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Phase Transitions in Ferromagnetic Spin Systems at Low Temperatures

W. Holsztynski[1] and J. Slawny[2]

[1] Department of Mathematics, The University of Western Ontario, London, NGA 5B9, Canada
[2] Department of Mathematics, Rutgers University, New Brunswick, New Jersey 08903, USA

Abstract. We consider the problem of the existence of first order phase transitions in ferromagnetic spin systems at low temperatures. A criterion is given for the existence of phase transitions in terms of an algebraic system canonically associated with any interaction. The criterion involves finding out if the greatest common divisor of few polynomials belongs to the ideal generated by these polynomials.

In connection with results published earlier, this work yields a description of all translation invariant (also of periodic and quasi-periodic) equilibrium states at low temperatures.

Introduction

A system exhibits first order phase transition if it has more than one invariant equilibrium state. One would like to be able to say if for given interaction (and external parameters) there is a phase transition, how many pure phases are there, and how to distinguish the phases. Furthermore, one would like to test the extremal invariant states against the breakdown of various symmetries and to discuss their clustering properties.

A solution of these problems for ferromagnetic spin $\frac{1}{2}$ systems at low temperatures on $\mathbb{Z}^\nu$ was described in our note [15] which can serve as an introduction to the present work. In Sect. 5 we prove a strengthened version of the conjecture of [15]. Extension of these results to higher spin systems is contained in [32].

The problem of phase transitions at low temperatures has received considerable attention. The case of Ising model and its perturbations was treated in the sixties by Griffiths [9] and Dobrushin [3]. This was generalized in [1, 8]; other relevant papers can be traced through [10, 11, 13].

More recently a theory of phase transitions for any finite number of ground states has been developed, first in case when the ground states are related by symmetries of the interaction [5] and then more complete theory in [25, 26]. In a number of cases existence of phase transitions has been proved with help of the Reflection Positivity method [6]. And ferromagnetic spin $\frac{1}{2}$ systems have been treated in [12, 13, 20–23].

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In case of a finite number of ground state configurations the phase diagram at low temperatures is obtained by perturbation of the zero-temperature phase diagram [26]. In particular, the number of pure phases and low temperatures is equal to the number of ground states. However, when the number of ground states is infinite – which occurs in quite a few systems of interest in physics – the picture can be more complicated and no general results have been obtained. Here we analyze completely one class of models, the translation invariant ferromagnetic discrete spin systems.

Phase transitions in ferromagnetic spin systems are intimately related to symmetry breakdown. Namely, the group $\mathscr{S}$ of flippings of the spins which leave the interaction invariant acts transitively on the ground state configurations [34]. To describe the phases at low temperatures it is enough to know the part $\mathscr{S}_+$ of $\mathscr{S}$ that is unbroken [31, 32][1]). (In fact this holds at almost all temperatures [19]). Having the complete description of $\mathscr{S}_+$ we easily find examples with infinite $\mathscr{S}$ and finite $\mathscr{S}/\mathscr{S}_+$ (finite number of phases at low temperatures), or with infinite $\mathscr{S}$ and infinite $\mathscr{S}/\mathscr{S}_+$ (infinite number of phases at low temperatures) [15, 32].

This is a third version of our paper on phase transitions in ferromagnetic spin systems at low temperatures. The first one (Princeton University, May 1975) is summarized in [15]. Section 2 in the present paper are as in the first version but both the decomposition property and the reduction theorem are proved there under stronger assumptions. The second version (CNRS, Marseille, December 1975) has Sects. 2 and 3 in the present form, only the reduction is not carried out there in full generality. It contains an independent discussion of trivial systems (here in Sect. 4.4), which contributed to the present proof of the reduction theorem, and an appendix with a detailed study of the groups of contours and cycles putting the decomposition property and the reduction in another perspective. This has been cut out to shorten the paper, but can be consulted if the present treatment is found to terse or incomprehensible. Some of this material appears in [34] where our results are extended to the general case with the translations no longer acting on the lattice in a transitive way and with the symmetry group having a gauge (localized) part. [35] contains also a detailed comparison of our work with [5, 13, 26].

1. Preliminaries

Our lattice $\mathbb{L}$ will be $\mathbb{Z}^\nu$. The configuration space of a spin $\frac{1}{2}$ system on $\mathbb{L}$ is

$$\mathscr{X} = \{-1, 1\}^{\mathbb{L}},$$

and for any finite $A \subset \mathbb{L}$

$$\sigma_A = \prod_{a \in A} \sigma_a$$

where σ_a, the spin at point a, is defined by

$$\sigma_a(X) = X_a, \qquad X \in \mathscr{X}.$$

An *interaction* is identified here with a real valued $\mathbb{Z}^\nu$-invariant function J on $\mathscr{P}_f(\mathbb{L})^2$; $K = \beta J$, where β is the inverse temperature, will be also named interaction.

1 In the case of Ising model, earlier argument of the type employed in [31] is due to D. Ruelle [29]

2 As usual, for any set $\mathbb{L}$, $\mathscr{P}_f(\mathbb{L})$ denotes the family of all finite subsets of $\mathbb{L}$

The support of J is denoted by $\mathscr{B}$. Elements of $\mathscr{B}$ are called *bonds*. We assume that there is a *finite fundamental family* of bonds ("finite range interaction") i.e. there is a finite family $\mathscr{B}_0$ of bonds such that any element of $\mathscr{B}$ is congruent with exactly one bond of $\mathscr{B}_0$. The energy corresponding to the interaction J is written as

$$U = -\sum_{B\in\mathscr{B}} J(B)\sigma_B .$$

The Gibbs state in a finite $\Lambda \subset \mathbb{L}$ corresponding to the configuration $+1$ outside Λ and to a temperature β^{-1} ascribes to a configuration X on Λ the probability

$$\varrho_\Lambda^+(X) = (Z_\Lambda^+)^{-1} \exp \sum_{B\cap\Lambda \neq \emptyset} K(B)\sigma_B(X^+)$$

where for $X \in \mathscr{X}_\Lambda = \{-1, 1\}^\Lambda$, X^+ is the element of $\mathscr{X}$ equal to X on Λ and to $+1$ outside of Λ.

An interaction J is *ferromagnetic* if $J(B) > 0$, all $B \in \mathscr{B}$. The *equilibrium state* ϱ^+ obtained as the limit of the net (ϱ_Λ^+) as $\Lambda \uparrow \mathbb{L}$ is of special interest in case of ferromagnetic systems[3]. Let

$$\mathscr{B}^+ = \{A \in \mathscr{P}_f(\mathbb{L}) : \varrho^+(\sigma_A) \neq 0\}$$

$\mathscr{B}^+$ depends on the temperature since ϱ^+ does; *our task here is to determine $\mathscr{B}^+$ for general ferromagnetic systems at low temperatures.* As shown in [30, 31, 15] at low temperatures $\mathscr{B}^+$ determines all invariant (and also quasi-invariant, cf. Appendix B) equilibrium states.

Let $\mathscr{X}_f$ be the family of all configurations that are "$+1$ at infinity":

$$\mathscr{X}_f = \{X \in \mathscr{X} : X_a = -1 \text{ at a finite number of points}\}.$$

For $X \in \mathscr{X}_f$ let the *contour* $\gamma(X)$ [22] be defined by

$$\gamma(X) = \{B \in \mathscr{B} : \sigma_B(X) = -1\}.$$

Then from the obvious identities:

$$\begin{aligned}\exp \sum_{B\cap\Lambda\neq\emptyset} K(B)\sigma_B(X^+) &= \exp\left[\sum_{B\cap\Lambda\neq\emptyset} K(B) + \sum_{B\cap\Lambda\neq\emptyset} K(B)(\sigma_B(X^+)-1)\right]\\ &= \left[\exp \sum_{B\cap\Lambda\neq\emptyset} K(B)\right] \exp -2 \sum_{B\in\gamma(X^+)} K(B)\end{aligned}$$

one obtains the following expression for ϱ_Λ^+ (Low Temperature Expansion, [35, 22]):

$$\varrho_\Lambda^+(X) = \Big(\sum_{X\in\mathscr{X}_\Lambda} \exp -2 \sum_{B\in\gamma(X^+)} K(B)\Big)^{-1} \exp -2 \sum_{B\in\gamma(X^+)} K(B).$$

2. $\varrho^+(\sigma_x) \neq 0$ if Contours Decompose

By a *graph* we understand a pair $G = (V, E)$ where V is a non-empty set and E is a family of two-element subsets of V; elements of V are called *vertices* and of E *edges* of

3 Cf. [4, 17] for the definition and properties of equilibrium state and to [7, 11], and also [31], for the proof of the existence and/or properties of ϱ^+

the graph G. A *path* in the graph G is a sequence

$$(v_0, \dots, v_n, e_1, \dots, e_n), \qquad v_i \in V, \qquad e_i \in E, \tag{2.1}$$

such that $e_k = \{v_{k-1}, v_k\}$ for $k = 1, \dots, n$. We say that the path (2.1) passes through the vertices $v_0, \dots, v_n$ and that it connects v_0 and v_n. The number n in (2.1) is called the *length* of the path; (v_0) is a path of *length* zero. We adopt the usual definition of *connectivity and components* of graphs, *trees*, etc.

2.1. Lemma. *Let $G = (V, E)$ be a connected graph with a finite number l of vertices. Then for each $v \in V$ there exists a path of length $2l - 2$ which starts at v and passes through all the vertices.*

For trees the lemma is proved easily by induction with respect to the number of vertices and the general case can be reduced to that of a tree.

2.2. We now place ourselves in the framework of Sect 1. We fix a (finite) fundamental family $\mathscr{B}_0$ of bonds.

The following notation and definitions will be employed:

For $x = (x_1, \dots, x_\nu) \in \mathbb{R}^\nu$, $|x| = \max\{|x_1|, \dots, |x_\nu|\}$; for a subset $B \subset \mathbb{R}^\nu$

$$\operatorname{diam} B = \sup\{|x - y| : x, y \in B\}$$

$\operatorname{mesh} \mathscr{B} = \operatorname{mesh} \mathscr{B}_0 = \sup\{\operatorname{diam} B : B \in \mathscr{B}\}$.

For two subsets $A, B \subset \mathbb{R}^\nu$ we let

$$\delta(A, B) = \inf\{|x - y| : x \in A,\ y \in B\}.$$

Let N be a natural number; we will say that a *family $\mathscr{C}$ of non-empty subsets of $\mathbb{R}^\nu$ is N-connected* if the graph

$$(\mathscr{C}, \{\{A, B\} : A, B \in \mathscr{C}, \delta(A, B) \leqq N\})$$

is connected.

2.3 Lemma. *Let l be a natural number and let $A \in \mathscr{B}$. Then the number of l-element subfamilies of B which are N-connected and contain A is not greater than*

$$[(2b + 2N + 1)^\nu b]^{2l-2}$$

where $b = \max(|\mathscr{B}_0|, \operatorname{mesh} \mathscr{B}_0)$.

Proof. It is not difficult to see that for any $B \in \mathscr{B}$ the number of $B' \in \mathscr{B}$ such that $\delta(B, B') \leqq N$ is not greater than

$$[2(\operatorname{mesh}(\mathscr{B}) + N) + 1]^\nu \cdot |\mathscr{B}_0| \leqq (2b + 2N + 1)^\nu \cdot b.$$

Let $[A, l, N]$ be the family of all $(2l - 1)$-element sequences $(B_1, B_2, \dots, B_{2l-1})$ of bonds such that $B_1 = A$ and $\delta(B_i, B_{i+1}) \leqq N$, $i = 1, \dots, 2l - 2$. By Lemma 2.1, the mapping from $[A, l, N]$ which to each sequence assigns the set of all bonds appearing in it maps $[A, l, N]$ onto the set of all subfamilies of our lemma. On the other hand the above estimation yields

$$\operatorname{card}[A, l, N] \leqq ((2b + 2N + L)^\nu \cdot b)^{2l-2}.$$

This proves our lemma.

We will now show that the contour mapping $X \to \gamma(X)$ from $\mathscr{X}_f$ to $\mathscr{P}_f(\mathscr{B})$, *is injective*[4]. Let X and Y be two different elements of $\mathscr{X}_f$ and let a be the first (in lexicographic order) element of $\mathbb{Z}^\nu$ for which $X_a \neq Y_a$; since X differs from Y at a finite number of lattice sites only, such $a \in \mathbb{Z}^\nu$ exists. It is easy to see that there exists $B \in \mathscr{B}$ for which a is the last element. Then B is either in $\gamma(X)$ or in $\gamma(Y)$ but not in both of these contours, which proves the assertion.

For any integers l, N we let

$$\mathscr{X}_f^{l,N} = \{X \in \mathscr{X}_f : X_0 = -1, \quad |\gamma(X)| = l, \quad \gamma(X) \text{ is } N\text{-connected}\}.$$

2.4 Lemma. *Let $A \in \mathscr{B}$ and let $0 \in A$. Then the number of the translates of A contained in*

$$\{\gamma(X) : X \in \mathscr{X}_f^{l,N}\}$$

is not greater than

$$[(2b-2)(b+N)+N+1]^\nu .$$

If $l=1$ then the number in question is 0 if there are no one-element bonds and 1 otherwise. Hence the estimation holds. Let a be the first element of A, let $X \in \mathscr{X}_f^{l,N}$ and let y be the last element of $\mathbb{Z}^\nu$ for which $X_y = -1$. Obviously, $B = y - a + A \in \gamma(X)$. Moreover, $x_1 > 0$ for each $x = (x_1, \dots, x_\nu) \in B$. Put

$$\alpha^- = \min\{x_1 : x \in A\}, \qquad \alpha^+ = \max\{x_1 : x \in A\}.$$

As is easy to see, if $x + A \in \gamma(X)$ then for $l \geqq 2$

$$x_1 > -[\alpha^+ + (l-2)b + (l-1)N]. \tag{2.3}$$

By symmetry

$$x_1 < \alpha^- + (l-2)b + (l-1)N. \tag{2.4}$$

Since the right hand sides of (2.3) and (2.4) do not depend on the choice of $X \in \mathscr{X}_f^{l,N}$ we conclude that there are at most $(2b-3)b + (l-1)N$ possible values of x_1 for which there exists $x = (x_1, \dots, x_\nu)$ such that $x + A \in \{\gamma(X) : X \in \mathscr{X}_f^{l,N}\}$. The same is true for any coordinate x_k, $k = 1, \dots, \nu$. This proves our lemma.

2.5 Proposition.

$$|\mathscr{X}_f^{l,N}| \leqq [(2b-3)(b+N)+N+1]^\nu \times [(2b+2N+1)^\nu b]^{2l-2}. \tag{2.5}$$

This is an immediate corollary of Lemma 2.3, Lemma 2.4 and the injectivity of γ for any non-empty fundamental family of bonds. One has also to take into account that for every $X \in \mathscr{X}_f$ and for each $B \in \mathscr{B}_0$ the translates of B appear in $\gamma(X)$.

2.6. We pass to the core of the argument. We will say that the interaction has the *decomposition property* if there exists a non-negative number N such that N-components of contours are again contours.

It is not hard to see, that a system has the decomposition property if and only if there exists an integer N such that for each $X \in \mathscr{X}_f$ there exists a natural number

4 This was noticed before in [12]

$n=n(X)$ and $X_1,\dots,X_n\in\mathscr{X}_f$ such that

(i) $\gamma(X_i)$ are N-connected, $i=1,\dots,n$.

(ii) $\gamma(X_i)\cap\gamma(X_j)=\emptyset$, $1\leqq i<j\leqq n$.

(iii) $\gamma(X_1)\cup\dots\cup\gamma(X_n)=\gamma(X)$.

If $\gamma(X)\subset\gamma(Y)$ we write $X\subset Y$.

Later we introduce in $\mathscr{P}_f(\mathbb{Z}^\nu)$ a multiplication operation making it into a ring and we relate algebraic properties of $\mathscr{B}$ to the decomposition property. We will show that a system has the decomposition property if and only if the greatest common divisor of $\mathscr{B}$ is trivial.

The decomposition property allows one to majorize the probability of occurrence of a contour $\gamma(X)$ by

$$\exp-2\sum_{B\in\gamma(X)}K(B). \tag{2.6}$$

Moreover with the above definitions and lemmas we have the following version of the classical Peierls argument.

2.7 Theorem. *If the system has the decomposition property then* $\varrho^+(\sigma_0)\neq 0$ *at low temperatures.*

It is enough to show that at low enough temperatures

$$\varrho_\Lambda^+(\{X\in\mathscr{X}_\Lambda:X_0=-1\})\leqq\varepsilon<\tfrac{1}{2}$$

for any large enough Λ.

Let $K=\min_{B\in\mathscr{B}_0}K(B)$, let N be an integer of the definition of the decomposition property and let

$$\mathscr{X}_f^N=\{X\in\mathscr{X}_f:X_0=-1,\ \gamma(X)\text{ is }N\text{-connected}\}.$$

Then $\mathscr{X}_f^N=\bigcup_l\mathscr{X}_f^{l,N}$ and (see the comments below)

$$\begin{aligned}
&\varrho_\Lambda^+(\{X\in\mathscr{X}_\Lambda:X_0=-1\})\leqq\sum_{Y\in\mathscr{X}_f^N}\varrho_\Lambda^+(\{X\in\mathscr{X}_\Lambda:Y\subset X^+\})\\
&\leqq\sum_{Y\in\mathscr{X}_f^N}\exp\left[-2\sum_{B\in\gamma(Y)}K(B)\right]\leqq\sum_{Y\in\mathscr{X}_f^N}\exp-2K\cdot|\gamma(Y)|\\
&=\sum_{l=1}^{\infty}\sum_{Y\in\mathscr{X}_f^{l,N}}\exp-2K\cdot|\gamma(Y)|\\
&\leqq\sum_{l=1}^{\infty}[(2l-3)(b+N)+N+1]^\nu\times[(2b+N+L)^\nu b]^{2l-2}\times e^{-2kl};
\end{aligned}\tag{2.7}$$

as easily seen directly, and also follows from the next section, if $\gamma(X)=\gamma(X_1)\cup\gamma(X_2)$ and $\gamma(X_1)\cap\gamma(X_2)=\emptyset$ then $X_a=X_{1,a}\cdot X_{2,a}$ for all $a\in\mathbb{L}$. Hence if $X_0=-1$ and $\gamma(X_1),\dots,\gamma(X_n)$ are the N-components of $\gamma(X)$ then there exist i, $1\leqq i\leqq n$, such that $X_{i,0}=-1$; this proves the first inequality of the chain. The next one is (2.6) and the last one Proposition 2.5.

Since the series (2.7) is Λ-independent and its sum tends to 0 as $K\to\infty$, the theorem is proved.

3. Cycles and the Decomposition Property

We introduce now the algebraic structure which allows us to study the decomposition property. First, in by now standard way (for finite systems cf. [7, 35, 22] and literature therein) we define in $\mathscr{X}_f$, $\mathscr{P}_f(\mathbb{L})$, $\mathscr{P}_f(\mathscr{B})$ an (abelian) group structure which makes γ into a homomorphism [22]. Then, to obtain deeper results in case $\mathbb{L}=\mathbb{Z}^\nu$ we define in $\mathscr{P}_f(\mathbb{L})$ the structure of a ring.

The group operation in $\mathscr{X}_f$ is defined pointwise, with $\{-1,1\}$ being the group with identity 1. For any set $\mathbb{L}$, $\mathscr{P}_f(\mathbb{L})$ has the symmetric difference

$$A,B \mapsto A+B=(A\backslash B)\cup(B\backslash A)$$

as the group operation; the empty set is the zero of $\mathscr{P}_f(\mathbb{L})$. The map

$$A,B \to \langle A,B\rangle=(-1)^{\mathrm{Card}(A\cap B)} \tag{3.1}$$

is a bicharacter of $\mathscr{P}_f(\mathbb{L})\times\mathscr{P}_f(\mathbb{L})$ which separates points in the sense that for any $A\in\mathscr{P}_f(\mathbb{L})$, $A\neq\emptyset$, there is $B\in\mathscr{P}_f(\mathbb{L})$ such that $\langle A,B\rangle\neq 1$.

The map

$$X\mapsto\{a\in\mathbb{L}:X_a=-1\}$$

defines an isomorphism $\mathscr{X}_f\to\mathscr{P}_f(\mathbb{L})$; we can therefore identify these two groups. As follows from the definition of σ_B (Sect. 1),

$$\sigma_B(X)=\langle\{a\in\mathbb{L}:X_a=-1\},B\rangle\,.$$

Thus σ_B is a character of $\mathscr{X}_f$.

$X\mapsto\gamma(X)$ is a homomorphism of $\mathscr{P}_f(\mathbb{L})$ into $\mathscr{P}_f(\mathscr{B})$ (cf. Sect. 1 for the definition of γ). For, since $\langle X+Y,B\rangle=\langle X,B\rangle\langle Y,B\rangle$, $B\in\gamma(X+Y)$ iff $B\in(\gamma(X)\backslash\gamma(Y))\cup(\gamma(Y)\backslash\gamma(X))$ which shows that $\gamma(X+Y)=\gamma(X)+\gamma(Y)$.

For $\beta\in\mathscr{P}_f(\mathscr{B})$, $\varepsilon(\beta)\in\mathscr{P}_f(\mathbb{L})$ is defined by

$$\varepsilon(\beta)=\sum_{B\in\beta}B\,,$$

the sum here being taken in $\mathscr{P}_f(\mathbb{L})$; we write sometimes $\bar\beta$ instead of $\varepsilon(\beta)$. $\varepsilon(\beta)$ is the set of all these $x\in\mathbb{L}$ which belong to an odd number of members of β. *$\beta\mapsto\varepsilon(\beta)$ is a homomorphism* of $\mathscr{P}_f(\mathscr{B})$ into $\mathscr{P}_f(\mathbb{L})$. It is not hard to see that the maps ε and γ are conjugated one to another in the sense that

$$\langle\gamma(X),\beta\rangle=\langle X,\varepsilon(\beta)\rangle \tag{3.2}$$

for $X\in\mathscr{P}_f(\mathbb{L})$, $\beta\in\mathscr{P}_f(\mathscr{B})$, [30].

3.1. Set:

$$\Gamma_f=\mathrm{Im}(\gamma)=\{\gamma(X):X\in\mathscr{P}_f(\mathbb{L})\}$$

$$\mathscr{K}_f=\mathrm{Ker}(\varepsilon)=\{\beta\in\mathscr{P}_f(\mathscr{B}):\varepsilon(\beta)=0\}\,.$$

Elements of Γ_f are called *contours* and elements of $\mathscr{K}_f$ are called *cycles*. $\bar{\mathscr{B}}=\varepsilon(\mathscr{P}_f(\mathscr{B}))$. Since γ and ε are homomorphisms Γ_f and $\mathscr{K}_f$ are subgroups of $\mathscr{P}_f(\mathscr{B})$.

Γ_f and $\mathscr{K}_f$ are mutually orthogonal subgroups of $\mathscr{P}_f(\mathscr{B})$ in the sense that

$$\langle\alpha,\beta\rangle=1 \quad \textit{for} \quad \alpha\in\mathscr{K}_f \quad \textit{and} \quad \beta\in\Gamma_f\,.$$

The orthogonality follows from (3.2).

3.2. Since $X+X=0$ for any $X\in\mathscr{P}_f(\mathbb{L})$, $\mathscr{P}_f(\mathbb{L})$ admits unique structure of a vector space over the two-element field $\mathbb{F}_2=\{0,1\}$. The same vector-space structure is obtained through the identification of $\mathscr{P}_f(\mathbb{L})$ with $\{0,1\}^{(\mathbb{L})}$ which identifies sets with their characteristic functions. Each subgroup of $\mathscr{P}_f(\mathbb{L})$ is a subspace of this vector space. $\mathbb{Z}^\nu$ through its action on $\mathbb{L}$ acts on $\mathscr{P}_f(\mathbb{L})$ and $\mathscr{P}_f(\mathscr{B})$ by automorphisms of these vector spaces. It is therefore possible, and very useful in the following, to consider the action of the group algebra $\mathbb{F}_2[\mathbb{Z}^\nu]$ on these spaces. We now recall the relevant facts, [18, 27].

3.3. Let k be a field and let G be a (commutative) group; in our case $k=\mathbb{F}_2$ and $G=\mathbb{Z}^\nu$. Let $k[G]$ be the k-vector-space of functions from G to k with finite support. The k-vector-space structure is defined pointwise and the multiplication by

$$F\cdot G(a)=\sum_{b\in G}F(a-b)G(b)\,,\qquad F,G\in k[G]\,.$$

Since G is an abelian group $k[G]$ is a commutative algebra.

Let ξ be a vector space over the field k and suppose that the group G acts by automorphisms of ξ:

$$a:v\mapsto a\cdot v\,.$$

Then there is a natural action of $k[G]$ by endomorphisms of ξ defined by

$$v\mapsto F\cdot v=\sum_{a\in G}F(a)a\cdot v\,,\qquad v\in\xi\,,\qquad F\in k[G]\,. \tag{3.3}$$

For $a\in G$ let X^a be the characteristic function of $\{a\}$:

$$\begin{aligned} X^a(b)&=1\in k \quad \text{if}\quad b=a\\ &=0 \qquad\ \text{if}\quad b\neq a\,. \end{aligned}$$

Then

$$X^aX^b=X^{a+b} \tag{3.4}$$

and X^0 is the unit element 1 of $k[G]$. Furthermore, let $\langle\,,\rangle$ be a k-bilinear form on $k[G]$ defined by

$$\langle F,G\rangle=\sum_{x\in G}F(x)G(x) \tag{3.5}$$

and let

$$I(F)(x)=F(-x)\,. \tag{3.6}$$

I is an involutive automorphism of $k[G]$:

$$I(I(F))=F\,,\qquad I(F\cdot G)=I(F)\cdot I(G)\,, \tag{3.7}$$

and

$$\langle F\cdot G,H\rangle=\langle F,I(G)\cdot H\rangle \tag{3.8}$$

for any $F,G,H\in k[G]$.

To prove (3.7) and (3.8) notice that both sides of these equalities are linear, bilinear or trilinear expressions in F, G, H and that therefore it is enough to verify them if $F=X^a$, $G=X^b$, $H=X^c$, $a, b, c \in G$, which is straightforward. [Physicists are familiar with formulae like (3.5)–(3.8) from the Fourier analysis, when $k=\mathbb{C}$ and $G=\mathbb{R}$ or $\mathbb{Z}$; $I(F)$ is written usually as $\check{F}$.]

Let now $G=\mathbb{Z}^\nu$ and let

$$X_i = X^{e_i} \quad \text{where} \quad e_i = (\delta_{1i}, \delta_{2i}, \ldots, \delta_{\nu i}) \in \mathbb{Z}^\nu .$$

Then for any $a=(a_1, \ldots, a_\nu) \in \mathbb{Z}^\nu$

$$X^a = X_1^{a_1} \ldots X_\nu^{a_\nu},$$

and since $\{X^a : a \in \mathbb{Z}^\nu\}$ are linearly independent, (3.4) shows that the subalgebra $k[X_1, \ldots, X_\nu]$ of $k[\mathbb{Z}^\nu]$ generated by $\{X_i : i=1, \ldots, \nu\}$ is isomorphic to the algebra of polynomials in ν variables with coefficients in k; we will call its elements polynomials.

By well known properties of polynomials with coefficients in a field, $k[X_1, \ldots, X_\nu]$ has the following properties:

(P_1) *It is an integral domain*, i.e. $P.Q=0$ or $Q=0$.

(P_2) *It is a unique factorization domain* (for any $P \in k[X_1, \ldots, X_\nu]$ there exist irreducible polynomials $P_1, \ldots, P_r$ and natural numbers $n_1, \ldots, n_r$ such that

$$P = P_1^{n_1} \ldots P_r^{n_r} .$$

$(P_i, n_i)_{i=1,\ldots,r}$ is unique up to a permutation and k-factors). Plainly, each element of $k[\mathbb{Z}^\nu]$ can be written as

$$X^a \cdot P, \qquad a \in \mathbb{Z}^\nu, \qquad P \in k[X_1, \ldots, X_\nu];$$

and since $\lambda 1$, $\lambda \in k$, $\lambda \neq 0$, are the only units (invertible elements) of $k[X_1, \ldots, X_\nu]$ it follows that λX^a, $\lambda \in k$, $\lambda \neq 0$, $a \in \mathbb{Z}^\nu$, are the only units of $k[\mathbb{Z}^\nu]$.

From the fact that any element of $k[\mathbb{Z}^\nu]$ is a unit times a polynomial one easily deduces that $k[\mathbb{Z}^\nu]$ has also the properties (P_1), (P_2) [the uniqueness of (P_2) holds "up to a unit"]. Then from (P_2) one deduces that for any family T of elements of $k[\mathbb{Z}^\nu]$ there exists unique, up to a unit, *greatest common divisor* [g.c.d. (T)]. We note the following important difference between the case $\nu=1$ and $\nu \neq 1$: if T is an ideal of $k[\mathbb{Z}^\nu]$ then in case $\nu=1$, g.c.d. $(T) \in T$ whereas this is not true in higher dimensions in general. This fact is responsible, from the point of view of this paper, for the existence of phase transitions if $\nu>1$ and their absence in the one-dimensional case.

3.4. Specializing to $k=\mathbb{F}_2$ yields the group algebra $\mathbb{F}_2[\mathbb{Z}^\nu]$ which is of interest here.

As a vector space over $\mathbb{F}_2$, $\mathbb{F}_2[\mathbb{Z}^\nu]$ can be identified with $\mathscr{P}_f(\mathbb{Z}^\nu)$: the identification mapping assigns to $F \in \mathbb{F}_2[\mathbb{Z}^\nu]$ its support, $\{a \in \mathbb{Z}^\nu : F(a)=1\}$. In other words, we identify $A \in \mathscr{P}_f[\mathbb{Z}^\nu]$ with

$$\sum_{a \in A} X^a .$$

This identification allows us to transfer the multiplication operation from $\mathbb{F}_2[\mathbb{Z}^\nu]$ to $\mathscr{P}_f(\mathbb{Z}^\nu)$. Multiplication by X^a, $a \in \mathbb{Z}^\nu$, corresponds to translation by a; more generally

$$A \cdot B = \sum_{a \in A} \tau_a(B), \qquad A, B \in \mathscr{P}_f(\mathbb{Z}^\nu)$$

the sum being taken here in $\mathscr{P}_f(\mathbb{Z}^\nu)$. Hence, $\bar{\mathscr{B}}$ as a translation invariant subgroup of $\mathscr{P}_f(\mathbb{Z}^\nu)$ becomes now an ideal. Note also that

$$I(A)=\{-a : a\in A\}$$

and that under the natural identification of the group $\{-1,1\}$ with the field $\mathbb{F}_2=\{0,1\}$ the bicharacter (3.1) is (in case $\mathbb{L}=\mathbb{Z}^\nu$) identical with the bilinear form (3.5).

Polynomials are identified with subsets of $\mathbb{Z}^\nu_+$; and that any element of $\mathbb{F}_2[\mathbb{Z}^\nu]$ is a unit times polynomial corresponds to the fact that any finite set can be shifted to $\mathbb{Z}^\nu_+$ by suitable translation. We also note that the greatest common divisor is unique up to translation.

The action of $k[G]$ on k-vector-spaces on which G operates by automorphisms, described at the beginning of the preceding section, yields here an action of $\mathbb{F}_2[\mathbb{Z}^\nu]$ on $\mathscr{P}_f(\mathscr{B})$, Γ_f and $\mathscr{K}_f$. The identification of $\mathscr{P}_f(\mathbb{Z}^\nu)$ with $\mathbb{F}_2[\mathbb{Z}^\nu]$ allows then to write expressions like

$$A\cdot\beta, \qquad A\in\mathscr{P}_f(\mathbb{Z}^\nu), \qquad \beta\in\mathscr{P}_f(\mathscr{B}),$$

and if $\mathscr{B}_0$ is a (finite) fundamental family of bonds then for any $\alpha\in\mathscr{P}_f(\mathscr{B})$ there are unique $P_B\in\mathscr{P}_f(\mathbb{Z}^\nu)$, $B\in\mathscr{B}_0$, such that

$$\alpha=\sum_{B\in\mathscr{B}_0} P_B\cdot\{B\}.$$

3.5 Lemma. *Let* $\alpha=\sum_{B\in\mathscr{B}_0} P_B\cdot\{B\}$ *and* $\beta=\sum_{B\in\mathscr{B}_0} Q_B\cdot\{B\}$ *be two elements of* $\mathscr{P}_f(\mathscr{B})$; α *is orthogonal to all translates of* β *if and only if*

$$\sum_{B\in\mathscr{B}_0} P_B\cdot I(Q_B)=0.$$

Obviously

$$\langle\alpha,\beta\rangle=\sum_{B\in\mathscr{B}_0}\langle P_B, Q_B\rangle. \tag{3.9}$$

α is orthogonal to all translates of β if and only if it is orthogonal to $R\cdot\beta$ for any $R\in\mathscr{P}_f(\mathbb{Z}^\nu)$. Applying (3.9):

$$\langle\alpha, R\beta\rangle=\sum_{B\in\mathscr{B}_0}\langle P_B, R\cdot Q_B\rangle$$

and by (3.8)

$$\sum_{B\in\mathscr{B}_0}\langle P_B, R\cdot Q_B\rangle=\sum_{B\in\mathscr{B}_0}\langle P_B\cdot I(Q_B), R\rangle=\left\langle\sum_{B\in\mathscr{B}_0} P_B\cdot I(Q_B), R\right\rangle.$$

The last expression is 0 for all $R\in\mathscr{P}_f(\mathbb{Z}^\nu)$ if and only if $\sum_{B\in\mathscr{B}_0} P_B I(Q_B)=0$. Hence the lemma is proved.

3.6. Every element of Γ_f is a combination of translates of

$$\gamma_0=\gamma(\{0\}).$$

On the other hand since

$$\gamma(\{0\}) = \{B \in \mathscr{B} : 0 \in B\}$$

is not hard to see that

$$\gamma_0 = \sum_{B \in \mathscr{B}_0} I(B) \cdot \{B\}. \tag{3.10}$$

Let $\mathscr{B}_0 = \{B_1, \ldots, B_n\}$ and let for $1 \leqq i < j \leqq n$

$$\beta_{ij} = B_i\{B_j\} + B_j \cdot \{B_i\}. \tag{3.11}$$

Since the homomorphism $\varepsilon : \mathscr{P}_f(\mathscr{B}) \to \mathscr{P}_f(\mathbb{Z}^\nu)$ of the $\mathbb{F}_2$-vector-spaces commutes with the actions of $\mathbb{Z}^\nu$ it commutes with the actions of $\mathbb{F}_2[\mathbb{Z}^\nu] (\cong \mathscr{P}_f(\mathbb{Z}^\nu))$. Therefore

$$\varepsilon(\beta_{ij}) = B_1 \cdot \varepsilon(\{B_j\}) + B_j \cdot \varepsilon(\{B_i\}) = B_i B_j + B_j B_i = 0.$$

This shows that $\beta_{ij} \in \mathscr{K}_f$.

3.7 Proposition. *If* g.c.d. $\mathscr{B} = 1$ *and* $\alpha \in \mathscr{P}_f(\mathscr{B})$ *is orthogonal to all translates of* β_{ij}, $1 \leqq i < j \leqq n$, *then* $\alpha \in \Gamma_f$.

Let $\alpha = \sum_{i=1}^{n} P_i \cdot \{B_i\}$, $P_i \in \mathscr{P}_f(\mathbb{Z}^\nu)$. By Lemma 3.6 the orthogonality of α to all translates of β_{ij} is equivalent to

$$P_i \cdot I(B_j) = P_j I(B_i), \qquad i, j = 1, \ldots, n. \tag{3.12}$$

Since I is an automorphism of the algebra $\mathscr{P}_f(\mathbb{Z}^\nu)$, g.c.d. $\{I(B) : B \in \mathscr{B}_0\} = 1$. By factorizing each of $I(B_i)$ into primes, it is easy to see that this together with (3.12) implies that P_i is divisible by $I(B_i)$, $i = 1, \ldots, n$. This means that there exist $P'_i \in \mathscr{P}_f(\mathbb{Z}^\nu)$ such that

$$P_i = P'_i \cdot I(B_i).$$

Substituting this into (3.12), we get

$$(P'_i - P'_j) \cdot I(B_i) \cdot I(B_j) = 0;$$

and since there are no zero-divisors in $\mathscr{P}_f(\mathbb{Z}^\nu)$ it follows that $P_i = P \cdot I(B_i)$. Hence

$$\alpha = P \cdot \sum_{B \in \mathscr{B}_0} I(B) \cdot \{B\}$$

which by (3.10) proves the proposition.

3.8 Theorem. *The system has the decomposition property if and only if* g.c.d. $\mathscr{B} = 1$.

We prove now that g.c.d. $\mathscr{B} = 1$ implies the decomposition property of Γ_f: only this part of Theorem is needed in this paper. The converse is proved in the next section.

Let

$$N = \max_{1 \leqq i < j \leqq n} \operatorname{diam}(\beta_{ij});$$

we claim that N-components of contours are contours.

For let $\alpha \in \Gamma_f$ and let α' be an N-component of α. If β is a translate of one of β_{ij}, $1 \leqq i < j \leqq n$, and β intersects α' then $\beta \cap (\alpha \setminus \alpha') = \emptyset$. This shows that $\langle \alpha', \beta \rangle = \langle \alpha, \beta \rangle$ and that therefore α' is orthogonal to β. To finish the proof, we apply Proposition 3.8.

3.9. Since for any $a \in \mathbb{Z}^\nu$, $\{a\}^2 = \{2a\}$ in the sense of $\mathscr{P}_f(\mathbb{Z}^\nu)$ $(\cong F_2[\mathbb{Z}^\nu])$ and $(A+B)^2 = A^2 + B^2$, raising $A = \sum_{a \in A} \{a\}$ to power 2^n we obtain

$$A^{2^n} = \{2^n a : a \in A\} \quad \text{for any} \quad A \in \mathscr{P}_f(\mathbb{Z}^\nu). \tag{3.13}$$

We will prove now that *if* $D = \text{g.c.d.}\ (\mathscr{B})$ *is not a unit*, i.e. if the greatest common divisor is non-trivial *then* Γ_f *does not have the decomposition property.*

Let

$$\gamma_1' = \sum_{B \in \mathscr{B}_0} I\left(\frac{B}{D}\right) \cdot \{B\} = \sum_{B \in \mathscr{B}_0} \frac{I(B)}{I(D)} \{B\},$$

here $\frac{B}{D}$ is the element of $\mathscr{P}_f(\mathbb{Z}^\nu)$ for which

$$\frac{B}{D} \cdot D = B$$

and $\frac{I(B)}{I(D)}$ makes sense since I is an automorphism of the algebra $\mathscr{P}_f(\mathbb{Z}^\nu)$. $\gamma_0' \notin \Gamma_f$, for from

$$\gamma_0' = P \cdot \gamma_0$$

it would follow that $I(D) \cdot P = 1$ in contradiction with the fact that D is not a unit. On the other hand

$$I(D)^{2^n} \cdot \gamma_0' = I(D)^{2^n - 1} \cdot \gamma_0$$

is in Γ_f. But it follows from (3.13) that for any fixed N we can find n such that the N-components of $I(D)^{2^n} \cdot \gamma_0'$ are translates of γ_0'. This finishes the proof.

4. Reduction

Let $\mathscr{B}$ be a translation invariant family of finite subsets of $\mathbb{L}$ with a finite fundamental subfamily $\mathscr{B}_0$, let $D \in \mathbb{F}_2[\mathbb{Z}^\nu]$ and let

$$\mathscr{B}' = \{D \cdot B : B \in \mathscr{B}\}.$$

Let K be ferromagnetic translation invariant interaction with bonds $\mathscr{B}$ and let

$$K'(D \cdot B) = K(B), \qquad B \in \mathscr{B}.$$

The theorem below tells that the systems with interactions K and K' are, in a sense, isomorphic.

4.1 Theorem. *If ϱ^+ and ϱ'^+ are the equilibrium states corresponding to the interactions K and K', respectively, then*

$$\varrho'^+(\sigma_{D\cdot A}) = \varrho^+(\sigma_A), \quad any \quad A \in \mathscr{P}_f(\mathbb{L}), \tag{4.1}$$

$$\varrho'^+(\sigma_A) = 0 \quad if \quad A \notin (D), \tag{4.2}$$

where

$$(D) = \{A \cdot D : A \in \mathscr{P}_f(\mathbb{L})\}.$$

Let for $A \in \mathscr{P}_f(\mathbb{L})$

$$A' = D \cdot A, \qquad \mathscr{B}_A = \{x \cdot A : x \in \mathbb{Z}^\nu\},$$

and let K_A be the interaction with bonds $\mathscr{B}_A$ such that

$$K_A(B) = 1, \qquad \forall B \in \mathscr{B}_A.$$

We will show now that

$$P(K + \lambda K_A) = p(K' + \lambda K_{A'}). \tag{4.3}$$

To compare the partition functions that yield the two pressures of (4.3) we use Heigh Temperature Expansion (HTE)[5]: Let Λ be a finite subset of the lattice, let $\mathscr{C}$ be a family of subsets of Λ, and let I be a function from $\mathscr{C}$ to real numbers. If we let

$$Z(I)_\Lambda = \sum_{X \in \mathscr{X}_\Lambda} \exp \sum_{C \in \mathscr{C}} I(C)\sigma_C(X)$$

then HTE reads:

$$Z(I)_\Lambda = 2^{|\Lambda|} \left[\prod_{C \in \mathscr{C}} ch I(C)\right] \sum_{\beta \in \mathscr{K}(\mathscr{C})} \prod_{B \in \beta} th I(B). \tag{4.4}$$

Let now $D \in \mathbb{F}_2[\mathbb{Z}^\nu]$, let

$$\mathscr{C}' = \{D \cdot C : C \in \mathscr{C}\}$$

and let

$$\beta' = \{C' : C \in \beta\}.$$

Then $\beta \mapsto \beta'$ is an isomorphism of $\mathscr{P}_f(\mathscr{C})$ onto $\mathscr{P}_f(\mathscr{C}')$ which commutes with ε:

$$\bar{\beta}' = D \cdot \bar{\beta}. \tag{4.5}$$

Since $A \mapsto D \cdot A$ is a monomorphism of $\mathscr{P}_f(\mathbb{L})$ into $\mathscr{P}_f(\mathbb{L})$ (this follows from the fact that $\mathscr{P}_f(\mathbb{L})$ is isomorphic as $\mathbb{F}_2[\mathbb{Z}^\nu]$-module to direct sum of a few copies of $\mathbb{F}_2[\mathbb{Z}^\nu]$, and from the fact that there are no zero divisors in $\mathbb{F}_2[\mathbb{Z}^\nu]$) (4.5) shows that $\beta \mapsto \beta'$ defines an isomorphism of $\mathscr{K}_f(\mathscr{C})$ onto $\mathscr{K}_f(\mathscr{C}')$. Defining an interaction I' in

$$'\Lambda = \bigcup_{x \in D} x \cdot \Lambda$$

by

$$I'(C') = I(C)$$

5 cf. [16, 7, 35, 13]; we employ the notation of the last of these references

we conclude from (4.4) that

$$Z(I')_{\Lambda} = 2^{|'\Lambda| - |\Lambda|} Z(I)_{\Lambda}. \tag{4.6}$$

For any interaction K and for any finite subset Λ of the lattice, let K_{Λ} be the restriction of K to

$$\mathscr{B}_{\Lambda} = \{B \in \mathscr{B} : B \subset \Lambda\}.$$

If (Λ) is a Van Hove net then so is $('\Lambda)$, and $Z(((K+\lambda K_A)_A)')_{'\Lambda}$ differs from $Z((K'+\lambda K_{A'})_{'A})_{'\Lambda}$ in a way that does not affect the thermodynamic limit, i.e.

$$\lim_{\Lambda} \frac{1}{|'\Lambda|} \log Z(((K+\lambda K_A)_A)')_{'\Lambda} = \lim_{\Lambda} \frac{1}{|'\Lambda|} \log Z((K'+\lambda K_{A'})_{'A})_{'\Lambda}.$$

The limit on the right hand side here is equal to $p(K'+\lambda K_{A'})$; applying (4.6) to $I = (K+\lambda K_A)_A$ we see that the left hand side is equal to $p(K+\lambda K_A)$. This proves (4.3).

On the other hand, as is not hard to deduce from the maximality of ϱ^+ (the Griffiths inequality $(G2)$ in [31]) for any ferromagnetic interaction K.

$$\varrho^+(\sigma_A) = \lim_{\lambda \downarrow 0} \lambda^{-1}(p(K+\lambda K_A) - p(K)).$$

This, together with (4.3), yields (4.1).

Let now A be any element of $\mathscr{P}_f(\mathbb{L})$. Then by (3.13) for large enough n

$$D^{2^n} \cdot A$$

consists of Card(D) copies of A, one separated from another by a distance of order 2^n. Since the "+" state is translation invariant and clustering it follows that

$$\lim_{n \to \infty} \varrho'^+(\sigma_{D^{2^n} \cdot A}) = [\varrho'^+(\sigma_A)]^{\mathrm{Card}(D)}.$$

On the other hand, by (4.1),

$$\varrho'^+(\sigma_{D^{2^n} \cdot A}) = \varrho^+(\sigma_{D^{2^n - 1} \cdot A})$$

Hence, with the notation:

$$A_n = D^{2^n - 1} \cdot A,$$

it is enough to show that if $A \notin (D)$ then $\varrho^+(\sigma_{A_n}) \to 0$ as $n \to \infty$.

The proof of the last statement will be divided in two steps: in Step 1 we show that if $A \notin (D)$ then Card$(A_n) \to \infty$ as $n \to \infty$; in Step 2 (Lemma 4.3) we show that for any ferromagnetic finite range interaction and any family A_n such that Card$(A_n) \to \infty$

$$\varrho^+(\sigma_{A_n}) \to 0 \quad \text{as} \quad n \to \infty.$$

4.2. Step 1. Let δ be the diameter of D (diam(D)):

$$\delta = \max\{|x-y| : x, y \in D\}.$$

Let $(C_i)_{i \in I}$ be the family of $(2\delta+1)$-components of A_n, i.e. components of the graph

$$(A_n, \{\{x, y\} : x, y \in A_n, |x-y| \leq 2\delta + 1\})$$

(cf. Sect. 2.1). $D \cdot C_i \cap D \cdot C_j = \emptyset$, for $i \neq j$, since the distance between C_i and C_j is larger than 2δ. If $2^n \geqq 2\,\mathrm{diam}(A)$ then $D \cdot A_n\ (= D^{2^n} \cdot A)$ consists of Card (D) translates of A with mutual distance $\geqq 2^n - 2\,\mathrm{diam}(A)$ [cf. (3.13)]:

$$\begin{aligned} &D \cdot A_n = \bigcup_{x \in D} A_x \\ &\mathrm{dist}(A_x, A_y) \geqq 2^n - 2\,\mathrm{diam}(A) \quad \text{for} \quad x \neq y, \end{aligned} \tag{4.7}$$

where

$$A_x = (2^n x) \cdot A .$$

Let

$$I(x) = \{ i \in I : D \cdot C_i \cap A_x \neq \emptyset \}$$

and let a be any element of D. If $I(a) \cap I(x) = \emptyset$ for all $x \in D$, $x \neq a$, then

$$A_x = \sum_{i \in I(a)} D \cdot C_i$$

and therefore $A \in (D)$, contrary to what was assumed. Hence there exist $i_0 \in I(a)$ and $b \in D$, $b \neq a$, such that $D \cdot C_{i_0}$ intersects both A_a and A_b. It follows now from (4.7) that at least two points of C_{i_0} are distant one from another by not less than

$$2^n - 2\,\mathrm{diam}(A) - 2\,\mathrm{diam}(D) .$$

And since C_{i_0} is $(2\delta + 1)$-connected its cardinality, and therefore the cardinality of A_n too, is at least

$$(2^n - 2\,\mathrm{diam}(A) - 2\,\mathrm{diam}(D))/(2\delta + 1) .$$

This proves our assertion.

4.3 Lemma. *For any ferromagnetic finite range interaction and for any sequence* (A_n) *of finite subsets of the lattice with* Card$(A_n) \to \infty$ *as* $n \to \infty$

$$\varrho^+(\sigma_{A_n}) \to 0 \quad as \quad n \to \infty .$$

If $(A_i)_{i \in I}$ is a family of subsets of the lattice with mutual distance exceeding the range of the interaction $[= \max\{\mathrm{diam}(B) : B \in \mathscr{B}\}]$ then

$$\varrho^+(\sigma_A) \leqq \prod_{i \in I} \varrho^+_{A_i}(\sigma_{A \cap A_i}) ; \tag{4.8}$$

on the right hand side here we have the expectation value in the state which is the tensor product of the $\varrho^+_{A_i}$-states in A_i and the "+" ground state in $\mathbb{L} \setminus \bigcup_i A_i$. To prove (4.8) it is enough to add to the interaction the term

$$H \sum_{i \in \mathbb{L} \setminus \bigcup_i A_i} \sigma_i , \qquad H > 0$$

which by GKS [7, 11] increases the expectation value of σ_A, and to let H go to $+\infty$. Then only contributions from configurations which are + outside of $\bigcup_i A_i$

survive in the limit as is most easily seen by considering the finite volume states first. But under the assumption on range of the interaction the limit yields the right hand side of (4.8).

Let now r be any integer larger than the range of the interaction, let $\mathbb{L}_0$ be a fundamental subset of $\mathbb{L}$ with respect to the action of

$$r\mathbb{Z}^\nu = \{r \cdot x : x \in \mathbb{Z}^\nu\}$$

and let A_i be the family of all the one point subsets of $r\mathbb{Z}^\nu \cdot p$, $p \in \mathbb{L}_0$; applying (4.8) we obtain

$$\varrho^+(\sigma_A) \leqq [\varrho^+_{\{p\}}(\sigma_{\{p\}})]^{|A \cap r\mathbb{Z}^\nu \cdot p|}.$$

Since there exists $p \in \mathbb{L}_0$ such that

$$|A \cap r\mathbb{Z}^\nu \cdot p| \geqq E(|A|/|\mathbb{L}_0|),$$

where $E(t)$ is the smallest integer $\geqq t$, we have

$$\varrho^+(\sigma_A) \leqq \alpha^{E(|A|/|\mathbb{L}_0|)}$$

where

$$\alpha = \max_{p \in \mathbb{L}_0} \varrho^+_{\{p\}}(\sigma_{\{p\}}).$$

Since α is strictly smaller than 1, Lemma follows.

This ends the proof of Theorem 4.1.

4.4 Trivial Systems. This is an amusing example of systems that can be completely analyzed with help of Theorem 4.1:

A system is called *trivial* if $\mathscr{K}_f = \{0\}$. By [30, Sect. 3.5] for any trivial system and at any temperature all translation invariant equilibrium states coincide on $\{\sigma_A : A \in \bar{\mathscr{B}}\}$. Again by [30, Sect. 3.5] if $\mathbb{L} = \mathbb{Z}^\nu$ a system is trivial if and only if all the bonds are translates of a one, say D. Now Theorem 4.1 shows that in case of a ferromagnetic trivial system on $\mathbb{Z}^\nu$

$$\varrho^+(\sigma_{D \cdot A}) = (\operatorname{th} K(D))^{|A|}, \tag{4.1'}$$

$$\varrho^+(\sigma_A) = 0 \quad \text{if} \quad A \notin (D); \tag{4.2'}$$

here (4.1') follows from (4.1) since the system with bonds $\{B/D : B \in \mathscr{B}\}$ has one point interaction of strength K. Since in the present case $\bar{\mathscr{B}} = (D)$, (4.2') shows that there is only one translation invariant (and also periodic and quasi-periodic) equilibrium state.

In fact one of us has shown recently that for some trivial system the equilibrium state, not only the translation invariant one, is unique at all temperatures.

5. The Main Theorem

Let J be any ferromagnetic, translation invariant interaction on $\mathbb{L} = \mathbb{Z}^\nu$. Then, according to [31],

$$\mathscr{B}^+ = \{A \in \mathscr{P}_f(\mathbb{Z}^\nu) : \varrho^+(\sigma_A) \neq 0\}$$

is a (temperature dependent) translation invariant subgroup of $\mathscr{P}_f(\mathbb{Z}^\nu)$, i.e. $\mathscr{B}^+$ is an ideal of $\mathbb{F}_2(\mathbb{Z}^\nu]$.

Suppose now that the interaction is of a *finite range* and let $D = \text{g.c.d.}(\mathscr{B})$. Let J' be the interaction with bonds $\mathscr{B}'$ obtained by factoring out D from the bonds of $\mathscr{B}$:

$$\mathscr{B} = \{D \cdot B : B \in \mathscr{B}'\},$$

and such that

$$J'(B) = J(D \cdot B).$$

Then, obviously, $\text{g.c.d.}(\mathscr{B}') = 1$. We can therefore apply Theorem 4.1 first and then Theorem 2.7 to the system with interaction J' ("reduced system"). This yields the following.

Theorem. *Let J, $\mathscr{B}$, D and J' be as above; at low enough temperatures $\mathscr{B}^+$ is the subgroup of $\mathscr{P}_f(\mathbb{Z}^\nu)$ generated by translates of D, or, which is the same, the ideal (D) of $\mathbb{F}_2[\mathbb{Z}^\nu]$ generated by D. For any temperature*

$$\varrho^+(\sigma_{D \cdot A}) = \varrho'^+(\sigma_A), \qquad \forall A \in \mathscr{P}_f(\mathbb{Z}^\nu)$$

$$\varrho^+(\sigma_A) = 0 \quad \textit{if} \quad A \notin (D)$$

where ϱ^+ is defined by J and ϱ'^+ by J'.

In fact most of the results are obtained under weaker hypotheses than those of the theorem:

Suppose that $\text{g.c.d.}\, \mathscr{B} = 1$. Let $\mathscr{B}_0$ be a fundamental family for $\mathscr{B}$ but instead of assuming that $\mathscr{B}_0$ is finite suppose that

$$\sum_{B \in \mathscr{B}_0} J(B) < \infty, \qquad J(B) > 0. \tag{5.1}$$

Since $\mathbb{F}_2[\mathbb{Z}^\nu]$ is a Noetherian ring, [27], there is a finite subfamily $\mathscr{B}'_0$ of $\mathscr{B}_0$ such that $\bar{\mathscr{B}}' = \bar{\mathscr{B}}$, where $\mathscr{B}'$ is the set of translates of the elements of $\mathscr{B}_0$. Obviously, $\text{g.c.d.}(\mathscr{B}') = 1$. Let J' be the restriction of J to $\mathscr{B}'$. Theorem 2.7 applies to J', and therefore at low enough temperature $\mathscr{B}'^+ = \mathscr{P}_f(\mathbb{Z}^\nu)$. On the other hand by the GKS inequalities, [7, 11], $\varrho^+(\sigma_A) \geqq \varrho'^+(\sigma_A)$ for all $A \in \mathscr{P}_f(\mathbb{Z}^\nu)$. Thus if $\text{g.c.d.}(\mathscr{B}) = 1$ the theorem holds without the assumption of the finiteness of the range of the interaction.

As for the reduction, the proof of Lemma 4.3 was the only place where finiteness of the range of the interaction was essential. *We therefore conjecture that the theorem holds under the condition* (5.1).

Appendix 1

We shall give a very explicit description of the group of finite cycles $\mathscr{K}_f$ of arbitrary system in $\mathbb{Z}^\nu$ with a two-element fundamental set $\mathscr{B}_0$.

(A1) Theorem[6]. *Let $\mathscr{B}_0 = \{A, B\}$ and $D = \text{g.c.d.}\,(A, B)$. Then $\mathscr{K}_f$ is a free $\mathbb{F}_2[\mathbb{Z}^\nu]$-*

6 This theorem of the first named author, was used in a proof that all systems with two fundamental bonds are self dual (cf. [33])

module freely generated by one element

$$\alpha = \frac{B}{D}\{A\} + \frac{A}{D}\{B\},$$

i.e. for each $\beta \in \mathscr{K}_f$ *there is unique* $R \in \mathbb{F}_2[\mathbb{Z}^\nu]$ *such that* $\beta = R \cdot \alpha$.

Proof. Let $\beta = P \cdot \{A\} + Q\{B\}$, $P, Q \in \mathbb{F}_2[\mathbb{Z}^\nu]$, be an arbitrary element of $\mathscr{K}_f$. Then

$$P \cdot A + Q \cdot B = 0.$$

Since $\mathbb{F}_2[\mathbb{Z}^\nu]$ is an integral domain hence

$$P \cdot \frac{A}{D} + Q \cdot \frac{B}{D} = 0.$$

Since $\mathbb{F}_2[\mathbb{Z}^\nu]$ is a unique factorization domain and

$$\text{g.c.d.}\left(\frac{A}{D}, \frac{B}{D}\right) = 1$$

hence

$$P = R \cdot \frac{B}{D} \quad \text{and} \quad Q = R \cdot \frac{A}{D}$$

for a unique $R \in \mathbb{F}_2[\mathbb{Z}^\nu]$. Thus $\beta = R \cdot \alpha$. We have shown that $\mathscr{K}_f$ is generated by α. Since $\mathscr{K}_f$ is free hence when it is generated by one element α, it is freely generated by such an element.

(A2) Corollary. *Let* $\mathscr{B}_0 = \{A, B\}$ *and* g.c.d.$(A, B) = 1$. *Then the group* $\mathscr{K}_f$ *is generated by translates of*

$$\alpha = B \cdot \{A\} + A \cdot \{B\}.$$

Appendix B

We shall show here that at low temperatures $\mathscr{B}^+$ determines not only $\mathbb{Z}^\nu$-invariant but also periodic and, more generally, quasi-periodic equilibrium states. This is an extension of the results of [32, Sect. 4.8]; we retain here the framework and notation of this paper.

Consider first the case of a periodic equilibrium state ϱ, i.e. assume that ϱ is invariant under a subgroup G of $\mathbb{Z}^\nu$ of a finite index. Since ϱ is G-invariant we can consider the action of $\mathbb{Z}^\nu/G$ on ϱ. Let

$$\bar{\varrho} = \frac{1}{\operatorname{Card}(\mathbb{Z}^\nu/G)} \sum_{x \in \mathbb{Z}^\nu/G} \varrho_x;$$

$\bar{\varrho}$ is a $\mathbb{Z}^\nu$-invariant equilibrium state and therefore $\bar{\varrho}(s_A) = \varrho^+(s_A)$ if $\bar{A} \in \bar{\mathscr{B}}$, [32, Theorem 4.5]. Since ϱ^+ is maximal on each s_A [32, inequality (G3)], it follows that each ϱ_x coincides with ϱ^+ on s_A, $\bar{A} \in \bar{\mathscr{B}}$. In particular $\varrho \in \Delta^+$.

Suppose now that, more generally, ϱ is $\mathbb{Z}^\nu$-quasi-periodic. That means that the $\mathbb{Z}^\nu$-orbit of ϱ is pre-compact in the norm topology of the Banach space $\mathfrak{A}'$. Let $[\varrho]$ be the norm-closure of the $\mathbb{Z}^\nu$-orbit of ϱ and let μ be a $\mathbb{Z}^\nu$-invariant measure on

$[\varrho]$; it is not hard to see that such a measure always exists. Then

$$\bar{\varrho} = \int\limits_{[\varrho]} \varrho_\xi(d\xi)$$

is a $\mathbb{Z}^\nu$-invariant equilibrium state and hence at low temperatures $\bar{\varrho}(s_A) = \varrho^+(s_A)$ if $\bar{A} \in \mathscr{B}$. From the maximality of ϱ^+ we conclude that for μ-almost all ξ, $\varrho_\xi \in \Delta^+$.

Since the orbit of ϱ is dense in $[\varrho]$, there exists a sequence $a_n \in \mathbb{Z}^\nu$ and $\omega \in \Delta^+$ such that

$$\|\tau_{a_n}(\varrho) - \omega\| \to 0 \quad \text{as} \quad n.$$

In particular for any A such that $\bar{A} \in \bar{\mathscr{B}}$

$$\sup\{|\varrho(s_{A' + a_n}) - \omega(s_{A'})| : A' \quad \text{is a translate of } A\} \to 0.$$

But by the translation invariance of ω on $\{s_A : \bar{A} \in \bar{\mathscr{B}}\}$ this last supremum is certainly majorizing $|\varrho(s_A) - \omega(s_A)|$ which shows that $\varrho(s_A) = \omega(s_A)$ if $\bar{A} \in \bar{\mathscr{B}}$ and that therefore $\varrho \in \Delta^+$. This is the result we wanted to prove.

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