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The Charged Sectors of Quantum Electrodynamics in a Framework of Local Observables

Jürg Fröhlich
Institut des Hautes Etudes Scientifiques, F-91440 Bures-sur-Yvette, France

Abstract. The construction of charged sectors in Quantum Electrodynamics (QED) is analyzed within a framework of algebras of local observables. It is argued that charged sectors arise by composing a vacuum state with charged * morphisms of an algebra of (neutral) quasi-local observables. Charged * morphisms, in turn, are obtained as weak limits of charge transfer cocycles. These are non-local elements of the algebra of all quasi-local observables obeying "topological" commutation relations with the local charge operators. It is shown that in this framework, charged sectors are invariant under the time evolution and satisfy the relativistic spectrum condition. The total charge operator is well defined and time-independent (conserved) on all charged sectors. Under an additional hypothesis the spectrum of the total charge operator is shown to be a discrete subgroup of the real line. A generalized Haag-Ruelle scattering theory for charged infra-particles is suggested, and some comments on non-abelian gauge theories are described.

0. Introduction

This paper is a continuation of the analysis presented in [1], hereafter referred to as I. In that paper we have investigated charged sectors in gauge theories with unconfined, abelian charges, in particular QED, from the points of view of a local, covariant formulation on an indefinite metric space and of collision theory, using as one basic input Buchholz' results [2]. Moreover, the construction of charged states in QED was analyzed heuristically, extrapolating procedures applicable in lattice gauge theories to the continuum theory. In this paper that analysis is replaced by a mathematically rigorous one, based on a few general, physical principles.

The main results of Paper I are as follows:

Asymptotic charged fields (if they exist; see Sect. 7 of this paper) are non-local relative to the asymptotic, electromagnetic field and are not covariant under Lorentz boosts.

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A rather complete characterization of "scattering representations" of the algebra generated by bounded functions of the asymptotic, electromagnetic field, in particular of its representations on charged sectors, was achieved.

Asymptotic, charged one-(infra)particle states were constructed.

Under reasonable hypotheses it was proven that the charged sectors of QED are *not* invariant under Lorentz boosts (breaking of the boost symmetry on charged sectors).

For detailed statements of these and other results we refer the reader to I.

This paper represents a preliminary attempt at extending the Doplicher-Haag-Roberts (DHR) theory [3] of superselection sectors in standard quantum field theories to QED – and other gauge theories with an unconfined, abelian charge – taking into account the conclusions of Paper I and trying to substantiate some of the hypotheses made there. Our approach is inspired by the general framework of Haag and Kastler [4] and DHR [3]. Some of the technical details in this paper are taken from [5] (where the main emphasis is placed on super-selection sectors labelled by topological charges, i.e. quantum solitons). Some knowledge of [3, 5, 6] might be helpful to understand the main concepts of the present paper.

The main physical hypotheses upon which the following analysis is based are:

A. Gauss' Law

$$\nabla \cdot \mathbf{E}(x) = \varrho(x),$$

where ϱ is the 0-component (charge density) of the local, locally conserved, electric current operator, and $\mathbf{E}(x) = (E_1(x), E_2(x), E_3(x))$ are the components of the quantized, electric field.

B. Covariance

Charged sectors are space-time translation invariant, i.e. a selfadjoint energy-momentum operator exists on charged sectors.

C. Additivity of the Electric Charge

Charged sectors can be composed, i.e. the electric charge is an additive quantum number.

D. Space-like Distant, Localized Charges are not Felt

Charges can be localized (in a sense explained in Sect. 5), and charged states arise from neutral states (via taking w^* limits) by removing a localized charge to space-like infinity.

Among some of the consequences of these hypotheses are:

The physical mass gap of QED is 0 [7].

Any representation of the algebra of all quasi-local observables determined by a charged state is disjoint from the vacuum representation, even when restricted to space-like distant regions; a consequence of Gauss' law. Technically, this implies that charged states cannot be obtained from the vacuum by strictly local * morphisms of the observable algebra. The DHR approach [3] must therefore be modified for QED and any gauge theory with unconfined charges (Sect. 2).

Charged fields (or field bundles) are *non-local* relative to the interpolating, electromagnetic field (Sect. 2).

The space-time translation covariance of charged states implies that "charged field bundles" (* morphisms of the observable algebra) uniquely determine unitary operators on the vacuum sector, space-time translation cocycles, which describe the transfer of a localized charge from, say, the origin to some point $a \in M^4$ (Sect. 3.1). "Topological" commutation relations between those cocycles and the local charge operators are derived (Sects. 5 and 7).

These so-challed charge transfer cocycles are non local relative to the interpolating, electromagnetic field (in a very strong sense) (Sect. 3.2).

On the basis of these results and assuming PCT-invariance we then propose a tentative framework for the description of charged sectors in QED. Our framework guarantees that charged states can be constructed as w^* limits of neutral states (vector states in the vacuum sector) by removing a localized charge to space-like infinity (Sects. 4 and 5).

Moreover, in that framework charged states are space-time translation covariant, and the relativistic spectrum condition holds on all sectors. We then prove that the total charge operator exists and is conserved on all sectors of the theory (it is zero on the vacuum sector). Under an additional hypothesis it is shown that charge transfer cocycles transfer a definite electric charge, and charged * morphisms carry a definite charge. Then the spectrum of the charge operator on the total Hilbert space is a discrete subgroup of the additive group of the real line (see Sects. 5 and 6). Our main results are in Sects. 3, 5, 6.2, and 6.3. The main purposes of a general framework for QED are:

I) To develop specific concepts and explicit procedures for the construction of sectors labelled by an abelian, unconfined charge in a gauge theory, in particular QED, the vacuum sector of which is supposed to be given, e.g. in the form of a sequence of Wightman distributions of gauge-invariant fields satisfying a suitably modified form of the Wightman axioms. This is attempted in Sect. 3–6.

II) To extend Buchholz' collision theory for massless bosons [2], in QED *only* applicable on the vacuum sector, to the electromagnetic field on the charged sectors of QED.

III) To complement and complete that analysis by constructing a collision theory for charged infra-particles (see Sect. 7).

Some relevant results can also be found in Sect. 3.3 and 3.6 of Paper I.

IV) To derive the principal hypotheses in Sect. 3.4, 3.5 (or the weaker ones in Sect. 3.3) of Paper I which would determine the structure of charged scattering states (generalized coherent states!) quite explicitly, from a few basic, dynamical hypotheses which are convenient to check in models.

A minimal result of this type is to show that charged sectors determine representations of the algebra, $\mathfrak{A}^{\mathrm{as}}$, generated by bounded functions of the asymptotic, electromagnetic field which are *disjoint* from the Fock representation (see Sect. 1 of Paper I). A somewhat stronger result containing that one would be to prove that the electric charge operator, Q, is affiliated with the von Neumann algebra generated by $\mathfrak{A}^{\mathrm{as}}$ in the physical representation.

The reader will find out that none of these goals is reached completely in this paper. We hope it at least clarifies the conceptual basis and the main difficulties met in the construction of charged states and supplies some useful first steps towards a more complete, general theory of the charge super-selection rule (see also [8, 3]). Readers who think that theorems with short proofs are necessarily trivial will find this paper trivial. Some of the experts in the field may share this feeling. We hope some of the ideas developed in the following will be useful.

1. Local Observables and Covariant States

Here we recall some basic notions and concepts of the Haag-Kastler frame-work [4], the basic theorem of Bisognano and Wichmann [9] and a result of [10] concerning the existence of local algebras satisfying the Haag-Kastler axioms in a Wightman field theory. Let $\mathcal{O}$ denote a double cone (the intersection of a forward with a backward light cone) in M^4, and let $\sim\mathcal{O}$ denote its causal complement (all space-time points which are space-like relative to $\mathcal{O}$).

Given a double cone $\mathcal{O}$, let $\mathfrak{A}(\mathcal{O})$ be a C^* – or von Neumann algebra containing at least all bounded functions of the interpolating, electromagnetic field, $F_{\mu\nu}(f^{\mu\nu})$, where the $f^{\mu\nu}$ are real-valued Schwartz space functions with support in $\mathcal{O}$, and possibly other local observables which are local relative to the electromagnetic field[1] (such observables have of necessity total charge 0; see [8] and Sect. 2). Let B be some general, open region in M^4. Let $\mathfrak{B}$ denote the family of all *bounded double cones* in M^4. We define $\mathfrak{A}(B)$ to be the norm closure of

$$\bigcup_{\substack{\mathcal{O}\in\mathfrak{B}\\ \mathcal{O}\subset B}} \mathfrak{A}(\mathcal{O})\,; \tag{1.1}$$

in particular, $\mathfrak{A}\equiv\mathfrak{A}(B=M^4)$ is the *algebra of all quasi-local observables* of the theory.

As usual, *locality* is expressed by the condition that, for arbitrary $A\in\mathfrak{A}(\mathcal{O})$ and arbitrary $B\in\mathfrak{A}(\sim\mathcal{O})$,

$$[A,B]=AB-BA=0\,. \tag{1.2}$$

We also assume that the Poincaré group, $\mathscr{P}_+^\uparrow$, is represented on the algebra $\mathfrak{A}$ by a (strongly continuous) * automorphism group, $\{\tau_\xi:\xi\in\mathscr{P}_+^\uparrow\}$, such that

$$\tau_\xi(\mathfrak{A}(\mathcal{O}))=\mathfrak{A}(\mathcal{O}(\xi))\,, \tag{1.3}$$

where $\mathcal{O}(\xi)$ is the image of the region $\mathcal{O}$ under a Poincaré transformation ξ (see [4]).

Sufficient conditions – which are quite efficient in models – for the existence of a net of local algebras $\{\mathfrak{A}(\mathcal{O})\}_{\mathcal{O}\in\mathfrak{B}}$ with all the properties, (1.2) and (1.3), listed above in a Wightman field theory are given in [10].

Given a state, ϱ, on $\mathfrak{A}$, the G.N.S. construction (see e.g. [11]) provides one with a Hilbert space, $\mathscr{H}_\varrho$, a representation, π_ϱ, of $\mathfrak{A}$ on $\mathscr{H}_\varrho$, and a cyclic unit vector $\Omega_\varrho\in\mathscr{H}_\varrho$, such that

$$\begin{aligned}&\mathscr{H}_\varrho=\{\pi_\varrho(A)\Omega_\varrho:A\in\mathfrak{A}\}^-\,,\\ &\varrho(A)=(\Omega_\varrho,\pi_\varrho(A)\Omega_\varrho)\,,\quad\text{for}\quad A\in\mathfrak{A}\,.\end{aligned} \tag{1.4}$$

In an unambiguous context, A will henceforth denote both, the abstract element of $\mathfrak{A}$ and the bounded operator $\pi_\varrho(A)$ on $\mathscr{H}_\varrho$ in a given representation π_ϱ of $\mathfrak{A}$ on $\mathscr{H}_\varrho$.

1 e.g., observables of the type of the "Wilson loops"

Let G be an arbitrary, topological group represented on $\mathfrak{A}$ by a (strongly continuous) group of * automorphisms, $\{\tau_g : g \in G\}$ of $\mathfrak{A}$.

Definition 1. A state, ϱ, on $\mathfrak{A}$ is said to be *G-covariant* iff there exists a continuous, unitary representation, U_ϱ, of G on $\mathscr{H}_\varrho$ such that, for all $A \in \mathfrak{A}$, $g \in G$,

$$\pi_\varrho(\tau_g(A)) = U_\varrho(g)^* \pi_e(A) U_\varrho(g) \quad \text{on} \quad \mathscr{H}_\varrho . \tag{1.5}$$

A *vacuum state*, ω, is a state on $\mathfrak{A}$ which is *Poincaré-invariant* (hence $\mathscr{P}_+^\uparrow$-covariant), so that

$$U_\omega(\xi)\Omega = \Omega , \quad \text{for all} \quad \xi \in \mathscr{P}_+^\uparrow , \tag{1.6}$$

where $\Omega \equiv \Omega_\omega$ is the physical vacuum, and the spectrum of the generators, $(H, \mathbf{P})$, the energy-momentum operator, of the translation subgroup $\{U_\omega(a) : a \in M^4\}$, $U_\omega(a) \equiv U_\omega(\xi = (1, a))$, is contained in the forward light cone $\bar{V}_+$. $\square$

Henceforth we may always assume that the physical vacuum is non-degenerate, i.e. ω is a pure state on $\mathfrak{A}$, without loss of generality. This is because of Araki's theorem [12]. Then the von Neumann algebra, $\pi_\omega(\mathfrak{A})''$, generated by $\pi_\omega(\mathfrak{A})$ on the *vacuum sector*, $\mathscr{H}_\omega$, coincides with the algebra of all bounded operators on $\mathscr{H}_\omega$, $B(\mathscr{H}_\omega)$. In the following we assume that we are given an arbitrary, but fixed, pure vacuum state ω on $\mathfrak{A}$ (but see [13, 12, 5]).

For the expert we now recall a basic theorem, due to Bisognano and Wichmann [9] which, we believe, is at least implicitly important in the following analysis. (The reader can skip this in first reading.) This theorem says that, under certain technical assumptions (in particular PCT invariance, which are guaranteed by the conditions of [10]), one can construct from the net $\{\mathfrak{A}(\mathcal{O})\}_{\mathcal{O} \in \mathfrak{B}}$ another net, $\{\bar{\mathfrak{A}}(\mathcal{O})\}_{\mathcal{O} \in \mathfrak{B}}$, of *local von Neumann algebras* on $\mathscr{H}_\omega$ such that

$$\bar{\mathfrak{A}}(\mathcal{O}) \supseteqq \mathfrak{A}(\mathcal{O}) \quad \text{and} \quad \bar{\mathfrak{A}}(\mathcal{O}) = \pi_\omega(\bar{\mathfrak{A}}(\sim \mathcal{O}))' \tag{1.7}$$

(the famous *duality condition*; see e.g. [3]), for all $\mathcal{O} \in \mathfrak{B}$.

In the following we shall imagine working with the net $\{\bar{\mathfrak{A}}(\mathcal{O})\}_{\mathcal{O} \in \mathfrak{B}}$, but we write again $\mathfrak{A}(\mathcal{O})$, instead of $\bar{\mathfrak{A}}(\mathcal{O})$. We only consider states on $\mathfrak{A}$ whose restriction to $\mathfrak{A}(\mathcal{O})$ is *normal*, for all $\mathcal{O} \in \mathfrak{B}$.

DHR consider those states, ϱ, on $\mathfrak{A}$ as relevant for particle physics which have the property that

$$\|(\varrho - \omega)/\mathfrak{A}(\sim \mathcal{O}_n)\| \to 0 , \quad \text{as} \quad n \to \infty , \tag{1.8}$$

for each sequence $\{\mathcal{O}_n\} \subset \mathfrak{B}$ increasing to M^4. Under suitable, technical conditions this property is equivalent to

$$\varrho = \omega \circ \sigma , \qquad (\varrho(A) = \omega \circ \sigma(A) \equiv \omega(\sigma(A)), A \in \mathfrak{A}) , \tag{1.9}$$

where σ is a * morphism of $\mathfrak{A}$[2] with the property that, for some bounded double cone $\mathcal{O}$, called the *support* of σ,

$$\sigma(A) = A , \quad \text{for all} \quad A \in \mathfrak{A}(\sim \mathcal{O}) . \tag{1.10}$$

Such morphisms are called *local* (see [3]).

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2 I.e. $\sigma(AB) = \sigma(A)\sigma(B)$, $\sigma(A^*) = \sigma(A)^*$, σ is linear and $\|\sigma(A)\| \leqq \|A\|$, for all A, B in $\mathfrak{A}$

Lemma 1. *Let σ be a local * morphism of $\mathfrak{A}$ and ω a vacuum state. Then*

$$\mathscr{H}_\omega(\sigma) \equiv \{\sigma(A)\Omega : A \in \mathfrak{A}\}^- = \mathscr{H}_\omega .$$

Proof. Let $\mathcal{O} \in \mathfrak{B}$ denote the support of σ. Then $\sigma(A) = A$, for $A \in \mathfrak{A}(\sim \mathcal{O})$. Thus

$$\mathscr{H}_\omega(\sigma) \supset \{A\Omega : A \in \mathfrak{A}(\sim \mathcal{O})\} . \tag{1.11}$$

By the Reeh-Schlieder property [4, 10, 14], the closure of the r.s. of (1.11) is $\mathscr{H}_\omega$ when ω is a vacuum state. □

Remark. The Reeh-Schlieder property has been derived from the Reeh-Schlieder theorem [14], under suitable conditions, in [10]. Lemma 1 is significant for the discussion presented in Sect. 3.

We now show that when ϱ is a charged state on $\mathfrak{A}$ and the charge satisfies Gauss' law (see condition A in the introduction) then properties (1.9) and (1.10) cannot be fulfilled, hence (1.8) must fail, too. This result is widely known [8, 3].

2. Consequences of Gauss' Law

We repeat here, in a more formal way, an argument showing why the DHR theory of super-selection sectors is not applicable to the charged sectors of QED.

First, we recall the definition of the electric charge operator, Q: Let $\alpha(t) \geqq 0$ be a test function on $\mathbb{R}$ of compact support, with $\int \alpha(t)dt = 1$. Let Σ be a simply connected, bounded region in $\mathbb{R}^3$ with smooth boundary $\partial\Sigma$. Let $f_\Sigma(x)$ be a test function on $\mathbb{R}^3$ with the properties

i) $0 \leqq f_\Sigma(\mathbf{x}) \leqq 1$;

ii) $f_\Sigma(\mathbf{x}) = 1$, for all $\mathbf{x}$ with the property that $(\mathbf{x}, t)$ is in the causal shadow of Σ (i.e. not space-like to Σ), for all $t \in \operatorname{supp} \alpha$;

iii) $\operatorname{supp} f_\Sigma$ compact.

We then define

$$Q_\Sigma = \int \varrho(\mathbf{x}, t) f_\Sigma(\mathbf{x}) \alpha(t) d^3x dt , \tag{2.1}$$

where $\varrho(\mathbf{x}, t)$ is the charge density operator. Gauss' law is expressed in the form

$$Q_\Sigma = \nabla \cdot \mathbf{E}(f_\Sigma \otimes \alpha) = -\mathbf{E}(\nabla f_\Sigma \otimes \alpha) , \tag{2.2}$$

where $\mathbf{E}$ is the electric field operator.

We assume that

$$\mathbf{E}(\nabla f_\Sigma \otimes \alpha) \quad \text{is affiliated with} \quad \mathfrak{A}(\sim \Sigma \cap \mathcal{O}) , \tag{2.3}$$

for some sufficiently large $\mathcal{O} \in \mathfrak{B}$, in accordance with the fact that, for a sufficiently large $\mathcal{O} \in \mathfrak{B}$, $\operatorname{supp}(\nabla f_\Sigma \otimes \alpha) \subset \sim \Sigma \cap \mathcal{O}$. Then the operator $\mathbf{E}(\nabla f_\Sigma \otimes \alpha)$ is a densely defined, selfadjoint operator in any locally normal representation of $\mathfrak{A}$. (Property (2.3) is true under the conditions of [10].)

If Σ is the ball $\{\mathbf{x} : |\mathbf{x}| \leqq R\}$ we denote Q_Σ by Q_R, and f_Σ by f_R.

From locality, (1.2), and (2.3) we get

Lemma 2.

$$n\text{-}\lim_{R \to \infty} e^{isQ_R} A e^{-isQ_R} = A , \quad \text{for all} \quad A \in \mathfrak{A}, \quad s \in \mathbb{R} .$$

We define the electric charge, Q, as the generator of

$$w\text{-}\lim_{R\to\infty} e^{isQ_R}, \quad s\in\mathbb{R}, \tag{2.4}$$

in any representation π of $\mathfrak{A}$ for which the limits (2.4) exist and are continuous in s. Then Q is affiliated with $\pi(\mathfrak{A})''$. This and Lemma 2 show that the *electric charge is a super-selection rule*. The analysis of this super-selection rule is the main purpose of this paper (see also [8]).

It is common to assume that

$$Q=0, \quad \text{on} \quad \mathscr{H}_\omega, \tag{2.5}$$

but see Lemma 14 (Sect. 5).

Proposition 3. *Let σ be a local * morphism of $\mathfrak{A}$. Then the sector $\mathscr{H}_\sigma\equiv\mathscr{H}_{\omega\circ\sigma}$ has the same electric charge as $\mathscr{H}_\omega$, i.e. if*

$$w\text{-}\lim_{R\to\infty} Q_R\Omega=Q\Omega=0 \tag{2.6}$$

then

$$Q\Psi=0, \quad \textit{for all} \quad \Psi\in\mathscr{H}_\omega \quad \textit{and all} \quad \Psi\in\mathscr{H}_\sigma.$$

Proof. Let $\Omega_\sigma\equiv\Omega_{\omega\circ\sigma}$. Then, for arbitrary A and B in $\bigcup_{\mathscr{O}\in\mathfrak{B}}\mathfrak{A}(\mathscr{O})$,

$$(A\Omega_\sigma, QB\Omega_\sigma)\equiv\lim_{R\to\infty}(A\Omega_\sigma, Q_RB\Omega_\sigma),$$

and by (3.8),

$$-(A\Omega_\sigma, Q_RB\Omega_\sigma)=(A\Omega_\sigma, \mathbf{E}(Vf_R\otimes\alpha)B\Omega_\sigma).$$

For R sufficiently large, $\mathbf{E}(Vf_R\otimes\alpha)$ is affiliated with $\mathfrak{A}(\sim\operatorname{supp}\sigma)$, moreover $\mathbf{E}(Vf_R\otimes\alpha)$ and B commute, since $B\in\bigcup_{\mathscr{O}\in\mathfrak{B}}\mathfrak{A}(\mathscr{O})$. Thus, using (1.9), we conclude that, for sufficiently large R,

$$\begin{aligned}(A\Omega_\sigma, &\mathbf{E}(Vf_R\otimes\alpha)B\Omega_\sigma)\\ &=(\sigma(A)\Omega\,;\sigma(B)\sigma(\mathbf{E}(Vf_R\otimes\sigma))\Omega)\\ &=(\sigma(B^*A)\Omega, \mathbf{E}(Vf_R\otimes\alpha)\Omega)\,.\end{aligned} \tag{2.7}$$

As $R\to\infty$, the r.h.s. of (2.7) tends to

$$(\sigma(B^*A)\Omega, Q\Omega)=0 \quad [\text{see (2.2) and (2.6)}]\,.$$

Therefore

$$(A\Omega_\sigma, QB\Omega_\sigma)=\lim_{R\to\infty}(A\Omega_\sigma, Q_RB\Omega_\sigma)=0\,. \quad \square$$

Thus, in QED charged states do *not* arise by composing the vacuum state ω with local morphisms. Should we give up the idea that charged states can be constructed by composing the vacuum state with * morphisms of $\mathfrak{A}$? Not only would such a radical proposal contradict the requirement that charged sectors can be composed (charged fields can be multiplied) and the electric charge is additive, but it would also make a general analysis too vague.

3. Translation Covariant Sectors and * Morphisms of $\mathfrak{A}$

We propose to regard those states ϱ on $\mathfrak{A}$ as relevant for QED which have the properties

P1) ϱ is space-time translation covariant;

P2) $\varrho = \omega \circ \sigma$, where ω is the vacuum state, and σ is a * morphism of $\mathfrak{A}$.

Remark. The results of Sects. 3.4 and 3.5 in Paper I have cautioned us not to assume that ϱ is Lorentz-covariant in case ϱ is a charged state. We may, however, assume that ϱ is also rotation covariant, but this is quite unimportant in the following. Translation covariance is crucial, because it guarantees the existence of an energy-momentum operator.

3.1. Transportable and Covariant Morphisms and Cocycles

Definition 2. Let G be a topological group, and $\{\tau_g : g \in G\}$ a representation of G by a strongly continuous * automorphism group of $\mathfrak{A}$. Let ω be a G-covariant state on $\mathfrak{A}$. A mapping $\Gamma : g \in G \to \Gamma(g)$, where $\Gamma(g)$ is a unitary operator on $\mathscr{H}_\omega$, is called a *G-cocycle* on $\mathscr{H}_\omega$ iff $\Gamma(g)$ is (weakly or strongly) continuous in g on $\mathscr{H}_\omega$, and

$$\Gamma(g_1 \cdot g_2) = \Gamma(g_1) U_\omega(g_1) \Gamma(g_2) U_\omega(g_1)^* . \tag{3.1}$$

A * morphism σ on $\mathfrak{A}$ is called G-transportable on $\mathscr{H}_\omega$ iff

$$\pi_\omega(\tau_{g^{-1}} \circ \sigma \circ \tau_g(A)) = \Gamma(g)^* \pi_\omega(\sigma(A)) \Gamma(g) , \tag{3.2}$$

where $\Gamma(g)$ is a G-cocycle on $\mathscr{H}_\omega$.

Remark. If $\pi_\omega \circ \sigma$ is an irreducible representation of $\mathfrak{A}$ then Eq. (3.2) alone implies that Γ is a G cocycle, *unique* up to a phase. This is not so if $\pi_\omega \circ \sigma$ is not irreducible. For simplicity, we require in general that Γ in (3.2) be a G-cocycle.

A * morphism σ on $\mathfrak{A}$ is called *G-covariant* iff $\omega \circ \sigma$ is a G-covariant state.

We define

$$\mathscr{H}_\omega(\sigma) \equiv \{\sigma(A)\Omega : A \in \mathfrak{A}\}^- . \tag{3.3}$$

Clearly $\mathscr{H}_\omega(\sigma) \subseteqq \mathscr{H}_\omega$. If σ is a local * morphism and ω the vacuum then by Lemma 1, $\mathscr{H}_\omega(\sigma) = \mathscr{H}_\omega$, but this is *not* so in general. □

Lemma 4. *If Γ is a G-cocycle on $\mathscr{H}_\omega$ then $V(g) \equiv \Gamma(g) U_\omega(g)$ is a continuous, unitary representation of G on $\mathscr{H}_\omega$.*

Proof. By the definition of G-cocycles, $V(g)$ is clearly unitary and continuous in g on $\mathscr{H}_\omega$. By (3.1)

$$\begin{aligned} V(g_1 \cdot g_2) &= \Gamma(g_1 \cdot g_2) U_\omega(g_1 \cdot g_2) \\ &= \Gamma(g_1) U_\omega(g_1) \Gamma(g_2) U_\omega(g_1)^* U_\omega(g_1 \cdot g_2) \\ &= \Gamma(g_1) U_\omega(g_1) \Gamma(g_2) U_\omega(g_2) \\ &= V(g_1) V(g_2), \quad \text{for all} \quad g_1, g_2 \quad \text{in} \quad G . \quad \square \end{aligned}$$

Theorem 5. *Let ω be a G-covariant state on $\mathfrak{A}$, and σ a G-covariant * morphism on $\mathfrak{A}$.*

Then there exists a G-cocycle $\Gamma_\sigma(g)$ *on* $\mathscr{H}_\omega$ *with the property that* $V_\sigma(g) \equiv \Gamma_\sigma(g) U_\omega(g)$ *leaves* $\mathscr{H}_\omega(\sigma)$ *invariant, and*

$$\pi_{\omega,\sigma}(\sigma \circ \tau_g(A)) = V_\sigma(g)^* \pi_{\omega,\sigma}(\sigma(A)) V_\sigma(g),$$

where $\pi_{\omega,\sigma}$ *is the representation of* $\sigma(\mathfrak{A})$ *on* $\mathscr{H}_\omega(\sigma)$. *Conversely, suppose that* ω *is G-covariant and* σ *G-transportable on* $\mathscr{H}_\omega$, *and assume that* $\mathscr{H}_\omega(\sigma)$ *is invariant under* $V_\sigma(g) \equiv \Gamma_\sigma(g) U_\omega(g)$, *for all* $g \in G$. *Then* σ *is G-covariant.*

Proof. We define an isometric isomorphism $T : \mathscr{H}_\omega(\sigma) \to \mathscr{H}_\sigma$, by

$$T\sigma(A)\Omega = A\Omega_\sigma, \quad \text{for} \quad A \in \mathfrak{A}. \tag{3.4}$$

One verifies immediately that T is isometric. Moreover, since Ω is cyclic in $\mathscr{H}_\omega(\sigma)$ for $\sigma(\mathfrak{A})$, T extends by continuity to all of $\mathscr{H}_\omega(\sigma)$. By (3.4) and the cyclicity of Ω_σ for $\mathfrak{A}$ in $\mathscr{H}_\sigma$, the range of T is $\mathscr{H}_\sigma$. Thus $T^{-1} = T^*$ exists and is an isometry from $\mathscr{H}_\sigma$ to $\mathscr{H}_\omega(\sigma)$.

If $\omega \circ \sigma$ is G-covariant there exists a continuous, unitary representation U_σ of G on $\mathscr{H}_\sigma$ such that

$$\pi_\sigma(\tau_g(A)) = U_\sigma(g)^* \pi_\sigma(A) U_\sigma(g). \tag{3.5}$$

Using (3.4) we conclude

$$U_\sigma(g) A\Omega_\sigma = U_\sigma(g) T\sigma(A)\Omega \in \mathscr{H}_\sigma, \quad \text{for all} \quad g \in G,$$

so that

$$T^* U_\sigma(g) A\Omega_\sigma = T^* U_\sigma(g) T\sigma(A)\Omega \in \mathscr{H}_\omega(\sigma),$$

for all $g \in G$. Since T and T^* are isometric isomorphisms, and U_σ is a continuous, unitary representation,

$$V_\sigma(g) = T^* U_\sigma(g) T \tag{3.6}$$

can be extended by continuity to all of $\mathscr{H}_\omega(\sigma)$ and is a continuous, unitary representation of G on $\mathscr{H}_\omega(\sigma)$. We can extend $V_\sigma(g)$ to all of $\mathscr{H}_\omega$ by setting e.g.

$$V_\sigma(g) = 1, \quad \text{on} \quad \mathscr{H}_\omega \ominus \mathscr{H}_\omega(\sigma).$$

We then define

$$\Gamma_\sigma(g) = V_\sigma(g) U_\omega(g)^*. \tag{3.7}$$

Since V_σ and U_ω are continuous, unitary representations of G on $\mathscr{H}_\omega$, $\Gamma_\sigma(g)$ is clearly a G-cocycle on $\mathscr{H}_\omega$ [in particular, (3.1) follows directly from (3.7)]. That

$$\pi_{\omega,\sigma}(\sigma \circ \tau_g(A)) = V_\sigma(g)^* \pi_{\omega,\sigma}(\sigma(A)) V_\sigma(g)$$

follows easily from

$$U_\sigma(g) T = T V_\sigma(g), \quad \text{on} \quad \mathscr{H}_\omega(\sigma). \tag{3.8}$$

This proves the first part of Theorem 5. To prove the second part, notice that, given V_σ,

$$U_\sigma(g) = T V_\sigma(g) T^*$$

defines a unitary group on $\mathscr{H}_\sigma$, since, by hypothesis, $V_\sigma(g)$ leaves $\mathscr{H}_\omega(\sigma)$ invariant, for all $g\in G$. Continuity in g of U_σ follows from the assumed continuity of Γ_σ. Furthermore, for $\Psi=B\Omega_\sigma$ and $\Phi=C\Omega_\sigma$,

$$\begin{aligned}(\Psi,\tau_g(A)\Phi)&=(\sigma(B)\Omega,\sigma(\tau_g(A))\sigma(C)\Omega)\\&=(V_\sigma(g)\sigma(B)\Omega,\sigma(A)V_\sigma(g)\sigma(C)\Omega)\\&=(U_\sigma(g)B\Omega_\sigma,AU_\sigma(g)C\Omega_\sigma)\\&=(\Psi,U_\sigma(g)^*AU_\sigma(g)\Phi).\end{aligned}$$

This completes the proof of the theorem. □

Corollary 6. *1) Suppose that, for each $\Psi\in\mathscr{H}_\omega$, the state $(\Psi,\sigma(\cdot)\Psi)$ on $\mathfrak{A}$ is G-covariant. Then there exists a G-cocycle $\Gamma_\sigma(g)$ on $\mathscr{H}_\omega$ such that*

$$\mathscr{H}_\Psi(\sigma)\equiv\{\sigma(A)\Psi:A\in\mathfrak{A}\}^-$$

is invariant under $V_\sigma(g)=\Gamma_\sigma(g)U_\omega(g)$, for all $g\in G$, for all Ψ, and

$$\pi_\omega(\sigma\circ\tau_g(A))=V_\sigma(g)^*\pi_\omega(\sigma(A))V_\sigma(g),\tag{3.9}$$

and σ is G-transportable.

2) Suppose that $\mathscr{H}_\omega(\sigma)=\mathscr{H}_\omega$, i.e. Ω is cyclic for $\sigma(\mathfrak{A})$. Then σ is G-covariant if and only if it is G-transportable.

*3) Suppose ω is a vacuum state and σ a local * morphism. Then σ is G-covariant if and only if it is G-transportable.*

Proof. 1) Defining $\omega'(A)=(\Psi,A\Psi)$, $\Psi\in\mathscr{H}_\omega$, one sees that ω' and σ satisfy the hypotheses of the first part of Theorem 3.1. Now, we first choose $\Psi=\Omega$. Then we choose $\Psi=\Psi_1\in\mathscr{H}_\omega\ominus\mathscr{H}_\omega(\sigma)$, then $\Psi=\Psi_2\in\mathscr{H}_\omega\ominus\mathscr{H}_\omega(\sigma)\ominus\mathscr{H}_{\Psi_1}(\sigma)$, etc. By iterating this procedure we obtain $\Gamma_\sigma(g)$ and $V_\sigma(g)$ such that (3.2) is satisfied on $\mathscr{H}_\omega$.

2) Since $\mathscr{H}_\omega(\sigma)=\mathscr{H}_\omega$, $\mathscr{H}_\omega(\sigma)$ is automatically invariant under $V_\sigma(g)=\Gamma_\sigma(g)U_\omega(g)$, for any G-cocycle Γ_σ. Thus 2) follows from Theorem 5.

3) This follows immediately from Lemma 1 and Corollary 6, 2). □

Remark. If $\pi_\omega\circ\sigma$ is irreducible then clearly $\mathscr{H}_\omega(\sigma)=\mathscr{H}_\omega$.

3.2. Localization Properties of Translation Cocycles

If G is the (space-time) translation subgroup of $\mathscr{P}_+^\uparrow$ and ω is a vacuum state on $\mathfrak{A}$ then Theorem 5 says that if a state $\varrho=\omega\circ\sigma$ is translation covariant there exists a translation cocycle, $\Gamma(a)=\Gamma_\sigma(a)$, $a\in M^4$, such that

$$\sigma\circ\tau_a(A)=U_\omega(a)^*\Gamma(a)^*\sigma(A)\Gamma(a)U_\omega(a),\tag{3.10}$$

on $\mathscr{H}_\omega(\sigma)$.

We propose to determine the *localization properties* of Γ for the case when $\omega\circ\sigma$ is a *charged state*. For this purpose we consider the *space-translation cocycles*. Let

$$B_\varepsilon=\{a=(\mathbf{x},0):|\mathbf{x}|=\varepsilon\}.$$

Let $\Gamma(a)$ be an arbitrary translation cocycle on $\mathscr{H}_\omega$. Suppose that, for some $\varepsilon>0$ and all $\mathbf{x}\in B_\varepsilon$, $\Gamma(\mathbf{x})\equiv\Gamma((\mathbf{x},0))\in\mathfrak{A}(\mathscr{O})$, for some *bounded* double cone $\mathscr{O}$.

Pick an arbitrary $\mathbf{y}\in\mathbb{R}^3$ with $|\mathbf{y}|=y\varepsilon$, $y=0,1,2,\ldots$, and set $\mathbf{n}=y^{-1}\mathbf{y}$, so that $|\mathbf{n}|=\varepsilon$. Then, by iterating the cocycle identity (3.1), we obtain

$$\Gamma(\mathbf{y})=\prod_{j=0}^{y-1} U_\omega(j\mathbf{n})\Gamma(\mathbf{n})U_\omega(j\mathbf{n})^*. \tag{3.11}$$

Hence

$$\Gamma(\mathbf{y})\in\mathfrak{A}(\mathcal{O}_\mathbf{y}), \tag{3.12}$$

where $\mathcal{O}_\mathbf{y}$ is the smallest, connected, convex union of double cones containing both $\mathcal{O}$ and $\mathcal{O}((\mathbf{y},0))$. Clearly the "transverse width" of $\mathcal{O}_\mathbf{y}$ is bounded uniformly in $\mathbf{y}$. As noted in [5] (Theorems 2.8 and 2.9), the cocycle identity (3.1) and some additional, more technical arguments (see also [3, 6]) then imply that there exists a bounded double cone $\mathcal{O}_\Gamma$ such that

$$\Gamma(a)\in\mathfrak{A}(\mathcal{O}_\Gamma\cup\mathcal{O}_\Gamma(a)), \quad \text{for all} \quad a\in M^4, \tag{3.13}$$

and (see Theorem 2.9 of [5] and [3, 6])

$$\sigma(A)=\lim_{a\to\infty}\Gamma(a)A\Gamma(a)^* \tag{3.14}$$

exists, for all $A\in\mathfrak{A}$, whenever a tends to ∞ in a space-like direction, and the limit is *independent* of that direction. Moreover σ is a *local * morphism* with support $\operatorname{supp}\sigma=\mathcal{O}_\Gamma$. By Proposition 3, $\omega\circ\sigma$ has the same charge as ω. Thus we have proven

Theorem 7. *Let ω be a vacuum state on $\mathfrak{A}$ of charge* 0. *Let σ be a * morphism on $\mathfrak{A}$ such that $\omega\circ\sigma$ is a charged, translation covariant state. Let $\Gamma(a)=\Gamma_\sigma(a)$ be the corresponding translation cocycle on $\mathcal{H}_\omega$ (constructed in Theorem 5).*

Then, for arbitrary $\varepsilon>0$, there exists no bounded double cone $\mathcal{O}$ such that $\Gamma((\mathbf{x},0))\in\mathfrak{A}(\mathcal{O})$, for all $\mathbf{x}$ with $|\mathbf{x}|=\varepsilon$.

Remarks. 1. Assume, in addition, that σ is space-rotation covariant (an assumption that is compatible with the conclusions of Sects. 3.4 and 3.5 of I). In this case, one can choose σ such that it commutes with the space-rotation automorphisms. It follows that, for an arbitrary space rotation R, $\Gamma(R\mathbf{x})=U_\omega(R)\Gamma(\mathbf{x})U_\omega(R)^*$. Combining this with Theorem 7 we conclude that for arbitrary $\mathbf{0}\neq\mathbf{x}$, $\Gamma(\mathbf{x})\notin\mathfrak{A}(\mathcal{O})$!

2. Theorem 7 remains true if $\{a:a=(\mathbf{x},0),|\mathbf{x}|=\varepsilon\}$ is replaced by $\{a:a\in\Sigma,|a|=\varepsilon\}$, where Σ is an arbitrary space-like hyperplane.

3. It is natural to view the translation cocycles $\Gamma(a)$ as the formal continuum limit of coherent superpositions of (charge transfer) string operators in lattice gauge theories. Theorem 7 then substantiates the claims made at the end of Sect. 3.5 of Paper I. The absence of localization properties of $\Gamma(a)$ might make the construction of these cocycles very difficult in models.

4. Assuming that charged, translation covariant states exist, we can refer to a result of Swieca [7] that says that in such a situation the physical mass gap of the theory is 0 (see also Sect. 2 of I). Assuming a sharper version of this result, namely that $F_{\mu\nu}$ couples the vacuum Ω to a 0-mass one-particle state, the photon, one concludes that the positive metric formalism developed here is incompatible with the existence of a local vector potential whose curl is $F_{\mu\nu}$. In addition, Proposition

3 and Theorem 7 prove that there are no charged fields and no charge transfer operators that are local relative to $F_{\mu\nu}$.

5. We summarize the main conclusions: If $\omega \circ \sigma$ is a charged, translation-covariant state on $\mathfrak{A}$ then σ is non-local and there exists a translation cocycle $\Gamma = \Gamma_\sigma$ on $\mathscr{H}_\omega$ with the property that $\Gamma_\sigma(a)$ is non-local in the sense of Theorem 7 and Remarks 1 and 2.

In this situation

1) it is *not* necessarily true that

$$\mathscr{H}_\omega(\sigma) = \mathscr{H}_\omega, \tag{3.15}$$

i.e. Ω need *not* be cyclic in $\mathscr{H}_\omega$ for $\sigma(\mathfrak{A})$. Therefore a *transportable* * morphism σ (i.e. one that is G-transportable, with G the translation group), does *not* necessarily give rise to a translation covariant state (see Theorem 5).

2) It is not necessarily true that

$$\sigma(A) = \text{"}\lim_{a \to \infty}\text{"}\, \Gamma(a) A \Gamma(a)^*, \quad \text{for} \quad A \in \mathfrak{A}, \tag{3.16}$$

with $a \to \infty$ in a space-like direction, as would be the case if σ were a local * morphism [see (3.14)].

3) (Composition Property) The composition of translation covariant * morphisms is not necessarily translation covariant.

However, the results of Paper I suggest that (3.15), (3.16) and the Composition Property which expresses the additivity of the electric charge are valid in QED. Therefore we propose to develop a framework for the construction of charged states in QED with the property that (3.15), (3.16) and the Composition Property are satisfied. [Condition (3.15) will be seen to be related to some locality properties of σ; Sect. 6.] The *relativistic spectrum condition* will then automatically be satisfied on the charged sectors. On these grounds a tentative theory of the "asymptotic statistics" of charged sectors will be outlined in the last section. It permits us to set up a generalized Haag-Ruelle scattering theory for charged infraparticles.

4. Transportable * Morphisms and Translation Cocycles

In this section we attempt to implement the ideas that charged states are weak * limits of states in the vacuum sector, as a localized charge is moved to space-like infinity, and that a charge localized in a space-like distant region has only a negligible effect on measurements done in a bounded space-time region $\mathcal{O}$. In the following, ω is a fixed, pure vacuum state on $\mathfrak{A}$.

Definition 3. A representation π of $\mathfrak{A}$ is a called *locally normal* iff $\pi(\mathfrak{A}(\mathcal{O}))'' \cong \pi_\omega(\mathfrak{A}(\mathcal{O}))''$, for all $\mathcal{O} \in \mathfrak{B}$. Let $\{A_\alpha\}$ be some net of operators in $\mathfrak{A}$. In the following "$w\text{-}\lim_{\alpha \to \infty} A_\alpha = A$" means that A_α converges weakly to A, as $\alpha \to \infty$, in every locally normal representation of $\mathfrak{A}$. A * morphism σ of $\mathfrak{A}$ is called locally normal iff $\pi_\omega \circ \sigma$ is locally normal.

In the following, "$a\to\infty$" means that a tends to ∞ in some space-like, asymptotic direction. A translation cocycle Γ on $\mathscr{H}_\omega$ is called *quasi-local* iff $\Gamma(a)A\Gamma(a)^*\in\mathfrak{A}$, for all $a\in M^4$, and for all $A\in\bigcup_{\mathcal{O}\in\mathfrak{B}}\mathfrak{A}(\mathcal{O})$,

$$(1)\qquad \sigma_\Gamma(A)=\underset{a\to\infty}{w\text{-lim}}\,\Gamma(a)A\Gamma(a)^*\quad\text{exists}\,, \tag{4.1}$$

is independent of the direction in which $a\to\infty$, and is a locally normal * morphism of $\mathfrak{A}$. (Notice that σ_Γ is automatically locally normal if the local algebras $\mathfrak{A}(\mathcal{O})$, $\mathcal{O}\in\mathfrak{B}$, are of type III and $\mathscr{H}_{\omega\circ\sigma_\Gamma}$ is separable; a theorem of Takesaki, see [15].)

$$(2)\qquad \underset{\substack{a\to\infty\\ a+b\to\infty}}{w\text{-lim}}\;\tau_{-a}(\Gamma(b)\tau_a(A)\Gamma(b)^*)=A\,. \tag{4.2}$$

We also define $\sigma_a\equiv\tau_{-a}\circ\sigma\circ\tau_a$, $a\in M^4$, where σ is an arbitrary * morphism of $\mathfrak{A}$. □

Remarks. 1. A translation cocycle Γ is called *quasi-local in norm* iff $\Gamma(a)A\Gamma(a)^*\in\mathfrak{A}$, for all $A\in\mathfrak{A}$, $a\in M^4$, and

$$\underset{\substack{a\to\infty\\ a+b\to\infty}}{n\text{-lim}}\;\tau_{-a}(\Gamma(b)\tau_a(A)\Gamma(b)^*)=A\,. \tag{4.3}$$

Then $\sigma_\Gamma(A)\equiv\underset{a\to\infty}{n\text{-lim}}\,\Gamma(a)A\Gamma(a)^*$ exists, is independent of the direction in which $a\to\infty$, and is a transportable * morphism of $\mathfrak{A}$ with the property that

$$\underset{a\to\infty}{n\text{-lim}}\,\sigma_{\Gamma,a}(A)=A\,,\quad\text{for all}\quad A\in\mathfrak{A}\,. \tag{4.4}$$

Conversely, suppose that σ is a transportable * morphism of $\mathfrak{A}$ with the properties that the corresponding translation cocycle Γ_σ obeys $\Gamma_\sigma(a)A\Gamma_\sigma(a)^*\in\mathfrak{A}$, for all $A\in\mathfrak{A}$, $a\in M^4$, and that

$$\underset{a\to\infty}{n\text{-lim}}\,\sigma_a(A)=A\,.$$

Then $\Gamma_\sigma(a)$ is quasi-local in norm, and $\sigma=\sigma_{\Gamma_\sigma}$. I.e. there is a 1–1-correspondence between * morphisms σ with the above properties and translation cocycles that are quasi-local in norm. The *proof* of this theorem is given in Appendix 1. At first sight, it seems to offer an attractive extension of the DHR theory. We have however good reasons to reject translation cocycles which are quasi-local in norm (which may be interesting e.g. for statistical mechanics) as a suitable framework for the description of charged sectors in QED, rather we base our analysis on the quasi-local cocycles introduced in Definition 3.

2. Clearly, the * morphisms σ_Γ arising from quasi-local translation cocycles, Γ, are in general not local, so that the states $\omega\circ\sigma_\Gamma$ may be charged (see Sect. 5). (This would even be so for morphisms arising from translation cocycles which are quasi-local in norm.)

Proposition 8. *Let Γ be a quasi-local translation cocycle. Then the * morphism σ_Γ of $\mathfrak{A}$ (see* (4.1)*) is transportable, with*

$$\sigma_{\Gamma,a}(a)=\Gamma(a)^*\sigma_\Gamma(A)\Gamma(a)\,, \tag{4.5}$$

and

$$\underset{a\to\infty}{w\text{-lim}}\,\sigma_{\Gamma,a}(A)=A\,. \tag{4.6}$$

Proof. By the cocycle identity (3.1)

$$\tau_{-a}(\Gamma(b)\tau_a(A)\Gamma(b)^*)=\Gamma(a)^*\Gamma(a+b)A\Gamma(a+b)^*\Gamma(a).$$

Taking $b\to\infty$ on both sides of this equation, applying (4.1), yields (4.5). In particular, σ_Γ is transportable. Moreover,

$$\begin{aligned}\sigma_{\Gamma,a}(A)&=\tau_{-a}\left(\underset{b\to\infty}{w\text{-lim}}\,\Gamma(b)^*\tau_a(A)\Gamma(b)\right)\\&=\underset{b\to\infty}{w\text{-lim}}\,\tau_{-a}(\Gamma(b)\tau_a(A)\Gamma(b)^*)\end{aligned}$$

which converges weakly to A, as $a\to\infty$, for all $A\in U\mathfrak{A}(\mathcal{O})$, by (4.2). $\square$

Theorem 9. *Let σ_1 and σ_2 be * morphisms of $\mathfrak{A}$ arising from quasi-local translation cocycles Γ_1 and Γ_2, as in* (4.1). *Suppose that $\Gamma_2(a)\in\mathfrak{A}$, for all $a\in M^4$.*

*Then $\sigma_1\circ\sigma_2$ is a transportable * morphism, and the corresponding translation cocycle, $\Gamma_{\sigma_1\circ\sigma_2}$, is given by*

$$\Gamma_{\sigma_1\circ\sigma_2}(a)=\sigma_1(\Gamma_2(a))\Gamma_1(a), \qquad a\in M^4.$$

Proof. Since $\Gamma_2(a)\in\mathfrak{A}$, for all $a\in M^4$, we have

$$\begin{aligned}\tau_{-a}\circ\sigma_1\circ\sigma_2\circ\tau_a(A)&=\tau_{-a}\circ\sigma_1\circ\tau_a(\tau_{-a}\circ\sigma_2\circ\tau_a(A))\\&=\tau_{-a}\circ\sigma_1\circ\tau_a(\Gamma_2(a)^*\sigma_2(A)\Gamma_2(a)), \quad \text{by Proposition 8},\\&=\Gamma_1(a)^*\sigma_1(\Gamma_2(a)^*\sigma_2(A)\Gamma_2(a))\Gamma_1(a), \quad \text{by Proposition 8}\\&=(\sigma_1(\Gamma_2(a))\Gamma_1(a))^*\sigma_1\circ\sigma_2(A)\sigma_1(\Gamma_2(a))\Gamma_1(a).\end{aligned}$$

To complete the proof we must show that $\sigma_1(\Gamma_2(a))\Gamma_1(a)$ is a cocycle: Continuity of $\sigma_1(\Gamma_2(a))\Gamma_1(a)$ in a follows from the continuity of $\Gamma_1(a)$ and $\Gamma_2(a)$ in a and the local normality of σ_1. By the cocycle identity (3.1) applied to Γ_2 and Γ_1,

$$\begin{aligned}&\sigma_1(\Gamma_2(a+b))\Gamma_1(a+b)\\&\quad=\sigma_1(\Gamma_2(a)\tau_{-a}(\Gamma_2(b)))\Gamma_1(a)\tau_{-a}(\Gamma_1(b))\\&\quad=\sigma_1(\Gamma_2(a))\Gamma_1(a)\{\Gamma_1(a)^*\sigma_1(\tau_{-a}(\Gamma_2(b)))\Gamma_1(a)\}\tau_{-a}(\Gamma_1(b))\\&\quad=\sigma_1(\Gamma_2(a))\Gamma_1(a)\tau_{-a}(\sigma_1(\Gamma_2(b))\Gamma_1(b)),\end{aligned}$$

since σ_1 is transportable. Recalling that $U_\omega(a)^*$ implements τ_{-a} on $\mathcal{H}_\omega$, we see that this equation is just the cocycle identity (3.1) for $\Gamma_{\sigma_1\circ\sigma_2}$. $\square$

5. Charged, Transportable * Morphisms, and Charge Transfer Cocycles

We are interested in those * morphisms σ which have the property that, when ω is a vacuum state $\omega\circ\sigma$ is a translation-covariant, charged state on $\mathfrak{A}$ [see P1) and P2), Sect. 3]. Such * morphisms are called *charged.*

Since we require, on physical grounds, that charged morphisms and the compositions of charged morphisms are transportable, and charged states are weak * limits of neutral states, as a localized charge is removed to space-like infinity, the analysis of Sect. 4 suggests to consider only those charged * morphisms, σ, which are of the form

$$\sigma=\sigma_{\Gamma_1}\circ\ldots\circ\sigma_{\Gamma_n}, \tag{5.1}$$

where $\Gamma_1, \dots, \Gamma_n$ are quasi-local translation cocycles, with $\Gamma_k(a) \in \mathfrak{A}$, for all $a \in M^4$, and $k = 1, \dots, n$. We must therefore isolate those quasi-local translation cocycles, Γ, with the property that $\omega \circ \sigma_\Gamma$ is a charged state. By Lemma 2, Sect. 2, we know that the representations π_ω and $\pi_{\omega \circ \sigma_\Gamma}$ of $\mathfrak{A}$ are disjoint. [In the terminology of [5] this means that the cocycle Γ is "non-trivial", in the sense defined in [5] for the case of cocycles generating the soliton sectors in two dimensional theories, i.e. $\Gamma(a)$ is *not* of the form $VU_\omega(a)VU_\omega(a)^*$, with $V \in B(\mathscr{H}_\omega)$.]

Let Σ be some simply connected, bounded region in $\mathbb{R}^3$ with piecewise smooth boundary $\partial\Sigma$. Let Q_Σ be the local-charge operator introduced in Sect. 2, with the property that $e^{isQ_\Sigma} \in \mathfrak{A}(\mathcal{O})$, for some sufficiently large $\mathcal{O} \supseteqq \Sigma$.

Let $\mathscr{P}(\mathbb{R}^3)$ be some family of bounded subsets of $\mathbb{R}^3$ with piecewise smooth boundaries, containing a covering of $\mathbb{R}^3$ by simply connected, disjoint sets and closed under finite unions.

In order to make our subsequent analysis more elegant, we assume henceforth that the test functions f_Σ and α in the definition of Q_Σ (Sect. 2) can be chosen such that for arbitrary, disjoint sets Σ_1 and Σ_2 in $\mathscr{P}(\mathbb{R}^3)$,

$$Q_{\Sigma_1 \cup \Sigma_2} = Q_{\Sigma_1} + Q_{\Sigma_2},$$

with (5.2)

$$e^{isQ_{\Sigma_1}} e^{itQ_{\Sigma_2}} = e^{itQ_{\Sigma_2}} e^{isQ_{\Sigma_1}}$$

for all real s and t.

Definition 4. A one-parameter family $\{\gamma_\Sigma(s) : s \in \mathbb{R}\}$ of unitary operators contained in $\mathfrak{A}$ is called a *local-charge cocycle* if $\gamma_\Sigma(s)$ is strongly continuous in s, in every locally normal representation of $\mathfrak{A}$, and the cocycle identity

$$\gamma_\Sigma(s+t) = \gamma_\Sigma(s) e^{isQ_\Sigma} \gamma_\Sigma(t) e^{-isQ_\Sigma} \tag{5.3}$$

is satisfied, for arbitrary real s and t. (Here $\Sigma \in \mathscr{P}(\mathbb{R}^3)$.) □

Remark. Let σ be some locally normal * morphism of $\mathfrak{A}$. Then

$$\gamma_\Sigma^\sigma(s) \equiv \sigma(e^{isQ_\Sigma}) e^{-isQ_\Sigma} \tag{5.4}$$

is clearly a local-charge cocycle. We define $\operatorname{supp}\sigma$ to be the smallest region in $\mathbb{R}^3$ belonging to $\mathscr{P}(\mathbb{R}^3)$ with the property that, for all $\Sigma \in \mathscr{P}(\mathbb{R}^3)$ with $\Sigma \subset (\operatorname{supp}\sigma)^c$ (the complement of $\operatorname{supp}\sigma$)

$$\gamma_\Sigma^\sigma(s) = 1, \quad \text{for all real} \quad s; \tag{5.5}$$

$\operatorname{supp}\sigma$ is called the "support of σ".

Let $\Sigma' = (\operatorname{supp}\sigma) \cup \Sigma$, $\Sigma \in \mathscr{P}(\mathbb{R}^3)$, $\Sigma \subset (\operatorname{supp}\sigma)^c$. Then, by (5.2) and (5.5)

$$\begin{aligned} \gamma_{\Sigma'}(s) &= \sigma(e^{isQ_{(\operatorname{supp}\sigma)\cup\Sigma}}) e^{-isQ_{(\operatorname{supp}\sigma)\cup\Sigma}} \\ &= \sigma(e^{isQ_{\operatorname{supp}\sigma}}) \sigma(e^{isQ_\Sigma}) e^{-isQ_\Sigma} e^{-isQ_{\operatorname{supp}\sigma}} \\ &= \sigma(e^{isQ_{\operatorname{supp}\sigma}}) \gamma_\Sigma^\sigma(s) e^{-isQ_{\operatorname{supp}\sigma}} \\ &= \gamma_{\operatorname{supp}\sigma}(s), \end{aligned} \tag{5.6}$$

i.e. for $\Sigma' \supset \operatorname{supp}\sigma$, $\Sigma' \in \mathscr{P}(\mathbb{R}^3)$, $\gamma_{\Sigma'}^\sigma(s) \equiv \gamma^\sigma(s)$ is *independent* of Σ'!

A simple calculation shows that for $a=(\mathbf{a},0)$, a space translation

$$\gamma_\Sigma^{\sigma_a}(s)=\tau_{-a}(\gamma_{\Sigma(a)}^\sigma(s)),$$

hence $\operatorname{supp}\sigma^a=(\operatorname{supp}\sigma)(-a)$, and

$$\gamma^{\sigma_a}(s)=\tau_{-a}(\gamma^\sigma(s)). \tag{5.7}$$

Lemma 10. *Let σ be a locally normal, transportable * morphism of $\mathfrak{A}$.*

1) *Let $\Sigma\supset\operatorname{supp}\sigma$. Let $a\in M^4$ be such that $\Sigma\subset(\operatorname{supp}\sigma_a)^c$. Then*

$$e^{isQ_\Sigma}\Gamma_\sigma(a)e^{-isQ_\Sigma}=\gamma^\sigma(s)^*\Gamma_\sigma(a). \tag{5.8}$$

2) *If $\Sigma\subset(\operatorname{supp}\sigma)^c$, $\Sigma\supset\operatorname{supp}\sigma_a$ then*

$$e^{isQ_\Sigma}\Gamma_\sigma(a)e^{-isQ_\Sigma}=\Gamma_\sigma(a)\gamma^{\sigma_a}(s). \tag{5.9}$$

3) *If $\Sigma\supset\operatorname{supp}\sigma$, $\Sigma\supset\operatorname{supp}\sigma_a$ then*

$$e^{isQ_\Sigma}\Gamma_\sigma(a)e^{-isQ_\Sigma}=\gamma^\sigma(s)^*\Gamma_\sigma(a)\tau_{-a}(\gamma^\sigma(s)). \tag{5.10}$$

4) *If $\Sigma\subset(\operatorname{supp}\sigma)^c$, $\Sigma\subset(\operatorname{supp}\sigma_a)^c$*

$$e^{isQ_\Sigma}\Gamma_\sigma(a)e^{-isQ_\Sigma}=\Gamma_\sigma(a). \tag{5.11}$$

Proof. By the definition of transportable * morphisms, we have

$$\sigma_a(A)=\Gamma_\sigma(a)^*\sigma(A)\Gamma_\sigma(a),$$

for $A\in\mathfrak{A}$ and $a\in M^4$. Thus

$$\Gamma_\sigma(a)^*\sigma(e^{isQ_\Sigma})e^{-isQ_\Sigma}e^{isQ_\Sigma}\Gamma_\sigma(a)e^{-isQ_\Sigma}$$
$$=\sigma_a(e^{isQ_\Sigma})e^{-isQ_\Sigma},$$

i.e.

$$\gamma_\Sigma^\sigma(s)e^{isQ_\Sigma}\Gamma_\sigma(a)e^{-isQ_\Sigma}=\Gamma_\sigma(a)\gamma_\Sigma^{\sigma_a}(s). \tag{5.12}$$

Under the hypothesis of 1), $\gamma_\Sigma^{\sigma_a}(s)=1$, and $\gamma_\Sigma^\sigma(s)=\gamma^\sigma(s)$, whence (5.8). The proof of 2) is similar. Under the hypotheses of 3) $\gamma_\Sigma^\sigma(s)=\gamma^\sigma(s)$ and $\gamma_\Sigma^{\sigma_a}(s)=\gamma^{\sigma_a}(s)=\tau_{-a}(\gamma^\sigma(s))$, by (5.7). Thus (5.10) follows. Finally, in the situation of 4) $\gamma^\sigma(s)=\gamma^{\sigma_a}(s)=1$. □

Next, we prove a converse to Lemma 10. For this purpose we consider a quasi-local translation cocycle, Γ, with the property that, for $a=(\mathbf{a},0)$, with $\mathbf{a}\in\mathbb{R}^3$, and some region $\Sigma_\Gamma\in\mathscr{P}(\mathbb{R}^3)$,

$$e^{isQ_\Sigma}\Gamma(a)e^{-isQ_\Sigma}=\gamma_\Sigma(s)^*\Gamma(a), \tag{5.13}$$

for some local-charge cocycle $\gamma_\Sigma(s)$ (see Definition 4) with the properties that

$$\gamma_\Sigma(s)=\gamma(s), \quad \text{independent of} \quad \Sigma,$$

if (5.14)

$$\Sigma_\Gamma\subset\Sigma \quad \text{and} \quad \Sigma_\Gamma(a)\subset\Sigma^c,$$

and

$$\gamma_\Sigma(s)=1 \tag{5.15}$$

if

$$\Sigma \subset \Sigma_\Gamma^c, \quad \Sigma \subset \Sigma_\Gamma(a)^c.$$

We call Σ_Γ the *charge support* of Γ.

Lemma 11. *If Γ is a quasi-local translation cocycle satisfying* (5.13)–(5.15) *then σ_Γ – defined in* (4.1) *– has the properties that*

$$\sigma_\Gamma(e^{isQ_\Sigma})e^{-isQ_\Sigma} = \gamma(s), \quad \text{for} \quad \Sigma \supset \Sigma_\Gamma, \tag{5.16}$$

and $(\operatorname{supp}\sigma_\Gamma) \subseteqq \Sigma_\Gamma$.

Proof. By (4.1), (5.13), and (5.14), we have, for $\Sigma \supset \Sigma_\Gamma$ and $a_0 = (\mathbf{a}_0, 0)$ with $|\mathbf{a}_0|$ large enough,

$$\begin{aligned} \gamma(s)^* &= e^{isQ_\Sigma}\Gamma(a_0)e^{-isQ_\Sigma}\Gamma(a_0)^* \\ &= \operatorname*{w\text{-lim}}_{a\to\infty} e^{isQ_\Sigma}\Gamma(a)e^{-isQ_\Sigma}\Gamma(a)^* \\ &= e^{isQ_\Sigma}\sigma_\Gamma(e^{-isQ_\Sigma}). \end{aligned}$$

Next, for all $\Sigma \in \mathscr{P}(\mathbb{R}^3)$ with $\Sigma \subset \Sigma_\Gamma^c$ and for $a_0 = (\mathbf{a}_0, 0)$ with $|\mathbf{a}_0|$ large enough [so that $\Sigma \subset \Sigma_\Gamma(a_0)^c$]

$$\begin{aligned} 1 &= e^{isQ_\Sigma}\Gamma(a_0)e^{-isQ_\Sigma}\Gamma(a_0)^* \\ &= \operatorname*{w\text{-lim}}_{a\to\infty} e^{isQ_\Sigma}\Gamma(a)e^{-isQ_\Sigma}\Gamma(a)^* \\ &= e^{isQ_\Sigma}\sigma_\Gamma(e^{-isQ_\Sigma}). \quad \square \end{aligned}$$

Lemma 12. *Let $\gamma_\Sigma(s)$ be a local-charge cocycle with the property that, for some bounded $\Sigma_\gamma \in \mathscr{P}(\mathbb{R}^3)$, $\gamma_\Sigma(s) = \gamma(s)$ is independent of Σ, for all $\Sigma \supset \Sigma_\gamma$. Then $\gamma(s)$ is a unitary one-parameter group in $\mathfrak{A}$. In particular, if σ is a locally normal * morphism of $\mathfrak{A}$ of compact support then $\gamma^\sigma(s) \in \sigma(\mathfrak{A})' \cap \mathfrak{A}$.*

Proof. By the cocycle identity

$$\gamma(s+t) = \gamma(s)e^{isQ_\Sigma}\gamma(t)e^{-isQ_\Sigma},$$

if $\Sigma \supset \Sigma_\gamma$. Using Lemma 2, Sect. 2, and the fact $\gamma(t) \in \mathfrak{A}$, for all t, we conclude that

$$\lim_{\substack{\Sigma \uparrow \mathbb{R}^3 \\ \Sigma \in \mathscr{P}(\mathbb{R}^3)}} e^{isQ_\Sigma}\gamma(t)e^{-isQ_\Sigma} = \gamma(t),$$

so that $\gamma(s+t) = \gamma(s)\gamma(t)$.

By Lemma 2 and the definition of $\gamma^\sigma(s)$,

$$\begin{aligned} \gamma^\sigma(s)\sigma(A)\gamma^\sigma(-s) &= \lim_{\Sigma \uparrow \mathbb{R}^3} e^{-isQ_\Sigma}\sigma(e^{isQ_\Sigma})\sigma(A)\sigma(e^{-isQ_\Sigma})e^{isQ_\Sigma} \\ &= \lim_{\Sigma \uparrow \mathbb{R}^3} e^{-isQ_\Sigma}\sigma(e^{isQ_\Sigma}Ae^{-isQ_\Sigma})e^{isQ_\Sigma} \\ &= \lim_{\Sigma \uparrow \mathbb{R}^3} e^{-isQ_\Sigma}\sigma(A)e^{isQ_\Sigma}, \quad \text{for} \quad A \in \bigcup_{\mathcal{O}\in\mathfrak{B}} \mathfrak{A}(\mathcal{O}), \\ &= \sigma(A), \quad \text{as} \quad \sigma(A) \in \mathfrak{A}, \quad \text{for} \quad A \in \mathfrak{A}. \end{aligned}$$

Since $\gamma^\sigma(s) \in \mathfrak{A}$, this implies that $\gamma^\sigma(s) \in \sigma(\mathfrak{A})' \cap \mathfrak{A}$. $\square$

Lemma 12 permits us to characterize locally normal * morphisms σ of compact support by the unitary group $\gamma^\sigma(s)$.

If σ is a * automorphism of $\mathfrak{A}$ then, clearly,

$$\sigma(\mathfrak{A})' \cap \mathfrak{A} = \{\lambda \cdot 1;\ \lambda \in \mathbb{C}\},$$

hence

$$\gamma^\sigma(s) = e^{isq}, \quad \text{for some} \quad q \in \mathbb{R}. \tag{5.17}$$

If σ is irreducible, i.e. $\pi_\omega \circ \sigma$ is an irreducible representation of $\mathfrak{A}$, then

$$\pi_\omega(\sigma(\mathfrak{A}))' = \{\lambda \cdot 1 : \lambda \in \mathbb{C}\},$$

so that $\pi_\omega(\gamma^\sigma(s)) = e^{isq}$, for some $q \in \mathbb{R}$.

Since π_ω is faithful,

$$\gamma^\sigma(s) = e^{isq}. \tag{5.18}$$

Next suppose that the action of σ is local in the sense that, given a double cone $\mathcal{O} \in \mathfrak{B}$, there exists some $\mathcal{O}_\sigma \in \mathfrak{B}$ such that $\sigma(\mathfrak{A}(\mathcal{O})) \subset \mathfrak{A}(\mathcal{O}_\sigma)$. Suppose, in addition, that $\operatorname{supp} \sigma$ is compact. Then for some bounded $\Sigma \in \mathscr{P}(\mathbb{R}^3)$

$$\gamma^\sigma(s) = \sigma(e^{isQ_\Sigma})e^{-isQ_\Sigma}.$$

By Sect. 2, $e^{isQ_\Sigma} \in \mathfrak{A}(\mathcal{O}_\Sigma)$, for some double cone $\mathcal{O}_\Sigma \in \mathfrak{B}$ containing Σ and all $s \in \mathbb{R}$. Thus

$$\gamma^\sigma(s) \in \mathfrak{A}(\mathcal{O}_{\Sigma,\sigma} \cup \mathcal{O}_\Sigma). \tag{5.19}$$

Suppose now that $\gamma^\sigma(s)$ is translation invariant, i.e.

$$\tau_a(\gamma^\sigma(s)) = \gamma^\sigma(s), \quad \text{for all} \quad a = (\mathbf{a}, 0), \quad \mathbf{a} \in \mathbb{R}^3. \tag{5.20}$$

Then

$$\gamma^\sigma(s) \in \mathfrak{A}(\tilde{\mathcal{O}}) \wedge \mathfrak{A}(\tilde{\mathcal{O}}(a)), \qquad \tilde{\mathcal{O}} \equiv \mathcal{O}_{\Sigma,\sigma} \cup \mathcal{O}_\Sigma.$$

Choosing $\mathbf{a}$ large enough, we conclude using locality and the fact that $\mathfrak{A}(\tilde{\mathcal{O}})$ is a factor that $\gamma^\sigma(s) = e^{isq}$, for some $q \in \mathbb{R}$.

In all three cases, the physical interpretation of q is the one of *total charge* of the * morphism σ, and we then say that σ is a *localized, charged * morphism of charge q*.

The analysis presented above proves that for $\gamma^\sigma(s)$ not to be of the form e^{isq}, $q \in \mathbb{R}$, it is necessary that σ be *not* irreducible and (assuming the action of σ is local) $\gamma^\sigma(s)$ be *not* translation invariant.

Next, we merely suppose that $\gamma^\sigma(s)$ is translation invariant. Then, for all $a = (\mathbf{a}, 0)$, $\mathbf{a} \in \mathbb{R}^3$,

$$U_\omega(a)^* \gamma^\sigma(s)\Omega = \tau_a(\gamma^\sigma(s))\Omega = \gamma^\sigma(s)\Omega,$$

i.e. $\gamma^\sigma(s)\Omega$ is a translation invariant vector in $\mathscr{H}_\omega$. Since we have assumed that the vacuum is non-degenerate, we conclude that $\gamma^\sigma(s)\Omega = e^{isq}\Omega$.

By Lemma 12,

$$\gamma^\sigma(s)\sigma(A)\Omega = \sigma(A)\gamma^\sigma(s)\Omega = e^{isq}\sigma(A)\Omega,$$

i.e.

$$\gamma^\sigma(s) \upharpoonright \mathscr{H}_\omega(\sigma) = e^{isq} \cdot 1_{\mathscr{H}_\omega(\sigma)}, \tag{5.21}$$

for some $q \in \mathbb{R}$.

If $\mathscr{H}_\omega(\sigma)=\mathscr{H}_\omega$ then (5.21) implies that $\gamma^\sigma(s)=e^{isq}$, and σ is a localized, charged * morphism of charge q (see also Sect. 6).

The following result relates the translation covariance properties of $\gamma^\sigma(a)$ to the cocycle Γ_σ.

Lemma 13. *Let σ be a transportable, localized * morphism with $\Gamma_\sigma(a)\in\mathfrak{A}$, for all a. Then*

$$\tau_{-a}(\gamma^\sigma(s))=\gamma^{\sigma_a}(s)=\Gamma_\sigma(a)^*\gamma^\sigma(s)\Gamma_\sigma(a),$$

for all $a=(\mathbf{a},0)$, $\mathbf{a}\in\mathbb{R}^3$.

Proof. Given $a=(\mathbf{a},0)$, we choose Σ so large that $\Sigma\supset\operatorname{supp}\sigma$ and $\Sigma(a)\supset\operatorname{supp}\sigma$. Then

$$\begin{aligned}\tau_{-a}(\gamma^\sigma(s))&=\tau_{-a}(\sigma(e^{isQ_\Sigma})e^{-isQ_\Sigma})\\&=\sigma_a(e^{isQ_{\Sigma(a)}})e^{-isQ_{\Sigma(a)}}\end{aligned}\tag{5.22}$$

is independent of Σ. Hence

$$\sigma_a(e^{isQ_{\Sigma(a)}})e^{-isQ_{\Sigma(a)}}=\gamma^{\sigma_a}(s)=\tau_{-a}(\gamma^\sigma(s)).$$

But

$$\begin{aligned}\sigma_a(e^{isQ_{\Sigma(a)}})e^{-isQ_{\Sigma(a)}}&=\Gamma_\sigma(a)^*\sigma(e^{isQ_{\Sigma(a)}})\Gamma_\sigma(a)e^{-isQ_{\Sigma(a)}}\\&=\Gamma_\sigma(a)^*[\sigma(e^{isQ_{\Sigma(a)}})e^{-isQ_{\Sigma(a)}}]e^{isQ_{\Sigma(a)}}\Gamma_\sigma(a)e^{-isQ_{\Sigma(a)}}\\&=\Gamma_\sigma(a)^*\gamma^\sigma(s)e^{-isQ_{\Sigma(a)}}\Gamma_\sigma(a)e^{-isQ_{\Sigma(a)}}.\end{aligned}\tag{5.23}$$

Since $\Gamma_\sigma(a)\in\mathfrak{A}$,

$$n\text{-}\lim_{\Sigma\uparrow\mathbb{R}^3}e^{isQ_{\Sigma(a)}}\Gamma_\sigma(a)e^{-isQ_{\Sigma(a)}}=\Gamma_\sigma(a),\tag{5.24}$$

by Gauss' law (see Lemma 2).

Combining (5.22)–(5.24) and letting $\Sigma\uparrow\mathbb{R}^3$, we arrive at

$$\tau_{-a}(\gamma^\sigma(s))=\gamma^{\sigma_a}(s)=\Gamma_\sigma(a)^*\gamma^\sigma(s)\Gamma_\sigma(a).\quad\square$$

Lemmas 10, 12, and 13 yield the following "topological" *commutation relations*: Let σ be a transportable, localized * morphism of $\mathfrak{A}$ of compact support, supp σ, with translation cocycle $\Gamma_\sigma(a)$. Then

$$\gamma^\sigma(s)=\lim_{\Sigma\uparrow\mathbb{R}^3}\sigma(e^{isQ_\Sigma})e^{-isQ_\Sigma}$$

exists and is a unitary one-parameter group in $\sigma(\mathfrak{A})'\cap\mathfrak{A}$ (the charge cocycle associated with σ), and

(A) $\quad e^{isQ_\Sigma}\Gamma_\sigma(a)e^{-isQ_\Sigma}=\gamma^\sigma(s)^*\Gamma_\sigma(a)\quad$ if $\quad\operatorname{supp}\sigma\subsetneqq\Sigma\subset\operatorname{supp}\sigma_a^c$

[see (5.8)];

(B) $\quad e^{isQ_\Sigma}\Gamma_\sigma(a)e^{-isQ_\Sigma}=\gamma^\sigma(s)\Gamma_\sigma(a)\quad$ if $\quad\operatorname{supp}\sigma_a\subsetneqq\Sigma\subset\operatorname{supp}\sigma^c$

[see (5.9) and Lemma 13];

(C) $\quad e^{isQ_\Sigma}\Gamma_\sigma(a)e^{-isQ_\Sigma}=\Gamma_\sigma(a),$

otherwise [see (5.10) and Lemma 13].

Lemma 11 is the converse to this.

Next, we attempt to construct a total charge operator, Q, on the sectors $\mathscr{H}_{\omega\circ\sigma}$ in the case where σ is a localized * morphism, with the help of the local-charge cocycle $\gamma^\sigma(s)$.

Lemma 14. 1) *Suppose that* $\underset{\Sigma\uparrow\mathbb{R}^3}{s\text{-lim}}\, Q_\Sigma\Omega$ *exists and* (5.2) *is valid. Then*

$$\underset{\Sigma\uparrow\mathbb{R}^3}{s\text{-lim}}\, e^{isQ_\Sigma}\Omega=\Omega\,.$$

2) *If* σ *is a localized * morphism and the hypotheses of* 1) *hold then* $\underset{\Sigma\uparrow\mathbb{R}^3}{s\text{-lim}}\, e^{isQ_\Sigma}\equiv e^{isQ}$ *exists on* $\mathscr{H}_{\omega\circ\sigma}$ *and is a unitary group in the center of* $\pi_{\omega\circ\sigma}(\mathfrak{A})''$.

3) *Under the same hypotheses, if* σ_1 *and* σ_2 *are localized, charged * morphisms of charge* q_1, q_2 *resp., with* $q_1\neq q_2$ *then the representations* $\pi_{\omega\circ\sigma_1}$ *and* $\pi_{\omega\circ\sigma_2}$ *of* $\mathfrak{A}$ *are disjoint.*

Proof. 1) By Duhamel's formula and (5.2),

$$(e^{isQ_\Sigma}-e^{isQ_{\Sigma'}})\Omega=\int_0^s e^{itQ_\Sigma}e^{i(s-t)Q_{\Sigma'}}(Q_\Sigma-Q_{\Sigma'})\Omega dt\,.$$

Hence

$$\|(e^{isQ_\Sigma}-e^{isQ_{\Sigma'}})\Omega\|\leqq|s|\,\|(Q_\Sigma-Q_{\Sigma'})\Omega\|\,,$$

which tends to 0, as Σ, $\Sigma'\uparrow\mathbb{R}^3$. Thus

$$\underset{\Sigma\uparrow\mathbb{R}^3}{s\text{-lim}}\, e^{isQ_\Sigma}\Omega\equiv\Psi_s\quad\text{exists, for all}\quad s\in\mathbb{R}.$$

Next

$$\begin{aligned}U_\omega(a)^*\Psi_s&=U_\omega(a)^*\underset{\Sigma\uparrow\mathbb{R}^3}{s\text{-lim}}\, e^{isQ_\Sigma}\Omega\\&=\underset{\Sigma\uparrow\mathbb{R}^3}{s\text{-lim}}\, U_\omega(a)^*e^{isQ_\Sigma}\Omega\\&=\underset{\Sigma\uparrow\mathbb{R}^3}{s\text{-lim}}\, \tau_a(e^{isQ_\Sigma})\Omega\\&=\underset{\Sigma\uparrow\mathbb{R}^3}{s\text{-lim}}\, e^{isQ_{\Sigma(-a)}}\Omega=\Psi_s\,,\end{aligned}$$

for all $a=(\mathbf{a},0)$, $\mathbf{a}\in\mathbb{R}^3$; i.e. Ψ_s is space-translation invariant. Since the vacuum Ω is unique,

$$\Psi_s=e^{i\Phi_s}\Omega,\ \Phi_s\ \text{real}.$$

Clearly

$$e^{isQ_\Sigma}\Omega-\Omega=\int_0^s e^{itQ_\Sigma}Q_\Sigma\Omega dt\,,$$

and the l.s. tends to $(e^{i\Phi_s}-1)\Omega$, as $\Sigma\uparrow\mathbb{R}^3$. Hence

$$|e^{i\Phi_s}-1|^2(\Omega,\Omega)=\lim_{\Sigma\uparrow\mathbb{R}^3}(e^{-i\Phi_s}-1)\int_0^s(e^{-itQ_\Sigma}\Omega,Q_\Sigma\Omega)dt\,. \qquad (5.25)$$

Next,

$$|(e^{-itQ_\Sigma}\Omega, Q_\Sigma\Omega)| \leqq \|Q_\Sigma\Omega\| < \text{const},$$

uniformly in Σ, since $\underset{\Sigma\uparrow\mathbb{R}^3}{s\text{-lim}}\, Q_\Sigma\Omega$ exists by hypothesis. Thus, for arbitrary $\varepsilon > 0$ and Σ large enough (depending on ε).

$$|((e^{-itQ_\Sigma} - e^{i\Phi_{-t}})\Omega, Q_\Sigma\Omega)| < \varepsilon,$$

by the strong convergence of $e^{-itQ_\Sigma}\Omega$ to $e^{i\Phi_{-t}}\Omega$. Therefore

$$\lim_{\Sigma\uparrow\mathbb{R}^3}(e^{-itQ_\Sigma}\Omega, Q_\Sigma\Omega) = e^{-i\Phi_{-t}}\lim_{\Sigma\uparrow\mathbb{R}^3}(\Omega, Q_\Sigma\Omega). \tag{5.26}$$

By Gauss' law,

$$-(\Omega, Q_\Sigma\Omega) = (\Omega, \mathbf{E}(\nabla f_\Sigma \otimes \alpha)\Omega). \tag{5.27}$$

Since Ω is Poincaré-invariant, the r.s. of (5.27) vanishes. Combining (5.25)–(5.27), we conclude that

$$|e^{i\Phi_s} - 1|^2(\Omega, \Omega) = \lim_{\Sigma\uparrow\mathbb{R}^3}(e^{-i\Phi_s} - 1)\int_0^s (e^{-itQ_\Sigma}\Omega, Q_\Sigma\Omega)dt = 0,$$

i.e. $e^{i\Phi_s} = 1$. This completes the proof of 1).

2) Using the operator $T: \mathscr{H}_\omega(\sigma) \to \mathscr{H}_{\omega\circ\sigma}$ constructed in the proof of Theorem 5, we have, for arbitrary $A \in \mathfrak{A}$,

$$e^{isQ_\Sigma}A\Omega_\sigma = e^{isQ_\Sigma}T\sigma(A)\Omega = T\sigma(e^{isQ_\Sigma})\sigma(A)\Omega = T\gamma^\sigma(s)e^{isQ_\Sigma}\sigma(A)\Omega,$$

if $\Sigma \supset \operatorname{supp}\sigma$. Moreover

$$e^{isQ_\Sigma}\sigma(A)\Omega = e^{isQ_\Sigma}\sigma(A)e^{-isQ_\Sigma}e^{isQ_\Sigma}\Omega,$$

and

$$\underset{\Sigma\uparrow\mathbb{R}^3}{n\text{-lim}}\, e^{isQ_\Sigma}\sigma(A)e^{-isQ_\Sigma} = \sigma(A), \quad \text{as} \quad \sigma(A) \in \mathfrak{A},$$

$$\underset{\Sigma\uparrow\mathbb{R}^3}{s\text{-lim}}\, e^{isQ_\Sigma}\Omega = \Omega, \quad \text{by 1)}.$$

Thus

$$\underset{\Sigma\uparrow\mathbb{R}^3}{s\text{-lim}}\, e^{isQ_\Sigma}A\Omega_\sigma = \underset{\Sigma\uparrow\mathbb{R}^3}{s\text{-lim}}\, T\sigma(e^{isQ_\Sigma})\sigma(A)\Omega = T\gamma^\sigma(s)\sigma(A)\Omega.$$

Notice that, by Lemma 12,

$$\gamma^\sigma(s)\sigma(A)\Omega = \sigma(A)\gamma^\sigma(s)\Omega,$$

i.e.

$$\underset{\Sigma\uparrow\mathbb{R}^3}{s\text{-lim}}\, e^{isQ_\Sigma}A\Omega_\sigma \equiv e^{isQ}A\Omega_\sigma = T\sigma(A)\gamma^\sigma(s)\Omega. \tag{5.28}$$

Since e^{isQ_Σ} is a continuous, unitary group, for all bounded Σ, so is e^{isQ}. Moreover, $e^{isQ_\Sigma} \in \mathfrak{A}$, for all bounded Σ and all $s \in \mathbb{R}$. Hence

$$e^{isQ} \in \pi_{\omega\circ\sigma}(\mathfrak{A})''. \tag{5.29}$$

By Lemma 2, e^{isQ} is also in $\pi_{\omega\circ\sigma}(\mathfrak{A})'$, hence it is in the center of $\pi_{\omega\circ\sigma}(\mathfrak{A})''$. This completes the proof of 2).

3) This follows from 2) and a standard theorem [11, 15]. □

Remark. Assuming only that $\underset{\Sigma\uparrow\mathbb{R}^3}{w\text{-lim}}\, e^{isQ_\Sigma}\Omega$ exists, one can prove that $\underset{\Sigma\uparrow\mathbb{R}^3}{w\text{-lim}}\, e^{isQ_\Sigma}$ exists on $\mathscr{H}_{\omega\circ\sigma}$, and if $\underset{\Sigma\uparrow\mathbb{R}^3}{w\text{-lim}}\, Q_\Sigma\Omega$ exists and $\gamma^\sigma(s)\Omega$ is differentiable then $\underset{\Sigma\uparrow\mathbb{R}^3}{w\text{-lim}}\, Q_\Sigma\Omega=0$ and

$$\underset{\Sigma\uparrow\mathbb{R}^3}{w\text{-lim}}\, Q_\Sigma A\Omega_\sigma = T\sigma(A)\frac{d}{ds}\gamma^\sigma(s=0)\Omega$$

(see also Sect. 2 of Paper I).

Definition 5. A * morphism σ of $\mathfrak{A}$ is called a *localized, charged * morphism* of charge q iff $\gamma^\sigma(s)=e^{isq}$. A quasi-local translation cocycle Γ is called a *charge-transfer cocycle of charge* q iff $\Gamma(a)\in\mathfrak{A}$, for all $a\in M^4$, and Γ satisfies (5.13)–(5.15), with Σ_Γ compact and $\gamma(s)=e^{isq}$. □

We summarize a part of our findings (Lemmas 10–14) in

Theorem 15. 1) *If Γ is a charge-transfer cocycle of charge q then σ_Γ (defined in* (4.1)*) is a localized, charged * morphism of $\mathfrak{A}$ of charge q (Lemma 11).*

2) *If σ is a transportable, localized * morphism with the property that $\Gamma_\sigma(a)\in\mathfrak{A}$, for all $a\in M^4$ then*

$$\tau_a(\gamma^\sigma(s))=\Gamma_\sigma(a)^*\gamma^\sigma(s)\Gamma_\sigma(a)$$

(Lemma 13), and if σ has charge q then Γ_σ is a charge-transfer cocycle of charge q (Lemma 10).

3) *If $\sigma_1,\ldots,\sigma_n$ are localized, charged * morphisms with charges $q_1,\ldots,q_n$ then $\sigma_1\circ\ldots\circ\sigma_n$ is a localized, charged * morphism with charge $q_1+\ldots+q_n$ (the proof is a simple exercise).*

4) *If $\underset{\Sigma\uparrow\mathbb{R}^3}{s\text{-lim}}\, Q_\Sigma\Omega$ exists (neutrality of the vacuum, Lemma 14) and σ has charge q then*

$$Q=q\cdot 1, \quad \text{on} \quad \mathscr{H}_{\omega\circ\sigma}, \quad \text{and}$$

if σ_1 and σ_2 have charge q_1, q_2, resp., with $q_1 \neq q_2$, then $\pi_{\omega\circ\sigma_1}$ and $\pi_{\omega\circ\sigma_2}$ are disjoint representations of $\mathfrak{A}$ (Lemma 14).

6. Space-Time Translation Covariant, Charged * Morphisms

In this section we study a class $\mathscr{C}$ of quasi-local translation cocycles with the property that, for $\Gamma_1,\ldots,\Gamma_n$ in $\mathscr{C}$, $n=1,2,3,\ldots$, the state $\omega\circ\sigma_{\Gamma_1}\circ\ldots\circ\sigma_{\Gamma_n}$ is space-time translation covariant (see Definition 1, Sect. 1), and to each $\Gamma\in\mathscr{C}$ there exists a conjugate cocycle $\bar{\Gamma}\in\mathscr{C}$ such that the representation $\pi_{\omega\circ\sigma_{\bar{\Gamma}}\circ\sigma_\Gamma}$ of $\mathfrak{A}$ contains the representation π_ω exactly once. This last property can be interpreted as PCT invariance of the theory (see [3]).

6.1. Translation Covariant * Morphisms

Let σ be an arbitrary, transportable * morphism of $\mathfrak{A}$ with translation cocycle Γ_σ. From Theorem 5 (Sect. 3) we know that $\omega\circ\sigma$ is translation covariant iff the subspace

$$\mathscr{H}_\omega(\sigma)=\{\sigma(A)\Omega : A\in\mathfrak{A}\}^- \subseteqq \mathscr{H}_\omega$$

is invariant under the group

$$V_\sigma(a) = \Gamma_\sigma(a)U_\omega(a), \quad a \in \mathbb{M}^4.$$

There are thus two approaches to proving translation covariance of $\omega \circ \sigma$:

1) Show that $\mathscr{H}_\omega(\sigma)$ is invariant under $V_\sigma(a)$.

2) Show that $\mathscr{H}_\omega(\sigma) = \mathscr{H}_\omega$.

It appears that approach 1) is the natural one. We try to elucidate this by the following discussion: Let $\mathcal{O}$ be a bounded double cone and T a positive number. We define

$$M_T \equiv \{a \in \mathbb{M}^4 : a = (a^0, \mathbf{a}), |a^0| \leqq T, |\mathbf{a}| \leqq T\},$$

and

$$\mathcal{O}^T = \bigcup_{a \in M_T} \mathcal{O}(a).$$

We now assume that the space-time translation automorphisms τ_a of $\mathfrak{A}$ have locally correct generators: Given $\mathcal{O} \in \mathfrak{B}$ and $T > 0$, there exist $\mathcal{O}(\mathcal{O}, T) \in \mathfrak{B}$ with $\mathcal{O}(\mathcal{O}, T) \supseteqq \mathcal{O}^T$ and operators $U_{\mathcal{O},T}(a) \in \mathfrak{A}(\mathcal{O}(\mathcal{O}, T))$ such that, for all $A \in \mathfrak{A}(\mathcal{O})$ and $a \in M_T$,

$$U_{\mathcal{O},T}(a)^* A U_{\mathcal{O},T}(a) = \tau_a(A). \tag{6.1}$$

The existence problem of operators $U_{\mathcal{O},T}(a)$ with these properties can be reduced to showing that, for each $\mathcal{O} \in \mathfrak{B}$ and $r > 1$, there exists a factor $N_{\mathcal{O},r}$ of type I_∞ such that

$$\mathfrak{A}(\mathcal{O}) \subset N_{\mathcal{O},r} \subset \mathfrak{A}(\sim {}^r\mathcal{O}), \tag{6.2}$$

where ${}^r\mathcal{O} = \{x \in \mathbb{M}^4 : r^{-1}x \in \mathcal{O}\}$, [16].

Property (6.2) has been established for the free, scalar field by Buchholz [17], but it is believed to be a general property of the local nets $\{\mathfrak{A}(\mathcal{O})\}_{\mathcal{O} \in \mathfrak{B}}$ of relativistic quantum field theories.

We now assume, in addition, that

$$\underset{\mathcal{O} \uparrow \mathbb{M}^4}{s\text{-lim}}\, U_{\mathcal{O},T}(a)\Omega = \Omega, \tag{6.3}$$

for all $a \in M_T$ and all $T < \infty$.

Next, we study those * morphisms σ of $\mathfrak{A}$ which have the property that

$$\underset{\mathcal{O} \uparrow \mathbb{M}^4}{s\text{-lim}}\, \sigma(U_{\mathcal{O},T}(a))U_{\mathcal{O},T}(a)^* \tag{6.4}$$

exists on $\mathscr{H}_\omega$, for all $T < \infty$. We leave it to the reader to check that the limit in (6.4) defines a translation cocycle in the sense of Definition 2, Sect. 3.

Lemma 16. *Assume (6.1) and (6.3). Let σ be a * morphism of $\mathfrak{A}$ with the property that (6.4) is satisfied. Then σ is transportable, with*

$$\Gamma_\sigma(a) = \underset{\mathcal{O} \uparrow \mathbb{M}^4}{s\text{-lim}}\, \sigma(U_{\mathcal{O},T}(a))U_{\mathcal{O},T}(a)^*, \tag{6.5}$$

for all $a \in M_T$; $V_\sigma(a) = \Gamma_\sigma(a)U_\omega(a)$ leaves $\mathscr{H}_\omega(\sigma)$ invariant, and $\omega \circ \sigma$ is space-time translation covariant.

Proof. Let $A\in\bigcup_{\mathcal{O}\in\mathfrak{B}}\mathfrak{A}(\mathcal{O})$. Then, for $a\in M_T$ and arbitrary T,

$$\tau_{-a}\circ\sigma\circ\tau_a(A)=n\text{-}\lim_{\mathcal{O}\uparrow\mathbb{M}^4} U_{\mathcal{O},T}(a)\sigma(U_{\mathcal{O},T}(a)^*AU_{\mathcal{O},T}(a))U_{\mathcal{O},T}(a)^*.$$

since $A\in\bigcup_{\mathcal{O}\in\mathfrak{B}}\mathfrak{A}(\mathcal{O})$, and $\sigma(\tau_a(A))\in\mathfrak{A}$.

The first part of Lemma 16 and (6.5) follow by writing out the r.s. of this equation and applying (6.4).

Next, we prove invariance of $\mathscr{H}_\omega(\sigma)$ under $V_\sigma(a)$: For all $A\in\mathfrak{A}$, $a\in M_T$,

$$\begin{aligned}\Gamma_\sigma(a)U_\omega(a)\sigma(A)\Omega&=\Gamma_\sigma(a)\tau_{-a}(\sigma(A))\Omega\\
&=s\text{-}\lim_{\mathcal{O}\uparrow\mathbb{M}^4}\sigma(U_{\mathcal{O},T}(a))U_{\mathcal{O},T}(a)^*\tau_{-a}(\sigma(A))\Omega\,,\quad\text{by (6.5)}\\
&=s\text{-}\lim_{\mathcal{O}\uparrow\mathbb{M}^4}\sigma(U_{\mathcal{O},T}(a))U_{\mathcal{O},T}(a)^*\tau_{-a}(\sigma(A))U_{\mathcal{O},T}(a)\Omega\,,\quad\text{by (6.3)}\\
&=s\text{-}\lim_{\mathcal{O}\uparrow\mathbb{M}^4}\sigma(U_{\mathcal{O},T}(a)A)\Omega\,.\end{aligned}\tag{6.6}$$

For all $\mathcal{O}\in\mathfrak{B}$, $T<\infty$, $\sigma(U_{\mathcal{O},T}(a)A)\Omega\in\mathscr{H}_\omega(\sigma)$. Since $\mathscr{H}_\omega(\sigma)$ is closed, (6.6) implies that

$$\Gamma_\sigma(a)U_\omega(a)\sigma(A)\Omega\in\mathscr{H}_\omega(\sigma)\,.$$

The space-time translation covariance of $\omega\circ\sigma$ now follows from Theorem 5. □

On the basis of Lemma 16 one might conjecture that, in general, $\mathscr{H}_\omega(\sigma)$ is invariant under $V_\sigma(a)$, whenever σ is a transportable * morphism.

We define $\mathscr{C}_1$ to be the class of all those quasi-local translation cocycles which have the property that, for $\Gamma_1,\ldots,\Gamma_n$ in $\mathscr{C}_1$, $n=1,2,3,\ldots$, $\mathscr{H}_\omega(\sigma_{\Gamma_1}\circ\ldots\circ\sigma_{\Gamma_n})$ is invariant under $V_{\sigma_{\Gamma_1}\circ\ldots\circ\sigma_{\Gamma_n}}(a)$, $a\in\mathbb{M}^4$, where

$$V_{\sigma_{\Gamma_1}\circ\ldots\circ\sigma_{\Gamma_n}}(a)=\sigma_{\Gamma_1}\circ\ldots\circ\sigma_{\Gamma_{n-1}}(\Gamma_n(a))\ldots\sigma_{\Gamma_1}(\Gamma_2(a))\Gamma_1(a)U_\omega(a)\tag{6.7}$$

(see Theorem 9, Sect. 4). By Lemma 16, it suffices that $\sigma_{\Gamma_1}\circ\ldots\circ\sigma_{\Gamma_n}$ satisfies (6.4).

Next, we discuss conditions which guarantee that

$$\mathscr{H}_\omega(\sigma)=\mathscr{H}_\omega\,.$$

Let σ be a localized * morphism of compact support, i.e.

$$\sigma(e^{isQ_\Sigma})e^{-isQ_\Sigma}=1\,,\tag{6.8}$$

for all $\Sigma\subset\operatorname{supp}\sigma^c$ (see Sect. 5).

Physically, Eq. (6.8) says that the charge carried by σ is localized in the compact region $\operatorname{supp}\sigma\subset\mathbb{R}^3$.

From Proposition 3 (Sect. 2) we know that this does *not* imply that σ is local in the sense of DHR [3], in the contrary, σ is not local unless its charge is 0.

However, one might expect that σ is quite close to acting trivially on $\mathfrak{A}(\mathcal{O})$, provided $\mathcal{O}$ is a *bounded* double cone which is space-like distant from $\operatorname{supp}\sigma$. [One might expect, moreover, that for such morphisms (6.4) is true.] A possible way of expressing that is as follows: There exists some compact region $\Sigma_\sigma\subset\mathbb{R}^3$, with $\Sigma_\sigma\supseteqq\operatorname{supp}\sigma$, such that for arbitrary $\mathcal{O}\in\mathfrak{B}$ with $\mathcal{O}\subset\sim\Sigma_\sigma$

$$\sigma(\mathfrak{A}(\mathcal{O}))=\mathfrak{A}(\mathcal{O})\,.\tag{6.9}$$

Clearly, (6.9) implies that $\mathscr{H}_\omega(\sigma)=\mathscr{H}_\omega$. (This follows from the Reeh-Schlieder property, as noted in the proof of Lemma 1, Sect. 1.)

If σ is transportable and $\mathcal{O}\subset\sim(\Sigma_\sigma\cup\Sigma_{\sigma_a})$, for some $a\in\mathbb{M}^4$, then by (6.9)

$$\mathfrak{A}(\mathcal{O})=\sigma(\mathfrak{A}(\mathcal{O}))=\sigma_a(\mathfrak{A}(\mathcal{O}))=\Gamma(a)^*\sigma(\mathfrak{A}(\mathcal{O}))\Gamma(a)=\Gamma(a)^*\mathfrak{A}(\mathcal{O})\Gamma(a).$$

[Note, however, that the condition

$$\mathfrak{A}(\mathcal{O})=\Gamma(a)^*\mathfrak{A}(\mathcal{O})\Gamma(a),$$

for $\mathcal{O}\subset\sim(\Sigma(\Gamma)\cup\Sigma(\Gamma)(a))$, for some compact region $\Sigma(\Gamma)\subset\mathbb{R}^3$ *only* implies

$$\left\{w\text{-}\lim_{a\to\infty}\Gamma(a)A\Gamma(a)^*:A\in\mathfrak{A}(\mathcal{O})\right\}\subseteqq\mathfrak{A}(\mathcal{O}),$$

i.e. it appears difficult to characterize those localized * morphisms σ_Γ which satisfy (6.9) entirely in terms of the cocycle Γ.]

We let $\mathscr{C}_2$ be the class of all those quasi-local translation cocycles, Γ, which have the property that

$$\sigma_\Gamma(\mathfrak{A}(\mathcal{O}))=\mathfrak{A}(\mathcal{O}),\tag{6.10}$$

for all $\mathcal{O}\in\mathfrak{B}$ with $\mathcal{O}\subset\sim\Sigma_{\sigma_\Gamma}$.

Let $\Gamma_1,\ldots,\Gamma_n$ be in $\mathscr{C}_2$ and let $\mathcal{O}\in\mathfrak{B}$, with $\mathcal{O}\subset\sim(\Sigma_{\sigma_{\Gamma_1}}\cup\ldots\cup\Sigma_{\sigma_{\Gamma_n}})$. Then

$$\sigma_{\Gamma_1}\circ\ldots\circ\sigma_{\Gamma_n}(\mathfrak{A}(\mathcal{O}))=\sigma_{\Gamma_1}\circ\ldots\circ\sigma_{\Gamma_{n-1}}(\mathfrak{A}(\mathcal{O}))=\ldots=\sigma_{\Gamma_1}(\mathfrak{A}(\mathcal{O}))=\mathfrak{A}(\mathcal{O}).\tag{6.11}$$

Hence, by the Reeh-Schlieder property,

$$\mathscr{H}_\omega(\sigma_{\Gamma_1}\circ\ldots\circ\sigma_{\Gamma_n})=\mathscr{H}_\omega,\tag{6.12}$$

for arbitrary $\Gamma_1,\ldots,\Gamma_n$ in $\mathscr{C}_2$, so that $\omega\circ\sigma_{\Gamma_1}\circ\ldots\circ\sigma_{\Gamma_n}$ is space-time translation covariant.

Next, we introduce a class $\mathscr{C}_3$ of quasi-local translation cocycles:

$$\Gamma\in\mathscr{C}_3\quad\text{iff}\quad\Gamma(a)\in\sigma_\Gamma(\mathfrak{A}),\tag{6.13}$$

for all $a\in\mathbb{M}^4$.

We say that a quasi-local translation cocycle Γ is *irreducible* iff $\Gamma(a)\in\pi_\omega(\sigma_\Gamma(\mathfrak{A}))''$. (Clearly, if $\Gamma\in\mathscr{C}_3$ then Γ is irreducible.)

Lemma 17. *Let Γ be a quasi-local translation cocycle.*

1) *If Γ is irreducible then σ_Γ is an irreducible * morphism of $\mathfrak{A}$, i.e. π_ω is an irreducible representation of $\sigma_\Gamma(\mathfrak{A})$.*

2) *If $\Gamma_1,\ldots,\Gamma_n$ are in $\mathscr{C}_3$ then $\sigma_{\Gamma_1}\circ\ldots\circ\sigma_{\Gamma_n}$ is irreducible.*

Remarks. The converse of 1) is of course trivial. By 2) we have $\mathscr{H}_\omega(\sigma_{\Gamma_1}\circ\ldots\circ\sigma_{\Gamma_n})=\mathscr{H}_\omega$, for arbitrary $\Gamma_1,\ldots,\Gamma_n$ in $\mathscr{C}_3$. Hence the state $\omega\circ\sigma_{\Gamma_1}\circ\ldots\circ\sigma_{\Gamma_n}$ is space-time translation covariant.

Proof of Lemma 17. 1) Since Γ is irreducible, $\Gamma(a)\in\pi_\omega(\sigma_\Gamma(\mathfrak{A}))''$, for all a. Hence, for all

$A\in\bigcup_{\mathcal{O}\in\mathfrak{B}}\mathfrak{A}(\mathcal{O})$,

$$\sigma_{\Gamma,a}(A)=\tau_{-a}\circ\sigma_\Gamma\circ\tau_a(A)=\Gamma(a)^*\sigma_\Gamma(A)\Gamma(a)\in\pi_\omega(\sigma_\Gamma(\mathfrak{A}))''.$$

By Proposition 8, Sect. 4,

$$w\text{-}\lim_{a\to\infty} \sigma_{\Gamma,a}(A) = A\,.$$

Since $\pi_\omega(\sigma_\Gamma(\mathfrak{A}))''$ is weakly closed, $A \in \pi_\omega(\sigma_\Gamma(\mathfrak{A}))''$. Hence $\pi_\omega(\sigma_\Gamma(\mathfrak{A}))'' \supset \bigcup_{\mathcal{O}\in\mathfrak{B}} \mathfrak{A}(\mathcal{O})$, and therefore

$$\pi_\omega(\sigma_\Gamma(\mathfrak{A}))'' = \pi_\omega(\mathfrak{A})''\,,$$

i.e. π_ω is an irreducible representation of $\sigma_\Gamma(\mathfrak{A})$.

2) If $\Gamma_1, \ldots, \Gamma_n$ are in $\mathscr{C}_3$ then

$$\sigma_{\Gamma_1} \circ \ldots \circ \sigma_{\Gamma_{n-1}} \circ \sigma_{\Gamma_n,a}(A) = \sigma_{\Gamma_1} \circ \ldots \circ \sigma_{\Gamma_{n-1}}(\Gamma_n(a)^* \sigma_{\Gamma_n}(A)\Gamma_n(a)) \in \sigma_{\Gamma_1} \circ \ldots \circ \sigma_{\Gamma_n}(\mathfrak{A})\,,$$

for all $A \in \bigcup_{\mathcal{O}\in\mathfrak{B}} \mathfrak{A}(\mathcal{O})$. As $a \to \infty$, we obtain, using Proposition 8 and the local normality of the morphisms $\sigma_{\Gamma_1}, \ldots, \sigma_{\Gamma_{n-1}}$,

$$\pi_\omega(\sigma_{\Gamma_1} \circ \ldots \circ \sigma_{\Gamma_{n-1}}(A)) \in \pi_\omega(\sigma_{\Gamma_1} \circ \ldots \circ \sigma_{\Gamma_n}(\mathfrak{A}))''\,,$$

i.e.

$$\pi_\omega(\sigma_{\Gamma_1} \circ \ldots \circ \sigma_{\Gamma_n}(\mathfrak{A}))'' \supseteq \pi_\omega(\sigma_{\Gamma_1} \circ \ldots \circ \sigma_{\Gamma_{n-1}}(\mathfrak{A}))\,.$$

Proceeding in this manner we conclude, after n steps, that

$$\pi_\omega(\sigma_{\Gamma_1} \circ \ldots \circ \sigma_{\Gamma_n}(\mathfrak{A}))'' \supseteq \pi_\omega(\mathfrak{A})\,,$$

i.e.

$$\pi_\omega \circ \sigma_{\Gamma_1} \circ \ldots \circ \sigma_{\Gamma_n}$$

is irreducible. □

Remarks. 1. Let Γ be a quasi-local translation cocycle. Then σ_Γ is a * automorphism if and only if

$$\Gamma(a) = \sigma_\Gamma(\Gamma'(a))\,, \tag{6.14}$$

for some quasi-local translation cocycle Γ'. (The proof is given in Appendix 2.) Clearly a cocycle Γ satisfying (6.14) is in $\mathscr{C}_3$.

2. The fact that, for $\Gamma_1, \ldots, \Gamma_n$ in $\mathscr{C}_3$, the * morphism $\sigma_{\Gamma_1} \circ \ldots \circ \sigma_{\Gamma_n}$ of $\mathfrak{A}$ is irreducible will imply that the sectors $\omega \circ \sigma_\Gamma$, $\Gamma \in \mathscr{C}_3$, have necessarily ordinary Fermi- or Bose-statistics, i.e. parastatistics is automatically excluded.

Thus, the hypothesis that the charged sectors of a theory be generated by all * morphisms $\{\sigma_\Gamma : \Gamma \in \mathscr{C}_3\}$ might be appropriate in QED, but cannot be valid in more general gauge theories with an unconfined, abelian charge and parastatistics.

Section 6.1 can be summarized as follows: Let $\Gamma_1, \ldots, \Gamma_n$ be quasi-local translation cocycles in one of the classes $\mathscr{C}_1$, $\mathscr{C}_2$, $\mathscr{C}_3$ [see (6.7), (6.10), (6.13), resp.]. Then $\omega \circ \sigma_{\Gamma_1} \circ \ldots \circ \sigma_{\Gamma_n}$ is space-time translation covariant.

Our discussion leaves the problem open to characterize those localized charged * morphisms which are space-time translation covariant entirely in terms of quasi-local translation cocycles.

6.2. Existence of Conjugate Sectors

In this section we discuss the following problem: Suppose Γ is a quasi-local translation cocycle. Does there exist a quasi-local translation cocycle $\bar{\Gamma}$ such that $\pi_{\omega\circ\sigma_{\bar{\Gamma}}\circ\sigma_\Gamma}$ contains π_ω precisely once? In the DHR theory, the answer to this is yes (see [3, 6]).

In our case, however, where the basic * morphisms, σ, generating the charged sectors are necessarily non-local, this is not clear, at all.

The first problem one meets is that, given σ, the existence of a left inverse to σ, Φ [i.e. $\Phi(\sigma(A))=A$, for all $A\in\mathfrak{A}$] is not automatic. One has only

Proposition 18. *Let Γ be a quasi-local translation cocycle with $\Gamma(a)\in\mathfrak{A}$, for all $a\in\mathbb{M}^4$. Then, for some sequence $\{a_n\}$, $a_n\to\infty$, as $n\to\infty$,*

$$\omega_{\Gamma^*}(A)=\lim_{n\to\infty}\omega(\Gamma(a_n)^*A\Gamma(a_n)) \text{ exists},$$

and $\omega_{\Gamma^*}(\sigma_\Gamma(A))=\omega(A)$, for all $A\in\mathfrak{A}$. If

$$w\text{-}\lim_{a\to\infty}(\Gamma(a+b)^*A\Gamma(a+b)-\Gamma(a)^*A\Gamma(a))=0, \tag{6.15}$$

for all $b\in\mathbb{M}^4$, then ω_{Γ^*} is a translation covariant state on $\mathfrak{A}$.

Remarks. The existence of ω_{Γ^*} follows from a general compactness argument. Next,

$$\begin{aligned}\omega_{\Gamma^*}(\sigma_\Gamma(A)) &= \lim_{n\to\infty}\omega(\Gamma(a_n)^*\sigma_\Gamma(A)\Gamma(a_n))\\ &=\lim_{n\to\infty}\omega(\sigma_{\Gamma,a_n}(A))=\omega(A),\end{aligned}$$

by Proposition 8. Finally, using the cocycle identity (3.1) and (6.15) one shows that

$$\omega_{\Gamma^*}(A\tau_a(B)C)=\omega_{\Gamma^*}(\Gamma(a)\tau_{-a}(A)B\tau_{-a}(C)\Gamma(a)^*),$$

so that the unitary group $U_{\omega_{\Gamma^*}}(a)$, defined on $\mathscr{H}_{\omega_{\Gamma^*}}$ by

$$U_{\omega_{\Gamma^*}}(a)A\Omega_{\omega_{\Gamma^*}}=\tau_{-a}(A)\Gamma(a)^*\Omega_{\omega_{\Gamma^*}},$$

implements τ_a. (Details of the proof of Proposition 18 are left to the reader.)

In the DHR theory [3]

$$n\text{-}\lim_{n\to\infty}\Gamma(a_n)^*A\Gamma(a_n)=\Phi(A)$$

exists always, for some sequence $\{a_n\}$, and Φ is a left inverse of σ (with the same support as σ).

Let Γ be as in Proposition 18. Suppose that $\omega_{\Gamma^*}=\omega\circ\Phi$, where Φ is a * morphism of $\mathfrak{A}$ and a left inverse of σ_Γ. Using (6.15) and the cocycle identity (3.1) one can show that

$$\tau_{-a}\circ\Phi\circ\tau_a(A)=\Phi(\Gamma(a)A\Gamma(a)^*)=\Phi(\Gamma(a))\Phi(A)\Phi(\Gamma(a))^*)).$$

Thus Φ is transportable, and $\Gamma_\Phi(a)=\Phi(\Gamma(a)^*)\in\Phi(\mathfrak{A})$. This motivates the study of the class $\mathscr{C}_3$ introduced in Sect. 6.2. (In the present case, σ_Γ and $\Phi=\sigma_\Gamma^{-1}$ are actually * automorphisms of $\mathfrak{A}$; see Appendix 2).

Lemma 19. *If Φ is a left inverse of a localized, charged * morphism σ then Φ is localized, with* supp Φ = supp σ, *and if σ has charge q then Φ has charge $-q$.*

Proof. Let $\Sigma \subset \operatorname{supp} \sigma^c$, i.e. $\gamma_\Sigma^\sigma(s)=1$. Since Φ is a left inverse of σ, we have

$$\begin{aligned} 1 &= \Phi(\sigma(e^{isQ_\Sigma}))e^{-isQ_\Sigma} \\ &= \Phi(\gamma_\Sigma^\sigma(s)e^{isQ_\Sigma})e^{-isQ_\Sigma} \\ &= \Phi(e^{isQ_\Sigma})e^{-isQ_\Sigma} \equiv \gamma_\Sigma^\Phi(s). \end{aligned}$$

Thus, $\operatorname{supp} \Phi = \operatorname{supp} \sigma$.

Next, let $\Sigma \supset \operatorname{supp} \sigma = \operatorname{supp} \Phi$, and suppose that $\gamma^\sigma(s)=e^{isq}$. Then

$$\begin{aligned} 1 &= \Phi(\sigma(e^{isQ_\Sigma}))e^{-isQ_\Sigma} \\ &= \Phi(\gamma_\Sigma^\sigma(s)e^{isQ_\Sigma})e^{-isQ_\Sigma} \\ &= \gamma^\Phi(s)e^{isq}, \quad \text{i.e.} \quad \gamma^\Phi(s)=e^{-isq}. \quad \square \end{aligned}$$

In contrast to the situation met in the DHR framework [3], neither the existence of left inverses nor the one of conjugate morphisms appear to be automatic, in the present framework. Therefore, in the absence of general results which guarantee that, given a quasi-local translation cocycle Γ, there exists a transportable * morphism $\bar{\sigma}$ such that the representation

$$\pi_{\omega \circ \bar{\sigma} \circ \sigma_\Gamma} \text{ of } \mathfrak{A} \text{ contains } \pi_\omega \text{ precisely once}, \tag{6.16}$$

one must attempt to formulate a plausible principle which ensures that (6.16) is valid. Such a principle is suggested by the "topological" commutation relations (A)–(C), subsequent to Lemma 13 (Sect. 5), and the identity

$$\gamma^{\sigma_{\Gamma_1} \circ \sigma_{\Gamma_2}}(s) = \sigma_{\Gamma_1}(\gamma^{\sigma_{\Gamma_2}}(s))\gamma^{\sigma_{\Gamma_1}}(s). \tag{6.17}$$

Henceforth we assume that

$$\operatorname*{s\text{-}lim}_{\Sigma \uparrow \mathbb{R}^3} Q_\Sigma \Omega \quad \text{exists}, \tag{6.18}$$

so that the total electric charge operator exists on all sectors generated by localized * morphisms of $\mathfrak{A}$ (see Lemma 14).

Conjugation Principle. 1) Let Γ be a quasi-local translation cocycle satisfying the "topological" commutation relations (A)–(C), for some local-charge cocycle γ^Γ.

Then there exists a quasi-local translation cocycle $\tilde{\Gamma}$ satisfying the commutation relations (A)–(C), for a local-charge cocycle $\gamma^{\tilde{\Gamma}}$ with the property that the unitary group

$$\{\sigma_{\tilde{\Gamma}}(\gamma^\Gamma(s))\gamma^{\tilde{\Gamma}}(s)\} \text{ has eigenvalue } 1. \tag{6.19}$$

2) All super-selection sectors of the theory of total electric charge 0 are generated by strictly *local*, transportable * morphisms of $\mathfrak{A}$, in the sense of DHR, [3]. $\square$

Using (6.19) and the fact that

$$\sigma_{\tilde{\Gamma}}(\gamma^\Gamma(s))\gamma^{\tilde{\Gamma}}(s) \in \mathfrak{A} \cap \sigma_{\tilde{\Gamma}} \circ \sigma_\Gamma(\mathfrak{A})',$$

for all s (see Lemma 12), one verifies easily that the representation $\pi_{\omega \circ \sigma_{\tilde{\Gamma}} \circ \sigma_\Gamma}$ of $\mathfrak{A}$ contains a subrepresentation of $\mathfrak{A}$ of total charge 0. Part 2) of the Conjugation Principle then says that that subrepresentation is of the form $\pi_{\omega \circ \sigma_{\mathrm{loc}}}$, where σ_{loc} is a local * morphism of $\mathfrak{A}$, in the sense of DHR. Their results then imply that there exists a * morphism $\bar{\sigma}_{\mathrm{loc}}$ such that $\pi_{\omega\circ\bar{\sigma}_{\mathrm{loc}}\circ\sigma_{\mathrm{loc}}}$ contains π_ω precisely once. Therefore, the morphism conjugate to σ_Γ is

$$\bar{\sigma}_\Gamma = \bar{\sigma}_{\mathrm{loc}} \circ \sigma_{\tilde{\Gamma}}. \tag{6.20}$$

Since $\bar{\sigma}_{\mathrm{loc}}$ and $\sigma_{\tilde{\Gamma}}$ are transportable, so is $\bar{\sigma}_\Gamma$ (see Theorem 9). Moreover, $\bar{\sigma}_\Gamma$ is uniquely determined by its translation cocycle $\Gamma_{\bar{\sigma}_\Gamma}$, given by

$$\Gamma_{\bar{\sigma}_\Gamma}(a) = \bar{\sigma}_{\mathrm{loc}}(\tilde{\Gamma}(a))\Gamma_{\bar{\sigma}_{\mathrm{loc}}}(a),$$

as is easy to check (see (4.1) and [3]).

Thus, the Conjugation Principle guarantees (6.16). If the morphisms σ_Γ and $\sigma_{\tilde{\Gamma}}$ are irreducible then

$$\gamma^\Gamma(s) = \gamma^{\sigma_\Gamma}(s) = e^{isq},$$

and

$$\gamma^{\tilde{\Gamma}}(s) = \gamma^{\sigma_{\tilde{\Gamma}}}(s) = e^{is\tilde{q}}, \tag{6.21}$$

for some q and $\tilde{q}$ in $\mathbb{R}$ [see Lemma 11 and (5.18)]. The Conjugation Principle then implies that

$$\gamma^{\tilde{\Gamma}}(s) = \gamma^\Gamma(s)^* = e^{-isq}, \tag{6.22}$$

because

$$\sigma_{\tilde{\Gamma}}(\gamma^\Gamma(s))\gamma^{\tilde{\Gamma}}(s) = 1, \quad \text{or} \quad \gamma^{\tilde{\Gamma}}(s) = \sigma_{\tilde{\Gamma}}(\gamma^\Gamma(s)^*). \tag{6.23}$$

In this case, the sectors $\mathscr{H}_{\omega\circ\sigma_{\tilde{\Gamma}}\circ\sigma_\Gamma}$ and $\mathscr{H}_{\omega\circ\sigma_\Gamma\circ\sigma_{\tilde{\Gamma}}}$ have both total electric charge 0.

Naturally, (6.23) suggests a converse problem. Suppose that

$$\gamma^{\tilde{\Gamma}}(s) = \sigma_{\tilde{\Gamma}}(\gamma^\Gamma(s)^*). \tag{6.24}$$

By Lemma 13,

$$\begin{aligned}\tau_{-a}(\gamma^{\tilde{\Gamma}}(s)) &= \tilde{\Gamma}(s)^*\gamma^{\tilde{\Gamma}}(s)\tilde{\Gamma}(a)\\ &= \tilde{\Gamma}(a)^*\sigma_{\tilde{\Gamma}}(\gamma^\Gamma(s)^*)\tilde{\Gamma}(a)\\ &= \sigma_{\tilde{\Gamma},a}(\gamma^\Gamma(s)^*).\end{aligned} \tag{6.25}$$

Proposition 8 gives

$$\underset{a\to\infty}{w\text{-lim}}\,\sigma_{\tilde{\Gamma},a}(\gamma^\Gamma(s)^*) = \gamma^\Gamma(s)^*, \quad \text{on} \quad \mathscr{H}_\omega.$$

Thus,

$$\underset{a\to\infty}{w\text{-lim}}\,\tau_{-a}(\gamma^{\tilde{\Gamma}}(s)) \quad \text{exists}. \tag{6.26}$$

Clearly,

$$\underset{a\to\infty}{w\text{-lim}}\,\tau_{-a}(\gamma^{\tilde{\Gamma}}(s)) \in \pi_\omega(\mathfrak{A})'. \tag{6.27}$$

But $\gamma^\Gamma(s)^* \in \mathfrak{A}$. Combining this with (6.25)–(6.27), we conclude, using the irreducibility of π_ω, that

$$\underset{a\to\infty}{w\text{-lim}}\,\tau_{-a}(\gamma^{\tilde{\Gamma}}(s)) \in \mathfrak{A}\cap\pi_\omega(\mathfrak{A})' = \{\lambda 1 : \lambda \in \mathbb{C}\},$$

i.e.

$$\gamma^\Gamma(s) = e^{isq} = \gamma^{\tilde{\Gamma}}(s)^*, \tag{6.28}$$

for some $q \in \mathbb{R}$. Therefore

$$\gamma^{\sigma_{\tilde{\Gamma}} \circ \sigma_\Gamma}(s) = \gamma^{\sigma_\Gamma \circ \sigma_{\tilde{\Gamma}}}(s) = 1,$$

i.e. $\sigma_{\tilde{\Gamma}} \circ \sigma_\Gamma$ and $\sigma_\Gamma \circ \sigma_{\tilde{\Gamma}}$ carry electric charge 0. Part 2) of the Conjugation Principle then implies

$$\sigma_{\tilde{\Gamma}} \circ \sigma_\Gamma = \sigma_{\text{loc}},$$

for some local * morphism σ_{loc} of $\mathfrak{A}$. By [3], there exists a conjugate morphism $\bar{\sigma}_{\text{loc}}$ such that $\pi_{\omega \circ \bar{\sigma}_{\text{loc}} \circ \sigma_{\text{loc}}}$ contains π_ω precisely once, and therefore, with $\bar{\sigma}_\Gamma = \bar{\sigma}_{\text{loc}} \circ \sigma_{\tilde{\Gamma}}$, one has that $\pi_{\omega \circ \bar{\sigma}_\Gamma \circ \sigma_\Gamma}$ contains π_ω precisely once, i.e. (6.16) holds.

Thus, we have proven

Lemma 20. *Suppose that the Conjugation Principle and Eq.* (6.24) *are true. Then*

$$\gamma^\Gamma(s) = e^{isq} = \gamma^{\tilde{\Gamma}}(s)^*,$$

*for some $q \in \mathbb{R}$, and there exists a local, transportable * morphism $\bar{\sigma}_{\text{loc}}$ such that $\bar{\sigma}_\Gamma \equiv \bar{\sigma}_{\text{loc}} \circ \sigma_{\tilde{\Gamma}}$ is conjugate to σ_Γ, in the sense of* (6.16). *Moreover $\bar{\sigma}_\Gamma$ is transportable.*

If the vacuum sector, $\mathscr{H}_\omega$, is the only super-selection sector of the theory of total electric charge 0, *then*

$$\pi_{\omega \circ \sigma_{\tilde{\Gamma}} \circ \sigma_\Gamma} \cong \pi_{\omega \circ \sigma_\Gamma \circ \sigma_{\tilde{\Gamma}}} \cong \pi_\omega, \tag{6.29}$$

*σ_Γ and $\sigma_{\tilde{\Gamma}}$ are * automorphisms of $\mathfrak{A}$, and $\tilde{\Gamma}$ can be so chosen that $\sigma_{\tilde{\Gamma}} = \sigma_\Gamma^{-1}$.*

Remarks. 1. To prove the last part of Lemma 20, we note that (6.29) follows from part 2) of the Conjugation Principle and the absence of non-trivial, local * morphisms (i.e. $\sigma_{\text{loc}} = \text{identity}$), and that, by (6.29), $\sigma_{\tilde{\Gamma}} \circ \sigma_\Gamma$ and $\sigma_\Gamma \circ \sigma_{\tilde{\Gamma}}$ are both * automorphisms of $\mathfrak{A}$ given by unitary operators on $\mathscr{H}_\omega$.

2. The situation expected in QED is that all * morphisms $\sigma_\Gamma, \sigma_{\tilde{\Gamma}}, \ldots$ satisfy (6.24), and that the vacuum sector is the only sector of total electric charge 0. In this case, the last part of Lemma 20 says that $\sigma_\Gamma, \sigma_{\tilde{\Gamma}}, \ldots$ are * *automorphisms* of $\mathfrak{A}$, i.e. *all sectors of* QED *are generated by charged * automorphisms of* $\mathfrak{A}$, a rather interesting conclusion! In this case covariance follows from transportability.

6.3. Relativistic Spectrum Condition, Charge Conservation and Additivity of the Electric Charge

We use the results of Sects. 6.1 and 6.2 as motivation for

Definition 6. A quasi-local translation cocycle Γ is said to be PCT *covariant* if and only if

$$\Gamma(a) \in \mathfrak{A}, \quad \text{for all} \quad a \in \mathbb{M}^4;$$

Γ is in one of the classes $\mathscr{C}_j$, $j = 1, 2, 3$, introduced in Sect. 6.1 (the *same* for all Γ); and

there exists a quasi-local translation cocycle $\bar{\Gamma}$ with the same properties as Γ such that $\pi_{\omega \circ \sigma_{\bar{\Gamma}} \circ \sigma_\Gamma}$ contains π_ω precisely once. □

We remark that, assuming the Conjugation Principle is valid, the analysis of Sect. 6.2 shows that the cocycle $\bar{\Gamma}$ is the translation cocycle $\Gamma_{\bar{\sigma}_\Gamma}$ of the morphism $\bar{\sigma}_\Gamma$ conjugate to σ_Γ which is given by

$$\Gamma_{\bar{\sigma}_\Gamma}(a) = \bar{\sigma}_{\text{loc}}(\tilde{\Gamma}(a)) \Gamma_{\bar{\sigma}_{\text{loc}}}(a) \tag{6.30}$$

(see Sect. 6.2). Definition 6 is a strengthened version of the Conjugation Principle in so far as it hypothesizes that $\bar{\Gamma}=\Gamma_{\bar{\sigma}_\Gamma}$ is a quasi-local translation cocycle in a class $\mathscr{C}_j$, so that $\bar{\sigma}_\Gamma=\sigma_{\bar{\Gamma}}$ is not only transportable, but space-time translation covariant.

We have

Theorem 21. 1) *If $\Gamma_1, \ldots, \Gamma_n$ are* PCT *covariant translation cocycles then $\omega \circ \sigma_{\Gamma_1} \circ \ldots \circ \sigma_{\Gamma_n}$ is space-time translation covariant, and the relativistic spectrum condition is satisfied on $\mathscr{H}_{\omega \circ \sigma_{\Gamma_1} \circ \ldots \circ \sigma_{\Gamma_n}}$, i.e. the spectrum of the generator of*

$$\{U_{\omega \circ \sigma_{\Gamma_1} \circ \ldots \circ \sigma_{\Gamma_n}}(a) : a \in \mathbb{M}^4\}$$

(see Theorem 5) is contained in $\bar{V}_+$.

2) *If $\Gamma_1, \ldots, \Gamma_n$ are* PCT *covariant charge transfer cocycles, with charges $q_1, \ldots, q_n$, and $\underset{\Sigma \uparrow \mathbb{R}^3}{\text{s-lim}}\, Q_\Sigma \Omega$ exists then $\underset{\Sigma \uparrow \mathbb{R}^3}{\text{s-lim}}\, e^{isQ_\Sigma} \equiv e^{isQ}$ exists on $\mathscr{H}_{\omega \circ \sigma_{\Gamma_1} \circ \ldots \circ \sigma_{\Gamma_n}}$, and $Q=(q_1+\ldots+q_n)1$, in particular, Q is conserved.*

3) *If the total, physical Hilbert space is spanned by the spaces $\mathscr{H}_{\omega \circ \sigma_{\Gamma_1} \circ \ldots \circ \sigma_{\Gamma_n}}$, where $\Gamma_1, \ldots, \Gamma_n$ are* PCT *covariant charge transfer cocycles, $n=0,1,2,\ldots$, then the total charge operator Q has pure point spectrum which is a discrete subgroup of the additive group of the real line.*

Proof. 1) Since $\Gamma_1, \ldots, \Gamma_n$ are PCT covariant, $\Gamma_1, \ldots, \Gamma_n$ are in a class $\mathscr{C}_j$, for some $j=1,2,3$. The space-time translation covariance of $\omega \circ \sigma_{\Gamma_1} \circ \ldots \circ \sigma_{\Gamma_n}$ can thus be inferred from Sect. 6.1. In particular, if Γ is PCT covariant then $\omega \circ \sigma_\Gamma$, $\omega \circ \sigma_{\bar{\Gamma}}$ and $\omega \circ \sigma_{\bar{\Gamma}} \circ \sigma_\Gamma$ are space-time translation covariant, and

$$\omega \circ \sigma_{\bar{\Gamma}} \circ \sigma_\Gamma = \lambda \omega + (1-\lambda)\varrho,$$

for some $\lambda \in (0,1]$, and some state ϱ with the property that π_ϱ is disjoint from π_ω. The proof of the spectrum condition is now as in [3].

2) By hypothesis, $\gamma^{\sigma_{\Gamma_j}}(s)=e^{isq_j}$, for some $q_j \in \mathbb{R}$, $j=1,\ldots,n$. If $\Sigma \subset \bigcap_{j=1}^{n} \operatorname{supp} \sigma_{\Gamma_j}^c$ then

$$\begin{aligned}
&\sigma_{\Gamma_1} \circ \ldots \circ \sigma_{\Gamma_n}(e^{isQ_\Sigma})e^{-isQ_\Sigma} \\
&\quad = \sigma_{\Gamma_1} \circ \ldots \circ \sigma_{\Gamma_{n-1}}(\gamma^{\sigma_{\Gamma_n}}(s)e^{isQ_\Sigma})e^{-isQ_\Sigma} \\
&\quad = \sigma_{\Gamma_1} \circ \ldots \circ \sigma_{\Gamma_{n-1}}(e^{isQ_\Sigma})e^{-isQ_\Sigma} \\
&= \ldots = 1,
\end{aligned}$$

i.e.

$$\operatorname{supp} \sigma_{\Gamma_1} \circ \ldots \circ \sigma_{\Gamma_n} \subseteqq \bigcup_{j=1}^{n} \operatorname{supp} \sigma_{\Gamma_j}.$$

If $\Sigma \supset \bigcup_{j=1}^{n} \operatorname{supp} \sigma_{\Gamma_j}$ then

$$\begin{aligned}
&\sigma_{\Gamma_1} \circ \ldots \circ \sigma_{\Gamma_n}(e^{isQ_\Sigma})e^{-isQ_\Sigma} \\
&\quad = \sigma_{\Gamma_1} \circ \ldots \circ \sigma_{\Gamma_{n-1}}(\gamma^{\sigma_{\Gamma_n}}(s)e^{isQ_\Sigma})e^{-isQ_\Sigma} \\
&\quad = \sigma_{\Gamma_1} \circ \ldots \circ \sigma_{\Gamma_{n-1}}(e^{isQ_\Sigma})e^{-isQ_\Sigma}e^{isq_n} \\
&\quad = \ldots = e^{is(q_1+\ldots+q_n)},
\end{aligned}$$

i.e.

$$\gamma^{\sigma_{\Gamma_1} \circ \ldots \circ \sigma_{\Gamma_n}}(s) = e^{is(q_1+\ldots+q_n)}.$$

By Lemma 14, 2), $e^{isQ} = e^{is(q_1 + \dots + q_n)}$ on $\mathscr{H}_{\omega \circ \sigma_{\Gamma_1} \circ \dots \circ \sigma_{\Gamma_n}}$. But $\omega \circ \sigma_{\Gamma_1} \circ \dots \circ \sigma_{\Gamma_n}$ is space-time translation covariant. Therefore Q is conserved.

3) From 2) it follows that the spectrum of Q is a discrete semigroup in $\mathbb{R}$. However, since the charge of $\bar{\Gamma}$ is opposite to that of Γ, it is in fact a discrete subgroup of $\mathbb{R}$. □

Remarks. Let $\Gamma_1, \dots, \Gamma_n$ be PCT covariant translation cocycles, and assume that $\underset{\Sigma \uparrow \mathbb{R}^3}{s\text{-lim}}\, Q_\Sigma \Omega$ exists. Then by Lemma 14, 2), e^{isQ} exists on $\mathscr{H}_{\omega \circ \sigma_{\Gamma_1} \circ \dots \circ \sigma_{\Gamma_n}}$ and is a continuous, unitary group in the center of $\pi_{\omega \circ \sigma_{\Gamma_1} \circ \dots \circ \sigma_{\Gamma_n}}(\mathfrak{A})''$. By Theorem 21, 1), the relativistic spectrum condition is fulfilled on $\mathscr{H}_{\omega \circ \sigma_{\Gamma_1} \circ \dots \circ \sigma_{\Gamma_n}}$. As a consequence, the center of $\pi_{\omega \circ \sigma_{\Gamma_1} \circ \dots \circ \sigma_{\Gamma_n}}(\mathfrak{A})''$ is space-time translation invariant; a general theorem of Borchers. Hence, the charge operator Q is *conserved.*

The charge cocycle of $\sigma_{\Gamma_1} \circ \dots \circ \sigma_{\Gamma_n}$ is given by

$$\gamma^{\sigma_{\Gamma_1} \circ \dots \circ \sigma_{\Gamma_n}}(s) = \sigma_{\Gamma_1} \circ \dots \circ \sigma_{\Gamma_{n-1}}(\gamma^{\sigma_{\Gamma_n}}(s)) \dots \sigma_{\Gamma_1}(\gamma^{\sigma_{\Gamma_2}}(s)) \gamma^{\sigma_{\Gamma_1}}(s), \tag{6.31}$$

as is easy to check.

Furthermore, if Γ is irreducible then σ_Γ is irreducible (Lemma 17), so that $\gamma^{\sigma_\Gamma}(s) = e^{isq}$, for some $q \in \mathbb{R}$ [see (5.18)]. If Γ is PCT covariant, and $\tau_a(\gamma^{\sigma_\Gamma}(s)) = \gamma^{\sigma_\Gamma}(s)$ then $\gamma^{\sigma_\Gamma}(s) = e^{isq}$, $q \in \mathbb{R}$ [see (5.19) and (5.20)].

Finally, Theorem 21 remains valid in theories in which charged sectors arise by composing the vacuum with charged * morphisms that may *not* arise from PCT covariant cocycles, provided covariance and the relativistic spectrum condition are known. We also recall that Proposition 3 and Theorem 7 say that if $\gamma^{\sigma_\Gamma}(s) \neq 1$ then σ_Γ and Γ are *non-local.*

In conclusion, we may tentatively view the problem of constructing the charged sectors in QED as the problem of constructing all possible, PCT covariant charge transfer cocycles.

The following problems remain to be analyzed:

1) What is the statistics of charged sectors? Is the spectrum of the total charge operator related to the statistics of the sectors?

2) Do the charged sectors determine well-defined representations of the algebras, $\mathfrak{A}^{\mathrm{as}}$, generated by bounded functions of the asymptotic, electromagnetic field (which Buchholz only constructs on the vacuum sector [2])? Are the charged representations of $\mathfrak{A}^{\mathrm{as}}$ disjoint from the Fock representation, i.e. are they "infrared representation"?

3) Is there a generalized Haag-Ruelle theory for charged (infra-)particles?

Some answers are sketched in the last section.

7. Generalized Haag-Ruelle Theory and a Remark on Non-abelian Gauge Theories

In this section we outline a collision theory for the theories described in Sects. 5 and 6, in particular QED. We assume that the photon is a stable, neutral particle of zero mass, i.e. in the vacuum sector $\mathscr{H}_\omega$ there are stable one-photon states.

Under these hypotheses, Buchholz [2] has constructed, on the vacuum sector $\mathscr{H}_\omega$, free, asymptotic, electromagnetic fields, $F^{\mathrm{as}}_{\mu\nu} = F^{\pm}_{\mu\nu}$, as strong limits of a family of local observables, as $t \to \pm\infty$.

Buchholz' construction only works on the vacuum sector or on sectors generated by strictly local (hence neutral) * morphisms of $\mathfrak{A}$, as is easy to check (see [2]). A priori, it does not apply to the charged sectors of the theory. However, under various, rather plausible technical conditions of dynamical character (e.g. a condition that says that, away from "small frequencies and momenta" charged representations of $\mathfrak{A}$ look like the vacuum representation) one may hope to extend Buchholz' collision theory to the charged sectors. Our starting point is as follows: Let $\mathscr{C}(\mathfrak{A})$ be the class of all PCT covariant cocycles. The analysis presented in Sects. 5 and 6 justifies defining the total Hilbert space, $\mathscr{H}$, of the theory as the smallest Hilbert space with the property that all states $\omega\circ\sigma_{\Gamma_1}\circ\ldots\circ\sigma_{\Gamma_n}$, $\Gamma_1,\ldots,\Gamma_n$ in $\mathscr{C}(\mathfrak{A})$, $n=0,1,2,\ldots$, are given by unit rays in $\mathscr{H}$.

By Theorem 21, there exists a densely defined, selfadjoint energy-momentum operator $(H,\mathbf{P})$ on $\mathscr{H}$ such that $\operatorname{spec}(H,\mathbf{P})\subseteqq\bar{V}_+$, and on $\mathscr{H}$

$$\tau_a(A)=e^{i(a^0H-\mathbf{a}\cdot\mathbf{P})}Ae^{-i(a^0H-\mathbf{a}\cdot\mathbf{P})}, \tag{7.1}$$

$a=(a^0,\mathbf{a})$, for all $A\in\mathfrak{A}$.

We note that the results of [7] and [2] imply that

$$\operatorname{spec}(H,\mathbf{P})=\bar{V}_+,$$

as

$$\operatorname{spec}(H,\mathbf{P})\restriction\mathscr{H}_\omega=\bar{V}_+. \tag{7.2}$$

We now assume that, on all sectors in $\mathscr{H}$, one can prove a strong convergence asymptotic condition for the electromagnetic field (see [2]) yielding free, asymptotic fields $F^{\mathrm{as}}_{\mu\nu}$, as $=+$ or $-$, with the following properties:

1) For $f^{\mu\nu}\in\mathscr{S}_{\mathrm{real}}(\mathbb{R}^4)$

$$\pm F^{\mathrm{as}}_{\mu\nu}(f^{\mu\nu})\leqq H+\Delta(f),$$

for some finite constant $\Delta(f)$ only depending on $f=\{f^{\mu\nu}\}$;

2) $F^{\mathrm{as}}_{\mu\nu}$ satisfies the standard free-field canonical commutation relations. [In this context, the results of [18] might be important; see also [2] and Paper I.]

As remarked in I (Proposition 3.1), the operators

$$\{e^{iF^{\mathrm{as}}_{\mu\nu}(f^{\mu\nu})}:f^{\mu\nu}\in\mathscr{S}_{\mathrm{real}}(\mathbb{R}^4)\}$$

then generate Weyl algebras $\mathfrak{A}^{\mathrm{as}}$, and

$$\tau_a(e^{iF^{\mathrm{as}}_{\mu\nu}(f^{\mu\nu})})=e^{iF^{\mathrm{as}}_{\mu\nu}(f^{\mu\nu}_a)}, \tag{7.3}$$

where

$$f^{\mu\nu}_a(x)=(2\pi)^{-2}\int d^4p\,e^{ip\cdot x}e^{i(a^0|\mathbf{p}|-\mathbf{a}\cdot\mathbf{p})}\tilde{f}^{\mu\nu}(p). \tag{7.4}$$

These results have been established for the boson field in models with infraparticles [19]. The whole circle of problems obviously requires further investigation.

In Theorem 3.2 of Paper I it is shown that the relativistic spectrum condition for $(H,\mathbf{P})$ and (7.3) and (7.4) permit to decompose $(H,\mathbf{P})$ into an energy-momentum operator $(H^{ph}_{\mathrm{as}},\mathbf{P}^{ph}_{\mathrm{as}})$ affiliated with the von Neumann algebra $\overline{\mathfrak{A}^{\mathrm{as}}}$

generated by $\mathfrak{A}^{\text{as}}$ on $\mathscr{H}$ which describes the dynamics of the asymptotic, electromagnetic field (i.e. the free time evolution of asymptotic fields), and an operator $(H^c_{\text{as}}, \mathbf{P}^c_{\text{as}})$ affiliated with $\overline{\mathfrak{A}^{\text{as}\prime}}$ which describes the dynamics of the asymptotic charge and of fields without electromagnetic interactions. [There is an explicit expressions of $(H^{ph}_{\text{as}}, \mathbf{P}^{ph}_{\text{as}})$ in terms of $F^{\text{as}}_{\mu\nu}$ (see I).] Moreover

$$\operatorname{spec}(H^{ph}_{\pm}, \mathbf{P}^{ph}_{\pm}) = \bar{V}_+, \qquad \operatorname{spec}(H^c_{\pm}, \mathbf{P}^c_{\pm}) \subseteqq \bar{V}_+. \tag{7.5}$$

We now specialize to theories, such as QED of electrons and positrons, which have the property that

$$\gamma^{\bar{\Gamma}}(s) = \sigma_{\bar{\Gamma}}(\gamma^{\Gamma}(s)^*) \tag{7.6}$$

and that the only sector of 0 electric charge is the vacuum sector. In Lemma 20, Sect. 6.2, we have shown that, under these hypotheses,

$$\gamma^{\bar{\Gamma}}(s) = e^{-isq} = \gamma^{\Gamma}(s)^*, \tag{7.7}$$

and σ_Γ is a * automorphism of $\mathfrak{A}$, for all $\Gamma \in \mathscr{C}(\mathfrak{A})$. For a suitable choice of $\bar{\Gamma}$,

$$\sigma_\Gamma^{-1} = \sigma_{\bar{\Gamma}}. \tag{7.8}$$

From Theorems 9 and 5 we then infer that there are isometries T_Γ which intertwine the representations $\pi_{\omega \circ \sigma_{\Gamma_1} \circ \dots \circ \sigma_{\Gamma_n}}$ of $\sigma_\Gamma(\mathfrak{A})$ with the representations

$\pi_{\omega \circ \sigma_{\Gamma_1} \circ \dots \circ \sigma_{\Gamma_n} \circ \sigma_\Gamma}$ of $\mathfrak{A}$

for arbitrary $\Gamma_1, \dots, \Gamma_n$ and Γ in $\mathscr{C}(\mathfrak{A})$. Since we have specialized to those σ_Γ which are * automorphisms of $\mathfrak{A}$, T_Γ maps $\mathscr{H}_{\omega \circ \sigma_{\Gamma_1} \circ \dots \circ \sigma_{\Gamma_n}}$ to $\mathscr{H}_{\omega \circ \sigma_{\Gamma_1} \circ \dots \circ \sigma_{\Gamma_n} \circ \sigma_\Gamma}$, and $T^*_\Gamma = T_{\bar{\Gamma}}$, by (7.8). Thus

$$\operatorname{domain}(T_\Gamma) = \operatorname{range}(T_\Gamma) = \mathscr{H}. \tag{7.9}$$

We define

$$T^{\text{as}}_\Gamma(x) = e^{i(x^0 H^c_{\text{as}} - \mathbf{x} \cdot \mathbf{P}^c_{\text{as}})} T_\Gamma e^{-i(x^0 H^c_{\text{as}} - \mathbf{x} \cdot \mathbf{P}^c_{\text{as}})}. \tag{7.10}$$

The second basic assumption of this section is that, for Γ and Γ' in $\mathscr{C}(\mathfrak{A})$,

$$[T^{\text{as}}_\Gamma(x), T^{\text{as}}_{\Gamma'}(y)]_{\delta(\Gamma,\Gamma')} = 0, \tag{7.11}$$

provided $(x-y)^2 \leqq d(\Gamma,\Gamma') < 0$, for some finite $d(\Gamma,\Gamma')$. Here $\delta(\Gamma,\Gamma') = \pm$, and $[\cdot,\cdot]_\pm$ denotes the commutator, resp. anti-commutator. We note that higher statistics (see DHR [3]) is automatically excluded, in this set up, because we have specialized to those σ_Γ which are * automorphisms of $\mathfrak{A}$. Moreover, because of (7.7) and Theorem 21, the spectrum of the total charge operator Q on $\mathscr{H}$ is a discrete subgroup of $\mathbb{R}$. (This is generally true if the statistics is ordinary Bose- or Fermi-statistics.) The *physical interpretation* of (7.10) and (7.11) is that when the *asymptotic* positions of the charges created by $T^{\text{as}}_\Gamma(x)$, $T^{\text{as}}_{\Gamma'}(y)$, resp., are space-like separated by at least a square distance $d(\Gamma,\Gamma')$, the field bundles $T^{\text{as}}_\Gamma(x)$ and $T^{\text{as}}_{\Gamma'}(y)$ commute, resp. anti-commute.

Assumptions (7.10) and (7.11) are in perfect agreement with the results of I and general wisdom concerning QED. But of course they are quite ad hoc and require further justification.

We regard (7.10) and (7.11) and the result below as a challenge to develop a theory of the "*asymptotic statistics*" of sectors, along the lines of [3], for general theories such as described in Sects. 5 and 6, where higher (para-Bose or para-Fermi) statistics is not excluded. [The main difficulties met in such attempts are that $\exp i(a^0 H^c_{\mathrm{as}} - \mathbf{a}\cdot\mathbf{P}^c_{\mathrm{as}})$ does not necessarily implement an automorphism group of $\mathfrak{A}$ and that the charged * morphisms $\{\sigma_\Gamma : \Gamma \in \mathscr{C}(\mathfrak{A})\}$ are not local.]

On the basis of (7.3)–(7.5) and (7.10) and (7.11) one can now construct a generalized Haag-Ruelle scattering theory for charged infra-particles (see also Sects. 3.2 and 3.6 of I):

Suppose that, for some $\Gamma_1, \ldots, \Gamma_n$ in $\mathscr{C}(\mathfrak{A})$, the operators $T^{\mathrm{as}}_{\Gamma_j}(x), j=1, \ldots, n$, have non-vanishing matrix elements between the vacuum Ω and one-(infra-)particle states, i.e., eigenstates of $(H^c_{\mathrm{as}})^2 - (\mathbf{P}^c_{\mathrm{as}})^2$ of eigenvalue m_j^2, ($m_j > 0$ is the mass of the infra-particle (see Proposition 3.4, Sect. 3.2 of Paper I). Typically, $\Gamma_1 = \ldots = \Gamma_k = \Gamma$, $\Gamma_{k+1} = \ldots = \Gamma_n = \bar{\Gamma}$, for some $\Gamma \in \mathscr{C}(\mathfrak{A})$, $m_j = m$, for $j = 1, \ldots, n$).

The standard spectral hypothesis is that the mass shells $\bar{V}_{m_j}$ are isolated in the spectrum of $(H^c_{\mathrm{as}}, \mathbf{P}^c_{\mathrm{as}})$.

Let $f_t^j(x) = (2\pi)^{-2} \int d^4p e^{ip\cdot x} \tilde{f}^j(p) e^{-it\sqrt{\mathbf{p}^2 + m_j^2}}$, where $\tilde{f}^j(p)$ is a test function the support of which has non-empty intersection with $\bar{V}_{m_j}$ but no intersection with $\operatorname{spec}(H^c_{\mathrm{as}}, \mathbf{P}^c_{\mathrm{as}}) \backslash \bar{V}_{m_j}$.

One then proves as in [14].

Theorem 22. *Under the hypotheses stated above*

$$\operatorname*{s\text{-}lim}_{t \to \pm\infty} e^{itH^c_{\pm}} \prod_{j=1}^{n} T^{\pm}_{\Gamma_j}(f_t^j)\Omega$$

exists.

Remarks. General scattering states are obtained by applying operators from $\mathfrak{A}^+$, resp. $\mathfrak{A}^-$, to the limits constructed in Theorem 22 (and taking the closure in $\mathscr{H}$).

One convinces oneself that the states so obtained can indeed be interpreted as the scattering states of the theory; e.g. the obvious intertwining relations are valid. The theory described here has one unconventional aspect: In general

$$T^+_\Gamma(f)\Omega \neq T^-_\Gamma(f)\Omega,$$

i.e. charged one-infra-particle states will in general scatter, due to the emission and absorption of photons.

Remark. Preliminary results (indicating that the total charge operator, Q, is in the center of the algebra $\overline{\mathfrak{A}^{\mathrm{as}}}$) suggest that, within a slight extension of the framework developped in this paper, one can prove that representations of $\mathfrak{A}^{\mathrm{as}}$ of different charge are disjoint, in particular, charged representations of $\mathfrak{A}^{\mathrm{as}}$ are disjoint from the Fock representation constructed in [2]. This would represent a stronger version of the result reported in Sect. 2 of Paper I.

The whole circle of problems touched upon in this section and some further results, involving deriving and applying the Maxwell equations in the framework developped in this paper, ought to be studied in a separate article.

We conclude with some comments concerning

Non-abelian Gauge Theories

Within the framework introduced in Sects. 1, 2, and 5 we consider an idealized, non-abelian gauge theory with gauge group some compact Lie group G. (For simplicity we assume that G is simple, but this is unimportant.) The center of G – which will turn out to play the main role – is denoted $\mathscr{Z}(G)$.

The theory is described in terms of an algebra $\mathfrak{A} = \overline{\bigcup_{\mathcal{O}\in\mathfrak{B}} \mathfrak{A}(\mathcal{O})}$ of quasi-local, neutral (i.e. uncoloured) observables with the general properties described in Sect. 1.

As usual, ω denotes some pure physical vacuum state on $\mathfrak{A}$, and $\mathscr{H}_\omega$ the vacuum sector, assumed to be given. The object of the study is the question whether there are * morphisms, σ, of $\mathfrak{A}$ with the property that the state $\omega\circ\sigma$ has colour, i.e. $\mathscr{H}_{\omega\circ\sigma}$ carries a non-trivial representation of G.

We start with some preliminary considerations concerning non-abelian, local charges.

For this purpose, we assume temporarily that, given any bounded, open set $\Sigma \subset \mathbb{R}^3$ [e.g. $\Sigma\in\mathscr{P}(\mathbb{R}^3)$, see Sect. 5] and an arbitrary space-time translation covariant morphism σ of $\mathfrak{A}$, there exists a representation $\{Q^a_\Sigma\}$ of the Lie algebra of G on the sector $\mathscr{H}_{\omega\circ\sigma}$ in terms of selfadjoint, local charges Q^a_Σ satisfying local Gauss laws; here the superscript a labels the elements of a basis in the Lie algebra of G. This assumption may be considered a part of the conventional lore about non-abelian theories. In a positive metric framework, it is however not on safe grounds, since the local charges Q^a_Σ cannot be elements of the observable algebra $\mathfrak{A}$, for all $\Sigma\in\mathscr{P}(\mathbb{R}^3)$ and all a, unless they vanish.

To see this we suppose that, on the sector $\mathscr{H}_{\omega\circ\sigma}$, the limits

$$w\text{-}\lim_{\Sigma\uparrow\mathbb{R}^3} e^{isQ^a_\Sigma} \equiv e^{isQ^a}, \tag{7.12}$$

exists, for all real s and all a.

By Gauss' law,

$$e^{isQ^a}\in\pi_{\omega\circ\sigma}(\mathfrak{A})', \quad \text{for all} \quad a. \tag{7.13}$$

In the abelian case, we have shown in Sect. 5 that e^{isQ} is in the center of $\pi_{\omega\circ\sigma}(\mathfrak{A})''$. In the non-abelian case this is only possible if the representation of G determined by $\{e^{isQ^a}\}$ on $\mathscr{H}_{\omega\circ\sigma}$ is the *trivial* representation (i.e. $\pi_{\omega\circ\sigma}$ has no "colour"), because the center of a von Neumann algebra is *abelian*, whereas the operators $\{e^{isQ^a}\}$ generate a *non-abelian* algebra whenever the representation of G they determine is non-trivial.

We say that $\pi_{\omega\circ\sigma}$ is a *coloured representation* of $\mathfrak{A}$ iff (7.12) holds and the representation of G on $\mathscr{H}_{\omega\circ\sigma}$ determined by $\{e^{isQ^a}\}$ is not trivial. In this case it then follows that $e^{isQ^a_\Sigma}$ cannot be in $\mathfrak{A}$, for all $\Sigma\in\mathscr{P}(\mathbb{R}^3)$, all s and all a. This proves our contention. By (7.13), $\pi_{\omega\circ\sigma}(\mathfrak{A})'$ contains a non-abelian algebra, whenever $\pi_{\omega\circ\sigma}$ is coloured, i.e. coloured representations of $\mathfrak{A}$ are necessarily *reducible*. Moreover, only the Casimir operators of G may be in $\mathfrak{A}$, but *not* the colour charges.

One expects, formally, that for a suitable choice of $\{Q^a_\Sigma\}$ and some $a_1, \ldots, a_m$, the operators $\{\exp 2\pi i Q^{a_j}_\Sigma\}_{j=1}^m$ generate a unitary representation of $\mathscr{Z}(G)$, for arbitrary $\Sigma\in\mathscr{P}(\mathbb{R}^3)$. Since these operators then commute with $e^{isQ^a_{\Sigma'}}$, for all a, $s\in\mathbb{R}$ and $\Sigma'\in\mathscr{P}(\mathbb{R}^3)$, they are "colourless", i.e. neutral. For this reason it is safe to assume that

$$\{\exp 2\pi i Q^{a_j}_\Sigma\}_{j=1}^m \subset \mathfrak{A}, \quad \text{for all} \quad \Sigma\in\mathscr{P}(\mathbb{R}^3).$$

Without loss of generality we now specialize to the case where $\mathscr{Z}(G)$ is generated by a single element, i.e. $m=1$ [e.g. $G=\mathrm{SU}(n)$, $\mathscr{Z}(G)=\mathbb{Z}_n$]. We abstract the discussion presented above, by simply assuming that, for each bounded region $\Sigma\in\mathscr{P}(\mathbb{R}^3)$, there exists an operator $Z_\Sigma\in\mathfrak{A}$ with the following properties:

1) Z_Σ generates a unitary representation of $\mathscr{Z}(G)$.

2) There exists a bounded double cone $\mathcal{O}_\Sigma\supset\Sigma$ such that

$$Z_\Sigma\in\mathfrak{A}(\sim\Sigma\cap\mathcal{O}_\Sigma)\quad\text{(Gauss' law)}. \tag{7.14}$$

3) If Σ_1 and Σ_2 are disjoint subsets of $\mathscr{P}(\mathbb{R}^3)$

$$Z_{\Sigma_1\cup\Sigma_2}=Z_{\Sigma_1}\cdot Z_{\Sigma_2}. \tag{7.15}$$

Let $\mathscr{Z}(G)=\{e,g,g^2,\ldots,g^{n-1},g^n=e\}$.

In analogy to the abelian case (see Sect. 5) we may now introduce cocycles $\{\gamma_\Sigma(g^m):\Sigma\in\mathscr{P}(\mathbb{R}^3)\}$ with the following properties: For all $\Sigma\in\mathscr{P}(\mathbb{R}^3)$

a) $\gamma_\Sigma(g^m)\in\mathfrak{A}$, for all $m=1,2,\ldots,n$.

b) $$\gamma_\Sigma(g^m)=\gamma_\Sigma(g^k)Z_\Sigma^k\gamma_\Sigma(g^{m-k})Z_\Sigma^{-k}, \tag{7.16}$$

for all $k=1,2,\ldots,m-1$, and

c) $\gamma_\Sigma(g^m)$ is independent of Σ, for all $\Sigma\supseteq\Sigma_\gamma$, where Σ_γ is some bounded set in $\mathscr{P}(\mathbb{R}^3)$, and all $m=1,2,\ldots,n$.

One then shows as in Sect. 5 that

$$\gamma(g^m)=n\text{-}\lim_{\Sigma\uparrow\mathbb{R}^3}\gamma_\Sigma(g^m)$$

exists, and

$$\gamma(g^m)=\gamma(g)^m\equiv\gamma^m,$$

for all $m=1,2,\ldots,n$, i.e. γ generates a unitary representation of $\mathscr{Z}(G)$ (see Lemma 12).

Let b be a path in $\mathbb{M}^4$ parametrized by a real variable $s\in[0,1]$, with end points $b(0),b(1)$. Given $x\in\mathbb{M}^4$, let Σ_x be the intersection of the light cone with vertex at x with the hyperplane $\{x=(x^0,\mathbf{x}):x^0=0\}$. Let $\Sigma_x^\lambda=\{\lambda\mathbf{x}:\mathbf{x}\in\Sigma_x\}$, for some $\lambda>1$. We define

$$\varepsilon_\lambda(b,\Sigma)=\begin{cases}1 & \text{if}\ \ \Sigma_{b(1)}^\lambda\subset\Sigma,\quad \Sigma_{b(0)}^\lambda\not\subset\Sigma\\ -1 & \text{if}\ \ \Sigma_{b(0)}^\lambda\subset\Sigma,\quad \Sigma_{b(1)}^\lambda\not\subset\Sigma\\ 0, & \text{otherwise.}\end{cases} \tag{7.17}$$

We now suppose that there are operators

$$\Gamma:b\to\Gamma(b),$$

with $\Gamma(b)\in\mathfrak{A}$, for arbitrary smooth, bounded paths b, which satisfy the following "*topological*" *commutation relations*:

For some finite $\lambda>1$ and arbitrary smooth, bounded paths $b\subset\mathbb{M}^4$,

$$Z_\Sigma\Gamma(b)Z_\Sigma^*=\gamma^{\varepsilon_\lambda(b,\Sigma)}\Gamma(b). \tag{7.18}$$

The operators $\Gamma(b)$ are the correct generalizations of the charge transfer cocycles $\{\Gamma(a):a\in\mathbb{M}^4\}$ studied in Sects. 5 and 6 to non-abelian theories. The problem of proving *confinement of* "*colour*" (in particular quark confinement) can now be formulated as follows:

Show that the topological commutation relations (7.18) do *not* admit any solution Γ such that Γ is a translation cocycle, i.e. $\Gamma(b)=\Gamma(b(1),b(0))$ only depends on the end points $b(0)$ and $b(1)$ of b.

One possible way of proving this would be to show that any solution Γ of (7.18) also solves 't Hooft's "topological" commutation relations [20] (expressing "electric-magnetic duality"), so that, for *closed* paths b, $\Gamma(b)\neq 1$ if b is not a point, so that Γ cannot be a translation cocycle.

Assuming, however, that (7.18) *does* have a solution Γ which is a (quasi-local) *translation cocycle* then all results of the present paper can be extended to this theory, in particular $\Gamma(a)$ is *non-local*, for all $a\in\mathbb{M}^4$, etc. .... Assuming, in addition, that (7.12) and (7.13) hold one concludes that the morphisms σ_Γ obtained from cocycles Γ obeying (7.18) are *necessarily reducible*. Hence $\Gamma\notin\pi_\omega(\sigma_\Gamma(\mathfrak{A}))''$ (see Lemma 17); in particular, Γ cannot be of class $\mathscr{C}_3$ (see Sect. 6.1). Applying moreover the results of Sect. 6.2 we arrive at the following

Alternative

Either the composition of σ_Γ with its conjugate morphism $\overline{\sigma_\Gamma}$ is *not* neutral (in particular not irreducible) [i.e. $\gamma^{\tilde{\Gamma}}\neq\sigma_{\tilde{\Gamma}}(\gamma^{\Gamma*})$, see (6.24)], *or* there must exist *non-trivial, neutral* (colourless) super-selection sectors disjoint from the vacuum sector, with higher (i.e. para) statistics.

Hence, even if colour were *not* confined, the resulting super-selection structure would presumably have rather unconventional features.

Compared to Wilson's confinement criterion [21], our confinement criterion, as formulated above, has the advantage of being mathematically precise and stating a necessary and sufficient condition for confinement, but the considerable disadvantage of not being very constructive. Our criterion strongly suggests that a proof or disproof of colour confinement is a dynamical, rather than a kinematical problem.

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Appendix 1

At the beginning of Sect. 4 we have introduced translation cocycles which are quasi-local in norm and stated their main properties. Here those properties are proven.

We recall that a translation cocycle, Γ, is said to be quasi-local in norm iff $\Gamma(a)A\Gamma(a)^*\in\mathfrak{A}$, for all $A\in\mathfrak{A}$ and $a\in\mathbb{M}^4$, and

$$n\text{-}\lim_{\substack{a\to\infty\\ a+b\to\infty}} \tau_{-a}(\Gamma(b)\tau_a(A)\Gamma(b)^*)=A\,.$$

A * morphism, σ, of $\mathfrak{A}$ is called quasi-local in norm iff σ is transportable, Γ_σ has the property that $\Gamma_\sigma(a)A\Gamma_\sigma(a)^*\in\mathfrak{A}$, for all $A\in\mathfrak{A}$ and $a\in\mathbb{M}^4$, and

$$n\text{-}\lim_{a\to\infty}\sigma_a(A)=A\,.$$

We propose to prove

Theorem A1. *If Γ is quasi-local in norm then*

$$\sigma_\Gamma(A) = n\text{-}\lim_{a\to\infty} \Gamma(a)A\Gamma(a)^* \quad \textit{exists},$$

*for all $A \in \mathfrak{A}$, and defines a transportable * morphism of $\mathfrak{A}$ which is quasi-local in norm, and $\Gamma_{\sigma_\Gamma} = \Gamma$. Conversely, if σ is quasi-local in norm then Γ_σ is quasi-local in norm, and $\sigma_{\Gamma_\sigma} = \sigma$.*

For the proof of this theorem we require

Lemma A2. *Let Γ be a translation cocycle. Then the following are equivalent*

1) $\displaystyle n\text{-}\lim_{\substack{a\to\infty\\ a+b\to\infty}} \tau_{-a}(\Gamma(b)\tau_a(A)\Gamma(b)^*) = A$,

2) $\displaystyle n\text{-}\lim_{\substack{a\to\infty\\ a+b\to\infty}} (\Gamma(a+b)A\Gamma(a+b)^* - \Gamma(a)A\Gamma(a)^*) = 0$,

3) $\displaystyle n\text{-}\lim_{a\to\infty} \Gamma(a)A\Gamma(a)^*$ *exists and is independent of the space-like, asymptotic direction in which $a\to\infty$.*

Proof. Since $\Gamma(a)$ is unitary, we have, using the cocycle identity (3.1),

$$\begin{aligned}
&\|\Gamma(a+b)A\Gamma(a+b)^* - \Gamma(a)A\Gamma(a)^*\| \\
&\quad = \|\Gamma(a)[\tau_{-a}(\Gamma(b)\tau_a(A)\Gamma(b)^*) - A]\Gamma(a)^*\| \\
&\quad = \|\tau_{-a}(\Gamma(b)\tau_a(A)\Gamma(b)^*) - A\|,
\end{aligned}$$

from which the equivalence of 1) and 2) follows. Next, we note that 3) clearly implies 2). Now we show the converse: If $a = \lambda e$, $b = \mu e$, where e is some fixed, space-like vector and λ and μ are e.g. positive integers then 2) implies that, for arbitrary $A \in \mathfrak{A}$, $\{\Gamma(\lambda e)A\Gamma(\lambda e)^*\}_{\lambda=1,2,3,\dots}$ is a Cauchy sequence in the operator norm. Thus

$$n\text{-}\lim_{a\to\infty} \Gamma(a)A\Gamma(a)^* \quad \text{exists for} \quad a = \lambda e,$$

$\lambda = 1, 2, 3, \dots$. Applying 2) once more, we now see that the limit is independent of the space-like asymptotic direction in which $a\to\infty$. □

Proof of Theorem A1. If Γ is quasi-local in norm then

$$n\text{-}\lim_{a\to\infty} \Gamma(a)A\Gamma(a)^* \quad \text{exists},$$

see Lemma A2, 1)⇒3). Since for all $a \in \mathbb{M}^4$ and arbitrary $A \in \mathfrak{A}$, $\Gamma(a)A\Gamma(a)^* \in \mathfrak{A}$,

$$n\text{-}\lim_{a\to\infty} \Gamma(a)A\Gamma(a)^* = \sigma_\Gamma(A) \in \mathfrak{A}.$$

Next $(\Gamma(a)A\Gamma(a)^*)^* = \Gamma(a)A^*\Gamma(a)^*$, so that $\sigma_\Gamma(A)^* = \sigma_\Gamma(A^*)$. Moreover,

$$(\Gamma(a)A\Gamma(a)^*)(\Gamma(a)B\Gamma(a)^*) = \Gamma(a)AB\Gamma(a)^*.$$

By taking norm limits on both sides of this equation we obtain

$$\sigma_\Gamma(A)\sigma_\Gamma(B) = \sigma_\Gamma(A\cdot B).$$

Finally, σ_Γ is obviously linear, and $\|\sigma_\Gamma(A)\| = \|A\|$. Thus σ_Γ is a * morphism of $\mathfrak{A}$.

Next

$$\begin{aligned}\sigma_{\Gamma,a} &= \tau_{-a}\circ\sigma\circ\tau_a(A)\\ &= \underset{b\to\infty}{n\text{-lim}}\, \tau_{-a}(\Gamma(b)\tau_a(A)\Gamma(b)^*)\\ &= \underset{b\to\infty}{n\text{-lim}}\, \Gamma(a)^*\Gamma(a+b)A\Gamma(a+b)^*\Gamma(a), \quad \text{by the cocycle identy (3.1),}\\ &= \Gamma(a)^*\sigma_\Gamma(A)\Gamma(a), \quad \text{i.e.} \quad \Gamma_{\sigma_\Gamma}=\Gamma. \end{aligned} \tag{A.1}$$

Finally, if $a\to\infty$ in some space-like, asymptotic direction e, let $b=\lambda e$. Then

$$\begin{aligned}\underset{a\to\infty}{n\text{-lim}}\,\sigma_{\Gamma,a}(A) &= \underset{a\to\infty}{n\text{-lim}}\ \underset{\lambda\to\infty}{n\text{-lim}}\,\tau_{-a}(\Gamma(b)\tau_a(A)\Gamma(b)^*)\\ &= \underset{\substack{a\to\infty\\ a+b\to\infty}}{n\text{-lim}}\ \tau_{-a}(\Gamma(b)\tau_a(A)\Gamma(b)^*)=A,\end{aligned}$$

by the definition of cocycles which are quasi-local in norm. Now we prove the second part of Theorem A1: We assume that σ is transportable and $\underset{a\to\infty}{n\text{-lim}}\,\sigma_a(A)=A$, for all $A\in\mathfrak{A}$. Using the unitarity of $\Gamma_\sigma(a)$, for all $a\in\mathbb{M}^4$, we get

$$\begin{aligned}&\|\Gamma_\sigma(a)A\Gamma_\sigma(a)^*-\sigma(A)\|\\ &\quad=\|\Gamma_\sigma(a)[A-\tau_{-a}\circ\sigma\circ\tau_a(A)]\Gamma_\sigma(a)^*\|\\ &\quad=\|A-\tau_{-a}\circ\sigma\circ\tau_a(A)\|=\|A-\sigma_a(A)\|.\end{aligned}$$

Thus

$$\underset{a\to\infty}{n\text{-lim}}\,\Gamma_\sigma(a)A\Gamma_\sigma(a)^*=\sigma(A) \quad \text{exists} \tag{A.2}$$

and is independent of the space-like, asymptotic direction in which $a\to\infty$. Applying Lemma A2, 3)$\Rightarrow$1) we conclude that Γ_σ is quasi-local in norm, and this and (A.2) show $\sigma=\sigma_{\Gamma_\sigma}$. □

Remark. Let Γ be quasi-local in norm, and $\Gamma(a)\in\sigma_\Gamma(\mathfrak{A})$, for all $a\in\mathbb{M}^4$. Then σ_Γ is a * automorphism.

Proof. By (A.1) and the hypothesis,

$$\sigma_{\Gamma,a}(A)=\Gamma(a)^*\sigma_\Gamma(A)\Gamma(a)\in\sigma_\Gamma(\mathfrak{A}).$$

By Theorem A1, σ_Γ is quasi-local in norm. Hence

$$\underset{a\to\infty}{n\text{-lim}}\,\sigma_{\Gamma,a}(A)=A, \quad \text{for all } A\in\mathfrak{A}.$$

Since $\sigma_\Gamma(\mathfrak{A})$ is closed in norm, we conclude that $\sigma_\Gamma(\mathfrak{A})\supseteqq\mathfrak{A}$. Thus σ_Γ is a * automorphism, which concludes the proof.

This remark is relevant for the understanding of the class $\mathscr{C}_3$ of cocycles introduced in Sect. 6.

Appendix 2

We propose to prove

Theorem A3. *Let Γ be a quasi-local translation cocycle, such that $\Gamma(a)=\sigma_\Gamma(\Gamma'(a)^*)$, for some quasi-local translation cocycle Γ' with $\Gamma'(a)\in\mathfrak{A}$, for all $a\in\mathbb{M}^4$. Then σ_Γ is a * automorphism of $\mathfrak{A}$, and $\sigma_\Gamma^{-1}=\sigma_{\Gamma'}$.*

Proof. Since σ_Γ is locally normal (see Definition 3, Sect. 4), we have in each locally normal representation of $\mathfrak{A}$

$$\begin{aligned}\sigma_\Gamma\circ\sigma_{\Gamma',a}(A)&=\sigma_\Gamma(\Gamma'(a)^*\sigma_{\Gamma'}(A)\Gamma'(a))\\&=\underset{b\to\infty}{w\text{-lim}}\,\sigma_\Gamma(\Gamma'(a)^*\Gamma'(b)A\Gamma'(b)^*\Gamma'(a))\\&=\underset{b\to\infty}{w\text{-lim}}\,\Gamma(a)\Gamma(b)^*\sigma_\Gamma(A)\Gamma(b)\Gamma(a)^*,\quad\text{since}\quad\sigma(\Gamma'(a)^*)=\Gamma(a),\\&=\underset{b\to\infty}{w\text{-lim}}\,\Gamma(a)\sigma_{\Gamma,b}(A)\Gamma(a)^*\\&=\Gamma(a)A\Gamma(a)^*,\quad\text{by Proposition 8, Sect. 4.}\end{aligned}$$

In particular, $\sigma_\Gamma\circ\sigma_{\Gamma'}=$ identity, since $\Gamma(0)=1$. Thus $\sigma_\Gamma(\sigma_{\Gamma'}(\Gamma(a)))=\Gamma(a)$, so that using $\Gamma(a)^*=\sigma_\Gamma(\Gamma'(a)^*)^*=\sigma_\Gamma(\Gamma'(a))$ we conclude that

$$\sigma_\Gamma(\sigma_{\Gamma'}(\Gamma(a))\Gamma'(a))=\Gamma(a)\Gamma(a)^*=1\,.$$

Multiplying both sides of this equation from the left by $\Gamma(b)^*$ and from the right by $\Gamma(b)$ we obtain

$$\sigma_{\Gamma,b}(\sigma_{\Gamma'}(\Gamma(a))\Gamma'(a))=\Gamma(b)^*\Gamma(b)=1\,,$$

and by taking the limit $b\to\infty$ (see Proposition 8)

$$\sigma_{\Gamma'}(\Gamma(a))\Gamma'(a)=1$$

i.e. (A.3)

$$\Gamma'(a)=\sigma_{\Gamma'}(\Gamma(a)^*)\,.$$

The first part of the proof thus implies that also

$$\sigma_{\Gamma'}\circ\sigma_\Gamma=\text{identity}\,,$$

hence $\sigma_\Gamma^{-1}=\sigma_{\Gamma'}$, so that σ_Γ is a * automorphism of $\mathfrak{A}$ and, by (A.3), $\Gamma'(a)=\sigma_\Gamma^{-1}(\Gamma(a)^*)$. □

Finally we wish to show that a cocycle $\Gamma(a)$ of class $\mathscr{C}_3$, i.e. $\Gamma(a)\in\sigma_\Gamma(\mathfrak{A})$, for all $a\in\mathbb{M}^4$, has the form

$$\Gamma(a)=\sigma_\Gamma(\Gamma'(a)^*)\,,$$

for some cocycle $\Gamma'(a)\in\mathfrak{A}$, for all a.

Proof. Since, for each a, $\Gamma(a)\in\sigma_\Gamma(\mathfrak{A})$, there exist operators $B_a^*\in\mathfrak{A}$, for all a, such that

$$\Gamma(a)=\sigma_\Gamma(B_a^*)\,.$$

By Proposition 8,

$$B_a^*=\underset{b\to\infty}{w\text{-lim}}\,\sigma_{\Gamma,b}(B_a^*)=\underset{b\to\infty}{w\text{-lim}}\,\Gamma(b)^*\sigma_\Gamma(B_a^*)\Gamma(b)\,.$$

Next

$$\begin{aligned}\sigma_{\Gamma,b}(B^*_{a+c}) &= \Gamma(b)^*\Gamma(a+c)\Gamma(b)\\ &= \Gamma(b)^*\Gamma(a)\tau_{-a}(\Gamma(c))\Gamma(b), \quad \text{by the cocycle identity}\\ &= [\Gamma(b)^*\Gamma(a)\Gamma(b)][\Gamma(b)^*\tau_{-a}(\Gamma(c))\Gamma(b)]\\ &= \sigma_{\Gamma,b}(B^*_a)\Gamma(b)^*\tau_{-a}(\sigma_\Gamma(B^*_c))\Gamma(b)\\ &= \sigma_{\Gamma,b}(B^*_a)\Gamma(b)^*\Gamma(a)^*\sigma_\Gamma(\tau_{-a}(B^*_c))\Gamma(a)\Gamma(b)\\ &= \sigma_{\Gamma,b}(B^*_a)\Gamma(b)^*\sigma_\Gamma(B_a\tau_{-a}(B^*_c)B^*_a)\Gamma(b)\\ &= \sigma_{\Gamma,b}(\tau_{-a}(B^*_c)B^*_a)\,.\end{aligned}$$

Thus, by taking the limit $b\to\infty$, we obtain

$$B^*_{a+c} = \tau_{-a}(B^*_c)B^*_a\,,$$

i.e. $\Gamma'(a)\equiv B_a$ satisfies the cocycle identity. □

Remark. It is unknown whether Γ' is a quasi-local translation cocycle.

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Note Added in Proof

1) The following reference relevant to this paper had escaped our attention: Roberts, J.E.: Perturbations of dynamics and group cohomology. In: Les méthodes mathématiques de la théorie quantique des champs. Guerra, F., Robinson, D.W., Stora, R. (eds.), éditions du C.N.R.S., Paris 1976

2) G. Morchio has pointed out that the hypothesis of Lemma 14 (1) is presumably too strong. We note, however, that the conclusions of this paper also follow from the much weaker assumptions stated in the Remark subsequent to Lemma 14 and the condition that $\gamma_{\Sigma}^{\sigma}(s)$ converges, as $\Sigma \uparrow \mathbb{R}^3$.

3) We wish to emphasize the following consequence of our results in Sects. 6.1 and 6.2:

Theorem. *Consider a theory with the properties that*

*A) all its charged sectors arise by composing the vacuum with transportable, charged * morphisms* σ *of the algebra* $\mathfrak{A}$ *with* $\gamma^{\sigma}(s) \neq 1$, *and to each morphism* σ *there exists a conjugate morphism* $\bar{\sigma}$ *with*

$$\gamma^{\bar{\sigma}}(s) = \bar{\sigma}(\gamma^{\sigma}(s)^*) \tag{CP}$$

*B) there are no non-trivial, neutral * morphisms of* $\mathfrak{A}$. *Then all morphisms* σ *are * automorphisms of* $\mathfrak{A}$, $\gamma^{\sigma}(s) = e^{isq_{\sigma}}$, *for some* $q_{\sigma} \in \mathbb{R}$, *and all these automorphisms and arbitrary compositions thereof are translation covariant. The relativistic spectrum condition is satisfied on all sectors of the theory and the charge operator has pure point spectrum.* □

This result shows that the understanding of the charged sectors of ordinary QED is satisfactory, up to a derivation of condition (CP) from first principles. It suggests the conjecture that, in general, *all* charges * morphisms σ are of the form $\sigma = \sigma_{\mathrm{loc}} \circ \varrho$, where ϱ is a transportable, charged * *automorphism*, and σ_{loc} is a transportable, localized * morphism, in the sense of DHR [3], hence neutral. Then σ satisfies the Conjugation Principle and is covariant.