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# A Description of Instantons

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**Abstract.** An explicit description is given for all self-dual Euclidean Yang-Mills fields and their parameter spaces in the theory with unitary gauge group of arbitrary rank.

## §0. Introduction

*0.1.* In [3–5] a geometric construction was given for all instantons, i.e. autodual solutions of the Yang-Mills equations in the compactified Euclidean space $S^4$ in the theory with arbitrary classical compact simple Lie group. This paper is an elaboration of [3]. Here we derive explicit expressions for the field potentials and parameters involved. For brevity we restrict ourselves to the unitary gauge group case. The orthogonal and symplectic groups can be treated in a similar way; this will be done in a subsequent publication. We first state the main results.

*0.2. Parameter Space.* Fix an integer $n \geqq 1$ and consider the set of matrix triples $(R_n, B, C)$ fulfilling the following conditions:

a) $B, C$ are complex $(n \times n)$-matrices;

$$R_n = \begin{pmatrix} \varrho_1 & & 0 \\ & \ddots & \\ 0 & & \varrho_n \end{pmatrix} = \mathrm{diag}(\varrho_1, \ldots, \varrho_n), \qquad 0 < \varrho_1 \leqq \varrho_2 \leqq \ldots \leqq \varrho_n .$$

Set $D = \begin{vmatrix} B & C \\ -C^+ & B^+ \end{vmatrix}$, $R = \begin{vmatrix} R_n & 0 \\ 0 & R_n \end{vmatrix}$, $Q = \begin{vmatrix} E_{2n} & D \\ D^+ & R \end{vmatrix}$. Then

b) $Q \geqq 0$ and moreover, $Q$ is strictly positive on a set of $4n$-vectors which will be explicitly described in §5 below.

Call two such triples $(R_n, B, C)$ and $(R'_n, B', C')$ equivalent if $R_n = R'_n$ and there exists a unitary matrix $U$ such that $U^{-1}R_nU = R_n$, $U^{-1}BU = B'$, $U^{-1}CU = C'$.

Then the set of such triples up to equivalence is a parameter space for $\mathrm{SU}(2n)$-instantons with topological charge $n$ up to gauge equivalence.

Note that an easy count shows that the dimension of the parameter space is $4n^2 + 1$, in agreement with general results of [6] and [10].

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*0.3. Field Potentials.* Fix a triple $(R_n, B, C)$ as in 0.2. We represent a point $x=(x_\mu)\in R^4$ by the $(2n\times 2n)$-matrix

$$X=\begin{vmatrix}(x_4+ix_3)E_n, & -(x_2-ix_1)E_n\\ (x_2+ix_1)E_n, & (x_4-ix_3)E_n\end{vmatrix}.$$

Set $S=R-DD^+$ and

$$A_\mu=-[(X+D)(X^++D^+)+S]^{-1}\partial_\mu X(X+D)^{-1}S,$$

$$\partial_\mu=\frac{\partial}{\partial x_\mu},\qquad \mu=1,\dots 4.$$

These expressions define a solution of the self-dual Yang-Mills equations $F_{\mu\nu}=\tilde{F}_{\mu\nu}$, where

$$F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu+[A_\mu,A_\nu],\quad \tilde{F}_{\mu\nu}=\tfrac{1}{2}\varepsilon^{\mu\nu\varrho\lambda}F_{\varrho\lambda}.$$

Its only singularities are at the points $x$ where $X+D$ is degenerate. They are non-physical and may be removed by a gauge transformation. Note that although the $A_\mu$ are not antihermitian they are gauge equivalent to an antihermitian field. In fact they represent a self-dual Yang-Mills field with topological charge $n$ for $SU(2n)$ corresponding to the point $(R_n, B, C)$ of the parameter space.

*0.4. Group Reduction.* The above solution is irreducible only if $S$ has the maximal rank $2n$. In the general case $r=\mathrm{rk}\,S\leqq 2n$ this solution can be reduced to the subgroup $SU(r)\subset SU(2n)$ imbedded in the standard way.

*0.5. The Completeness of the Solution Set.* An arbitrary $n$-instanton field for $SU(r)$ is equivalent to one of those described above in the following sense: for $r>2n$ it reduces to a $SU(2n)$-field by means of the standard inclusion $SU(2n)\subset SU(r)$ and for $r\leqq 2n$ by means of $SU(r)\subset SU(2n)$.

*0.6. Gauge Equivalence.* $n$-instantons corresponding to the triples $(R_n, B, C)$ and $(R'_n, B', C')$ are gauge equivalent iff these triples are equivalent.

*0.7.* We use here the following geometric definition of an $n$-instanton in $SU(r)$-theory: a differentiable hermitian vector bundle $L$ over $S^4$ with self-dual connection $\nabla$, of rank $r$ and Pontryagin number $n$ (see [2, 8]). The hermitian metric $\langle\ ,\ \rangle$ on the fibres of $L$ is assumed to be $\nabla$-horizontal. That is, if $\nabla(\partial)$ is the covariant derivative along a local vector field $\partial$ on $S^4$ and $\varphi,\psi$ are two local sections of $L$, then

$$\partial\langle\varphi,\psi\rangle=\langle\nabla(\partial)\varphi,\psi\rangle+\langle\varphi,\nabla(\partial)\psi\rangle.$$

The sphere $S^4$ is a conformal compactification of the Euclidean 4-space $R^4: S^4=R^4\cup\{\infty\}$. If we choose Euclidean coordinates $(x_\mu)$ on $R^4$ and an orthonormal basis of sections of $L$, then for any section $\psi=\begin{pmatrix}\psi_1\\ \vdots\\ \psi_r\end{pmatrix}$ and $\nabla_\mu=\nabla(\partial_\mu)$, $\partial_\mu=\dfrac{\partial}{\partial x_\mu}$ we obtain

$$\nabla_\mu\psi=\partial_\mu\psi+A_\mu\psi.$$

The connection coefficients $A_\mu$ are the Yang-Mills field potentials corresponding to $(L, \nabla, \langle\ ,\ \rangle)$ in an antihermitian gauge. Change of basis in $L$ corresponds to a local gauge transformation of the $A_\mu$'s. The connection form of $\nabla$ is $\sum A_\mu dx_\mu$, the curvature form is $\sum F_{\mu\nu} dx_\mu \wedge dx_\nu$. Global gauge equivalence of instantons is defined as an isomorphism of the corresponding vector bundles preserving metrics and connections.

*0.8.* The geometric construction of instantons and the derivation of the results in 0.2–0.4 are given here with complete proofs. We do not, however, prove the statements of 0.5 and 0.6, for which a completely different technique is needed: see [4] for a sketch.

The paper is structured a follows. In Sects. 1 and 2 we present the method of "complexification" of our problem due to R. S. Ward and M. F. Atiyah and prove the fundamental Atiyah-Ward-Belavin-Zakharov lemma. Section 3 is devoted to a description of the SU(2)-instanton of Belavin et al. and its parameter space in a convenient geometric framework. Section 4 deals with a generalization of this construction to the $n$-instanton case, and in Sect. 5 explicit calculations in carefully chosen bases are presented. These are the main new results of our paper. Section 6 comments on the relation between this presentation and that in notes [3] and [4]. Finally, in Sect. 7 we discuss a possible application of our technique to the problem of approximate decomposition of an $n$-instanton into a combination of 1-instantons. We show that our method supplies a kind of "spectral analysis" and Bäcklund transforms for instanton fields.

*0.9. Notation.* $R$ (resp. $C$) denotes the real (resp. complex) numbers; $*$ means complex conjugation; $+$ denotes hermitian conjugation; $t$ denotes matrix transpose.

## §1. Compactification and Complexification of $R^4$

*1.1. Notation.* $R^4$ is Euclidean 4-space with a fixed orthonormal basis; $(x_\mu)$, $\mu = 1, \ldots, 4$ are the coordinates of a point $x$ in this basis. The point $x$ is conveniently represented by the complex $2\times 2$-matrix $X$ defined by

$$X^t = \sum_{a=1}^{3} i x_a \sigma_a + x_4 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} x_4 + i x_3, & x_2 + i x_1 \\ -x_2 + i x_1, & x_4 - i x_3 \end{pmatrix},$$

where the $\sigma_a$ are Pauli matrices. We also consider the bispinor space $C^4$ of complex 4-vectors $z = (z_\mu)$, $\mu = 1, \ldots, 4$ and denote by $j: C^4 \to C^4$ the map $j(z_1, z_2, z_3, z_4) = (-z_2^*, z_1^*, -z_4^*, z_3^*)$. We will also let $j$ denote the map $\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \mapsto \begin{pmatrix} -z_2^* \\ z_1^* \end{pmatrix} = -i\sigma_2 \begin{pmatrix} z_1^* \\ z_2^* \end{pmatrix}$ and similarly for $\begin{pmatrix} z_3 \\ z_4 \end{pmatrix}$. Each point $x \in R^4$ determines a complex plane $P_x \subset C^4$:

$$P_x = \left\{ \begin{pmatrix} z_1 \\ \vdots \\ z_4 \end{pmatrix} \middle| \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = X^+ \begin{pmatrix} z_3 \\ z_4 \end{pmatrix} \right\}. \tag{1}$$

We also set

$$P_\infty = \left\{ \begin{pmatrix} z_1 \\ \vdots \\ z_4 \end{pmatrix} \middle| z_3 = z_4 = 0 \right\}.$$

**1.2. Lemma.** *a)* $j(P_x) = P_x$ *for all* $x \in R^4 \cup \{\infty\}$.

*b) Each complex plane* $P \subset C^4$ *with* $j(P) = P$ *coincides with exactly one* $P_x$, $x \in R^4 \cup \{\infty\}$.

*c)* $C^4 = \cup P_x$, *all* $x \in R^4 \cup \{\infty\}$, *and* $P_x \cap P_y = \{0\}$, *if* $x \neq y$.

*Proof*. By definition (1), $(z_\mu) \in P_x$ iff

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} (x_4 - ix_3)z_3 + (x_2 - ix_1)z_4 \\ -(x_2 + ix_1)z_3 + (x_4 + ix_3)z_4 \end{pmatrix}.$$

Hence

$$j\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} -z_2^* \\ z_1^* \end{pmatrix} = \begin{pmatrix} (x_2 - ix_1)z_3^* - (x_4 - ix_3)z_4^* \\ (x_4 + ix_3)z_3^* + (x_2 + ix_1)z_4^* \end{pmatrix}$$
$$= X^+ j\begin{pmatrix} z_3 \\ z_4 \end{pmatrix}.$$

It follows that $j(P_x) \subset P_x$; the inverse inclusion and the case $x = \infty$ are obvious.

It follows easily from (1) that $P_x \neq P_y$ if $x \neq y$. Moreover, then $P_x \cap P_y = \{0\}$, because there are no $j$-invariant lines in $C^4$: if $a\begin{pmatrix} -z_2^* \\ z_1^* \end{pmatrix} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$, then $z_1 = -az_2^* = -a(az_1^*)^* = -|a|^2 z_1$ whence $z_1 = z_2 = 0$ and similarly $z_3 = z_4 = 0$. Every point $(z_\mu) \neq 0$ lies in the $j$-invariant plane $Cz_\mu + Cj(z_\mu)$, which proves b).

Finally, a point $z \in C^4 \backslash P_\infty$ lies in the plane $P_x$, where $x$ is defined by the equations

$$\begin{aligned} x_2 + ix_1 &= \frac{-z_2 z_3^* + z_4 z_1^*}{|z_3|^2 + |z_4|^2}, \\ x_4 + ix_3 &= \frac{z_2 z_4^* + z_3 z_1^*}{|z_3|^2 + |z_4|^2}. \end{aligned} \tag{2}$$

One easily cheeks this by substituting (2) into (1).

*1.3*. We let $\pi$ denote the map constructed above;

$$\pi : C^4 \backslash \{0\} \to R^4 \cup \{\infty\}, \quad \pi(z) = x \quad \text{iff} \quad z \in P_x.$$

We also identify $R^4 \cup \{\infty\}$ with the 4-sphere

$$S^4 = \left\{ (\xi_1, \dots, \xi_5) \middle| \sum_{a=1}^{5} \xi_a^2 = 1 \right\}$$

by means of the stereographic projection

$$\xi_\mu = \frac{2x_\mu}{1 + |x|^2}, \quad \mu = 1, \dots, 4; \quad \xi_5 = \frac{1 - |x|^2}{1 + |x|^2};$$
$$x_\mu = \frac{\xi_\mu}{1 + \xi_5}.$$

The center of projection (0000–1) goes to $\infty$; the natural metric on $R^4$ and $S^4$ are conformally equivalent: $\sum_{\mu=1}^{4} dx_\mu^2 = \frac{1}{(1+\xi_5)^2} \sum_{a=1}^{5} d\xi^2$.

Since the self-duality condition on the curvature form is conformally invariant, we may and will check it in the $R^4$-coordinates.

## §2. Atiyah-Ward-Belavin-Zakharov Lemma

*2.1.* Let $L$ be a vector bundle over $R^4$ or $S^4$, $\omega$ a 2-form on $R^4$ or $S^4$ with values in $L$, $\pi^*(\omega)$ its inverse image with values in $\pi^*(L)$. Ward and Atiyah [2] and independently Belavin and Zakharov [1] have discovered the following important fact.

**2.2. Lemma.** *The form $\omega$ is self-dual iff the form $\pi^*(\omega)$ is of type* $(1,1)$.

*Proof.* One easily sees that it suffices to prove this for forms with values in functions. We consider the following basis of the space of self-dual forms on $R^4$:

$$\begin{aligned}&dx_1 \wedge dx_2 + dx_3 \wedge dx_4\,; \qquad dx_1 \wedge dx_3 - dx_2 \wedge dx_4\,;\\ &dx_1 \wedge dx_4 + dx_2 \wedge dx_3\,.\end{aligned} \tag{3}$$

Writing simply $dx_\mu$ instead of $\pi^*(dx_\mu)$ etc. we find from Eqs. (1):

$$dX^+ \begin{pmatrix} z_3 \\ z_4 \end{pmatrix} = \begin{pmatrix} dz_1 \\ dz_2 \end{pmatrix} - X^+ \begin{pmatrix} dz_3 \\ dz_4 \end{pmatrix} \overset{\text{def}}{=} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}, \tag{4}$$

where $\omega_1, \omega_2$ are forms of $(1,0)$-type on $C^4$. We will not use their explicit expressions. It will suffice to note that $\omega_1 \wedge \omega_2 \neq 0$ because $dz_1 \wedge dz_2$ occurs in $\omega_1 \wedge \omega_2$.

Adding (4) to its complex-conjugate equations, we are then able to express $dx_\mu$ in terms of $z_\mu, z_\mu^*$ and $\omega_i, \omega_i^*$ and subsequently to calculate the $(2,0)+(0,2)$-components of the inverse images of the forms (3) on $C^4$.

Omitting the easy details, we state the result. Let $Z = 2(|z_3|^2 + |z_4|^2)$. Then

$$\begin{aligned}
[iZ^2 dx_1 \wedge dx_2]^{(2,0)+(0,2)} &= [-iZ^2 dx_3 \wedge dx_4]^{(2,0)+(0,2)}\\
&= 2z_3^* z_4^* \omega_1 \wedge \omega_2 - 2z_3 z_4 \omega_1^* \wedge \omega_2^*.\\
[Z^2 dx_1 \wedge dx_3]^{(2,0)+(0,2)} &= [Z^2 dx_2 \wedge dx_4]^{(2,0)+(0,2)}\\
&= (z_3^{*2} + z_4^{*2})\omega_1 \wedge \omega_2 + (z_3^2 + z_4^2)\omega_1^* \wedge \omega_2^*\,;\\
[iZ^2 dx_1 \wedge dx_4]^{(2,0)+(0,2)} &= [-iZ^2 dx_2 \wedge dx_3]^{(2,0)+(0,2)}\\
&= (-z_3^{*2} + z_3^{*2})\omega_1 \wedge \omega_2 + (z_4^2 - z_3^2)\omega_1^* \wedge \omega_2^*\,.
\end{aligned}$$

One sees directly from these identities that the forms (3) in the $(z_\mu, z_\mu^*)$-coordinates have no $(2,0)+(0,2)$-components.

Now we must check that if $\pi^*(\omega) = [\pi^*(\omega)]^{(1,1)}$, then $\omega$ is self-dual. This amounts to verifying the linear independence of the $(2,0)+(0,2)$-components of the forms (3) over the functions on $R^4$. But $\omega_1 \wedge \omega_2 \neq 0$, as we noted earlier, so this follows from the linear independence of $z_3 z_4$, $z_3^2 + z_4^2$, $z_3^2 - z_4^2$ over the functions of $x$, which is evident because one sees from (1) that for any given $x$ there is in $P_x$ a point with arbitrary given coordinates $(z_3, z_4)$.

A more general result with a more conceptual proof is given in [6].

The main application of Lemma 2.2 consists in a reduction of the instanton classification problem to an algebrogeometric problem (see [2, 6]). We will state it here without proof, because the final presentation of our construction, although based on this reduction, can be stripped of algebraic geometry, except for the completeness part.

We note first of all that every 1-subspace of $C^4$ is contained in a unique $j$-invariant plane $P_x$. Thus, the map $\pi: C^4\backslash\{0\} \to S^4$ can be factored through $CP^3 = \{\text{projective space of lines in } C^4\} \to S^4$. This last map will also be denoted $\pi$.

**2.3. Corollary.** *Let $L$ be a differentiable complex vector bundle on $S^4$ with a self-dual connection $\nabla$. Then the vector bundle $\pi^*(L)$ can be endowed with a unique complex-analytic and hence algebraic structure in which a local section $\psi$ of $\pi^*(L)$ is holomorphic iff* $[\pi^*(\nabla)]^{(0,1)}\psi = 0$.

For computational purposes we will need another corollary of Lemma 2.2 which, except for notation, coincides with the original Belavin-Zakharov observation. Put in earlier notation

$$\nabla_a = \pi^*(\nabla)\left(\frac{\partial}{\partial z_a}\right), \qquad \nabla_{\bar{a}} = \pi^*(\nabla)\left(\frac{\partial}{\partial z_a^*}\right):$$

these are covariant connection derivatives of $\pi^*(L)$ on $C^4\backslash\{0\}$.

**2.4. Corollary.** *The connection $\nabla$ on $L$ is self-dual iff*

$$[\nabla_a, \nabla_b] = [\nabla_{\bar{a}}, \nabla_{\bar{b}}] = 0 \quad \textit{for all} \quad a, b = 1, \ldots, 4.$$

In fact, the vanishing of these commutators is equivalent to the vanishing of the $(2,0)+(0,2)$-component of the curvature form on $C^4\backslash\{0\}$.

## § 3. 1-Instanton for SU (2)

*3.1.* In this section we describe a geometric construction of the SU(2)-instanton introduced for the first time in [7]. Our construction conveniently generalizes to the $n$-instanton case in Sect. 4.

*3.2. Vector Bundle M.* The map $\pi: C^4\backslash\{0\} \to S^4$, defined in Sect. 1, can be completed to a differentiable vector bundle of rank two with fibre $P_x$ over a point $x \in S^4$. We let $M$ denote this bundle. An instanton bundle $L$ over $S^4$ will then be constructed as the orthogonal complement of $L$ in $S^4 \times C^4$ relative to a hermitian metric on $C^4$ fulfilling some complementary conditions which we will describe now.

*3.3. Metric on $C^4$.* Consider a hermitian metric $\langle\,,\rangle$ on $C^4$ which is antilinear in the first argument and has the following properties:

a) $\langle z, z\rangle > 0$ for all $z \in C^4\backslash\{0\}$, that is, $\langle\,,\rangle$ is non-degenerate and positive.

b) $\langle z, j(z)\rangle = 0$ for all $z \in C^4$.

The metric is completely described by its matrix $Q$ in the standard basis of $C^4$. An easy calculation (performed in a more general situation below in Sect. 4) shows that condition b) is equivalent to the following symmetry property of $Q$:

$$Q=\begin{vmatrix} \tau & 0 & b^* & -c \\ 0 & \tau & c^* & b \\ b & c & \varrho & 0 \\ -c^* & b^* & 0 & \varrho \end{vmatrix}, \tag{5}$$

where $\varrho$, $\tau\in R$; b) $c\in C$. Condition a) is then equivalent to the inequalities

$$\varrho>0, \quad \tau>0; \quad \varrho\tau-(|b|^2+|c|^2)>0. \tag{6}$$

*3.4. Vector Bundle L.* Denote by $L$ the vector bundle on $S^4$ whose fibre $L_x$ over a point $x$ is the orthogonal complement $M_x^\perp=P_x^\perp$ of $M_x$ in $C^4$ relative to $\langle\,,\rangle$.

The metric $\langle\,,\rangle$ induces a hermitian metric on $L$.

*3.5. Connection on L.* The bundle $L$ is a subbundle of the trivial bundle $S^4\times C^4$. The latter is endowed with a trivial connection $\nabla^0$ relative to which all vectors in $C^4$, considered as constant sections of $S^4\times C^4$, are horizontal. Setting $\nabla^0_\mu=\nabla^0\left(\frac{\partial}{\partial x_\mu}\right)$ we have

$$\nabla^0_\mu\begin{pmatrix}\psi_1(x)\\ \vdots \\ \psi_4(x)\end{pmatrix}=\begin{pmatrix}\frac{\partial}{\partial x_\mu}\psi_1(x)\\ \vdots \\ \frac{\partial}{\partial x_\mu}\psi_4(x)\end{pmatrix}.$$

Denote by $\nabla$ the connection on $L$ which is the orthogonal projection on $L$ of $\nabla^0$:

$$\nabla\psi=\mathrm{pr}_L(\nabla^0\psi)$$

for all local sections $\psi$ of $L$.

We will now calculate the connection matrices of $\nabla$ in certain bases of sections of $L$, and we will see that they coincide with the usual 1-instanton matrices.

First note that $(L,\nabla)$ does not change if we consider any $\alpha\langle\,,\rangle$ instead of $\langle\,,\rangle$, with $\alpha>0$. So we may and will assume that $\tau=1$ in (5) and (6).

*3.6. Bases of Sections of L.* A basis of sections of $M$ over $R^4$ written in the basis of standard constant sections of $S^4\times C^4$ consists of rows of the matrix

$$|X^*E|=\begin{vmatrix} x_4-ix_3, & -(x_2+ix_1), & 1,0 \\ x_2-ix_1, & x_4+ix_3, & 0,1\end{vmatrix}. \tag{7}$$

In fact, one easily checks that these rows fulfill the Eq. (1). Set $D=\begin{vmatrix} b & c \\ -c^* & b^*\end{vmatrix}$, $R=\begin{vmatrix}\varrho & 0\\ 0 & \varrho\end{vmatrix}$, so that $Q=\begin{vmatrix} E & D^+ \\ D & R\end{vmatrix}$. A basis of sections of $L$ is given by the rows of any matrix $|Z_1^t, Z_2^t|$ of rank 2 where $Z_1$, $Z_2$ are two $2\times 2$-matrices depending on $x$ and fulfilling the orthogonality condition

$$|XE|\begin{vmatrix} E & D^+ \\ D & R\end{vmatrix}\begin{vmatrix} Z_1 \\ Z_2\end{vmatrix}=0 \tag{8}$$

that is

$$(X+D)Z_1+(XD^++R)Z_2=0. \tag{9}$$

Setting here $X+D=Y$ and $S=R-DD^+$, one obtains

$$XD^++R=(Y-D)D^++R=YD^++S,$$

and (8) and (9) may be conveniently written as

$$Z_1+(D^++Y^{-1}S)Z_2=0. \tag{10}$$

Each solution $(Z_1, Z_2)$ of (10) with $\operatorname{rk}|Z_1^t, Z_2^t|=2$ will represent a basis of sections of $L$. Here are the two simplest solutions:

$$Z_1=D^++Y^{-1}S, \quad Z_2=-E, \tag{11}$$

$$Z_1=-E, \quad Z_2=(D^++Y^{-1}S)^{-1}. \tag{12}$$

The basis (12) is regular everywhere on $R^4$, and the basis (11) is regular everywhere except at points $X=-D$.

3.7. *Gram's Matrix of Sections of L.* This is the "scalar product" of $|Z_1^t, Z_2^t|$ and $\begin{vmatrix} Z_1 \\ Z_2 \end{vmatrix}$ relative to $Q$, that is,

$$G=(Z_1^++Z_2^+D)Z_1+(Z_1^+D^++Z_2^+R)Z_2. \tag{13}$$

3.8. *Connection Matrices.* In a chosen basis we have

$$\nabla_\mu|Z_1^t, Z_2^t|=A_\mu^t|Z_1^t, Z_2^t|,$$

where the $A_\mu$ are connection matrices. Setting

$$\begin{aligned} H&=|\partial_\mu Z_1^+, \partial_\mu Z_2^+|Q\begin{vmatrix} Z_1 \\ Z_2 \end{vmatrix} \\ &=\partial_\mu Z_1^+(Z_1+D^+Z_2)+\partial_\mu Z_2^+(DZ_1+RZ_2) \end{aligned} \tag{14}$$

and comparing (13) and (14), we obtain:

$$A_\mu=(G^+)^{-1}H. \tag{15}$$

Now calculate $G$, $H$, and $A_\mu$ in the basis (11):

$$\begin{aligned} G&=(D+S(Y^+)^{-1}-D)(D^++Y^{-1}S) \\ &\quad-(D+S(Y^+)^{-1})D^++R=S(YY^+)^{-1}S+S. \end{aligned} \tag{16}$$

The matrices $S$ and $YY^+$ are scalar here:

$$S=s\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad s=\varrho-|b|^2-|c|^2,$$

$$YY^+=|x+d|^2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

where

$$d=(\operatorname{Im} c, -\operatorname{Re} c, \operatorname{Im} b, \operatorname{Re} b)\in R^4$$

and $|x+d|^2$ is the square of the length of $x+d$. It follows that (11) represents an orthogonal basis, and both of its sections have the length

$$\sqrt{s+s^2|x+d|^{-2}}\,.$$

Similarly one finds

$$H=\partial_\mu Z_1^+(Z_1-D^+)=S\partial_\mu(Y^+)^{-1}\cdot Y^{-1}S \tag{17}$$

and then, substituting (16) and (17) into (15),

$$A_\mu=-\frac{s}{s+|x+d|^2}\partial_\mu X\cdot(X+D)^{-1}\,. \tag{18}$$

Comparing this with the standard t' Hooft's formulae [6, 9], one sees that this is a self-dual 1-instanton Yang-Mills field for the gauge group $U(2)$. The point $x=-d$ is the instanton center and the parameter $s$ defines its scale.

Finally, note that Eq. (9) reveals a simple geometric meaning of the instanton center: this is the unique point at which the fibre of $L$ is orthogonal to the fibre $L_\infty$, that is, coincides with the space $M_\infty$ generated by the rows of $\begin{vmatrix}1&0&0&0\\0&1&0&0\end{vmatrix}$, in view of (7).

## § 4. A Geometric Construction of $n$-Instantons for SU($2n$)

*4.1.* This section presents a generalization of the previous construction to the case of arbitrary $n\geqq 1$. In addition to the previous data, choose an $n$-dimensional complex space $I$. All vector bundles to be constructed will be subbundles of the trivial bundle $S^4\times\left(I\bigotimes_C C^4\right)$. In the case $n=1$ our construction will reduce to that of Sect. 3.

*4.2. Vector Bundle M.* This is the subbundle of $S^4\times\left(I\bigotimes_C C^4\right)$ with fibre $I\otimes P_x$ over a point $x\in S^4$. The rank of $M$ is $2n$.

*4.3. Metric on $I\otimes C^4$.* Let $\varepsilon_1=(1,0,0,0),\ldots,\varepsilon_4=(0,0,0,1)$ be the standard basis of $C^4$. Set $I_\mu=I\otimes\varepsilon_\mu$, so that $I\otimes C^4$ is the direct sum $I_1\oplus\ldots I_4$. For a point $z\in C^4$ and a vector $e\in I$ we set

$$e_\mu=e\otimes\varepsilon_\mu\in I_\mu\,,\qquad ez=\sum_{\mu=1}^{4}e_\mu z_\mu\,,\qquad I_z=\{ez|e\in I\}\,.$$

It is clear that $I_z$ is an $n$-dimensional subspace of $I\otimes C^4$ if $z\neq 0$.

Choose a hermitian metric $\langle\,,\rangle$ on $I\otimes C^4$ which is antilinear in the first argument and fulfills the following conditions.

a) $\langle\,,\rangle\geqq 0;\quad \langle ez,ez\rangle>0$ for all $e\in I,\quad z\in C^4,\quad ez\neq 0$.

It is essential, in contrast with the case $n=1$, that the metric is permitted to be degenerate. The corresponding instantons will be reducible to a subgroup SU($r$)$\subset$SU($2n$).

b) $\langle ez,fj(z)\rangle=0$ for all $e,f\in I,\quad z\in C^4$.

*4.4. Vector Bundle N.* We denote by $N$ the vector bundle over $S^4$ whose fibre over a point $x \in S^4$ is the orthogonal complement $M_x^\perp$ of $M_x$ relative to the metric $\langle\, , \rangle$. To show that $N$ is well-defined, some verifications are needed.

First of all, if $z \in P_x$, $z \neq 0$, $x \neq \infty$, $e \in I$, we have in view of $(I)$:

$$ez = (e_1 e_2)\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + (e_3 e_4)\begin{pmatrix} z_3 \\ z_4 \end{pmatrix} = [(e_1 e_2)X^+ + (e_3 e_4)]\begin{pmatrix} z_3 \\ z_4 \end{pmatrix}. \tag{19}$$

This shows that $M_x = I_z + I_{j(z)}$; this is also true for $x = \infty$, $M_\infty = I_1 + I_2$.

Now it follows from Condition 4.3a) that the restrictions of $\langle\, , \rangle$ to $I_z$ and $I_{j(z)}$ are positive definite, and it follows from Condition 4.3b) that $I_z$ and $I_{j(z)}$ are orthogonal. Hence the metric on all $M_x$'s is nondegenerate, so that $\dim N_x = 4n - \dim M_x = 2n$ and $N$ is actually a vector bundle.

Notice by the way, that the rank of the metric lies between $2n$ and $4n$.

*4.5. Vector Bundle L.* Denote the rank of $\langle\, , \rangle$ by $2n + r$, $0 \leqq r \leqq 2n$. Let $K$ be the null-space of this metric in $I \otimes C^4$, of dimension $2n - r$. Obviously $K \subset N_x$ for all $x \in S^4$. We denote by $L$ the vector bundle of rank $r$ whose fibre over $x \in S^4$ is $N_x/K$. The induced metric on the fibres of $L$ is nondegenerate.

*4.6. The Connection on N.* As before we denote by $\nabla^0$ the trivial connection on $S^4 \times (I \otimes C^4)$. All vectors in $I \otimes C^4$ define $\nabla^0$-horizontal sections of this trivial bundle. We also denote by $\nabla$ the projected connection on $N$. A little care is needed in defining the projection, as $N_x$ contains the null-space of the metric. However, $M_x \cap K = \{0\}$ for all $x$, hence $\mathrm{pr}_M$ is well-defined, and we may set $\mathrm{pr}_N(\psi) = \psi - \mathrm{pr}_M(\psi)$ for any section $\psi$ of $N$. The induced metric on $N$ is $\nabla$-horizontal. In fact, if $\varphi$, $\psi$ are local sections of $N$, we have

$$\begin{aligned}
\partial_\mu \langle \varphi, \psi \rangle &= \langle \nabla^0(\partial_\mu)\varphi, \psi \rangle + \langle \varphi, \nabla^0(\partial_\mu)\psi \rangle \\
&= \langle (\mathrm{id} - \mathrm{pr}_M)\nabla^0(\partial_\mu)\varphi, \psi \rangle + \langle \varphi, (\mathrm{id} - \mathrm{pr}_M)\nabla^0(\partial_\mu)\psi \rangle \\
&= \langle \nabla_\mu \varphi, \psi \rangle + \langle \varphi, \nabla_\mu \psi \rangle .
\end{aligned}$$

**4.7. Proposition.** *The connection $\nabla$ on $N$ is self-dual.*

*Proof.* We will use Corollary 2.4 and will show that

$$[\nabla_a, \nabla_b] = [\nabla_{\bar a}, \nabla_{\bar b}] = 0$$

on $\pi^*(N)$, where, as before, $\nabla_a = \pi^*(\nabla)\left(\dfrac{\partial}{\partial z_a}\right)$, $\nabla_{\bar a} = \pi^*(\nabla)\left(\dfrac{\partial}{\partial z_a^*}\right)$; $a, b = 1, \ldots, 4$.

Take a section $\varphi$ of $\pi^*(N)$ and a vector $e \in I$. Since

$$ez \in I_z \subset I_z + I_{j(z)} = (\pi^* N)_z$$

and the metric on $N$ is horizontal, we have

$$0 = \frac{\partial}{\partial z_a^*} \langle \varphi, ez \rangle = \langle \nabla_a^0 \varphi, ez \rangle + \langle \varphi, \nabla_{\bar a}^0 (ez) \rangle = \langle \nabla_a^0 \varphi, ez \rangle .$$

Therefore, the section $\nabla_a^0 \varphi$ of $(C^4 \backslash \{0\}) \times (I \otimes C^4)$ is orthogonal to the sections of the subbundle with fibre $I_z$ over a point $z \in C^4 \backslash \{0\}$ and so is a section of the subbundle $I_{j(z)} + \pi^*(N)$. This means that

$$\nabla_a^0 \varphi = \nabla_a \varphi + \text{a section of } I_{j(z)} .$$

Now the subbundle $I_{j(z)}$ is $\nabla_b^0$-invariant. Hence

$$\nabla_b^0 \nabla_a^0 = \nabla_b^0 \nabla_a \varphi + \text{a section of } I_{j(z)}.$$

Interchanging $a$ and $b$ here and substracting the results, we obtain, using $[\nabla_a^0, \nabla_b^0] = 0$:

$$(\nabla_a^0 \nabla_b - \nabla_b^0 \nabla_a)\varphi = \text{a section of } I_{j(z)}.$$

As $I_{j(z)}$ is orthogonal to $(\pi^* N)_z$ (see 4.4) we finally obtain $[\nabla_a, \nabla_b]\varphi = 0$, as desired. A similar argument shows that $[\nabla_{\bar{a}}, \nabla_{\bar{b}}] = 0$.

*4.8. The Connection on L.* Recall that $L = N/\tilde{K}$, where $\tilde{K}$ is the constant subbundle of $N$ with fibre $K$, the metric's null-space. Obviously, $\nabla_a^0(\tilde{K}) \subset \tilde{K}$. In addition, $\mathrm{pr}_M(\tilde{K}) = 0$ so that $\nabla^0$ coincides with $\nabla$ on $\tilde{K}$. Therefore, $\nabla$ induces a connection $\nabla_L$ on the quotient bundle $L$ which is also self-dual. All of this means simply that the instanton for SU(2$n$) represented by $(N, \nabla)$ reduces to an instanton for SU($r$), represented by $(L, \nabla_L)$, where $r = 2n - \dim K = \mathrm{rk}\langle\,,\rangle - 2n$.

*4.9. Topological Charge.* Since our bundles $M$ and $N$ are explicitly imbedded into a trivial bundle, easy topological considerations allow us to calculate their Pontryagin indices. We omit this calculation, which shows that this index is $n$ for $N$ and $L$ as soon as $r \geqq 2$. So we have in fact constructed $n$-instantons. Another argument may be based on the results of the following section, from which one can deduce that $N$ can be continuously deformed into the direct sum of $n$ 1-instantons for SU(2). Since the topological charge is additive and deformation-stable, one again sees that it is equal to $n$.

## § 5. Computation of Connection Coefficients

*5.1.* In this section we fix the space $I \otimes C^4$ with hermitian metric fulfilling the conditions of 4.3 and derive explicitly the parameters of this metric and the Yang-Mills potentials $A_\mu$ in a convenient gauge.

*5.2. A Standard Basis in I.* Recall that $I \otimes C^4 = I_1 \oplus \ldots \oplus I_4$, where $I_\mu = I \otimes \varepsilon_\mu$. The Condition 4.3a for $z = \varepsilon_1$ and $\varepsilon_3$ shows that the induced metrics on $I_1$ and $I_3$ are positive definite. Identifying $I$ with $I_\mu$ via $e \mapsto e \otimes \varepsilon_\mu$, we can consider them on $I$. A classical theorem on simultaneous reduction of two hermitian forms on a space shows that there exists a basis $(e_1, \ldots, e_n)$ in $I$ relative to which the first form is given by the identity matrix and the second one by the matrix $\mathrm{diag}(\varrho_1, \ldots, \varrho_n)$, where $0 < \varrho_1 \leqq \varrho_2 \leqq \ldots \leqq \varrho_n$. The numbers $\varrho_i$ do not depend on the choice of basis. The basis itself can be changed into $(e_1, \ldots, e_n)U$ where $U$ is an arbitrary unitary matrix with $U \mathrm{diag}(\varrho_i) U^{-1} = \mathrm{diag}(\varrho_i)$. We fix once and for all such a basis $(e_i)$ and set $e_{i\mu} = e_i \otimes \varepsilon_\mu \in I_\mu$; $i = 1, \ldots, n$; $\mu = 1, \ldots, 4$. The vectors $(e_{i\mu})$ constitute a basis of $I \otimes C^4$ which we order as follows: $(e_{11}, \ldots, e_{n1}; \ldots; e_{14}, \ldots, e_{n4})$.

**5.3. Proposition.** *The Gram matrix of $(e_{i\mu})$ for the metric $\langle\,,\rangle$ is of the form*

$$Q_{4n} = \begin{vmatrix} E_{2n} & D_{2n}^+ \\ D_{2n} & R_{2n} \end{vmatrix}, \tag{20}$$

*where*

$$D_{2n}=\begin{vmatrix} B_n & C_n \\ -C_n^+ & B_n^+ \end{vmatrix}, \quad R_{2n}=\begin{vmatrix} R_n & 0 \\ 0 & R_n \end{vmatrix}, \quad R_n=\operatorname{diag}(\varrho_1,\ldots,\varrho_n) \tag{21}$$

*$B,C$ being complex $(n\times n)$-matrices. Conversely, take any matrix $Q_{4n}$ of the form* (20) *and* (21) *and suppose that $Q\geqq 0$. Then the metric $\langle\ ,\ \rangle$ defined by $Q_{4n}$ in the basis $(e_{i\mu})$ of $I\otimes C^4$ fulfills all the conditions a) and b) of n 4.3 with the possible exception of the conditions $\langle ez,ez\rangle>0$ for all $ez\neq 0$. This last requirement cuts out a certain boundary subset of* (20) *and* (21).

*Proof*. We need only check that the symmetry properties (20) and (21) are equivalent to the condition $\langle ez, fj(z)\rangle=0$. In the above notation we must have for $e=e_i$, $f=e_k$; $i,\kappa=1,\ldots,n$:

$$\langle e_{i1}z_1+e_{i2}z_2+e_{i3}z_3+e_{i4}z_4,$$
$$-e_{\kappa 1}z_2^*+e_{\kappa 2}z_1^*-e_{\kappa 3}z_4^*+e_{\kappa 4}z_3^*\rangle=0.$$

The vanishing of the $z_1^*z_2^*$, $z_1^{*2}$, and $z_2^{*2}$ coefficients means that

$$\langle e_{i1},e_{\kappa 1}\rangle=\langle e_{i2},e_{\kappa 2}\rangle; \quad \langle e_{i1},e_{\kappa 2}\rangle=0.$$

Hence, the metrics on $I$ induced from $I_1$ and $I_2$ coincide, and $I_1$ is orthogonal to $I_2$. Since in our basis the matrix of $\langle\ ,\ \rangle$ on $I_1$ is the identity, this shows that the upper left $(2n\times 2n)$-block in $Q_{4n}$ is $E_{2n}$.

Similarly, the vanishing of the $z_3^*z_4^*$, $z_3^{*2}$, and $z_4^{*2}$ coefficients means that

$$\langle e_{i3},e_{\kappa 3}\rangle=\langle e_{i4},e_{\kappa 4}\rangle; \quad \langle e_{i3},e_{\kappa 4}\rangle=0.$$

Hence, the metrics on $I$ induced from $I_3$ and $I_4$ coincide, and $I_3$ is orthogonal to $I_4$. This shows that the lower right $(2n\times 2n)$-block in $Q_{4n}$ is $R_{2n}$.

Finally, the vanishing of the $z_1^*z_3^*$ and $z_1^*z_4^*$ coefficients means that

$$\langle e_{i1},e_{\kappa 4}\rangle+\langle e_{i3},e_{\kappa 2}\rangle=0; \quad \langle e_{i1},e_{\kappa 3}\rangle=\langle e_{i4},e_{\kappa 2}\rangle.$$

Since $\langle\ ,\ \rangle$ is hermitian, the coefficients of $z_2^*z_3^*$ and $z_2^*z_4^*$ then vanish automatically, and the blocks $D,D^+$ are of the form (21). This finishes the proof.

Note, incidentally, that the change of basis $(e_1,\ldots,e_n)U$ results in the change

$$B\mapsto U^{-1}BU, \quad C\mapsto U^{-1}CU$$

in agreement with 0.2.

In what follows, we will represent the sections of all our subbundle of $S^4\times(I\otimes C^4)$ in the basis $(e_{i\mu})$ of constant sections of this bundle.

*5.4. Basis of Sections of M*. Set

$$X_{2n}^*=X^*\otimes E_n=\begin{vmatrix}(x_4-ix_3)E_n, & -(x_2+ix_1)E_n \\ (x_2-ix_1)E_n, & (x_4+ix_3)E_n\end{vmatrix}.$$

Formulae (1), (7), and 4.2 show that the rows of the matrix $|X_{2n}^*, E_{2n}|$ represent a basis of sections of $M$.

From this point on we shall omit the subscripts showing the dimension of a matrix. In particular, we will write $R,X,X^*$ etc. instead of $R_{2n},X_{2n},X_{2n}^*$.

*5.5. Gram's Matrix for the Basis of Sections of M.* It is

$$|XE|\begin{vmatrix}E & D^+\\ D & R\end{vmatrix}\begin{vmatrix}X^t\\ E\end{vmatrix}=X(X^++D^+)+DX^++R.$$

Setting $S=R-DD^+$, as in 3.6, we can rewrite it as

$$(X+D)(X^++D^+)+S. \tag{22}$$

Since the rank of $|X^*E|$ is $2n$ at any point of $R^4$ and the metric is nondegenerate on all fibres of $M$, the matrix (22) is positive definite everywhere.

*5.6. Basis of Sections of N.* As in 3.6. we represent a basis of $N$ by the rows of a matrix $|Z_1^t, Z_2^t|$, where $Z_i$ are $(2n\times 2n)$-matrices depending on $x$. The condition of orthogonality to $M$ looks formally the same as (9) and (10):

$$Z_1+(D^++Y^{-1}S)Z_2=0, \tag{23}$$

where $Y=X+D$. We will choose a basis of type (11):

$$Z_1=D^++Y^{-1}S, \quad Z_2=-E. \tag{24}$$

As in Sect. 3, this is singular at the points $x\in R^4$ where $\det(X+D)=0$. But, unlike in the case $n=1$, this set in general is not a point or even a finite set of points. In fact, it is a closed bounded subset of $R^4$ which even for $n=2$ may have dimension 3. We will call this set the *central Zone* of our instanton. Note that it can be defined in invariant geometric terms as in the case $n=1$: it consists of those points $x\in R^4$ for which $\dim(N_x\cap N_\infty^\perp)>0$. It seems that the lagrangian density of the instanton is mainly concentrated around its central Zone. See further comments in Sect. 7.

This important distinction between the case $n=1$ and $n>1$ is also consequence of the fact that $YY^+$ and $S$ are not scalar matrices anymore, except for certain degenerate cases.

*5.7. Gram's Matrix for the Basis* (24). The same calculation as in 3.7 and 3.8 shows that it is equal to

$$S(YY^+)^{-1}S+S=S((YY^+)^{-1}S+E). \tag{25}$$

The matrix $(YY^+)^{-1}S+E$ tends to $E$ as $|x|\to\infty$. Hence (25) is almost everywhere of rank $\operatorname{rk}S=r$ in the notation of 4.5.

*5.8. Null-Space.* Since $K\subset N_x$ for all $x\in R^4$, we represent elements of $K$ in a basis of sections of $N$. Then $K$, or rather sections of $\tilde{K}$, can be identified with the space of vectors annihilated by $S$ in the basis (24). In fact, (25) shows that these vectors are orthogonal to all of $N$, and so also to $N\oplus M=S^4\times(I\otimes C^4)$. On the other hand, the dimension of the kernel of $S$ is equal to $2n-r$, that is, to the dimension of $K$.

*5.9. Connection Matrices.* We noted in 4.6 that to calculate the connection matrices for $\nabla$ in the basis $|Z_1^t, Z_2^t|$, *we should first project* $\partial_\mu|Z_1^t, Z_2^t|$ onto $M$. To calculate this projection in the basis of 5.4, first of all we set

$$\mathrm{pr}_M\partial_\mu|Z_1^t, Z_2^t|=B_\mu|X^*, E|.$$

Calculating the scalar product of both parts with the basis of sections of $M$, we find, using (22):

$$B_\mu^* = [\partial_\mu Z_1^+(X^+ + D^+) + \partial_\mu Z_2^+(DX^+ + R)] \cdot [(X+D)(X^+ + D^+) + S]^{-1}. \tag{26}$$

Furthermore,

$$\begin{aligned} \mathrm{pr}_N \partial_\mu |Z_1^t, Z_2^t| &= \partial_\mu |Z_1^t, Z_2^t| - \mathrm{pr}_M \partial_\mu |Z_1^t, Z_2^t| \\ &= |\partial_\mu Z_1^t, \partial_\mu Z_2^t| - |B_\mu X^*, B_\mu| \\ &= |\partial_\mu Z_1^t - B_\mu X^*, \partial_\mu Z_2^t - B_\mu|. \end{aligned} \tag{27}$$

The rows of the last matrix represent the corresponding projections in the initial basis $(e_{i\mu})$ of constant sections of $S^4 \times (I \otimes C^4)$. On the other hand, the connection matrices $A_\mu$ are determined from the relations

$$\mathrm{pr}_N \partial_\mu |Z_1^t, Z_2^t| = A_\mu^t |Z_1^t, Z_2^t|. \tag{28}$$

Comparing the last blocks of (27) and (28) we obtain

$$\partial_\mu Z_2^t - B_\mu = A_\mu^t Z_2^t.$$

Finally, using (26) and (24):

$$\begin{aligned} A_\mu = B_\mu^t &= [(X+D)(X^+ + D^+) + S]^{-1}(X+D)\partial_\mu (X+D)^{-1} S \\ &= -[(X+D)(X^+ + D^+) + S]^{-1} \partial_\mu X \cdot (X+D)^{-1} S. \end{aligned}$$

This is the formula given in 0.3.

## 6. Why All Instantons Have Been Constructed

*6.1.* In this section we will show briefly that all SU-instantons were constructed in Sects. 4 and 5, and that gauge equivalence was properly described in 0.6. In order to do this we will rely heavily on the corresponding results announced in [3] and [5], with outlines of proofs given in [4]. So we will explain only how the presentation given here can be reduced to that given in [3].

*6.2.* In [3] the following variant of the construction of $n$-instantons for $\mathrm{SU}(r)$ was introduced. Consider a diagram of linear spaces $I \subset H \otimes C^4$, where $\dim_{\mathbb{C}} I = n$, $\dim_{\mathbb{C}} H = 2n + r$, and $H$ is endowed with a positive hermitian metric with the properties corresponding to those of 4.3. Let a point $x \in S^4$ correspond to the orthogonal complement in $H$ of the sum of the images $(\mathrm{id}_H \otimes l)I$ where $l$ runs through all linear functions on $C^4$ which vanish on $P_x$. This gives a hermitian vector bundle $L$ on $S^4$ imbedded into $S^4 \times H$, of rank $r$. The instanton connection on it is the projection of the trivial connection onto $L$. Since the sum of the images $(\mathrm{id}_H \otimes l)I$ in $H$ is of dimension $\leqq 4n$, each $n$-instanton constructed in this way can be reduced to an $\mathrm{SU}(2n)$-instanton. So we can assume $\dim H = 2n + r \leqq 4n$.

To make the transition to our present description, we let the diagram $I \subset H \otimes C^4$ correspond to the pair $\{I \otimes (C^4)^*$, hermitian metric induced by

$I \otimes (C^4)^* \to H\}$. Here $(C^4)^*$ is the dual space of our earlier $C^4$, but as we do not need the latter anymore, we write $C^4$ instead of $(C^4)^*$. Under this identification, $P_x$ corresponds to forms which vanish on $P_x$. So the two pictures are essentially the same.

Now in [4] it was shown that this construction gives all instantons in an essentially unique way, that is, gauge equivalent instantons correspond to isomorphic diagrams. But in Sect. 5 we characterized each diagram by the set of its invariants $(R, B, C)$ up to unitary equivalence. This completes our discussion.

## §7. Final Remarks

*7.1. The Problem of Superposition.* In the physical literature [9, 10] it was argued that an $n$-instanton might be considered as a sort of nonlinear superposition of $n$1-instantons for SU(2). For example, in [9] it was suggested that if we take $n$1-instantons for SU(2) in t'Hooft's gauge ($A_\mu$'s singular in the center) and then sum up these $A_\mu$'s, we will get an approximate value for the $n$-instanton $A_\mu$. This gives the right number of parameters: center scales and relative gauge orientations of 1-instantons. A similar dimension count supported by a calculation for t'Hooft's solutions was presented in [10] for an arbitrary simple gauge group.

Since we have a complete description of $n$-instantons, it would be important to see this picture in our setting. It seems reasonable to conjecture that the centers of the 1-instanton constituents of our $n$-instanton should be points in its central Zone.

Since in general the latter is not 0-dimensional, one should probably look at its singular points or points where $\mathrm{rk}|X+D| \leqq 2n-2$. Relative gauge orientations might be connected with a sort of "matrix residue" of $(X+D)^{-1}$ at these points. On the other hand, it is conceivable that a non-negligeable part of the euclidean action is also concentrated around other points of the central Zone. A. S. Schwartz has informed us about his calculations in the 2-instanton case, which seem to support this conjecture.

*7.2. Bäcklund Transforms.* The geometric picture of Sect. 4 can be used to introduce a sort of Bäcklund transform or "non-linear spectral analysis" on $n$-instantons. Namely, take a space $I \otimes C^4$ with hermitian metric $\langle\ ,\ \rangle$. Then for each subspace $I' \subset I$ we have the induced metric on $I' \otimes C^4$. If the Conditions 4.3a) and b) are fulfilled for the initial metric, they are obviously fulfilled for the induced one. This means that each $n$-instanton determines a family of $m$-instantons connected with all $m$-dimensional subspaces of $I$. Note that in general these $m$-instantons will be of rank $2m$ even if the initial instanton was of smaller rank, say two. We do not see any simple way to relate these new instantons with the initial one using only the vector bundle description or the connection coefficients, although in our construction this transform looks perfectly natural.

The case $m=1$ is especially interesting. Actually, in this case we have $n$ uniquely determined 1-instantons corresponding to the subspaces $Ce_i \subset I$, at least if $\varrho_i \neq \varrho_j$ when $i \neq j$. These $n$ instantons constitute another plausible condidate for the superposition conjecture.

7.3. *Open Problems.* Besides the superposition conjecture, we would like to mention two problems which should be accessible now.

a) To understand the geometry of the parameter space described in 0.2. In particular, do the boundary components corresponding to a given $r<2n$ form a connected set? There is a striking similarity between this problem and some geometric questions in the theory of Cartan symmetric spaces.

b) To derive conservation laws for self-dual or even complete Yang-Mills equations.

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