that neither the origin nor the points on the four stable arcs are in the basin of attraction of the cycle of length 24.

For $a = 1.0768$ the graphs reveal that there are two distinct positive values for the characteristic exponent depending on initial conditions and therefore two distinct strange attractors which persist at least until $a = 1.08$. An examination of the (x, y)-plane shows that in the interval $[1.0768, 1.08]$ we have disjoint permuted arcs of period four and six, and we also find stable cycles having period 20 and 18.

In Figs. 2a, b we have reproduced the phase plots of the two different attractors associated with the differing values of the characteristic exponent, $a = 1.08$.

We offer a possible explanation of what we are observing in the subinterval mentioned above. Recall that as early as $a = 1.0721$ our phase space contains two different attractors, stable period 12 and four stable arcs. As we increased the parameter the cycle of period 12 undergoes repeated bifurcations (period 24 when $a = 1.0752$) until by $a = 1.0768$ all of the periodic points associated with cycles of type $3 \cdot 2^n$ are unstable, but it seems likely that these points still have one-dimensional stable manifolds. This stable "block" of points then gains and transfers stability just as a periodic orbit might. However, it remains unclear why such blocks should act in an apparently cooperative manner. We conclude this subsection by remarking that the two different attractors having positive exponents persist until $a = 1.0806$ at which point the stable block having period six becomes unstable and does not reappear in that form.

b) Frequency Spectrum

In computing the frequency spectrum for Hénon's transformation we have used the following procedure: Iterate T 500,000 times while simultaneously computing lag covariances, Γ_k, for separations as large as 1000, then perform the cosine transform. By taking k as large as 1000 we are able to resolve frequencies as small as 1/500.

In distinct contrast to the characteristic exponent for parameter values in the interval described in the last subsection, the frequency spectrum remains flat and uninteresting with peaks corresponding to periods two, four or six until around $a = 1.23$. For $a = 1.23$ there seems to be the first occurrence of an odd period, period seven. Below this parameter value the behavior of the dynamical system is confined to something having period 2^n or 2^n with "noise", and a small range of parameter values where there is a period $3 \cdot 2^n$ phenomenon.

Feit has found that there are entire subintervals where $a < 1.23$ on which the characteristic exponent is positive. In particular, for $a = 1.15$ the characteristic exponent associated with the attractor is greater than zero, the attractor for this value consists of two disjoint pieces of arc. A closer examination of this limit set

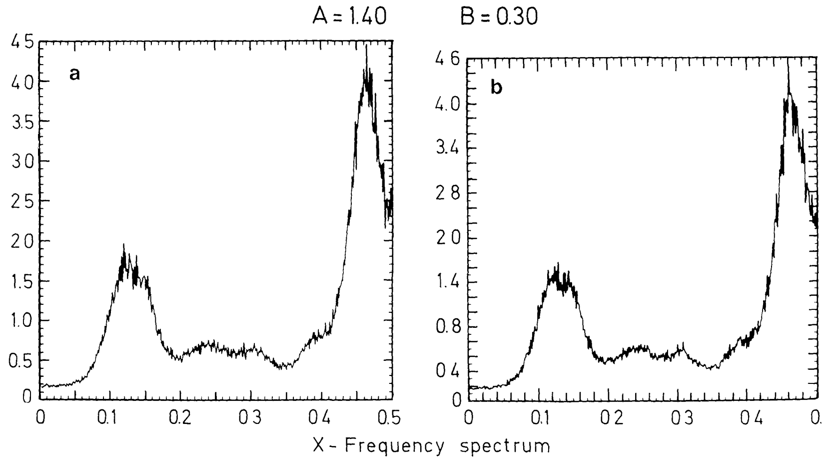

Fig. 3. **a** Frequency spectrum as computed on the CRAY-1 computer. **b** Frequency spectrum as computed on the CDC 7600 computer

reveals that the arcs are images of each other and the overall behaviour is a period two with noise.

A possible explanation of why one does not observe a continuous band of frequencies for T when $a = 1.15$ has been provided by Bowen in [1]. In that article Bowen is interested in modeling the Couette flow data of Gollub and Swinney [3]. In this context he points out that even though a dynamical system may undergo a qualitative change, a quantitative observation may be insensitive to that change. Our numerical experiments suggest that Bowen's comments are valid for Hénon's transformation for a less than 1.23 and the observable is the frequency spectrum.

The result of our frequency spectrum calculations are summarized in Table 1. There we present only the behaviour of the x-covariances and spectra, the corresponding y behavior are similar

Table 1[a]

a	Covariance	Power spectrum
1.30 period 7	period 7	peaks corresponding to periods 7, 14 and 21
1.31 attractor	decay to zero	periods 7, 14, 21
1.32 attractor	decay to zero	periods 7, 14 and broad band
1.33 attractor	decay to zero	periods 7, 14 and broad band
1.34 attractor	decay to zero	periods 8, 14 and broad band
1.35 attractor	decay to zero	periods 7, 14 and broad band
1.36 attractor	decay to zero	periods 7, 14 and broad band
1.37 attractor	decay to zero	periods 6 and broad band
1.38 attractor	decay to zero	broad band
1.39 attractor	decay to zero	broad band
1.40 attractor	decay to zero	broad band

[a] In each experiment summarized here, T was iterated 500,000 times and we resolved frequencies as small as 1/500, $b = 0.3$ throughout

It is immediately clear from the table that the time correlations tend to zero in all except one of the cases reported. Further, a cursory study of the correlation function suggests that the rate of decay is exponential; hence, we have evidence for an exponential rate of mixing for certain parameter values. Because of the decay of the time correlations it is not unexpected that for those values of a we should get a broad frequency band in the spectrum.

To complete this subsection we present Figs. 3a, b. In these figures we see the x-frequency spectra for the parameter values studied by Hénon. These graphs were the results of spectral calculation from the two different machines mentioned in the introduction of this article. Even though it is certain that rounding errors are significant as far as predictability is concerned, these figures provide evidence that the Hénon attractor has some of the statistical properties of the Axiom A systems of Smale [7]. Specifically, there seems to be a lack of sensitivity to the effects of small random perturbation (rounding error) on the statistics derived from iterating T a high number of times [6].

c) Unstable Manifold

In this subsection we present strong evidence that there is a Cantor set in the trapping region of Hénon.

In order to show the existence of a Cantor set in M, we make use of a theorem of Smale. First we recall a definition from [7]. Let $W^s(p)$, $W^u(p)$ denote the stable and unstable manifolds of a periodic point p. By a homoclinic point of T, a diffeomorphism of M, we mean a point $x \in W^s(p) \cap W^u(p)$. If $W^s(p)$ and $W^u(p)$ are transversal at x, then x is called a transverse homoclinic point. Smale proved the following

Theorem. *Suppose x is a transverse homoclinic point of a diffeomorphism T of M. Then there is a Cantor set $\Lambda \subset M$, $x \in \Lambda$, and m a positive integer such that $T^m(\Lambda) \subset \Lambda$ and T^m restricted to Λ is topologically a shift automorphism.*

It follows from the above theorem that there is a dense set of periodic orbits contained in Λ.

The above theorem suggests that we look for transverse homoclinic points in the trapping region M. Since it is straightforward to find the fixed points of T which lie in M, the associated stable and unstable direction, and their eigenvalues. we performed the following experiment:

1) Generate a segment of the stable manifold by constructing a line having the appropriate slope, containing 250 points, centered at the fixed point and having total length 0.02 units.

2) Follow the same procedure as in 1) for the unstable manifold; the only difference being in this case we generate a line containing 1000 points. Now using T^{-1} iterate the segment of stable manifold three iteration while saving the 250 initial points and all intermediate iterates. Then using the mapping T, iterate the section of unstable manifold forward to determine if there is a nontrivial transversal intersection of the unstable manifold with the segment of stable manifold. Because of the computational errors inherent in iterating T on a machine, we hope to see a transversal intersection well before iteration sixty.

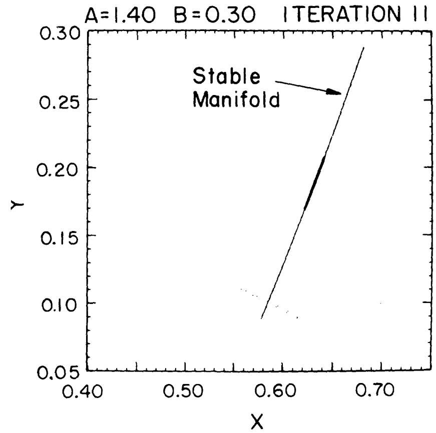

Fig. 4. Iteration 11 of the unstable manifold. Several intersections of stable and unstable manifold can be seen in this figure.

For the specific values investigated in [4], we found that by the 10th iterate of the initial points on the unstable manifold there was an intersection with the segment of stable manifold. By iteration 11 (Fig. 4) a section of the unstable manifold crosses the stable manifold within 0.01 units of the fixed point; we can also see several other such intersections in this picture.

Though Fig. 4 is extremely strong numerical evidence that the stable and unstable manifolds of the fixed point do have a nontrivial transversal intersection, what is required is an estimate which will show that by iterate eleven the computational error is not too great. In the appendix to this paper, we give the required estimates. Those estimates indicate that, for the case studied by Hénon, the absolute error is no larger than 10^{-5} for single precision arithmetic and 10^{-15} for double precision.

In Fig. 5 we have iterated a section of the unstable manifold 12 times. In this figure it is possible to see those parts of the unstable manifold which bend back on itself.

If we iterate the points on the unstable manifold four additional times, we find a phase portrait which is virtually identical to Fig. 2 of [4]. This proves, if one does the computations in double precision, that the unstable manifold is contained in a cylinder of small radius and provides additional support to the speculation that the Hénon attractor is the closure of its unstable manifold.

We note that the value of the parameter a ($b=0.3$) for which the first nontrivial transverse intersection of stable and unstable manifolds of the fixed point occur is a_c where $1.15 < a_c \leqq 1.16$. We conclude this subsection by remarking that as a is increased beyond the value studied by Hénon, the unstable manifold approaches the set, Q, of divergent points of Feit. For the critical value $a = 1.4272$ the unstable manifold of the fixed point in the first quadrant intersects Q. Because of the presumed mixing of the system, all points on the unstable manifold are eventually mapped into Q, and once in that set they tend to infinity. For those parameter values for which the unstable manifold intersects Q we expect to find no attractor present.

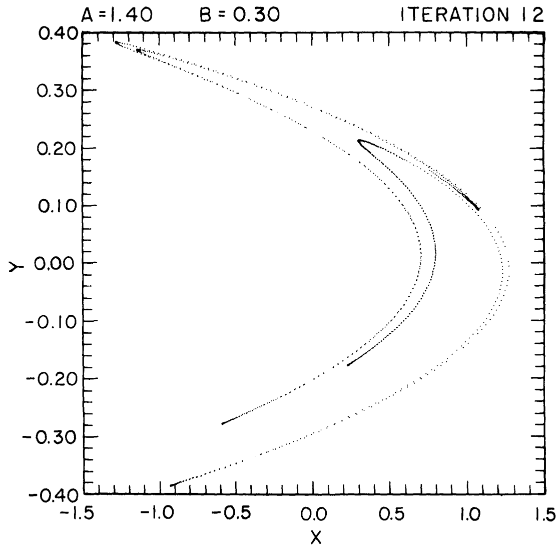

Fig. 5. Iteration 12 of the full unstable manifold. The unstable manifold bends back on itself

4. Conclusion

In this article we have presented the results of several of our numerical experiments on Hénon's transformation. In our study we have made use of the characteristic exponent, frequency spectrum, and a theorem of Smale. By using these tools we have found parameter values which give rise to two different attractors depending on initial conditions, both of which have positive characteristic exponents. Our analysis of the frequency spectrum suggests that for the parameter values where two attractors exist, the spectra calculations are insensitive to the qualitative difference between a stable period four and a stable block having period four. For higher parameter values we found continuous spectra and time correlations which apparently tend to zero at an exponential rate. Hence for the parameter values studied in [4] the motion on the attractor is most likely mixing. We have found that there is a transversal homoclinic point in the trapping region of Hénon. This provides a partial explanation of the graphics in [4]. However, though it is certainly the case that there is a Cantor set in the region, it is not the case that such Cantor sets are attractors. The calculations reported here provide evidence that the statistical quantities we have measured are stable under the systematic perturbations caused by rounding error.

Finally there remains at least one major unanswered question: Did Hénon find a very long periodic orbit or a strange attractor? The numerical experiments of Feit and this author provide support for the existence of a strange attractor. It is shown in the appendix to this paper that the absolute error committed by iteration 60 to T is order one, this suggests that it will be difficult, if not impossible, to prove using a machine that what Hénon found is nothing more than a very long periodic orbit.

Acknowledgment. We happily acknowledge the helpful comments and suggestions of D. Ruelle. Thanks to S. D. Feit for providing a preprint of her work and R. Easton for several stimulating discussions concerning the nature of homoclinic phenomena. Special thanks to O. E. Lanford for helpful comments and insightful remaks which we have incorporated into the Appendix. This work was supported in part by NSF Grant 77 10093ATM.

Appendix

In this appendix we do the estimates necessary to prove that by the eleventh iteration of Hénon's transformation the computational error is not significant. However, the argument which we present falls short of being a proof due to certain technicalities which we shall describe at the end of this section. The main sources of computational error produced when iterating a transformation are due to the method by which a machine internally represents each number and how it does its arithmetic. We shall call these two types of error "rounding errors". Therefore we want to prove that by the eleventh iteration of Hénon's transformation the rounding error is small. We shall do the estimates only for T since those for T^{-1} are similar. In what follows we shall denote the approximate floating point operations of addition, subtraction, and multiplication as performed in the machine by $\oplus$, $\ominus$, and $\otimes$.

We find it convenient to write T in the following form:

$$x_{i+1} = 1 + y_i - ax_i^2 ,$$
$$y_{i+1} = bx_i .$$

Further we shall also assume that $0 \leqq a \leqq 1.5$, $0 < b < 1$ and that $\max |x_i| \leqq 1.5$. This last condition is certainly true for the case studied in [4].

In the computer the above formula is represented as

$$\tilde{x}_{i+1} = 1 \oplus \tilde{y}_i \ominus \tilde{a} \otimes \tilde{x}_i \otimes \tilde{x}_i ,$$
$$\tilde{y}_{i+1} = \tilde{b} \otimes \tilde{x}_i .$$

Here we have used a "$\sim$" to denote the machine representation of the associated exact numbers.

We seek an estimate of the magnitude of

$$|x_{i+1} - \tilde{x}_{i+1}| \quad \text{and} \quad |y_{i+1} - \tilde{y}_{i+1}| .$$

It is enough to establish an estimate of the magnitude of

$$|x_{i+1} - \tilde{x}_{i+1}| \quad \text{since} \quad |y_{i+1} - \tilde{y}_{i+1}| = |bx_i - \tilde{b} \otimes \tilde{x}_i| .$$

Now if we let u denote the absolute error in representing a number in the machine then when the machine represents $\tilde{b} \otimes \tilde{x}_i$ it makes an absolute error no larger than $|\tilde{b} \otimes \tilde{x}_i| u$.

Therefore,

$$\begin{aligned}|y_{i+1}-\tilde{y}_{i+1}| &\leqq |bx_i-\tilde{b}\otimes\tilde{x}_i|+|\tilde{b}\otimes\tilde{x}_i|\,u\\ &\leqq |bx_i-b\tilde{x}_i|+|b\tilde{x}_i-\tilde{b}\otimes\tilde{x}_i|+|\tilde{b}\otimes\tilde{x}_i|u\\ &\leqq b|x_i-\tilde{x}_i|+|\tilde{x}_i|\,|b-\tilde{b}|+|\tilde{b}\otimes\tilde{x}_i|u\\ &\leqq b|x_i-\tilde{x}_i|+3u\end{aligned}$$

This last inequality is a consequence of our assumptions.

Because of the last inequality we have

$$|x_{i+1}-\tilde{x}_{i+1}|\leqq b|x_{i-1}-\tilde{x}_{i-1}|+|ax_i^2-\tilde{a}\otimes\tilde{x}_i\otimes\tilde{x}_i|+3u\ .$$

The error made in evaluating the number $\tilde{a}\otimes\tilde{x}_i\otimes\tilde{x}_i$ is no greater than

$$|\,\tilde{a}\otimes\tilde{x}_i\otimes_i\tilde{x}_i|\cdot u\ .$$

Therefore

$$\begin{aligned}|x_{i+1}-\tilde{x}_{i+1}| &\leqq b|x_{i-1}-\tilde{x}_{i-1}|+|ax_i^2-\tilde{a}\otimes\tilde{x}_i\otimes\tilde{x}_i|+|\tilde{a}\otimes\tilde{x}_i\otimes\tilde{x}_i|\cdot u+3u\\ &\leqq b|x_{i-1}-\tilde{x}_{i-1}|+|ax_i^2-\tilde{a}\otimes\tilde{x}_i\otimes\tilde{x}_i|+7u\ .\end{aligned}$$

Hence

$$\begin{aligned}|x_{i+1}-\tilde{x}_{i+1}| &\leqq b|x_{i-1}-\tilde{x}_{i-1}|+|ax_i^2-\tilde{a}\otimes\tilde{x}_i\otimes\tilde{x}_i|+7u\\ &\leqq b|x_{i-1}-\tilde{x}_{i-1}|+a|x_i^2-\tilde{x}_i\otimes\tilde{x}_i|+7u\\ &\leqq b|x_{i-1}-\tilde{x}_{i-1}|+a|x_i^2-\tilde{x}_i\otimes\tilde{x}_i|+7u\\ &\leqq b|x_{i-1}-\tilde{x}_{i-1}|+a|x_i-\tilde{x}_i|\,|x_i+\tilde{x}_i|+10u\\ &\leqq b|x_{i-1}-\tilde{x}_{i-1}|+3a|x_i-\tilde{x}_i|+10u\ .\end{aligned}$$

If we define $K=3a$, $\eta=10u$, $a_0=\max\,[|x_1-\tilde{x}_1|, |x_0-\tilde{x}_0|]$ and $a_{i+1}=(K+b)a_i+\eta$ then it follows that $|x_{i+1}-\tilde{x}_{i+1}|\leqq a_{i+1}$. If we now make use of the recursive definition of a_{i+1} in terms of a_i we find that

$$|x_{i+1}-\tilde{x}_{i+1}|\leqq(K+b)^i a_0+\eta\left[\frac{(K+b)^i-1}{(K+b)-1}\right]\leqq(K+b)^i\left[a_0+\frac{\eta}{(K+b)-1}\right].$$

We now compute an upper bound on a_0 and find that $a_0<(4a+5)u$. Therefore,

$$|x_{i+1}-\tilde{x}_{i+1}|\leqq(K+b)^i\left[(4a+5)u+\frac{\eta}{(K+b)-1}\right].$$

Now $u=10^{-14}$ on the machine which we did our computations, therefore

$$|x_{i+1}-\tilde{x}_{i+1}|\leqq(5.5)^i\left[11u+\frac{10u}{(3a+b)-1}\right].$$

In particular if $i+1=11$ and $a=1.4$, $b=0.3$

$$|x_{11}-\tilde{x}_{11}|<10^{-5}$$

for single precision arithmetic, and

$$|x_{11}-\tilde{x}_{11}|<10^{-15}$$

for double precision arithmetic.

Despite the error estimates given above we still have not proved that there is a transverse homoclinic point in the trapping region of Hénon. The reasons that our arguments fall short include the following:

(a) In locating the fixed point in the trapping region we have made use of formula (3) of Sect. 2. This formula involves doing floating point operations and computing a square root. The floating point operations are, as we have seen, not exact and computing a square root introduces additional uncertainties into the computation.

(b) In Fig. 4 we have a sequence of dots which should lie very near the stable and unstable manifolds. If these dots are connected in any reasonable manner to form the smooth curves, they will cross. But it is not obvious that by iteration 11 the stable and unstable manifolds are close enough to being straight so that they do indeed cross transversally.

(c) Finally, the accuracy of floating point operations on the CDC 7600 and CRAY–1 depends among other things on what the complier decides to do about normalizing intermediate results when evaluating complicated expressions.

Because our analysis has not taken (a)–(c) into account we do not have proof that there is a transverse homoclinic point in the trapping region of Hénon. Indeed, because of (c) it seems likely that the best one can do is prove that given a specific complier on a specific machine that estimates of the above sort are valid.

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Communicated by J. Lebowitz

Received July 25, 1978; in revised form February 25, 1979