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Vacuum Energy in $g\varphi_d^4$-Theory for $g\to\infty$

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Abstract. In the nonlocal $g\varphi_d^4(d\geqq 1)$ and local $g\varphi_2^4$ theory the S-matrix is obtained in a form of the functional integral which is proved to exist. The density of vacuum energy

$$E(g) = -\lim_{V\to\infty}\frac{1}{V}\ln\langle 0|S_V(g)|0\rangle$$

is investigated. It is proved to be analytic through the whole complex g-plane except for the negative real axis and point $g=0$. Its asymptotic behaviour for $g\to\infty$ is found.

1. Statement of the Problem

The φ^4-theory is rather popular; and quite a number of papers are devoted to investigations of various aspects of this theoretical model. Without exaggeration, one can say that it is just this model that tests majority of theoretical methods and approaches. It is difficult to report all contributions of investigations of different aspects of the φ^4-theory.

In the given paper, the $g\varphi^4$-theory is applied to study the density of vacuum energy

$$E(g) = -\lim_{V\to\infty}\frac{1}{V}\ln\langle 0|S_V(g,\varphi)|0\rangle\,. \tag{1.1}$$

The function $E(g)$ in (1.1) is finite in the local φ_d^4-theory for space-time dimensions $d=1$ (anharmonic oscillator) and $d=2$ (the so called φ_2^4-theory) only. For $d>2$ the function $E(g)$ does not exist at all in the local theory because of ultraviolet divergences. However, $E(g)$ is finite in the nonlocal theory when the causal Green functions $\tilde{D}(k^2)$ of the scalar field φ decreases rather fast in the Euclidean direction $(k^2=k_0^2-\mathbf{k}^2\to-\infty)$.

In the given paper we will consider all these cases and obtain the analytical properties in the complex g-plane and the asymptotical behaviour of the function $E(g)$ for $g\to\infty$.

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We will examine the theory of one component scalar field $\varphi(x)$ described by the Lagrangian density

$$\mathscr{L}(x)=\tfrac{1}{2}\varphi(x)(\Box-m^2)\varphi(x)-g[K(l^2\Box)\varphi(x)]^4 \tag{1.2}$$

The operator $K(l^2\Box)$ is nonlocal and satisfies the conditions

i) $K(z)$ is an entire analytical function of an order of $\varrho\geqq\frac{1}{2}$ in the complex z-plane,

ii) $K(z)$ decreases rapidly enough when $z=l^2k^2=l^2(k_0^2-\mathbf{k}^2)\to-\infty$,

iii) $K(-l^2m^2)=1$,

iv) l is a parameter characterizing the region of nonlocal interaction.

The local theory is the limit $l\to 0$. The finite unitary S-matrix in perturbation theory for the Lagrangian (1.2) is constructed in [1].

In paper [2] the representation of the S-matrix as a functional integral was given for the case of nonpolynomial interactions. In this paper we use the representation obtained in [2] for investigation of the theory described by the Lagrangian (1.2).

The S-matrix as a functional of the scalar field $\varphi(x)$ in the Euclidean space of dimension d is defined in the form of the functional integral

$$S_V(g,\varphi)=\int d\sigma_u \exp\{-g\int_V d^dx[\Phi(u,x)+\varphi(x)]^4\}. \tag{1.3}$$

Here the following notations are introduced.

The system is supposed to be in a finite folume $V\subset\mathbb{R}_d$. In the volume V there is an orthonormal system of functions $\{g_s(x)\}$ $(s=1,2,3,\ldots)$ such that

$$\begin{aligned}&\int_V d^dx g_s(x)g_{s'}(x)=\delta_{ss'},\\ &\sum_{s=1}^{\infty} g_s(x)g_s(x')=\delta^{(d)}(x-x').\end{aligned} \tag{1.4}$$

The volume V and the system of functions $\{g_s\}$ can be chosen in the following way

$$\begin{aligned}&V=\{x:-L\leqq x_j\leqq L, j=1,2,\ldots,d\},\\ &\{g_s(x)\}=\{g_{s_1,\ldots,s_d}(x_1,\ldots,x_d)\}=\left\{\prod_{j=1}^{d} f_{s_j}(x_j)\right\},\\ &f_s(x)=\begin{cases}\dfrac{1}{\sqrt{2L}}\cos\dfrac{\pi}{L}\left(\dfrac{s-1}{2}\right)x, & s \text{ is odd}\\[2mm] \dfrac{1}{\sqrt{2L}}\sin\dfrac{\pi}{L}\dfrac{s}{2}x, & s \text{ is even}.\end{cases}\end{aligned} \tag{1.5}$$

The measure $d\sigma_u$ is defined as

$$d\sigma_u=\prod_{s=1}^{\infty}\frac{du_s}{\sqrt{2\pi}}\exp\left\{-\frac{1}{2}u_s^2\right\}$$

so that

$$\int d\sigma_u = \prod_{s=1}^{\infty} \int_{\infty}^{\infty} \frac{du_s}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}u_s^2\right\} = 1 .$$

Sometimes we will use the notation

$$\delta u = \prod_{s=1}^{\infty} \frac{du_s}{\sqrt{2\pi}} .$$

The function $\Phi(u,x)$ is defined as

$$\Phi(u,x) = \sum_{s=1}^{\infty} \mathscr{D}_s(x) u_s . \tag{1.6}$$

where

$$\begin{aligned} \mathscr{D}_s(x) &= \int_V dy \mathscr{D}_V(x,y) g_s(y) \\ \mathscr{D}_V(x,y) &= h_V(x)\mathscr{D}(x-y)h_V(y), \\ \mathscr{D}(x) &= \int \frac{d^dk}{(2\pi)^d} \frac{K(-l^2k^2)}{\sqrt{m^2+k^2}} e^{ikx} . \end{aligned} \tag{1.7}$$

A function $h_V(x) \in D(V)$, i.e. this function is infinitely differentiable and positive, has the support V and

$$\lim_{V\to\infty} h_V(x) = 1 .$$

The function $\mathscr{D}(x)$ is connected with the causal function $D(x)$ in the following way

$$\begin{aligned} D_V(x,x') &= \int_V dy \mathscr{D}_V(x,y)\mathscr{D}(y,x'), \\ \lim_{V\to\infty} D_V(x,x') &= D(x-x') \\ &= \int dy \mathscr{D}(x-y)\mathscr{D}(y-x') = \int \frac{d^dk}{(2\pi)^d} \frac{[K(-l^2k^2)]^2}{m^2+k^2} e^{ik(x-x')} \end{aligned} \tag{1.8}$$

The system $\{g_s\}$ is chosen in such a way that

$$\begin{aligned} &\sum_{s=1}^{\infty} \mathscr{D}_s^2(x) = D_V(x,x) \xrightarrow[V\to\infty]{} D(0) = \int \frac{d^dk}{(2\pi)^d} \frac{[K(-l^2k^2)]^2}{m^2+k^2} < \infty ; \\ &\sum_{s=1}^{\infty} s^M \mathscr{D}_s^2(x) < \infty , \quad \forall M > 0, \quad V < \infty . \end{aligned} \tag{1.9}$$

The functional integral in (1.3) is defined as

$$\begin{aligned} S_V(g,\varphi) &= \lim_{N\to\infty} S_V^{(N)}(g,\varphi), \\ S_V^{(N)}(g,\varphi) &= \int d\sigma_u^{(N)} \exp\left\{-g\int_V d^dx[\Phi^{(N)}(u,x)+\varphi(x)]^4\right\}, \end{aligned} \tag{1.10}$$

where

$$d\sigma_u^{(N)}=\prod_{s=1}^{N}\frac{du_s}{\sqrt{2\pi}}\exp\left\{-\frac{1}{2}u_s^2\right\}, \tag{1.11}$$

$$\Phi^{(N)}(u,x)=\sum_{s=1}^{N}\mathscr{D}_s(x)u_s. \tag{1.12}$$

The choice of the interaction in (1.3) or (1.10) corresponds to the so called usual but not normal product of operators. We can do it in the nonlocal theory and for the anharmonic oscillator. In the case of the local φ_2^4-theory we have to take the normal product of operators (see Sect. 6).

In this paper we will study the density of vacuum energy defined as (1.1) where

$$\langle 0|S_V(g,\varphi)|0\rangle=S_V(g,0)=S_V(g)$$

is given by (1.10). Our investigation proceeds as follows. We prove that:

i) the functional integral (1.3) given by limit (1.10) does exist for any $V<\infty$ and defines an analytical function in the complex g-plane singular at point $g=0$;

ii) the function $E(g)$ in (1.1) is analytical throughout the complex g-plane except the negative real axis including the point $g=0$;

iii) the upper and lower bounds are found for the function $E(g)$ when $g\to\infty$ in the nonlocal theory as a function of dimension d and an ultraviolet behaviour of the causal Green function $D(k^2)$ in the Euclidean region;

iv) the representation of $E(g)$ is obtained as a dispersion integral in the complex g-plane;

v) the upper and lower estimations in the local $g\varphi_2^4$-theory for the function

$$E(g,\Phi)=-\lim_{V\to\infty}\frac{1}{V}\ln S_V(g,\Phi)$$

are found where $\Phi=\text{const}$. The minimum of $E(g,\Phi)$ for $g\to\infty$ is at the points

$$\Phi_\pm=\pm\sqrt{\frac{\eta}{2\pi}\ln g}\,,\qquad(\eta\simeq 1)\,.$$

2. Existence of the Functional Integral

Here we show that the functional integral (2.3) does exist, i.e. the limit in (1.10) exists for all complex $g\neq 0$.

First let us consider the right half plane $\operatorname{Re} g>0$. In this region we have

$$|S_V^{(N)}(g,\varphi)|\leq 1\,,\qquad(\forall N>1)\,. \tag{2.1}$$

Further consider the difference

$$\begin{aligned}\Delta_N^{N+M}&=S_V^{(N+M)}(g,\varphi)-S_V^{(N)}(g,\varphi)\\&=\int d\sigma_n^{(N+M)}\left[\exp\left\{-g\int_V dx[\varphi(x)+\Phi^{(N+M)}(n,x)]^4\right\}\right.\\&\quad\left.-\exp\left\{-g\int_V dx[\varphi(x)+\Phi^{(N)}(u,x)]^4\right\}\right].\end{aligned} \tag{2.2}$$

Transform this difference in the following way

$$\Delta_N^{N+M}=\int_0^1 d\xi(1-\xi)\int d\sigma_u^{(N+M)}\frac{\partial^2}{\partial\xi^2}$$

$$\cdot\exp\left\{-g\int_V dx[\varphi(x)+\Phi^{(N)}(u,x)+\xi\Psi^{(N,M)}(u,x)]^4\right\}$$

$$=\int_0^1 d\xi(1-\xi)\int d\sigma_u^{(N+M)}\exp\{-gC(u,\xi)\}$$

$$\left\{-12g\int_V dx[\Psi^{(N,M)}(u,x)]^2[\varphi(x)+\Phi^{(N)}(u,x)+\xi\psi^{(N,M)}(u,x)]^2\right.$$

$$\left.+16g^2\left[\int_V dx\Psi^{(N,M)}(u,x)[\varphi(x)+\Phi^{(N)}(u,x)+\xi\psi^{(N,M)}(u,x)]^3\right|^2\right\},$$

where

$$\Psi^{(N,M)}(u,x)=\Phi^{(N+M)}(u,x)-\Phi^{(N)}(u,x),$$

$$C(u,x)=\int_V dx[\varphi(x)+\Phi^{(N)}(u,x)+\xi\Psi^{(N,M)}(u,x)]^4.$$

Making use of the Hölder inequalities for integrals

$$|\int d\sigma A_1A_2|\leqq\left[\int d\sigma|A_1|^{\frac{1}{p_1}}\right]^{p_1}\left[\int d\sigma|A_2|^{\frac{1}{p_2}}\right]^{p_2},$$

where $p_1+p_2=1$, one can obtain

$$|\Delta_N^{N+M}|\leqq\int_0^1 d\xi(1-\xi)\int d\sigma_u^{(N,M)}\exp\{-\mathrm{Re}\,g\cdot C(u,\xi)\}$$

$$\cdot\{12|g|C^{1/2}(u,\xi)+16|g|^2C^{3/2}(u,\xi)\}\left[\int_V dx(\Psi^{(N,M)}(u,x))^4\right]^{1/2}.$$

Further using the obvious inequalities

$$C^\gamma e^{-\mathrm{Re}g\cdot C}\leqq[\mathrm{Re}\,g]^{-1}\gamma^\gamma e^{-\gamma},$$

$$\int d\sigma(\psi)^\alpha\leqq[\int d\sigma\psi]^\alpha,\qquad(\alpha<1,\int d\sigma=1),$$

one can obtain finally

$$|\Delta_N^{N+M}|\leqq\mathrm{const}\cdot\left\{\int d\sigma_u^{(N+M)}\int_V dx(\psi^{(N,M)}(u,x))^4\right\}^{1/2}$$

$$=\mathrm{const}\cdot\left\{\int_V dx\left(\sum_{s=N+1}^{N+M}\mathscr{D}_s^2(x)\right)^2\right\}^{1/2}. \tag{2.3}$$

Because the series $\sum\mathscr{D}_s^2(x)$ converges well, for any $\varepsilon>0$ there exists N_0 such that for any $M>0$ and $N>N_0$ the inequality

$$|\Delta_N^{N+M}|<\varepsilon, \tag{2.4}$$

is valid. This means that the sequence $S_V^{(N)}(g,\varphi)$ for $V<\infty$ and $\mathrm{Re}\,g>0$ is fundamental and bounded.

Thus the limit in (1.10) does exist. This limit defines the functional integral (1.3) for any $V<\infty$ and $\mathrm{Re}\,g>0$.

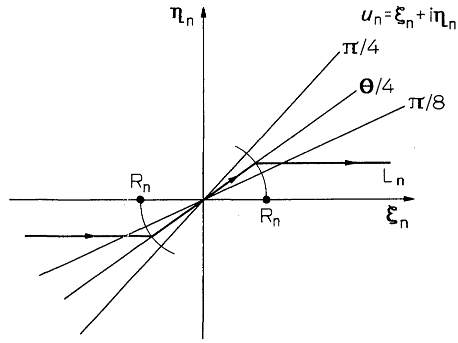

Fig. 1

Now consider the region $\mathrm{Re}\, g \leqq 0$. The integrals in (1.11) are not defined for $\mathrm{Re}\, g < 0$, therefore we have to do the analytical continuation of the function $S_V^{(N)}(g, \varphi)$ in the region $\mathrm{Re}\, g < 0$, that is not a complicated problem. In order to do the continuation into the region

$$g \to g e^{i\theta} (-\pi \leqq \theta \leqq \pi)$$

the contours of integrations over u_s should be transformed as follows

$$u_s \to u_s e^{-i\frac{\theta}{4}}.$$

Then we get

$$S_V^{(N)}(g e^{i\theta}, \varphi) = \prod_{s=1}^{N} \left(e^{-i\frac{\theta}{4}} \int_{-\infty}^{\infty} \frac{du_s}{\sqrt{2\pi}} \exp\left\{ -\frac{1}{2} e^{-i\frac{\theta}{2}} u_s^2 \right\} \right)$$
$$\cdot \exp\left\{ -g \int_V dx \left[e^{i\frac{\theta}{4}} \varphi(x) + \Phi^{(N)}(u, x) \right]^4 \right\}. \tag{2.5}$$

Now it should be proved that the limit $N \to \infty$ of this functions does exist for $-\pi \leqq \theta \leqq \pi$. To this end let us go over to the integration in (2.5) over each variable u_s along the contours L_s as shown in Fig. 1. Then we have

$$|S_V^{(N)}(g e^{i\theta}, \varphi)| \leqq \prod_{s=1}^{N} \int_{L_s} \left| \frac{du_s}{\sqrt{2\pi}} \right| \left| \exp\left\{ -\frac{1}{2} e^{-i\frac{\theta}{2}} u_s^2 \right\} \right|$$
$$\cdot \left| \exp\left\{ -g \int_V dx \left[e^{i\frac{\theta}{4}} \varphi(x) + \Phi^{(N)}(u, x) \right]^4 \right\} \right|. \tag{2.6}$$

Consider the integral

$$I_s = \int_{L_s} \left| \frac{du_s}{\sqrt{2\pi}} \right| \left| \exp\left\{ -\frac{1}{2} e^{-i\frac{\theta}{2}} u_s^2 \right\} \right|$$
$$\cdot \left| \exp\left\{ -g \int_V dx [\Phi(x) + \mathscr{D}_s(x) u_s]^4 \right\} \right|$$
$$= I_{1s} + I_{2s},$$

$$I_{1s}=\int_{-R_s}^{R_s}\frac{d\xi_s}{\sqrt{2\pi}}e^{-\frac{1}{2}\xi_s^2}\left|\exp\left\{-g\int_V dx\left[\Phi(x)+\mathscr{D}_s(x)\xi_s e^{i\frac{\theta}{4}}\right]^4\right\}\right|$$
$$=\left|\exp\left\{-g\int_V dx\left[\Phi(x)+\mathscr{D}_s(x)\xi_{0s}e^{i\frac{\theta}{4}}\right]^4\right\}\right|\left(1+O\left(e^{-\frac{1}{2}R_s^2}\right)\right),$$

where $-R_s\leqq\xi_{0s}\leqq R_s$. Here the mean value theorem was used. Further

$$I_{2s}=\int_0^\infty\frac{d\eta_s}{\sqrt{2\pi}}\left|\exp\left\{-\frac{1}{2}e^{-i\frac{\theta}{2}}\left(R_s e^{i\frac{\theta}{4}}+\eta_s\right)^2\right\}\right|$$
$$\cdot\left\{\left|\exp\left\{-g\int_V dx\left[\Phi(x)+\mathscr{D}_s(x)\left(R_s e^{i\frac{\theta}{4}}+\eta_s\right)\right]^4\right\}\right|+(\Phi\to-\Phi)\right\}$$
$$=\left\{\left|\exp\left\{-g\int_V dx\left[\Phi(x)+\mathscr{D}_s(x)\left(R_s e^{i\frac{\theta}{4}}+\eta_{1s}\right)\right]^4\right\}\right|\right.$$
$$\left.+\left|\exp\left\{-g\int_V dx\left[\Phi(x)-\mathscr{D}_s(x)\left(R_s e^{i\frac{\theta}{4}}+\eta_{2s}\right)\right]^4\right\}\right|\right\}O\left(e^{-\frac{1}{2}R_s^2}\right),$$

where $\eta_{1s},\eta_{2s}>0$. Finally we have

$$I_s=\left|\exp\left\{-g\int_V dx\left[\Phi(x)+\mathscr{D}_s(x)\xi_{0s}e^{i\frac{\theta}{4}}\right]^4\right\}\right|\cdot\left[1+O\left(e^{-\frac{1}{2}R_s^2}\right)\right].$$

Making use of this estimation for each integral in (2.6) we obtain

$$|S_V^{(N)}(ge^{i\theta},\varphi)|$$
$$\leqq\prod_{s=1}^N\left|\exp\left\{-g\int_V dx\left[\varphi(x)e^{i\frac{\theta}{4}}+\sum_{s_1=1}^N\mathscr{D}_{s_1}(x)\xi_{0s_1}e^{i\frac{\theta}{4}}\right]^4\right\}\right|$$
$$\cdot\left[1+O\left(e^{-\frac{1}{2}R_s^2}\right)\right].$$

Choosing $R_s=s^b\,(b>\frac{1}{2})$ we have

$$\prod_{s=1}^\infty\left(1+O\left(e^{-\frac{1}{2}s^b}\right)\right)=C<\infty$$

and

$$\left|\sum_{s=1}^\infty\mathscr{D}_s(x)\xi_{0s}\right|\leqq\sum_{s=1}^\infty|\mathscr{D}_s(x)|s^b<\infty\,.$$

Therefore

$$|S_V^{(N)}(ge^{i\theta},\varphi)|\leqq C\exp\left\{-\mathrm{Re}\,g\int_V dx\left[\varphi(x)+\sum_{s=1}^N\mathscr{D}_s(x)\xi_{0s}\right]^4\right\}$$

or

$$|S_V^{(N)}(ge^{i\theta},\varphi)|\leqq Ce^{-\mathrm{Re}\,g\cdot\mathrm{const}}\,. \tag{2.7}$$

The analogous estimations can be applied to the difference Δ_N^{N+M} in (2.2), and an estimation of the type (2.3) can be obtained.

Thus the functional integral (1.3) defined as the limit (1.10) does exist for any $V<\infty$ and defines an analytical function in the complex g-plane.

3. Analytical Properties of $E(g)$ in the Complex g-Plane

The following theorem will be a basis of further analytical investigations [3].

Theorem. *Let functions* $\{f_n(z)\}$ $(n=1,2,\ldots)$ *be regular in a region* $G\subset\mathbb{C}$ *and*

$$\lim_{n\to\infty} f_n(z)=f(z)\not\equiv 0$$

uniformly in each compact $K\subset G$. *Let* M *is a set of zeroes of the functions* $f_n(z)$ *in* G. *Then the zeroes of* $f(z)$ *in* G *coincide with the limiting points of set* M *lying in* G.

Let us consider the function

$$B_V(g)=\ln S_V(g). \tag{3.1}$$

The singularities of $B_V(g)$ in the complex g-plane are defined, first, by the singularities of $S_V(g)$ and, second, by zeroes of $S_V(g)$ in the g-plane.

We will study the position of singularities and zeroes of the function

$$S_V^{(N)}(g)=\prod_{s=1}^{N}\int_{-\infty}^{\infty}\frac{du_s}{\sqrt{2\pi}}\exp\left\{-\tfrac{1}{2}\sum_{n=1}^{N}\mathrm{u}_n^2-\mathrm{g}\int_V \mathrm{dx}\left(\sum_{n=1}^{N}\mathscr{D}_n(\mathrm{x})\mathrm{u}_n\right)^4\right\} \tag{3.2}$$

and following the above theorem determine the position of singularities of the function

$$\ln S_V(g)=\lim_{N\to\infty}\ln S_V^{(N)}(g). \tag{3.3}$$

So let us consider the function $S_V^{(N)}(g)$ in (3.2). Introducing the new variables

$$u_s=g^{-1/4}v_s$$

we obtain

$$S_V^{(N)}(g)=g^{-N/4}\left(\prod_{s=1}^{N}\int_{-\infty}^{\infty}\frac{dv_s}{\sqrt{2\pi}}\right)$$
$$\cdot\exp\left\{-\tfrac{1}{2}g^{-1/2}\sum_{n=1}^{N}v_n^2-\int_V dx\left(\sum_{n=1}^{N}\mathscr{D}_n(x)v_n\right)^4\right\}.$$

One can see from this representation that $S_V^{(N)}(g)$ has an essential singularity at the point $g=0$ and no other singularities in the complex g-plane.

Now let us show that the function $S_V^{(N)}(g)$ has zeroes on the negative real axis in the g-plane only. We calculate the increment of the argument of $S_V^{(N)}(g)$ along the contour C in Fig. 2. For this aim it is convenient to use a different representation of the function $S_V^{(N)}(g)$. Introducing in the integral (3.2) the new variables

$$u_s=\frac{1}{\sqrt{g}}v_s$$

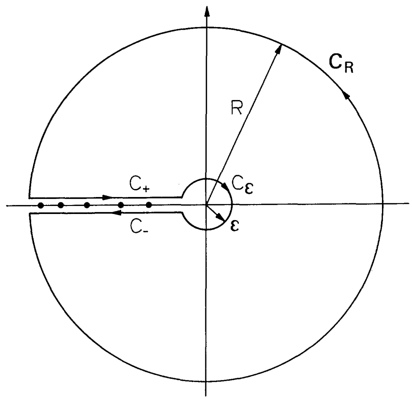

Fig. 2

and making simple transformations we obtain

$$S_V^{(N)}(g)=g^{-N/2}\int\frac{d^Nv}{(\sqrt{2\pi})^N}\exp\left\{-\frac{1}{g}\left[\frac{1}{2}\sum_{n=1}^N v_n^2+\int_V dx\left(\sum_{n=1}^N \mathscr{D}_n(x)v_n\right)^4\right]\right\}$$
$$=g^{-N/2}\int_0^\infty dte^{-\frac{1}{g}t}F_V^{(N)}(t), \tag{3.4}$$

where

$$F_V^{(N)}(t)=\int\frac{d^Nv}{(\sqrt{2\pi})^N}\delta\left(t-\frac{1}{2}\sum_{n=1}^N v_n^2-\int_V dx\left(\sum_{n=1}^N \mathscr{D}_n(x)v_n\right)^4\right)$$
$$=\int_0^\infty dRR^{N-1}\int\frac{d\Omega_N}{(\sqrt{2\pi})^N}\delta(t-\tfrac{1}{2}R^2-R^4A(\Omega)). \tag{3.5}$$

Here we go over to the spherical variables in the space $\mathbb{R}_N$. The function $A(\Omega)$ in (3.5) is

$$A(\Omega)=\int_V dx\left(\sum_{n=1}^N \mathscr{D}_n(x)\eta_n\right)^4,$$
$$\sum_{n=1}^N \eta_n^2=1, \tag{3.6}$$

where η_n are the spherical variables. It is important that

$$0<A_-\leqq A(\Omega)\leqq A_+<\infty.$$

Performing the integration over R in (3.5) one can obtain after simple transformations

$$F_V^{(N)}(t)=\int_{b_-}^{b_+}\frac{db(\sqrt{b+t}-\sqrt{b})^{\frac{N-2}{2}}}{\sqrt{b+t}}\cdot Q_V^{(N)}(b), \tag{3.7}$$

where

$$Q_V^{(N)}(b)=2^{N-2}(\sqrt{b})^{\frac{N}{2}+1}\int\frac{d\Omega_N}{(\sqrt{2\pi})^N}\delta\left(\frac{1}{b}-16A(\Omega)\right),$$

$$b_\pm=(A_\mp)^{-1}.$$

It is essential that the dependence on t in (3.7) is picked out in the explicit form, the integral over b is taken over the finite region

$$0<b_-\leqq b\leqq b_+<\infty$$

and the function $Q_V^{(N)}(b)$ is integrable.

Thus, the function $S_V^{(N)}(g)$ can be represented in the form

$$S_V^{(N)}(g)=g^{-N/2}\int_0^\infty dte^{-\frac{1}{g}t}\int_{b-}^{b+}db\frac{(\sqrt{b+t}-\sqrt{b})^{\frac{N-2}{2}}}{\sqrt{b+t}}\cdot Q_V^{(N)}(b). \tag{3.8}$$

This representation is convenient for our aim.

From the representation (3.8) it is easy to find the asymptotical behaviour of the function $S_V^{(N)}(g)$ for $|g|\to 0$ and $|g|\to\infty$. Introducing the variables

$$g=re^{i\varphi}\quad\text{and}\quad t=\varrho e^{i\varphi}$$

one can get

$$S_V^{(N)}(re^{i\varphi})=r^{-N/2}e^{-i\varphi\left(\frac{N}{2}-1\right)}\int_0^\infty d\varrho e^{-\varrho/r}$$

$$\left\{\cdot\int_{b_-}^{b_+}\frac{db(\sqrt{b+\varrho e^{i\varphi}}-\sqrt{b})^{\frac{N-2}{2}}}{\sqrt{b+\varrho e^{i\varphi}}}\cdot Q_V^{(N)}(b)\right\} \tag{3.9}$$

and then

$$S_V^{(N)}(g)=\begin{cases} g^{-N/4}\cdot\Gamma\left(\frac{N}{4}\right)\int_{b_-}^{b_+}dbQ_V^{(N)}(b)\left[1+O\left(\frac{1}{g}\right)\right], & |g|\to\infty,\\ 2\Gamma\left(\frac{N}{2}\right)\int_{b_-}^{b_+}\frac{db}{(2\sqrt{b})^{N/2}}Q_V^{(N)}(b)[1+O(g)], & |g|\to 0.\end{cases} \tag{3.10}$$

Making use of this asymptotical behaviour (3.10) it is easy to calculate the increment of the argument of $S_V^{(N)}(g)$ along the contours C_R and C_ε in Fig. 2:

$$\Delta\arg S_V^{(N)}(g)|_{C_R}=-2\pi\frac{N}{4},\qquad \Delta\arg S_V^{(N)}(g)|_{C_\varepsilon}=0.$$

Whence

$$\Delta\arg S_V^{(N)}(g)|_{C_R+C_\varepsilon}=-2\pi\frac{N}{4}. \tag{3.11}$$

Now let us consider the increment of the argument of $S_V^{(N)}(g)$ along the contours C_+ and C_- in Fig. 2. For this aim the function $S_V^{(N)}(g)$ for $g=re^{\pm i\pi}$ can be written in the form

$$S_V^{(N)}(re^{\pm i\pi})=-r^{-N/2}e^{\mp i\pi(N/2)}\int_{b_-}^{b_+} dbQ_V^{(N)}(b)$$
$$\cdot\left\{\int_0^b d\varrho e^{-\varrho/2}\frac{(\sqrt{b-\varrho}-\sqrt{b})^{\frac{N-2}{2}}}{\sqrt{b-\varrho}}+\int_b^\infty d\varrho e^{-\varrho/r}\frac{(\pm i\sqrt{\varrho-b}-\sqrt{b})^{\frac{N-2}{2}}}{\pm i\sqrt{\varrho-b}}\right\}.$$

Let us consider the even $N=2n$. Without loss of generality we can consider the limit

$$S_V(g)=\lim_{n\to\infty}S_V^{(2n)}(g).$$

The function $S_V^{(2n)}(re^{\pm i\pi})$ can be represented in the form

$$S_V^{(2n)}(re^{\pm i\pi})=A_V^{(n)}(r)\pm iB_V^{(n)}(r)=R_V^{(n)}(r)e^{\pm i\theta_V^{(n)}(r)}.$$

Let us consider

$$B_V^{(n)}(r)=\operatorname{Im}S_V^{(2n)}(re^{i\pi})=R_V^{(n)}(r)\sin\theta_V^{(n)}(r)$$
$$=(-)^n\sqrt{\pi}r^{n-1/2}\int_{b_-}^{b_+} dbQ_V^{(2n)}(b)(\sqrt{b})^{n-1}e^{-b/r}\,T_{\left[\frac{n-1}{2}\right]}\left(\frac{r}{4b}\right),$$

$$T_{\frac{n-1}{2}}\left(\frac{r}{4b}\right)=\sum_{q=0}^{\left[\frac{n-1}{2}\right]}(-)^q\left(\frac{r}{4b}\right)^q\frac{1}{q!}\frac{(n-1)!}{(n-1-2q)!}.$$

The function $B_V^{(n)}(r)$ can change its sign not more than $\left[\frac{n-1}{2}\right]$ times because the polynomial $T_{\left[\frac{n-1}{2}\right]}$ is of degree $\left[\frac{n-1}{2}\right]$. It means that the function $S_V^{(2n)}(re^{i\pi})$ along the contour C_+ can increase its phase not more than by $\pi\left(\left[\frac{n-1}{2}\right]+1\right)$. Thus we have

$$\Delta\arg S_V^{(2n)}(g)|_{C_++C_-}\leqq 2\pi\left(\left[\frac{n-1}{2}\right]+1\right). \tag{3.12}$$

Collecting all estimations (3.11) and (3.12) we obtain

$$\Delta\arg S_V^{(2n)}(g)|_C\leqq\pi.$$

The increment of the argument of $S_V^{(2n)}(g)$ cannot be negative because $S_V^{(2n)}(g)$ has no poles. Besides it is always a number multiple of 2π because $S_V^{(2n)}$ has zeroes only. Finally we get

$$\Delta\arg S_V^{(2n)}(g)|_C=0\qquad(\forall\, n>0). \tag{3.13}$$

Consequently the function $S_V^{(2n)}(g)$ has no zeroes in the complex g-plane except the negative real axis, and on this axis it has $\left[\frac{n-1}{2}\right]$ zeroes. The number of zeroes increases when $n\to\infty$.

Thus, according to the Theorem the function $S_V(g)$ in (3.3) has zeroes on the negative real axis only in the complex g-plane for any $V<\infty$. It means that $\ln S_V(g)$ has singularities at point $g=0$ and on the negative real axis. Consequently according to the same Theorem the function $E(g)$ in (1.13) has singularities at point $g=0$ and on the negative real axis in the complex g-plane.

4. Upper and Lower Bounds of $E(g)$ in Nonlocal Theory

In this section we obtain the upper and lower estimations for real positive g on the density of the vacuum energy in nonlocal theory when the causal Green function $\tilde{D}(k^2)$ decreases rapidly enough in the Euclidean direction.

First, the lower bound of $E(g)$ will be obtained. Let us consider the function

$$S_V(g)=\int d\sigma_u \exp\left\{-g\int_V dx\Phi^4(u,x)\right\}. \tag{4.1}$$

The inequality

$$\int_V dx\Phi^4(u,x)\geqq\frac{1}{V}\left[\int_V dx\Phi^2(u,x)\right]^2$$

gives

$$S_V(g)\leqq\int d\sigma_u \exp\left\{-\frac{g}{V}\left[\int_V dx\Phi^2(u,x)\right]^2\right\}. \tag{4.2}$$

This integral can be calculated in the following way. It can be represented in the form

$$S_V(g)\leqq\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} dt e^{-t^2}F_V(g,t), \tag{4.3}$$

where

$$F_V(g,t)=\int d\sigma_u \exp\left\{-2it\sqrt{\frac{g}{V}}\int_V dx\Phi^2(u,x)\right\}.$$

The last integral is calculated in the Appendix. When V is large enough we get

$$F_V(g,t)=\exp\left\{-\tfrac{1}{2}V\int\frac{d^dk}{(2\pi)^d}\ln\left(1+4it\sqrt{\frac{g}{V}}\tilde{D}(k^2)\right)\right\}. \tag{4.4}$$

Substituting this function $F_V(g,t)$ into (4.3) and introducing the new variable $t=u\sqrt{V}$ one can obtain

$$S_V(g)=\sqrt{\frac{V}{\pi}}\int_{-\infty}^{\infty} du$$

$$\cdot\exp\left\{-V\left[u^2+\tfrac{1}{2}\int\frac{d^dk}{(2\pi)^d}\ln(1+4iu\sqrt{g}\tilde{D}(k^2))\right]\right\}. \tag{4.5}$$

When $V\to\infty$ this integral can be calculated by the method of steepest descents. The saddle point is on the negative imaginary axis in the $(u+iv)$-plane. Putting $u_0=-iv_0$ we have the equation

$$\frac{d}{dv_0}\left[v_0^2-\tfrac{1}{2}\int\frac{d^dk}{(2\pi)^d}\ln(1+4v_0\sqrt{g}\tilde{D}(k^2))\right]=0$$

or

$$v_0=\sqrt{g}\int\frac{d^dk}{(2\pi)^d}\frac{\tilde{D}(k^2)}{1+4v_0\sqrt{g}\tilde{D}(k^2)}. \tag{4.6}$$

Thus the following estimation on $E(g)$ is valid

$$\begin{aligned}E(g)&=-\lim_{V\to\infty}\frac{1}{V}\ln S_V(g)\geqq E_-(g),\\ E_-(g)&=-v_0^2+\tfrac{1}{2}\int\frac{d^dk}{(2\pi)^d}\ln(1+4v_0\sqrt{g}\tilde{D}(k^2)),\end{aligned} \tag{4.7}$$

where v_0 is determined by (4.6). The formulas (4.7) can be written on the basis of Eq. (4.6) in a different form

$$E_-(g)=v_0\sqrt{g}\int\frac{d^dk}{(2\pi)^d}\frac{\left[-\frac{4}{d}k^2\tilde{D}'(k^2)-\tilde{D}(k^2)\right]}{1+4v_0\sqrt{g}\tilde{D}(k^2)}. \tag{4.8}$$

Now let us obtain the behaviour of $E_-(g)$ for $g\to0$ and $g\to\infty$. When $g\to0$

$$v_0=\sqrt{g}\int\frac{d^dk}{(2\pi)^d}\tilde{D}(k^2)=\sqrt{g}D(0)$$

and

$$E_-(g)=gD^2(0).$$

When $g\to\infty$ the behaviour of v_0 as a function of g is determined by the ultraviolet behaviour of the causal function $\tilde{D}(k^2)$. Further let us consider particular cases. First, the case when for $k^2\to\infty$

$$\tilde{D}(k^2)=O((k^2)^{-1-a}), \tag{4.9}$$

where

$$2(1+a)>d$$

as a consequence of the condition $D(0)<\infty$ in (1.9). In this case from (4.6) for $g\to\infty$

$$v_0=\text{const}\cdot g^{\frac{1}{2}\,\frac{d}{4(1+a)-d}}$$

follows, whence

$$E_-(g)=C_-g^{\frac{d}{4(1+a)-d}}, \tag{4.10}$$

where C_- is a constant which can be determined if necessary.

In the case of anharmonic oscillator [$d=1$ and $a=1$ in (4.10)] the causal Green function is

$$\tilde{D}(k^2)=\frac{1}{1+k^2}$$

(in units $m=1$). In this case the function $E_-(g)$ in (4.7) can be calculated in the explicit form:

$$E_-(g)=\max_u\left\{-\frac{u^2}{g}+\tfrac{1}{2}(\sqrt{1+4u}-1)\right\}.$$

This expression can be written after simple transformations

$$E_-(g)=\frac{v(16+18v+9v^2)}{8(1+v)(3v+4)}, \tag{4.11}$$

where

$$v(1+v)(2+v)=2g. \tag{4.12}$$

Hence it appears that in the limit of large and small g

$$E_-(g)=\begin{cases}\dfrac{g}{2}, & g\to 0,\\[2mm] \dfrac{3}{8}(2g)^{1/3}=0.474g^{1/3}, & g\to\infty.\end{cases} \tag{4.13}$$

Second, the case when

$$\tilde{D}(k^2)=O(\exp(-k^{2\gamma})) \tag{4.14}$$

for $k^2\to\infty$. The solution v_0 of the equation (4.6) is

$$v_0=\mathrm{const}\,(\ln g)^{\frac{d}{4\gamma}}$$

in the limit $g\to\infty$ so that

$$E_-(g)=C_-(\ln g)^{\frac{d}{2\gamma}+1}\left[1+O\left(\frac{\ln\ln g}{\ln g}\right)\right]. \tag{4.15}$$

Here C_- is a constant which can be found.

Now let us obtain the upper bound to the function $E(g)$ for real positive g. The following obvious inequality is valid for the function $S_V(g)$ in (4.1):

$$S_V(g)\geqq\int\delta u\exp\left\{-\frac{1}{2}\sum_{s=1}^{\infty}(1+q_s)u_s^2-g\int_V dx\left(\sum_{s=1}^{\infty}\mathscr{D}_s(x)u_s\right)^4\right\}$$

$$=\prod_{n=1}^{\infty}(1+q_n)^{-1/2}\int\delta u\exp\left\{-\frac{1}{2}\sum_{s=1}^{\infty}u_s^2-g\int_V dx\left(\sum_{s=1}^{\infty}\frac{\mathscr{D}_s(x)}{\sqrt{1+q_s}}u_s\right)^4\right\}. \tag{4.16}$$

Here q_n are positive numbers. Then it is easy to get the following upper estimation for $E(g)$:

$$E(g) = -\lim_{V\to\infty}\frac{1}{V}\ln S_V(g) \leqq \frac{1}{2}\lim_{V\to\infty}\frac{1}{V}\sum_s \ln(1+q_s)$$

$$-\lim_{V\to\infty}\frac{1}{V}\ln\int d\sigma_n \exp\left\{-g\int_V dx\left(\sum_s\frac{\mathscr{D}_s(x)}{\sqrt{1+q_s}}u_s\right)^4\right\}$$

$$\leqq \frac{1}{2}\lim_{V\to\infty}\frac{1}{V}\sum_s\ln(1+q_s) + 3g\lim_{V\to\infty}\frac{1}{V}\int_V dx\left(\sum_s\frac{\mathscr{D}_s^2(x)}{1+q_s}\right)^2 .$$

The numbers q_s can be chosen in such a way that

$$\lim_{V\to\infty}\frac{1}{V}\sum_s\ln(1+q_s) = \int\frac{d^dk}{(2\pi)^d}\ln(1+q(k^2)),$$
$$\lim_{V\to\infty}\sum_s\frac{\mathscr{D}_s^2(x)}{1+q_s} = \int\frac{d^dk}{(2\pi)^d}\cdot\frac{\tilde{D}(k^2)}{1+q(k^2)} . \tag{4.17}$$

For example in the case of the orthonormal system (1.5) these numbers should be taken as

$$q_s = q_{s_1,\dots,s_d} = q\left(\frac{\pi^2}{L^2}(s_1^2 + \dots + s_d^2)\right)$$

and

$$V = (2L)^d .$$

Because the numbers q_s and consequently the function $q(k^2)$ are arbitrary, we have $E(g) \leqq E_+(g)$,

$$E_+(g) = \min_{q>0}\left\{\frac{1}{2}\int\frac{d^dk}{(2\pi)^d}\ln(1+q(k^2)) + 3g\left|\int\frac{d^dk}{(2\pi)^d}\frac{\tilde{D}(k^2)}{(1+q(k^2))}\right|^2\right\} . \tag{4.18}$$

When $g\to 0$ the minimum is for $q(k^2)=0$. In this limit we obtain

$$E_+(g) = 3gD^2(0) , \tag{4.19}$$

i.e. the lowest perturbation order when the interaction Lagrangian is not taken in the normal form.

The behaviour of $E_+(g)$ for $g\to\infty$ is defined by the ultraviolet asymptotic of the Green function $\tilde{D}(k^2)$. In the case (4.9) the function

$$q(k^2) = \left(\frac{A^2}{k^2}\right)^\beta , \tag{4.20}$$

where A and $\beta > \frac{d}{2}$ are parameters, can be chosen to determine the function $E_+(g)$ in (4.18).

Substituting (4.20) into (4.18) and putting $g \to \infty$ one can see that the minimum is realized for large A. We have

$$E_+(g) \leqq \min_{\beta > \frac{d}{2}} \min_A \left[C_1(\beta) A^d + C_2(\beta) \frac{g}{A^{4(1+a)-2d}} \right],$$

where

$$C_1(\beta) = \frac{1}{2} \int \frac{d^d k}{(2\pi)^d} \ln\left(1 + \frac{1}{k^{2\beta}}\right),$$

$$C_2(\beta) = 3 \left[\int \frac{d^d k}{(2\pi)^d} \frac{k^{2\beta}}{k^{2(1+a)}(k^{2\beta}+1)} \right]^2.$$

Then it follows

$$E_+(g) \leqq C_+ \cdot g^{\frac{d}{4(1+a)-d}}, \tag{4.21}$$

where

$$C_+ = \min_\beta C_1(\beta) \left(1 + \frac{dC_2(\beta)}{2(2(1+a)-d)}\right) \left[\frac{2(2(1+a)-d)C_2(\beta)}{dC_1(\beta)}\right]^{4(1+a)-d}$$

This constant C_+ can be calculated.

Thus two estimations (4.15) and (4.21) show that in the nonlocal theory when $\tilde{D}(k^2) = O((k^2)^{-1-a})$ for $k^2 \to \infty$ in the limit $g \to \infty$ the vacuum energy $E(g)$ is

$$E(g) = C g^{\frac{d}{4(1+a)-d}}, \tag{4.22}$$

where C is a constant satisfying $C_- < C < C_+$. The formula (4.22) can be rewritten in the form

$$E(g) = C g^{\alpha(d,a)},$$
$$\alpha(d,a) = \frac{d}{4(1+a)-d} = \frac{1 - \left[1 - \frac{d}{2(1+a)}\right]}{1 + \left[1 - \frac{d}{2(1+a)}\right]} < 1 \tag{4.23}$$

because of $2(1+a) > d$ according to the condition $D(0) < \infty$.

Note once more that the behaviour $E(g)$ for $g \to \infty$ is defined by the dimension of space-time d and the ultraviolet asymptotic of the causal Green function $\tilde{D}(k^2)$ for $k^2 \to \infty$.

The function $E_+(g)$ in (4.18) for the anharmonic oscillator reads:

$$E_+(g) = \min_\beta \min_A \left\{ \frac{1}{4\pi} \int_{-\infty}^{\infty} dk \ln\left(1 + \left(\frac{A^2}{k^2}\right)^\beta\right) + 3g \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{dk}{(1+k^2)\left(1 + \left(\frac{A^2}{k^2}\right)^\beta\right)} \right]^2 \right\}.$$

In the limit $g\to\infty$ it is easy to obtain

$$E_+(g)=\min_\beta \min_A \left[\frac{A}{2\sin\frac{\pi}{2\beta}}+\frac{3g}{A^2}\cdot\frac{1}{\left(2\beta\sin\frac{\pi}{2\beta}\right)^2}\right]$$

$$=g^{1/3}\left\{\frac{3}{2\pi\left[\max\limits_u \frac{\sin^2 u}{u}\right]}\right\}^{2/3}=0.757g^{1/3}. \qquad (4.24)$$

Collecting the estimations (4.13) and (4.24) we obtain

$$0.474g^{1/3}<E(g)<0.757g^{1/3}. \qquad (4.25)$$

The exact calculations which can be done for the anharmonic oscillator making use of the Schrödinger equation give [4]

$$E(g)=0.6680g^{1/3}.$$

One can see that our estimations are rather accurate.

At last let us consider $E_+(g)$ for $g\to\infty$ in the case (4.14). This estimation can be obtained by choosing $q(k^2)=A\exp\{-k^{2\gamma}\}$. After simple calculations one can get in the limit $g\to\infty$

$$E_+(g)=C_+(\ln g)^{\frac{d}{2\gamma}+1}\left[1+O\left(\frac{\ln\ln g}{\ln g}\right)\right]. \qquad (4.26)$$

Thus in the φ_d^4-theory with the propagator $\tilde{D}(k^2)$ (4.14) we have in the limit of strong coupling

$$E(g)=C(\ln g)^{\frac{d}{2\gamma}+1}\left[1+O\left(\frac{\ln\ln g}{\ln g}\right)\right]. \qquad (4.27)$$

In paper [5] the φ^4-theory with the propagator $\tilde{D}(k^2)=e^{-k^2}$ [i.e. $\gamma=1$ in (4.14)] was considered. The authors have calculated eight perturbation orders for different characteristics of this model (energy levels, Green functions). The knowledge of exact asymptotics (4.27) should help for different methods of asymptotic summation.

5. Dispersion Representation of $E(g)$

After the investigations performed in the previous sections we prove the vacuum energy $E(g)$ defined by (1.1) has the following properties:

i) $E(g)$ is analytic in the complex g-plane except the negative real axis,

ii) for $|g|\to\infty$

$$|E(g)|\leqq \text{const}\,|g|, \qquad (5.1)$$

iii) for $g\to+\infty$

$$E(g)=\text{const}\, g^{\alpha(d,a)}, \qquad (5.2)$$

where

$$\alpha(d,a)=\frac{d}{4(1+a)-d}<1,$$

iv) for $\operatorname{Re} g>0$

$$E(g)\infty\sum_{n=1}^{\infty}g^n E_n, \tag{5.3}$$

where $E_n=O(n!)$, i.e. $E(g)$ can be developed into an asymptotical series.

It follows from (5.1) and (5.2) that for $|g|\to\infty$

$$|E(g)|\leqq\text{const}\,|g|^{\alpha(d,a)}. \tag{5.4}$$

Further it is possible to write the following dispersion representation

$$\frac{E(g)}{g}=\frac{1}{2\pi i}\int_C\frac{d\zeta E(\zeta)}{\zeta(\zeta-g)}, \tag{5.5}$$

where C is the contour shown in Fig. 2. The integration over this contour C in (5.5) gives

$$\frac{E(g)}{g}=\frac{1}{\pi}\int_{-\infty}^{0}\frac{d\zeta}{\zeta(\zeta-g)}\cdot\frac{1}{2i}[E(\zeta e^{i\pi})-E(\zeta e^{-i\pi})]$$

whence

$$E(g)=g\int_0^{\infty}\frac{d\xi\sigma\left(\frac{1}{\xi}\right)}{\xi(\xi+g)}$$

or introducing the new variable $u=\frac{1}{\xi}$

$$E(g)=g\int_0^{\infty}\frac{du\sigma(u)}{1+gu}. \tag{5.6}$$

According to the conditions (5.3) and (5.4) the function $\sigma(u)$ satisfies

$$\sigma(u)=\begin{cases}O(u^{-\alpha}), & u\to 0,\\ O(e^{-u}), & u\to\infty.\end{cases} \tag{5.7}$$

The representation (5.6) can be written in the other form

$$E(g)=g\int_0^{\infty}ds\int_0^{\infty}du\sigma(u)e^{-s-gsu}=g\int_0^{\infty}dse^{-s}B(gs), \tag{5.8}$$

where

$$B(t)=\int_0^{\infty}du\sigma(u)e^{-tu}.$$

This function $B(t)$ according to the estimations (5.7) is analytical for $\mathrm{Re}\,t > -1$ and for $t\to\infty$

$$B(t)=O(t^{-1+\alpha}). \tag{5.9}$$

Thus the perturbation series in the nonlocal φ^4-theory is summarizable by the Boral method to the exact expression.

6. $E(g)$ in the Local φ_2^4-Theory

A great number of papers (see [6]) are devoted to the local φ_2^4-theory. The upper and lower estimations were obtained for $E(g)$ [6–8] but these estimations are quite rough. Here, by using our representation (1.3) for $S_V(g,\varphi)$, we will get more exact estimations for

$$E(g,\Phi)=-\lim_{V\to\infty}\frac{1}{V}\ln S_V(g,\Phi) \tag{6.1}$$

in the limit $g\to\infty$ and for an arbitrary constant field $\Phi=\mathrm{const}$.

The function $S_V(g,\Phi)$ in the local limit can be represented as

$$S_V(g,\Phi)=\lim_{l\to\infty}\int d\sigma_u \exp\{gW_4^{(l)}(u,\Phi)\}, \tag{6.2}$$

where the interaction Lagrangian is taken in the normal form

$$\begin{aligned}\mathrm{W}_4^{(l)}(\mathrm{u},\Phi)&=-\int_V \mathrm{d}^2\mathrm{x}:[\Phi_l(u,x)+\Phi]^4:\\ &=-\int_V d^2x\,[(\Phi_l(u,x)+\Phi)^4-6D_l(0)(\Phi_l(u,x)+\Phi)^2+3D_l^2(0)]\,.\end{aligned} \tag{6.3}$$

In the two-dimensional space-time the limit $l\to 0$ in (6.2) does exist for any elements of the S-matrix. We will study the vacuum energy $E(g,\Phi)$ in (6.1).

First we obtain the upper estimation $E_+(g,\Phi)$ in (6.1). Let us write down the following obvious inequality

$$\begin{aligned}S_V(g,\Phi)\geqq&\lim_{l\to 0}\int\delta u\\ &\cdot\exp\left\{-\frac{1}{2}\sum_s u_s^2-\frac{1}{2}\sum_s q_s(u_s-v_s)^2-g\int_V dx:\left[\sum_s \mathcal{D}_s(x)u_s+\Phi\right]^4:\right\},\end{aligned} \tag{6.4}$$

where q_s are positive and v_s are real numbers. After simple transformations one can get

$$\begin{aligned}S_V(g,\Phi)\geqq&\lim_{l\to\infty}\prod_s(1+q_s)^{-1/2}\cdot\exp\left\{\frac{1}{2}\sum_s\frac{q_s}{1+q_s}v_s^2\right\}\int\delta u\exp\left\{-\frac{1}{2}\sum_s u_s^2\right\}\\ &\cdot\exp\left\{-g\int_V dx\left[\left(\sum_s\frac{\mathcal{D}_s(x)}{\sqrt{1+q_s}}u_s+\Phi_1\right)^4-6D_l(0)\left(\sum_s\frac{\mathcal{D}_s(x)}{\sqrt{1+q_s}}u_s+\Phi_1\right)^2\right.\right.\\ &\left.\left.+3D_l^2(0)\right]\right\},\end{aligned}$$

where

$$\Phi_1=\Phi+\sum_s \frac{\mathscr{D}_s(x)q_s}{1+q_s}v_s.$$

Then the following inequality is valid for $E(g,\Phi)$

$$E(g,\Phi)\leq\frac{1}{2}\lim_{V\to\infty}\frac{1}{V}\sum_s \ln(1+q_s)+\frac{1}{2}\lim_{V\to\infty}\frac{1}{V}\sum_s \frac{q_s}{1+q_s}v_s^2$$
$$+\lim_{l\to 0}\lim_{V\to\infty}\frac{g}{V}\int_V d^2x\int d\sigma_u\left[\left(\sum_s \frac{\mathscr{D}_s(x)}{\sqrt{1+q_s}}u_s+\Phi_1\right)^4\right.$$
$$\left.-6D_l(0)\left(\sum_s \frac{\mathscr{D}_s(x)}{\sqrt{1+q_s}}u_s+\Phi_1\right)^2+3D_l^2(0)\right]. \tag{6.5}$$

The numbers q_s and v_s can be chosen in such a way that

$$\lim_{V\to\infty}\frac{1}{V}\sum_s \ln(1+q_s)=\int\frac{d^2k}{(2\pi)^2}\ln(1+q(k^2)),$$

$$\lim_{V\to\infty}\frac{1}{V}\sum_s \frac{q_s}{1+q_s}v_s^2=\frac{q(0)}{1+q(0)}v^2,$$

$$\lim_{V\to\infty}\sum_s \frac{\mathscr{D}_s(x)}{1+q_s}v_s=\frac{\tilde{\mathscr{D}}(0)q(0)}{1+q(0)}v=\frac{q(0)}{1+q(0)}v,$$

$$\lim_{V\to\infty}\frac{1}{V}\sum_s \frac{\mathscr{D}_s^2(x)}{1+q_s}=\int\frac{d^2k}{(2\pi)^2}\cdot\frac{\tilde{D}_l(k^2)}{1+q(k^2)}$$
$$=\int\frac{d^2k}{(2\pi)^2}\frac{[K(-l^2k^2)]^2}{(1+k^2)(1+q(k^2))}=D_{l,1}.$$

Here $q(k^2)$ and v are arbitrary, $m=1$.

For example, in the case of the system (1.5) it can be done in the following way

$$q_s=q_{s_1s_2}=q\left(\frac{\pi^2}{L^2}(s_1^2+s_2^2)\right),$$

$$v_s=v_{s_1s_2}=\sqrt{(2L)^2}\,\delta_{s_1 1}\delta_{s_2 1}v.$$

Then the estimation (6.5) can be written

$$E(g,\Phi)\leq\frac{1}{2}\int\frac{d^2k}{(2\pi)^2}\ln(1+q(k^2))+\frac{1}{2}\frac{q(0)}{1+q(0)}v^2$$
$$+g\lim_{l\to 0}[3\Delta_l^2-6\Delta_l\Phi_1^2+\Phi_1^4],$$

where $\Phi_1=\Phi+\frac{q(0)}{1+q(0)}v$,

$$\Delta_l=D_l-D_{l,1}=\int\frac{d^2k}{(2\pi)^2}\frac{[K(-l^2k^2)]^2q(k^2)}{(1+k^2)(1+q(k^2))}.$$

One can see that

$$\lim_{l\to 0}\Delta_l=\Delta[q]=\int\frac{d^2k}{(2\pi)^2}\frac{q(k^2)}{(1+k^2)(1+q(k^2))}.$$

Introducing the new variable $\psi=\frac{q(0)}{1+q(0)}v-\Phi$ we obtain finally

$$E(g,\Phi)\leqq E_+(g,\Phi),$$

$$E_+(g,\Phi)=\min_{q>0,\psi}\left\{\frac{1}{2}\int\frac{d^2k}{(2\pi)^2}\ln(1+q(k^2))\right.$$
$$\left.+\frac{1+q(0)}{q(0)}(\psi-\Phi)^2+g[3\Delta^2[q]-6\psi^2\Delta[q]+\psi^4]\right\}. \tag{6.6}$$

Let us choose

$$q(k^2)=\left(\frac{A}{k^2}\right)^2,$$

then

$$\int\frac{d^2k}{(2\pi)^2}\ln\left(1+\left(\frac{A}{k^2}\right)^2\right)=\frac{A}{4},$$

$$\Delta[q]=\Delta(A)=\int\frac{d^2k}{(2\pi)^2}\cdot\frac{1}{1+k^2}\cdot\frac{A^2}{A^2+k^2}$$

$$=\frac{1}{4\pi}\cdot\frac{A^2}{1+A^2}\left[\frac{\pi}{2A}+\ln A\right]=\begin{cases}\frac{A}{8}, & A\to 0;\\ \frac{1}{4\pi}\ln A, & A\to\infty.\end{cases}$$

We have for (6.6)

$$E_+(g,\Phi)=\min_{A,\psi}\left\{\frac{A}{8}+\frac{1}{2}(\psi-\Phi)^2+g[3\Delta^2(A)-6\psi^2\Delta(A)+\psi^4]\right\}. \tag{6.7}$$

In the limit $g\to\infty$ one can obtain putting $\psi=\operatorname{sign}\Phi\cdot\sqrt{3\Delta(A)}$ and calculating the minimum with respect to A

$$E_+(g,\Phi)=-\frac{3}{8\pi^2}g\left[\left(\ln g+\ln\frac{3}{\pi^2}B\right)^2-B\right]$$
$$+\frac{1}{2}\left(|\Phi|-\sqrt{\frac{3}{4\pi}\left(\ln g+\ln\frac{3}{\pi^2}B\right)}\right)^2, \tag{6.8}$$

where

$$B=2\left[\ln g+\ln\frac{3}{\pi^2}B\right]=2\ln g\left[1+O\left(\frac{\ln\ln g}{\ln g}\right)\right].$$

One can see from this estimation that the minimum vacuum energy $E_+(g,\Phi)$ in (6.7) is at the points

$$\Phi_\pm=\pm\sqrt{\frac{3}{4\pi}\ln g\left[1+O\left(\frac{\ln\ln g}{\ln g}\right)\right]} \tag{6.9}$$

but not for $\Phi=0$.

Now let us study the lower estimation $E_-(g,\Phi)$ in (6.1). This estimation is obtained in a more complicated way.

First of all let us introduce the regularized field

$$\Phi^{(\Lambda)}(u,x)=\sum_s \mathscr{D}_s^{(\Lambda)}(x)u_s$$

for which

$$\mathscr{D}^{(\Lambda)}(x)=\frac{1}{(2\pi)^2}\int\frac{d^2k}{\sqrt{1+k^2}}\frac{\Lambda^2-1}{\Lambda^2+k^2}e^{ikx},$$

where

$$\Lambda\geqq 1.$$

Then introducing the notation

$$\langle\Phi^{(\Lambda)}(u,x)\Phi^{(\Lambda')}(u,x')\rangle=\int d\sigma_u\Phi^{(\Lambda)}(u,x)\Phi^{(\Lambda')}(u,x')$$

we get in the limit $V\to\infty$ and $l\to 0$

$$D(x)=\langle\Phi(u,x)\Phi(u,0)\rangle=\int\frac{d^2k}{(2\pi)^2}\frac{e^{ikx}}{1+k^2}=\frac{1}{2\pi}K_0(t),$$

$$\begin{aligned}D_\Lambda(x)&=\langle\Phi^{(\Lambda)}(u,x)\Phi(u,0)\rangle=\int dy\mathscr{D}^{(\Lambda)}(x-y)\mathscr{D}(y)\\&=\int\frac{d^2k}{(2\pi)^2}\frac{(\Lambda^2-1)e^{ikx}}{(1+k^2)(\Lambda^2+k^2)}=\frac{1}{2\pi}[K_0(t)-K_0(\Lambda t)],\end{aligned} \tag{6.10}$$

$$\begin{aligned}D_{2,\Lambda}(x)&=\langle\Phi^{(\Lambda)}(u,x)\Phi^{(\Lambda)}(u,0)\rangle=\int dy\mathscr{D}^{(\Lambda)}(x-y)\mathscr{D}^{(\Lambda)}(y)\\&=\int\frac{d^2k}{(2\pi)^2}\frac{(\Lambda^2-1)^2e^{ikx}}{(1+k^2)(\Lambda^2+k^2)^2}\\&=\frac{1}{2\pi}\left[K_0(t)-K_0(\Lambda t)-\left(1-\frac{1}{\Lambda^2}\right)\frac{\Lambda t}{2}K_1(\Lambda t)\right],\end{aligned}$$

where $K_\nu(z)$ is the Mac-Donald function, $t=\sqrt{x^2}$.

Let us introduce the interaction function

$$H_4^{(\Lambda)}(u,\Phi)=-\int_V d^2x:[\Phi^{(\Lambda)}(u,x)+\Phi]^4:$$
$$=-\int_V d^2x\{[\Phi^{(\Lambda)}(u,x)+\Phi]^4-6D_{2,\Lambda}(0)[\Phi^{(\Lambda)}(u,x)+\Phi]^2+3D_{2,\Lambda}^2(0)\},$$

where

$$D_{2,\Lambda}(0)=\frac{1}{4\pi}\left(\ln\Lambda^2-1+\frac{1}{\Lambda^2}\right)=\frac{1}{4\pi}r(\Lambda^2). \tag{6.11}$$

Making use of the following inequality

$$-(\varphi^4-6D\varphi^2+3D^2)\leqq-2\alpha D(\varphi^2-D^2)+C_\alpha D^2, \tag{6.12}$$

where $C_\alpha=6+4\alpha+\alpha^2$, we have

$$-:[\Phi^{(\Lambda)}]^4:\leqq-2\alpha D_{2,\Lambda}(0):[\Phi^{(\Lambda)}]^2:+C_\alpha D_{2,\Lambda}^2(0),$$
$$:[\Phi^{(\Lambda)}]^2:=[\Phi^{(\Lambda)}]^2-D_{2,\Lambda}(0). \tag{6.13}$$

Now let us transform $S_V(g,\Phi)$ in (6.2)

$$S_V(g,\Phi)=\lim_{l\to0}\int d\sigma_u\exp\{g[W_4^{(l)}-2\alpha D_{2,\Lambda}(0)H_2^{(\Lambda)}(u,\Phi)]\}$$
$$\cdot\exp\{2g\alpha D_{2,\Lambda}(0)H_2^{(\Lambda)}(u,\Phi)\}$$
$$=\int_{-\infty}^{\infty}dte^{gt}\left(-\frac{\partial}{\partial t}\right)\sigma(t)=g\int_{-\infty}^{\infty}dte^{gt}\sigma(t),$$

where

$$H_2^{(\Lambda)}(u,\Phi)=-\int_V d^2x:[\Phi^{(\Lambda)}(u,x)+\Phi]^2:,$$
$$\sigma(t)=\lim_{l\to0}\int d\sigma_u\exp\{2\alpha gD_{2,\Lambda}(0)H_2^{(\Lambda)}(u,\Phi)\}$$
$$\cdot\theta(W_4^{(l)}(u,\Phi)-2\alpha D_{2,\Lambda}(0)H_2^{(\Lambda)}(u,\Phi)-t). \tag{6.14}$$

Let us consider the representation

$$S_V(g,\Phi)=T_{1V}(g,\Phi)+T_{2V}(g,\Phi), \tag{6.15}$$

where

$$T_{1V}(g,\Phi)=g\int_{-\infty}^{0}dte^{gt}\sigma(t),$$
$$T_{2V}(g,\Phi)=g\int_{0}^{\infty}dte^{gt}\sigma(t). \tag{6.16}$$

For the function $T_{1V}(g,\Phi)$ the following estimation is valid

$$T_{1V}(g,\Phi)\leqq\lim_{l\to0}\int d\sigma_u\exp\{2\alpha gD_{2,\Lambda}(0)H_2^{(\Lambda)}(u,\Phi)\}.$$

This integral can be calculated according to (A.1):

$$T_{1V}(g,\Phi)\leqq\exp\left\{-\frac{V}{2}\int\frac{d^2k}{(2\pi)^2}[\ln(1+\gamma\tilde{D}_{2\Lambda}(k^2))-\gamma\tilde{D}_{2,\Lambda}(k^2)]\right.$$
$$\left.-\frac{V}{2}\Phi^2\frac{\gamma}{1+\gamma\tilde{D}_{2,\Lambda}(0)}\right\}, \tag{6.17}$$

where $\gamma=4g\alpha D_{2,\Lambda}(0)$ is an arbitrary number. In the limit $\Lambda\to\infty$ when γ is fixed we have

$$T_{1V}(g,\Phi)\leqq\exp\left\{\frac{V}{8\pi}[(1+\gamma)\ln(1+\gamma)-\gamma]-\frac{V}{2}\Phi^2\frac{\gamma}{1+\gamma}\right\}. \tag{6.18}$$

Now consider the function $T_{2V}(g,\Phi)$ in (6.16). We want to get an estimation for the function $\sigma(t)$ in (6.16). For this aim let us consider the region

$$W_4^{(l)}(u,\Phi)-2\alpha D_{2,\Lambda}(0)H_2^{(\Lambda)}(u,\Phi)\geqq V(b+C_\alpha D_{2,\Lambda}^2(0)),$$

where $b>0$. It is easy to see that this region is smaller than the region

$$W_4^{(l)}(u,\Phi)-H_4^{(\Lambda)}(u,\Phi)\geqq Vb$$

because of

$$W_4^{(l)}(u,\Phi)-H_4^{(\Lambda)}(u,\Phi)\geqq Vb$$
$$+[VC_\alpha D_{2,\Lambda}^2(0)+2\alpha D_{2,\Lambda}(0)H_2^{(\Lambda)}(u,\Phi)-H_4^{(\Lambda)}(u,\Phi)]\geqq Vb \tag{6.19}$$

according to (6.13).

Therefore putting

$$t=V(b+C_\alpha D_{2,\Lambda}^2(0))$$

one can obtain the following inequality for $\sigma(t)$ in (6.15) using (6.16) in the limit $l\to0$

$$\sigma(t)\leqq\min_{p>1}\int d\sigma_u\exp\{2g\alpha D_{2,\Lambda}(0)H_2^{(\Lambda)}(u,\Phi)\}$$
$$\cdot\left[\frac{1}{Vb}(W_4(u,\Phi)-H_4^{(\Lambda)}(u,\Phi))\right]^{2p}.$$

Further using the Hölder inequality we obtain

$$\sigma(t)\leqq\min_{p>1}\left\{\int d\sigma_u\left[\frac{1}{Vb}(W_4(u,\Phi)-H_4^{(\Lambda)}(u,\Phi))\right]^{2q'p}\right\}^{\frac{1}{q'}}$$
$$\cdot\{\int d\sigma_u\exp\{2qg\alpha D_{2,\Lambda}(0)H_2^{(\Lambda)}(u,\Phi)\}\}^{\frac{1}{q}}, \tag{6.20}$$

where $\frac{1}{q}+\frac{1}{q'}=1$, $q>1$, $q'>1$.

The integral

$$I_V=\int d\sigma_u\exp\{2qg\alpha D_{2,\Lambda}(0)H_2^{(\Lambda)}(u,\Phi)\}$$

can be calculated in the explicit form. According to (A.1) we get

$$I_V=\exp\left\{V\left[\frac{1}{8\pi}((1+\beta)\ln(1+\beta)-\beta)-\frac{1}{2}\frac{\beta}{1+\beta}\Phi^2\right]\right\},$$

where

$$\beta=4qg\alpha D_{2,\Lambda}(0)=q\gamma$$

according to (6.17).

The integral

$$J_{V,N}=\int d\sigma_n\left[\frac{1}{Vb}(W_4(u,\Phi)-H_4^{(\Lambda)}(u,\Phi))\right]^{2N} \tag{6.21}$$

is the $2N$-th perturbation order for the interaction

$$W_4(u,\Phi)-H_4^{(\Lambda)}(u,\Phi)=-\int_V d^2x[:(\Phi(u,x)+\Phi)^4:-:(\Phi^{(\Lambda)}(u,x)+\Phi)^4:].$$

When $V\to\infty$ the main contribution to the integral (6.21) comes from the two-points coupling

$$\langle(W_4-H_4^{(\Lambda)})^2\rangle=\int d\sigma_u[W_4(u,x)-H_4^{(\Lambda)}]^2=VG(\Lambda,\Phi),$$

where in the limit $V\to\infty$

$$G(\Lambda,\Phi)=24\int d^2x[\Delta_4(x)+4\Delta_3(x)\Phi^2+3\Delta_2(x)\Phi^4],$$
$$\Delta_s(x)=D^s(x)-2D_\Lambda^s(x)+D_{2,\Lambda}^s(x).$$

Making use of the formulas (6.10) one can get for $\Lambda\to\infty$

$$G(\Lambda,\Phi)=\frac{21}{4\pi^3}\cdot\frac{(\ln\Lambda^2)^2}{\Lambda^2}\cdot\left[1+4\left(\frac{2\pi\Phi^2}{\ln\Lambda^2}\right)+2\left(\frac{2\pi\Phi^2}{\ln\Lambda^2}\right)^2\right]. \tag{6.22}$$

Turning to the integral (6.21) we obtain for large V and N

$$J_{V,N}=(2N-1)!!\left[\frac{G(\Lambda,\Phi)}{Vb^2}\right]^N\left(1+O\left(\frac{1}{NV}\right)\right).$$

Since we study the behaviour of all functions in the limit $V\to\infty$ then we have (from the Stirling formula)

$$\min_{p>1}\{J_{V,2q'p}\}^{\frac{1}{q'}}$$
$$=\min_{p>1}\exp\{p\ln q'p-p\}\left[\frac{2G(\Lambda,\Phi)}{Vb^2}\right]^p=\exp\left\{-V\frac{b^2}{2q'G(\Lambda,\Phi)}\right\}.$$

Thus the following inequality for $\sigma(t)$ is valid

$$\sigma(t)=\sigma(V(b+C_\alpha D_{2,\Lambda}^2(0)))\leqq\sigma_+(t)$$
$$=\exp\left\{-\frac{V}{2q}\Phi^2\frac{\beta}{1+\beta}+\frac{V}{8\pi q}[(1+\beta)\ln(1+\beta)-\beta]-\frac{Vb^2}{2q'G(\Lambda,\Phi)}\right\}.$$

In the limit $g\to+\infty$ the main contribution to the vacuum energy gives $T_{2V}(g,\Phi)$ because $\beta=q\gamma$ is arbitrary. Then we have for $g\to\infty$

$$\begin{aligned}
-E(g,\Phi)&=\lim_{V\to\infty}\frac{1}{V}\ln S_V(g,\Phi)\\
&=\lim_{V\to\infty}\frac{1}{V}\ln(T_{1V}(g,\Phi)+T_{2V}(g,\Phi))\\
&=O(1)+\lim_{V\to\infty}\frac{1}{V}\ln T_{2V}(g,\Phi)\\
&\leqq\max_t\left[\frac{g}{V}t+\frac{1}{V}\ln\sigma_+(t)\right]\\
&=\max_{\Lambda>1}\left[g(b+C_\alpha D^2_{2,\Lambda}(0))-\frac{b^2}{2q'G(\Lambda,\Phi)}\right.\\
&\quad\left.-\frac{1}{2q}\cdot\frac{\beta}{1+\beta}\Phi^2+\frac{1}{8\pi q}[(1+\beta)\ln(1+\beta)-\beta]\right].
\end{aligned}\tag{6.23}$$

This expression should be minimized over the parameters b and β. Making use of (6.12) and (6.18) we obtain

$$gC_\alpha D^2_{2,\Lambda}(0)=6gD^2_{2,\Lambda}(0)+\frac{\beta}{q}D_{2,\Lambda}(0)+\frac{\beta^2}{16q^2g}.$$

Finally we can write changing the sign in (6.23)

$$\begin{aligned}
&E(g,\Phi)\geqq E_-(g,\Phi),\\
&E_-(g,\Phi)=\max_{\beta,b}\min_{\Lambda}\left\{-g(b+6D^2_{2,\Lambda}(0))\right.\\
&\qquad+\frac{b^2}{2q'G(\Lambda,\Phi)}+\frac{1}{2q}\frac{\beta}{1+\beta}\Phi^2-\frac{\beta}{q}D_{2,\Lambda}(0)\\
&\qquad\left.-\frac{1}{8\pi q}[(1+\beta)\ln(1+\beta)-\beta]-\frac{\beta^2}{16q^2g}\right\}.
\end{aligned}\tag{6.24}$$

Now let us get the asymptotical behaviour $E_-(g,\Phi)$ for $g\to\infty$. Using the formulas (6.11) and (6.22), changing the variables $b=\frac{3}{8\pi^2}\varrho$ and $\Lambda^2=A$ and limiting ourselves to logarithmic terms $\ln A$ we obtain

$$\begin{aligned}
E_-(g,\Phi)=\max_{\beta,\varrho}\min_{A}&\left\{-\frac{3}{8\pi^2}\left[g(\varrho+(\ln A)^2-2\ln A)\right.\right.\\
&\left.-\frac{\pi\varrho^2A}{28q'(\ln A)^2F(A,\Phi)}\right]\\
&\left.+\left[\frac{\beta}{1+\beta}\frac{\Phi^2}{2q}-\frac{\beta}{q}\frac{\ln A}{4\pi}-\frac{1}{8\pi q}[(1+\beta)\ln(1+\beta)-\beta]-\frac{\beta^2}{16q^2g}\right]\right\},
\end{aligned}\tag{6.25}$$

where

$$F=F(A,\Phi)=1+4\left(\frac{2\pi\Phi^2}{\ln A}\right)+2\left(\frac{2\pi\Phi^2}{\ln A}\right)^2.$$

The supremums with respect to A, ϱ, and β can be performed and the result in our approximation is

$$E_-(g,\Phi)=-\frac{3}{8\pi^2}g[\ln g+\ln B]^2+\varepsilon(g,\Phi), \tag{6.26}$$

where $B=\frac{7}{2\pi}q'F\cdot(\ln g+\ln B)$

$$\begin{aligned}\varepsilon(g,\Phi)&=\frac{1}{2q}\min_\beta\left[\frac{\beta}{1+\beta}\Phi^2-\frac{\beta}{2\pi}\ln g\right]\\&=\frac{1}{2q}\left(|\Phi|-\sqrt{\frac{1}{2\pi}\ln g}\right)^2,\end{aligned} \tag{6.27}$$

where $\beta_{\min}^2\sim\frac{\Phi^2}{\ln g}$. Therefore the inequality (6.23) proves its value in the limit $g\to\infty$.

Finally we obtain in the limit $g\to\infty$

$$\begin{aligned}E_-(g,\Phi)=&-\frac{3}{8\pi^2}g[\ln g+\ln B]^2\\&+\frac{1}{2q}\left(|\Phi|-\sqrt{\frac{1}{2\pi}(\ln g+\ln B)^2}\right)^2,\end{aligned} \tag{6.28}$$

where $q>1$.

We can see that the extimations (6.7) and (6.28) are close one to another in the limit $g\to\infty$. They can be jointed, the result for $E(g,\Phi)$ in (6.1) is

$$\begin{aligned}E(g,\Phi)=&-\frac{3}{8\pi^2}g(\ln g)^2\left[1+O\left(\frac{\ln\ln g}{\ln g}\right)\right]\\&+\tfrac{1}{2}\left(|\Phi|-\sqrt{\frac{\eta}{2\pi}\ln g\left(1+O\left(\frac{\ln\ln g}{\ln g}\right)\right)}\right)^2,\end{aligned} \tag{6.29}$$

where $\eta\simeq 1$.

It follows from this formula that the minimum of the vacuum energy in the $g\varphi_2^4$-theory in the limit $g\to\infty$ is at points

$$\Phi_\pm=\pm\sqrt{\frac{\eta}{2\pi}\ln g}\,. \tag{6.30}$$

It means that in this system there is a phase transition for $g\to\infty$. It should be noted that our estimations (6.8) and (6.28) do not prove, strictly speaking, the existence of the phase transition in the φ_2^4-theory. This was proved in [7] (see also

[9–11]). The reason is that the accuracy of the first term in (6.29) is $O(g\ln g\ln\ln g)$, what is much more than the second term which is $O(\ln g)$. Nevertheless our estimations certainly indicate that the phase transition takes place and give the points $\Phi_{\pm}$ where vacuum energy has minimum.

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Appendix

Let us show that for $V\to\infty$

$$\begin{aligned}
Q_V &= \int\delta\Lambda\exp\left\{-\tfrac{1}{2}\int_V d^dx\Lambda^2(x) - ia\int_V d^dx\left[\int_V d^dy\mathscr{D}_V(x,y)\Lambda(y)\right]^2 + i\int_V d^dx\alpha(x)\Lambda(x)\right\} \\
&= \int\delta u\exp\left\{-\tfrac{1}{2}\sum_s u_s^2 - ia\int_V d^dx\left[\sum_s\mathscr{D}_s(x)u_s\right]^2 + i\sum_s\alpha_s u_s\right\} \\
&= \exp\left\{-\frac{V}{2}\int\frac{d^dk}{(2\pi)^d}\ln(1+2ia\tilde{D}(k^2))\right. \\
&\quad\left.-\tfrac{1}{2}\int d^dx\int d^dx'\alpha(x)G(x-x')\alpha(x')\right\},
\end{aligned} \tag{A.1}$$

where

$$G(x-x') = \frac{1}{(2\pi)^d}\int\frac{d^dk e^{ik(x-x')}}{1+2ia\tilde{D}(k^2)}. \tag{A.2}$$

In the integral (A.1) let us change the variable

$$u_s = \sum_n U_{sn}v_n, \tag{A.3}$$

where

$$\begin{aligned}
&\sum_n U^T_{sn}U_{ns'} = \delta_{ss'}, \\
&\sum_{nn'} U^T_{sn}D_{nn'}U_{n's'} = d_s\delta_{ss'}, \\
&D_{nn'} = \int_V\int_V dydy'g_n(y)D_V(y,y')g_{n'}(y') \\
&\qquad = \sum_s U_{ns}d_sU^T_{sn'}.
\end{aligned} \tag{A.4}$$

Then we have

$$\begin{aligned}
&\sum_s u_s^2 = \sum_n v_n^2, \\
&\sum_{ss'} u_sD_{ss'}u_{s'} = \sum_n d_nv_n^2, \\
&\sum_s\alpha_su_s = \sum_n\beta_nv_n, \qquad \beta_n = \sum_s U^T_{ns}\alpha_s.
\end{aligned}$$

The integral in the new variables can be calculated:

$$Q_V=\int\delta v\exp\left\{-\tfrac{1}{2}\sum_n(1+2iad_n)v_n^2+i\sum_n\beta_n v_n\right\}$$
$$=\exp\left\{-\tfrac{1}{2}\sum_n\ln(1+2iad_n)+\tfrac{1}{2}\sum_n\frac{\beta_n^2}{1+2iad_n}\right\}. \quad (A.5)$$

Now let us consider

$$\sum_n\ln(1+2iad_n)=\sum_{k=1}^{\infty}\frac{(-)^{k+1}}{k}(2i\sigma)^k\sum_n d_n^k. \quad (A.6)$$

Making use of the formulas (A.4) one can get

$$\sum_n d_n^k=\sum_{n_1,\ldots,n_k}D_{n_1n_2}D_{n_2n_3}\cdot\ldots\cdot D_{n_kn_1}$$
$$=\int_V dy_1\cdot\ldots\cdot\int_V dy_k D_V(y_1\cdot y_2)D_V(y_2,y_3)\cdot\ldots\cdot D_V(y_n,y_1).$$

When V is large enough this function is

$$\sum_n d_n^k=V\int\frac{d^dk}{(2\pi)^d}[\tilde{D}(k^2)]^k+O(\exp\{-\mathrm{const}\,V^{1/d}\})$$

whence

$$\sum_n\ln(1+2iad_n)=V\int\frac{d^dk}{(2\pi)^d}\ln(1+2ia\tilde{D}(k^2)). \quad (A.7)$$

Further consider

$$\sum_n\frac{\beta_n^2}{1+2iad_n}=\sum_{k=0}^{\infty}(-2ia)^k\sum_n\beta_n d_n^k\beta_n,$$
$$\sum_n\beta_n d_n^k\beta_n=\sum_{n_1,\ldots,n_{k+1}}\alpha_{n_1}D_{n,n_2}\cdot\ldots\cdot D_{n_kn_{k-1}}\alpha_{n_{k+1}}$$
$$=\int_V dy_1\ldots\int_V dy_{k+1}\alpha(y_1)D_V(y_1,y_2)\cdot\ldots\cdot D_V(y_k,y_{k+1})\alpha(y_{k+1}).$$

In this expression it is possible to take the limit $V\to\infty$

$$\lim_{V\to\infty}\sum_n\beta_n d_n^k\beta_n=\int dy_1\ldots\int dy_{k+1}\alpha(y_1)D(y_1-y_2)\cdot\ldots\cdot D(y_k-y_{k+1})\alpha(y_{k+1})$$
$$=\int\int dx dx'\int\frac{d^dk}{(2\pi)^d}[\tilde{D}(k^2)]^k e^{ik(x-x')}.$$

Substituting this formula into (A.8) we obtain

$$\lim_{V\to\infty}\sum_n\frac{\beta_n^2}{1+2iad_n}=\int dx\int dx'\alpha(x)G(x-x')\alpha(x'), \quad (A.9)$$

where $G(x-x')$ is defined by the formula (A.2).

Thus substituting (A.7) and (A.9) into (A.5) one can obtain (A.1).

References

1. Efimov, G.V.: Nonlocal interactions of quantized fields. Moscow: Nauka 1977
2. Efimov, G.V.: Commun. math. Phys. **57**, 235 (1977)
3. Polya, G., Szegö, G.: Aufgaben und Lehrsätze aus der Analysis, Bd. 1. Berlin: Springer 1925
4. Parisi, G.: Phys. Lett. **69** B, 329 (1977)
5. Beruillier, C., Dronffe, J.M., Godreche, C., Zinn-Instin, J.: Comparison between large order estimated and perturbation series in a scalar field theory with Gaussian propagator. Preprint, Sacley (1977)
6. Simon, B.: The $P(\varphi)_2$ Euclidean (Quantum) field theory. Princeton, New Jersey: Princeton University Press 1974
7. Glimm, J., Jaffe, A., Spencer, T.: Commun. math. Phys. **45**, 203 (1975)
8. Guerra, F., Rosen, L., Simon, B.: Ann. Inst. Henri Poincaré **25** A, 931 (1976)
9. Chang, S.J.: Phys. Rev. D **13**, 2778 (1977)
10. Fröhlich, J.: Acta Phys. Austriaca, Suppl. **15**, 133–269 (1976)
11. Glimm, J., Jaffe, A., Spencer, T.: Ann. Phys. **101**, 610 (1976); **101**, 631 (1976)

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