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Models of Gross-Neveu Type are Quantization of a Classical Mechanics with Nonlinear Phase Space*

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Abstract. It is shown that the expansion in powers of $\frac{1}{N}$ characteristic of Gross-Neveu type models is of quasiclassical nature, $\frac{1}{N}$ taking part of the Planck constant. The limity classical mechanics has curved phase space that does not admit introduction of naturally canonically conjugated coordinates.

1. Introduction

During the recent the years a number of papers appeared which were based on the following scheme.

1. One considers a field theoretical model A with the Hamiltonian $H(\varphi, \psi, \ldots) = H_0(\varphi, \psi, \ldots) + H_{\text{int}}(\varphi, \psi, \ldots)$ wherein H_0 and H_{int} are operators which may be expressed through quadratic combinations of the fields. For instance

$$\begin{aligned} H_0 &= \int \left(\pi^2 + \sum \left(\frac{\partial \varphi}{\partial x_k} \right)^2 + m^2 \varphi^2 \right) d^3x \\ H_{\text{int}} &= g \int \varphi^2(x) \varphi^2(x) d^3x \,. \end{aligned} \tag{1.1}$$

2. One considers a Hilbert space $\mathscr{H}_{A_N}$ which is the product (the graded product in the Fermi case) of N specimens of the state space $\mathscr{H}_A$ of the model A. In the space $\mathscr{H}_{A_N}$ one considers fields $\varphi_k, \psi_k, \ldots$ which are the copies of the fields $\varphi, \psi, \ldots$ of the model A. For different k's these fields commute for the Bose case and anticommute for the Fermi case. To each operator $C(\varphi, \psi, \ldots)$ in the space $\mathscr{H}_A$ which is quadratic in the fields $\varphi, \psi, \ldots$ one puts into correspondence the operator

$$C_N = \frac{1}{N} \sum_1^N C(\varphi_k, \psi_k, \ldots) \,. \tag{1.2}$$

To each operator in the space $\mathscr{H}_A$ which may be expressed as a function of quadratic operators $F = F(C^1, \ldots, C^s)$ one puts into correspondence the operator F_N in $\mathscr{H}_{A_N}$ which is the same function of the operators $C_N^k : F_N = F(C_N^1, \ldots, C_N^s)$.

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In particular, one defines the Hamiltonian H_N as the operator that corresponds in the above sense to the Hamiltonian H of the model A. We denote as A_N the model which has thus arise.

For example with the operator (1.1) the operator $H_N = H_{0,N} + H_{\text{int},N}$ is associated where

$$
\begin{aligned}
H_{0,N} &= \frac{1}{N} \int \sum_{k=1}^{N} \left(\pi_k^2 + \sum_{s=1}^{3} \left(\frac{\partial \varphi_k}{\partial x_s} \right)^2 + m^2 \varphi_k^2 \right) d^3 x \,, \\
H_{\text{int},N} &= \frac{g}{N^2} \int \left(\sum_{k=1}^{N} \varphi_k^2(x) \right)^2 d^3 x \,.
\end{aligned}
\tag{1.3}
$$

3. One investigates the asymptotical behaviour of the model A_N as $N \to \infty$, namely that of the spectrum of the Hamiltonian H_N, of the effective potential etc. Such scheme was, in all appearance, first considered by Gross and Neveu in [1].

The Hamiltonian $\tilde{H}_N$ studied in [1] differs from (1.3) by a factor: $\tilde{H}_N = N H_N$. The corresponding evolution operator is the same in the both cases:

$$
\exp(t\tilde{H}_N = \exp(tNH_N) = \exp\left(\frac{1}{\varkappa} H_N\right) \quad \text{where} \quad \varkappa = \frac{1}{N}
$$

In the present paper it is established that the limiting theory $A_\infty = \lim_{N \to \infty} A_N$ is a classical field theory.

The parameter $\varkappa = \frac{1}{N}$ plays for this limiting transition the same role as the Planck constant h does in the usual quasiclassics. The phase space of the limiting theory is not flat. In particular no natural canonically conjugate variables exists in this space. Therefore the results obtained earlier within the Gross-Neveu-type models may be referred to as a quasiclassical approximation for the original model A although with a peculiar classical limit and peculiar quantization. The finitedimensional counterpart of these constructions was studied in detail in [3] and [2].

To avoid possible misunderstanding it is worth emphasizing that the quantity $\varkappa = \frac{1}{N}$ has nothing to do with the Planck constant h which is involved into the commutation relations. The value of h is kept constant while the limiting transition $\varkappa \to 0$ is being carried out. Therefore the limiting classical mechanics retains h as a parameter. It is classical in the sense that the observables are functions on the phase space and the dynamics is given by the Poisson brackets. At the end of Section 5 we discuss the further limiting transition, when $h \to 0$, too, and compare the obtained results with the ordinary quasiclassics.

2. Manifolds F_n and B_n and Their Infinite-Dimensional Analogs

1. Definition of the Manifolds F_n and B_n

Internal Description. Let $\mathscr{H}$ be the Fock space of states for the Bose or Fermi system with n degrees of freedom. We normalize the commutation relations as follows

$$
[\hat{a}(p), \hat{a}^*(q)]_\pm = h\delta_{p,q} \,,
\tag{2.1}
$$

where h is the Planck constant.

Denote as G the group of homogeneous linear canonical transformations. The group G may be conveniently represented as a group of $2n \times 2n$ matrices of the special form $\mathscr{g} = \begin{pmatrix} \Phi & \Psi \\ \overline{\Psi} & \overline{\Phi} \end{pmatrix}$, where the bar denotes the complex conjugation. Remind that the matrix $\mathscr{g}$ of this form is a matrix for a canonical transformation iff

$$\mathscr{g}^{-1} = \begin{pmatrix} \Phi^* & \varepsilon \Psi' \\ \varepsilon \Psi^* & \Phi' \end{pmatrix}, \qquad \varepsilon = \begin{cases} -1 & \text{for Bose case} \\ +1 & \text{for Fermi case}, \end{cases} \tag{2.2}$$

$*$ is a sign for hermithean conjugation, $'$ is a sign for matrix transposition.

Denote as $U_{\mathscr{g}}$, $\mathscr{g} = \begin{pmatrix} \Phi & \Psi \\ \bar{\Psi} & \bar{\Phi} \end{pmatrix} \in G$, the unitary transformation in $\mathscr{H}$ which perform the canonical transformation by means of the matrix $\mathscr{g}$:

$$\begin{aligned} &U_{\mathscr{g}} \hat{a}(p) U_{\mathscr{g}}^{-1} = \sum (\varphi(p,q)\hat{a}(q) + \psi(p,q)\hat{a}^*(q)), \\ &\Phi = \|\varphi(p,q)\|, \quad \Psi = \|\psi(p,q)\|. \end{aligned} \tag{2.3}$$

Let Ψ_0 the vacuum vector: $\hat{a}(p)\Psi_0 = 0$ for all p. The family of vectors of the form $U_{\mathscr{g}}\Psi_0$ spans a manifold embedded into $\mathscr{H}$ which we denote as F_n for the Fermi case and as B_n for the Bose case.

Let the matrix $\Phi = \|\varphi(p,q)\|$ be inversible (for the Bose case this assumption is always fulfilled). In this case it follows from (2.3) that

$$\begin{aligned} &U_{\mathscr{g}}^{-1}\Psi_0 = c(\mathscr{g})\Psi_{\bar{z}}, \\ &\Psi_{\bar{z}} = \exp\left\{-\frac{1}{2h}\hat{a}^*\bar{z}\hat{a}^*\right\}\Psi_0, \end{aligned} \tag{2.4}$$

where $\bar{z} = \Phi^{-1}\Psi = \|\bar{z}(p,a)\|$, $\hat{a}^*\bar{z}\hat{a}^* = \sum a^*(p)\bar{z}(p,q)a^*(q)$ and $c(\mathscr{g})$ is a normalizing factor. The matrices Φ, Ψ obey the relation $\Phi\Psi' + \varepsilon\Psi\Phi' = 0$. Hence it follows that the matrix z is skewsymmetrical in the Fermi and symmetrical in the Bose case. In the Bose case it follows, besides, from (2.2) that $\Phi\Phi^* - \Psi\Psi^* = 1$, whence

$$z\bar{z} < 1 \tag{2.5}$$

(i.e. all the eigenvalues of the matrix $z\bar{z}$ are less that 1).

The groups G act on the manifolds F_n and B_n in a transitive way, the stability subgroup G_0 for the vector Ψ_0 being the group of canonical transformations, which does not mix the creation and annihilation operators. Therefore the groups G and G_0 coincide, respectively with the groups of motion and the stability subgroups of the complex symmetrical spaces which were denoted in [3] as M_n^{III} and Ω_n^{II}. M_n^{III} and Ω_n^{II} correspond to the Fermi and Bose cases, respectively: $M_n^{III} = F_n$, $\Omega_n^{II} = B_n$. The manifold F_n is compact due to the compactness of the group G in the Fermi case. Contrariwise, the manifold B_n is not compact, as it is seen from (2.5). It is sometimes called Siegel circle. The matrix elements $z(p,q)$ of the matrix z involved in Equation (2.4) are complex coordinates on B_n for the Bose case, and for the Fermi case, on the set $\tilde{F}_n$ which is obtained from F_n by removing the submanifold of a lesser dimension.

Consider the automorphism in the group G, given as

$$\mathscr{g} = \begin{pmatrix} \Phi & \Psi \\ \bar{\Psi} & \bar{\Phi} \end{pmatrix} \to \mathscr{g}^{\sigma} = \begin{pmatrix} \bar{\Phi} & -\bar{\Psi} \\ -\Psi & \Phi \end{pmatrix}. \tag{2.6}$$

It follows from the definition of the vector $\Psi_{\bar{z}}$ that

$$U_{g^\sigma}\Psi_{\bar{z}}=c(g,z)\Psi_{\overline{gz}}\,, \tag{2.7}$$

where $c(g,z)$ is a normalizing factor,

$$gz=(\Phi z+\Psi)(\bar{\Psi}z+\bar{\Phi})^{-1}\,. \tag{2.8}$$

2. External Description of the Manifolds F_n and B_n

Note that the vector $\Psi_{\bar{z}}$ belong to the even subspace $\mathscr{H}''$ of the Fock space. It will be shown in Section 4 that they form the so called generalized overcomplete family.

Let $\hat{R}$ be an arbitrary operator in $\mathscr{H}''$. Following the general theory of overcomplete familier we put the covariant symbol

$$R(z,\bar{z})=\frac{(\hat{R}\Psi_{\bar{z}},\Psi_{\bar{z}})}{(\Psi_{\bar{z}},\Psi_{\bar{z}})}$$

into correspondence with $\hat{R}$. Consider for $\hat{R}$ the operators

$$\begin{aligned}&\hat{A}(p,q)=\hat{a}^*(p)\hat{a}^*(q)\,,\qquad \hat{A}^*(p,q)=\hat{a}(q)\hat{a}(p)\,,\\&\hat{B}(p,q)=\hat{a}^*(p)\hat{a}(q)\,.\end{aligned} \tag{2.9}$$

Some simple computation shows that their covariant symbols are resp. equal to $A(p,q)$, $\bar{A}(p,q)$, $B(p,q)$, where $A(p,q)$, $B(p,q)$ are the elements of the matrices

$$\begin{aligned}&A=-hz(1-\bar{z}z)^{-1}\\&B=hz(1-\bar{z}z)^{-1}\bar{z}\,.\end{aligned} \tag{2.10}$$

Note that the matrices A,B obey the relations

$$\begin{aligned}&\left(B-\frac{h}{2}\right)^2-A\bar{A}=\left(\frac{h}{2}\right)^2\,,\qquad BA=A\bar{B}\,,\\&B=B^*\,,\qquad A'=-\varepsilon A\,,\end{aligned} \tag{2.11}$$

where ε is the same as in (2.2). In the Bose case, besides, the inequality

$$B\geqq 0 \tag{2.12}$$

holds.

The inverse statement is also true: if matrices A,B satisfy Equations (2.11) and A^{-1} exists then there exists such a matrix $z, z+\varepsilon z'=0$ in terms of which the matrices A,B may be expressed as (2.10). To show this we put $z=-\bar{A}^{-1}\bar{B}$. It follows from the second, the third and the fourth equations in (2.11) that $z+\varepsilon z'=0$. By combining the first and the third equation of (2.10) we find that $0=BB^*-B^*h-A\bar{A}=A\bar{z}z\bar{A}-A\bar{A}-z\bar{A}h$, hence $A=-hz(1-\bar{z}z)^{-1}$, $B=-A\bar{z}=hz(1-\bar{z}z)^{-1}\bar{z}$.

In the Bose case (2.5) follows from (2.12), besides.

Denote as $\mathscr{G}$ the Lie algebra of the group G. The elements of $\mathscr{G}$ have the form

$$i\begin{pmatrix}C & A\\ -\bar{A} & -\bar{C}\end{pmatrix}, \tag{2.13}$$

where A is a complex and C is a Hermithean matrix, $A+\varepsilon A'=0$. By substituting here $C=B-\frac{h}{2}$ and expressing A, B in terms of $z, \bar{z}$ by means of (2.10) we get the embedding of the manifolds F_n and B_n into $\mathscr{G}$. With this embedding the image of F_n and B_n depends on h and we shall designate it by $F_n(h)$ and $B_n(h)$, resp. Let $ix=ix(z,\bar{z})\in F_n(h)$ or $ix\in B_n(h)$. By obvious transformations one establishes that

$$\mathscr{g}x(z,\bar{z})\mathscr{g}^{-1}=x(\mathscr{g}z,\overline{\mathscr{g}z})\,, \tag{2.14}$$

where $\mathscr{g}=\begin{pmatrix}\Phi & \Psi\\ \bar{\Psi} & \bar{\Phi}\end{pmatrix}\in G$, and $\mathscr{g}z$ is defined by Equation (2.8).

Therefore the manifolds $F_n(h)$ and $B_n(h)$ are orbits of the adjoint representation. Relations (2.10) may be rewritten as

$$(x(z,\bar{z}))^2=\left(\frac{h}{2}\right)^2 I\,, \tag{2.15}$$

where I is the unit matrix.

Note, that in the Fermi case the matrix x is Hermitean. Therefore it follows from (2.15) that at $h=0$ the manifold $F_n(h)$ degenerates down to a point, while this is not true in the Bose case; one can easily see that the manifold $B_n(0)$ lies on the boundary of the Siegel circle. [The unitary matrices z, $z\bar{z}=1$, correspond to $B_n(0)$.] We shall see below that this difference is due to the fact that in the Bose case the quantum dynamics turns into classical one when $h\to 0$ whereas in the Fermi case the $h\to 0$ limit does not exist in the usual sense. (The formal limiting transition $h\to 0$ leads in the Fermi case to the Grassmanian classical mechanics. See [6] for details.)

3. Infinite-Dimensional Analogs of the Spaces F_n and B_n

For infinite number of degrees of freedom we denote the manifolds analogous to F_n and B_n as F and B, resp. They are defined literally in the same way as F_n and B_n. One must near in mind, however, that canonical transformation g must be proper. This leads to the fact that the matrix z, apart from the properties described above, must be the matrix of a Hilbert-Schmidt operator

$$\operatorname{sp} zz^* < \infty\ . \tag{2.16}$$

Apart from the manifolds F and B it is also useful, in the case of infinite number of degrees of freedom to consider wider manifolds $\tilde{F}$ and $\tilde{B}$ which are parametrized by the matrizes of bounded operators. The manifolds $\tilde{F}$ and $\tilde{B}$ consist of the limiting points for the manifolds F and B respectively in the sense of strong operator topology. For the manifolds F, B, $\tilde{F}$, $\tilde{B}$ the description by means of the matrices A, B matrices A, B which obey relations (2.11), (2.16) is also valid. In the case of the manifolds F, B the matrices A, B are those of Hilbert-Schmidt operators, while for $\tilde{F}, \tilde{B}$ they are matrices of bounded operators.

I hope to demonstrate in the future publications that the manifolds $\tilde{F}, \tilde{B}$ are useful for studying improper canonical transformations and the renormalizations associated with them.

3. Classical Mechanics on the Manifolds F_n, B_n, F, B

1. General Properties of the Kähler Manifolds

The manifolds F_n and B_n belong to the so called Kähler manifolds. Remind that a manifold M is called Kähler manifold if it possesses a complex structure, is Riemannian, and the Riemann metrics has the following special form in the local coordinates

$$ds^2 = \sum \frac{\partial^2 f}{\partial z^\alpha \partial \bar{z}^\beta} dz^\alpha d\bar{z}^\beta , \tag{3.1}$$

where $f = f(z, \bar{z})$ is a local function called the Kähler potential.

The metrics (3.1) is invariant under a transformation g of manifold M if the potential satisfies the condition

$$f(gz, \overline{gz}) = f(z, \bar{z}) + \alpha(g, z) + \overline{\alpha(g, z)} , \tag{3.2}$$

where $\alpha(g, z)$ is an analytic function of z^α (i.e. it does not depend on $\bar{z}^\alpha$).

There always exists the Poisson bracket on the Kähler manifold. It is given as

$$[f_1, f_2] = i \sum g^{\alpha\dot{\beta}} \left(\frac{\partial f_1}{\partial z^\alpha} \frac{\partial f_2}{\partial \bar{z}^\beta} - \frac{\partial f_1}{\partial \bar{z}^\alpha} \frac{\partial f_2}{\partial z^\beta} \right) , \tag{3.3}$$

where $g^{\alpha\dot{\beta}}$ is the matrix inverse to the matrix

$$\|g_{\dot{\beta}\alpha}\| = \left\| \frac{\partial f}{\partial \bar{z}^\beta \partial z^\alpha} \right\| ; \quad \sum g^{\alpha\dot{\gamma}} g_{\dot{\gamma}\beta} = \delta^\alpha_\beta .$$

The Laplace-Beltrami operator on the Kähler manifold has the form

$$\Delta = \sum g^{\alpha\dot{\beta}} \frac{\partial^2}{\partial z^\alpha \partial \bar{z}^\beta} , \tag{3.4}$$

where $\|g^{\alpha\dot{\beta}}\|$ is the same matrix as in (3.3).

2. Metrics and Poisson Bracket on F_n and B_n

Let $z \in F_n$ or $z \in B_n$. Consider the function

$$F(z, \bar{z}) = \det(1 - z\bar{z}) . \tag{3.5}$$

It follows from (2.8) that

$$F(gz, \overline{gz}) = F(z, \bar{z}) \alpha(g, z) \overline{\alpha(g, z)} , \tag{3.6}$$

where $\alpha(g, z) = \det(\bar{\Phi} + \bar{\Psi} z)^{-1}$.

Consequently $f(z, \bar{z}) = \frac{1}{2} \varepsilon \ln F(z, \bar{z})$ is the Kähler potential of an invariant metrics with any $\varepsilon = \text{const}$. We fix ε the same as in (2.2).

The general expression (3.1) may be written in the form

$$ds^2 = \frac{\partial^2}{\partial t \partial \bar{t}} f(z + t dz, \bar{z} + \bar{t} d\bar{z})|_{t = \bar{t} = 0} . \tag{3.7}$$

Hence

$$ds^2 = -\frac{\varepsilon}{2}\frac{\partial^2}{\partial t\partial\bar{t}}\,\mathrm{sp}\sum_n \frac{1}{n}\,[(z+t\,dz)(\bar{z}+\bar{t}\,d\bar{z})]^n|_{t=\bar{t}=0}$$

$$= -\frac{\varepsilon}{2}\sum_{k,l}\mathrm{sp}\,dz(\bar{z}z)^k d\bar{z}(z\bar{z})^l\ .$$

Finally one has

$$ds^2 = -\frac{\varepsilon}{2}\,\mathrm{sp}(dz(1-\bar{z}z)^{-1}d\bar{z}(1-z\bar{z})^{-1})\ . \tag{3.8}$$

Introduce the scalar product $(x,y)=\mathrm{sp}(xy^*)$ in the linear space of the $n\times n$ matrices. In terms of this scalar product Equation (3.8) may be rewritten in the form

$$ds^2=(dz,A\,dz)\,,\quad A\,dz=\tfrac{1}{2}c\,dz\bar{c}\,,\quad C=(1-z\bar{z})^{-1}\ . \tag{3.9}$$

To find the Laplace-Beltrami operator and the Poisson bracket one should inverse the operator A in the space of matrices. It is evident that $A^{-1}\xi=2c^{-1}\xi\bar{c}^{-1}$. Both the operators A and A^{-1} leave the subspaces of symmetrical and skew symmetrical matrices invariant. Therefore, in accord with the general formulae (3.3) and (3.4) the Laplace-Beltrami operators and the Poisson bracket on the manifolds F_n and B_n have the form

$$\Delta = -2\varepsilon\,\mathrm{sp}\left[\frac{\partial}{\partial\bar{z}}(1-\bar{z}z)\frac{\partial}{\partial z}(1-z\bar{z})\right]\,, \tag{3.10}$$

$$[f_1,f_2]=2\frac{\varepsilon}{i}\,\mathrm{sp}\left[\left(\frac{\partial}{\partial\bar{z}}f_1\right)(1-\bar{z}z)\left(\frac{\partial}{\partial z}f_2\right)(1-z\bar{z})\right.$$
$$\left.-\frac{\partial}{\partial\bar{z}}f_2(1-\bar{z}z)\left(\frac{\partial}{\partial z}f_1\right)(1-z\bar{z})\right]\,, \tag{3.11}$$

where the operators $\frac{\partial}{\partial z}$, $\frac{\partial}{\partial\bar{z}}$ in the Fermi and Bose cases are, respectively

$$\frac{\partial}{\partial z}=\frac{1}{2}\begin{pmatrix} 0 & \frac{\partial}{\partial z_{12}}\cdots\frac{\partial}{\partial z_{1n}} \\ \cdots\cdots & \cdots\cdots \\ -\frac{\partial}{\partial z_{1n}} & -\frac{\partial}{\partial z_{2n}}\cdots 0 \end{pmatrix}\,,$$

$$\frac{\partial}{\partial z}=\begin{pmatrix} \frac{\partial}{\partial z_{11}} & \frac{1}{2}\frac{\partial}{\partial z_{12}}\cdots\frac{1}{2}\frac{\partial}{\partial z_{1n}} \\ \cdots\cdots & \cdots\cdots \\ \frac{1}{2}\frac{\partial}{\partial z_{1n}} & \frac{1}{2}\frac{\partial}{\partial z_{2n}}\cdots\frac{\partial}{\partial z_{nn}} \end{pmatrix}\,,$$

$\frac{\partial}{\partial\bar{z}}$ are complex conjugate.

[Here we abandon for a time being the notation $z(p, q)$ and come back to the more customary way of writing z_{ij}.]

The operators $\frac{\partial}{\partial z_{ij}}, \frac{\partial}{\partial \bar{z}_{ij}}$ in (3.10) are meant not to act on the functions embraced in the square bracket. [Equation (3.10) may be found in [7].)

Let $e_{\mu\nu}$ be the matrix whose elements are $(e_{\mu\nu})_{pq}=\frac{1}{2}(\delta_{\mu p}\delta_{\nu q}-\varepsilon\delta_{\mu q}\delta_{\nu p})$. Note that both in the Fermi and Bose cases $(e_{\mu\nu})_{pq}=(e_{pq})_{\mu\nu}$, $\frac{\partial}{\partial z}z_{pq}=(e_{\mu\nu})_{pq}$. Besides, if A is an arbitrary matrix subject to the condition $A+\varepsilon A'=0$ then $\mathrm{sp}(e_{\mu\nu}A)=A_{\nu\mu}=\varepsilon A_{\mu\nu}$.

Therefore the dynamical equations with the Poisson bracket (3.11) and the Hamiltonian function H may be written in the matrix form

$$\begin{aligned}\frac{dz}{dt}&=\frac{2}{i}(1-z\bar{z})\frac{\partial H}{\partial \bar{z}}(1-\bar{z}z)\\ \frac{d\bar{z}}{dt}&=-\frac{2}{i}(1-\bar{z}z)\frac{\partial H}{\partial z}(1-z\bar{z})\,.\end{aligned}\tag{3.12}$$

3. General Information About the Algebra of Poisson Brackets Induced by an Arbitrary Lie Algebra

Let $\mathscr{G}$ be the Lie algebra of an arbitrary Lie group G, e_i be a basis in $\mathscr{G}$ and c_{ij}^k be the corresponding structural constants. Let $\mathscr{G}'$ be the space of linear forms on $\mathscr{G}$, e'^i be the basis in $\mathscr{G}'$ which is byorthogonal for e_i, i.e. if $a=\sum a^i e_i \in \mathscr{G}$, and $x=\sum x_i e'^i \in \mathscr{G}'$ than

$$\langle a, x\rangle=\sum a^i x_i\,. \tag{3.13}$$

Let us define the bracket

$$[f_1, f_2]_p=\sum c_{ij}^k \frac{\partial f_1}{\partial x_i}\frac{\partial f_2}{\partial x_j} \tag{3.14}$$

for any two smooth functions f_1, f_2 on $\mathscr{G}'$. The known properties of the structural constants c_{ij}^k result in the properties of the bracket (3.14) which allow to call it the Poisson bracket. These are the antisymmetricity and the Jacobi identity [11]:

$$\begin{aligned}&[f_1, f_2]=-[f_2, f_1]\,,\\ &[f_1,[f_2, f_3]]+[f_3,[f_1, f_2]]+[f_2,[f_3, f_1]]=0\,.\end{aligned}$$

The bracket (3.14) is more convenient to be rewritten without the use of coordinates. To this end define $\operatorname{grad} f(x) \in \mathscr{G}$ by the relation

$$\frac{d}{dt}f(x+tg)|_{t=0}=\langle \operatorname{grad} f(x), y\rangle\,. \tag{3.15}$$

The scalar product in the right-hand side is defined by Equation (3.13). One can easily see that Equation (3.14) may be written as

$$[f_1, f_2]_p=\langle [\operatorname{grad} f_1, \operatorname{grad} f_2], x\rangle\,, \tag{3.16}$$

where the scalar product in the right-hand side is the same as in (3.13), $[\operatorname{grad} f_1, \operatorname{grad} f_2]$ is the commutator in $\mathscr{G}$.

In the space $\mathscr{G}'$ the representation of the algebra $\mathscr{G}$ acts which is given by the matrices

$$[\mathrm{ad}'(y)]^i_k = \sum y^j c^i_{jk}, \qquad y = \sum y^i e_i \in \mathscr{G} . \tag{3.17}$$

This representation is called codjoint. With it the coadjoint representation $\mathrm{A'd}(g)$ of the group G is associated. Let $P(x)$ be an invariant of the coadjoint representation of $G: P(\mathrm{A'd}(g)x) = P(x)$. The corresponding infinitesimal condition has due to (3.17) the form

$$\sum_{i,k} \frac{\partial P}{\partial x_i} c^k_{ij} x_k = 0 . \tag{3.18}$$

By comparing it with (3.14) one finds that

$$[P, f]_P = 0 \tag{3.19}$$

for any f.

Consequently, P is invariant under any dynamics induced by the Poisson bracket (3.14). The surfaces $P(x) = \text{const}$ are invariant with any dynamics. One can show that the equation of any closed, typical orbit of the coadjoint representation may be written in the form $P(x) = \text{const}$, where $P(x)$ are smooth functions invariant under the coadjoint representation.

The above facts acquire a new colouring in the case when in the Lie algebra there exists a nondegenerate invariant scalar product. Denote this product as (a, b). With its aid we identify $\mathscr{G}$ with $\mathscr{G}'$ in the following way. With each $x \in \mathscr{G}'$ we associate $\tilde{x} \in \mathscr{G}$ which is uniquely determined by the equation $\langle x, a \rangle = (\tilde{x}, a)$ for any $a \in \mathscr{G}$.

This identification enables us to build the Poisson bracket algebra of the functions on the algebra $\mathscr{G}$ itself

$$[f_1, f_2]_P = ([\operatorname{grad} f_1, \operatorname{grad} f_2], x) , \qquad x \in \mathscr{G} , \tag{3.20}$$

where grad f is defined analogously to (3.15):

$$\frac{d}{dt} f(x + ty)|_{t=0} = (\operatorname{grad} f, y) . \tag{3.21}$$

The condition of invariance of the scalar product has the form

$$([c, a], b) + (a, [c, b]) = 0$$

with its aid we transform (3.20) as

$$[f_1, f_2]_P = -(\operatorname{grad} f_2, [\operatorname{grad} f_1, x]) . \tag{3.22}$$

Put $f_1(x) = H(x)$, $f_2(x) = x^i$. Note that $(\operatorname{grad} x^i, a) = a^i$ due to (3.21). Therefore $[x^i, H]_P = [\operatorname{grad} H, x]^i$. By multiplying the both sides of these equations by e_i and summing, we conclude that the equation of the Hamiltonian dynamics in the case under consideration reduces to the form

$$\frac{dx}{dt} = [H, x]_P = [\operatorname{grad} H, x] . \tag{3.23}$$

Equation (3.23) has the specific Lax form. The role of the L, A pair is played by the elements x and $\operatorname{grad} H$ of the algebra $\mathscr{G}$.

Many important equations of the mathematical physics are reduced to the form (3.23). To this class obviously belong the Euler equations for the motion of a solid body with a fastened point (here $\mathscr{G}$ is the algebra of real skew-symmetrical matrices of the third order). Elsewhere we shall show that the famous Korteweg-de-Vries equation also reduces to the (3.23) form, $\mathscr{G}$ being an infinite-dimensional analog of the real simplectic algebra.

In the next subsection we show that the dynamical equations the manifolds we are interested in reduce to the form (3.23).

4. Poisson Brackets on the Manifolds F_n and B_n. Another Description

Let $x = \mathrm{i}\begin{pmatrix} C & A \\ -\bar{A} & -\bar{C}\end{pmatrix} \in \mathscr{G}$, where $\mathscr{G}$ is the Lie algebra of the group of Fermi or Bose linear canonical transformations. (C is Hermitean matrix for both the cases, while A is skew-symmetrical in the Fermi case and symmetrical in the Bose one.) There exists the invariant scalar product in $\mathscr{G}$:

$$(x, y) = \tfrac{1}{2}\operatorname{sp}(xy)\,. \tag{3.24}$$

Let $H = H(A, \bar{A}, C, \bar{C})$ be some function on $\mathscr{G}$.

Due to (3.24) one has

$$\operatorname{grad} H = -2i\begin{pmatrix} \dfrac{\partial H}{\partial C} & -\dfrac{\partial H}{\partial \bar{A}} \\ \dfrac{\partial H}{\partial A} & -\dfrac{\partial H}{\partial \bar{C}} \end{pmatrix}, \tag{3.25}$$

where

$$\left(\frac{\partial H}{\partial C}\right)(p, q) = \frac{\partial H}{\partial C(q, p)}\,, \qquad \left(\frac{\partial H}{\partial A}\right)(p, q) = \frac{\partial H}{\partial A(q, p)}\,.$$

Thus the equations of dynamics have in the present case the form

$$i\frac{d}{dt}\begin{pmatrix} C & A \\ -\bar{A} & -\bar{C}\end{pmatrix} = 2\left[\begin{pmatrix} \dfrac{\partial H}{\partial C} & -\dfrac{\partial H}{\partial \bar{A}} \\ \dfrac{\partial H}{\partial A} & -\dfrac{\partial H}{\partial \bar{C}} \end{pmatrix}, \begin{pmatrix} C & A \\ -\bar{A} & -\bar{C}\end{pmatrix}\right]. \tag{3.26}$$

Now we show that the manifolds $F_n(h)$, $B_n(h)$ are invariant under the dynamics (3.26). (Function H is assumed to be real.) Let the matrices $A(t)$, $C(t)$ satisfy Equation (3.26). Consider the matrices

$$K(t) = C^2 - A\bar{A} - \varepsilon\frac{h^2}{4}, \quad L(t) = AC' - CA\,, \tag{3.27}$$

where ε is the same as in (2.2). We shall show that if $K(0)=L(0)=0$ then $K(t)\equiv L(t)\equiv 0$. With the help of (3.26) we get

$$\frac{i}{2}\frac{dK}{dt}=\left[K,\frac{\partial H}{\partial C}\right]+\frac{\partial H}{\partial \bar{A}}\bar{L}-L\frac{\partial H}{\partial A}$$

$$\frac{i}{2}\frac{dL}{dt}=\frac{\partial H}{\partial C}L+L\frac{\partial H}{d\bar{C}}+\frac{\partial H}{\partial \bar{A}}\bar{K}-K\frac{\partial H}{\partial \bar{A}}\,.$$

Therefore elements of the matrices $K(t)$, $L(t)$ satisfy a linear homogeneous set of differential equations. The statement we need follows from the theorem on the existence and uniqueness for such sets.

Now we show that the dynamics (3.26) coincides with the dynamics (3.12) on the manifolds $F_n(h)$, $B_n(h)$. Multiply the first equation in (3.12) by $\bar{z}(1-z\bar{z})^{-1}$ on the right and by $(1-z\bar{z})^{-1}$ on the left, while the second one by $(1-z\bar{z})^{-1}z$ and $(1-z\bar{z})^{-1}$, respectively. After subtracting them one from another we get

$$\begin{aligned}\frac{d}{dt}(1-z\bar{z})^{-1}&=\frac{1}{h}\frac{d}{dt}(h+B)\\ &=\frac{1}{h}\frac{dc}{dt}=\frac{2}{i}\left(\frac{\partial H}{\partial \bar{z}}\bar{z}-z\frac{\partial H}{\partial z}\right).\end{aligned} \tag{3.28}$$

Multiply Equation (3.28) by z on the right and the first equation in (3.12) by $(1-z\bar{z})^{-1}$ on the left. By adding the relations thus obtained we are led to

$$\frac{d}{dt}\left[(1-z\bar{z})^{-1}z\right]=-\frac{1}{h}\frac{dA}{dt}=\frac{2}{i}\left(\frac{\partial H}{\partial \bar{z}}-z\frac{\partial H}{\partial z}z\right). \tag{3.29}$$

Note now that

$$\frac{\partial H}{\partial z_{\mu\nu}}=\operatorname{sp}\left(\frac{\partial H}{\partial A}\frac{\partial A}{\partial z_{\mu\nu}}+\frac{\partial H}{\partial \bar{A}}\frac{\partial \bar{A}}{\partial z_{\mu\nu}}+\frac{\partial H}{\partial C}\frac{\partial C}{\partial z_{\mu\nu}}+\frac{\partial H}{\partial \bar{C}}\frac{\partial \bar{C}}{\partial z_{\mu\nu}}\right). \tag{3.30}$$

Note farther that $\dfrac{\partial z}{\partial z_{\mu\nu}}=e_{\mu\nu}$, where $(e_{\mu\nu})_{pq}=\frac{1}{2}(\delta_{\mu p}\delta_{\nu q}-\varepsilon\delta_{\mu q}\delta_{\nu p})$. Hence

$$\begin{aligned}\frac{\partial A}{\partial z_{\mu\nu}}&=-h(e_{\mu\nu}(1-\bar{z}z)^{-1}+z(1-\bar{z}z)^{-1}\bar{z}e_{\mu\nu}(1-\bar{z}z)^{-1})\\ &=-[e_{\mu\nu}(h+\bar{B})+Be_{\mu\nu}(1+h^{-1}\bar{B})]\,,\end{aligned}$$

$$\frac{\partial \bar{A}}{\partial z_{\mu\nu}}=-h\bar{z}(1-z\bar{z})^{-1}e_{\mu\nu}\bar{z}(1-z\bar{z})^{-1}=-h^{-1}\bar{A}e_{\mu\nu}\bar{A}\,,$$

$$\begin{aligned}\frac{\partial B}{\partial z_{\mu\nu}}&=h[e_{\mu\nu}(1-\bar{z}z)^{-1}\bar{z}+z(1-\bar{z}z)^{-1}\bar{z}e_{\mu\nu}(1-\bar{z}z)^{-1}\bar{z}]\\ &=-e_{\mu\nu}\bar{A}-h^{-1}Be_{\mu\nu}\bar{A}\,,\end{aligned}$$

$$\begin{aligned}\frac{\partial \bar{B}}{\partial z_{\mu\nu}}&=h[\bar{z}(1-z\bar{z})^{-1}e_{\mu\nu}\bar{z}(1-z\bar{z})^{-1}z+\bar{z}(1-z\bar{z})^{-1}e_{\mu\nu}]\\ &=-h^{-1}\bar{A}e_{\mu\nu}\bar{B}-\bar{A}e_{\mu\nu}\,.\end{aligned}$$

Note, that $\mathrm{sp}(Xe_{\mu\nu})=X_{\nu\mu}$ for any matrix X with appropriate symmetry condition. Hence

$$\frac{\partial H}{\partial z_{\mu\nu}}=-\left[(h+\bar{B})\frac{\partial H}{\partial A}+(1+h^{-1}\bar{B})\frac{\partial H}{\partial A}B+h^{-1}\bar{A}\frac{\partial H}{\partial \bar{A}}\bar{A}\right.$$
$$\left.+\bar{A}\frac{\partial H}{\partial B}+h^{-1}\bar{A}\frac{\partial H}{\partial B}B+\frac{\partial H}{\partial \bar{B}}\bar{A}+h^{-1}\bar{B}\frac{\partial H}{\partial \bar{B}}\bar{A}\right]_{\nu\mu}$$

or, what is the same

$$\frac{\partial H}{\partial z}=-\left[(1+h^{-1}\bar{B})\frac{\partial H}{\partial A}(h+B)+h^{-1}\bar{A}\frac{\partial H}{\partial \bar{A}}\bar{A}\right.$$
$$\left.+\bar{A}\frac{\partial H}{\partial B}(1+h^{-1}B)+(1+h^{-1}B)\frac{\partial H}{\partial \bar{B}}\bar{A}\right]. \tag{3.31}$$

By going over to the complex conjugated quantities we find analogous expression for $\dfrac{\partial H}{\partial \bar{z}}$.

Now we use the identities which follow from (2.10):

$$z\bar{A}=A\bar{z}=-B, \qquad \bar{z}(1+h^{-1}B)=(1+h^{-1}\bar{B})\bar{z}=-h^{-1}\bar{A}.$$

The substitution of (3.31) into (3.28) after some obvious transformations results in

$$\begin{aligned}\frac{1}{h}\frac{\partial C}{\partial t}&=-\frac{2}{i}\left(\frac{\partial H}{\partial \bar{A}}\bar{A}+\frac{\partial H}{\partial C}C-A\frac{\partial H}{\partial A}-C\frac{\partial H}{\partial C}\right)\\ \frac{1}{h}\frac{\partial A}{\partial t}&=\frac{2}{i}\left(A\frac{\partial H}{\partial \bar{C}}+\frac{\partial H}{\partial C}A+C\frac{\partial H}{\partial \bar{A}}+\frac{\partial H}{\partial \bar{A}}\bar{C}\right).\end{aligned} \tag{3.32}$$

Equations (3.32) coincide with (3.26) if H in (3.26) is replaced by hH.

Thus the dynamics (3.26) in $\mathscr{G}$ with the Hamiltonian function $H(A,\bar{A},B,\bar{B})$ induces on the manifolds $F_n(h)$, $B_n(h)$ the dynamics (3.12) with the Hamiltonian function

$$H_1(z,\bar{z})=\frac{1}{h}H(A(z,\bar{z}),\bar{A}(z,\bar{z}),B(z,\bar{z}),\bar{B}(z,\bar{z})).$$

All the results of this section are extended without changes to the infinite dimensional manifolds F and B.

4. Quantization of Classical Mechanics on the Manifolds F_n, B_n, F, B

1. Spaces $\mathscr{F}_\varkappa(M_n)$

Let M_n be either F_n or B_n, $\tilde{M}_n\subset M_n$ be the subset in which elements $z(p,q)$ of the matrix z serve as a coordinate system. (Remind that in the Fermi case $M_n\backslash\tilde{M}_n$ is a submanifold of the dimension lesser than M_n, while in the Bose case $\tilde{M}_n=M_n$.)

Denote as $\mathscr{F}_\varkappa(M_n)$ the Hilbert space of analytical functions on $\tilde{M}_n$ with the scalar product

$$(f_1, f_2) = c_n(\varkappa) \int f_1(z) f_2(\bar{z}) \det(1 - z\bar{z})^{-\frac{\varepsilon}{2\varkappa}} d\mu_n(z, \bar{z}) , \tag{4.1}$$

where ε has the same sense as in (2.2),

$$d\mu_n(z, \bar{z}) = \frac{1}{\det(1 - z\bar{z})^{n-\varepsilon}} \prod \frac{dz_{ij} d\bar{z}_{ij}}{2\pi} \tag{4.2}$$

is the invariant measure on M_n and the factor $c_n(\varkappa)$ is fixed by the condition $(f_0, f_0) = 1$, where $f_0(z) \equiv 1$. Due to [3, 7] one has in the Bose case

$$c_n(\varkappa) = 2^{-n} \frac{\Gamma\left(\frac{1}{\varkappa} - 1\right) \Gamma\left(\frac{1}{\varkappa} - 2\right) \dots \Gamma\left(\frac{1}{\varkappa} - n\right)}{\Gamma\left(\frac{1}{\varkappa} - 2\right) \Gamma\left(\frac{1}{\varkappa} - 4\right) \dots \Gamma\left(\frac{1}{\varkappa} - 2n\right)} \tag{4.3}$$

and in the Fermi case

$$c_{n+1}(\varkappa) = \frac{\Gamma\left(\frac{1}{\varkappa} + n + 1\right) \Gamma\left(\frac{1}{\varkappa} + n + 2\right) \dots \Gamma\left(\frac{1}{\varkappa} + 2n\right)}{\Gamma\left(\frac{1}{\varkappa} + 1\right) \Gamma\left(\frac{1}{\varkappa} + 3\right) \dots \Gamma\left(\frac{1}{\varkappa} + 2n - 1\right)} . \tag{4.4}$$

In the Bose case the integral in (4.1) converges at $\frac{1}{\varkappa} \geqq 2(n+1)$ while at $\varkappa > \frac{1}{2(n+1)}$ it is understood as the analytical continuation. The integral (4.1) understood in this way defines a nonnegative scalar product if $\varkappa$ belongs to the set formed by a segment (into which the domain of convergence is included) and the separate points located to the right of it [3], [13]:

$$0 < \varkappa \leqq \frac{1}{n-1}, \quad \varkappa = \frac{1}{k}, \quad k = 0, 1, 2, \dots, n-1 . \tag{4.5}$$

In the Fermi case integral (4.1) is of interest for the theory of quantization at

$$\varkappa = \frac{1}{k}, \qquad k = 0, 1, 2, \dots . \tag{4.6}$$

The values (4.5) and (4.6) of $\varkappa$ we call *admissible* and will deal only with them in what follows. Consider the vectors

$$\Phi_{\bar{v}}(z) = \det(1 - z\bar{v})^{\frac{\varepsilon}{2\varkappa}} \tag{4.7}$$

in the space $\mathscr{F}_\varkappa(M_n)$. It was shown in [3] that for any $f \in \mathscr{F}_\varkappa(M_n)$ the following equality holds

$$(f, \Phi_{\bar{z}}) = f(z) . \tag{4.8}$$

Equality (4.8) indicates that the vectors $\Phi_{\bar{z}}$ for $\varkappa \leqq \frac{1}{2(n+1)}$ in the Bose case and for all admissible $\varkappa$'s in the Fermi case form the over-complete family of states in

Klauder's sense [4]. In the case when the scalar product (f_1, f_2) is not given by an integral over a measure, it is natural to call the set of vectors which possess the property (4.8) the *generalized overcomplete family* of states. The families of the sort were met earlier in [8].

Thus the vectors (4.7) in the Bose case for $\varkappa = \frac{1}{k}$, $k = 0, 1, \ldots, n-1$ form the generalized overcomplete family of states.

The parameter $\varkappa$ is related to the parameter h used in [2, 3] as $h = 2(n+\varepsilon)\varkappa$. $\varkappa$ is more convenient as far as extension of the results of [2, 3] to infinite dimensional manifolds is concerned.

2. *Algebra of Covariant Symbols*

Let $\hat{A}$ be an operator in the space $\mathscr{F}_\varkappa(M_n)$, whose domain includes the vectors $\Phi_{\bar{z}}$. The expectation value of $\hat{A}$ on $\Phi_{\bar{z}}$

$$A(z, \bar{z}) = \frac{(\hat{A}\Phi_{\bar{z}}, \Phi_{\bar{z}})}{(\Phi_{\bar{z}}, \Phi_{\bar{z}})} \tag{4.9}$$

is called the covariant symbol of the operator $\hat{A}$ [5]. $A(, \bar{z})$ admits the apparent analytical continuation $A(z, v)$ as holomorphic function on the product $\tilde{M}_n \times \tilde{M}_{\bar{n}}$.

It is shown in [2, 3] that the operator $\hat{A}$ is uniquely restored once its covariant symbol is given, i.e.

$$(\hat{A}f)(z) = c_n(\varkappa) \int A(z, \bar{v}) f(v) \left[\frac{\det(1 - z\bar{v})}{\det(1 - v\bar{v})}\right]^{\frac{\varepsilon}{2\varkappa}} d\mu_n(v, \bar{v}) \,. \tag{4.10}$$

If $\hat{A} = \hat{A}_1 \hat{A}_2$, the corresponding covariant symbols are connected by the relation

$$\begin{aligned} A(z, \bar{z}) &= (A_1 * A_2)(z, \bar{z}) \\ &= c_n(\varkappa) \int A_1(z, \bar{v}) A_2(v, \bar{z}) \left(\frac{\det[(1 - z\bar{v})(1 - \bar{z}v)]}{\det[(1 - z\bar{z})(1 - v\bar{v})]}\right)^{\frac{\varepsilon}{2\varkappa}} d\mu_n(v, \bar{v}) \,. \end{aligned} \tag{4.11}$$

In the Bose case for $\varkappa > \dfrac{1}{2(n+1)}$ Equations (4.10) and (4.11) should be referred to as analytical continuations in $\varkappa$. Denote as $\mathscr{A}_\varkappa$ the set of covariant symbols of such operators in $\mathscr{F}_\varkappa(\mathrm{M}_n)$ whose powers are defined on the vectors $\Phi_{\bar{z}}$. It is evident that $\mathscr{A}_\varkappa$ forms an associative algebra with the multiplication law (4.11). The multiplication (4.11) was shown in [2, 3] to have the property

$$\begin{aligned} &\lim_{\varkappa \to 0} A_1 * A_2 = A_1 A_2 \\ &\lim_{\varkappa \to 0} \frac{1}{\varkappa}(A_1 * A_2 - A_2 * A_1) = \frac{1}{i}[A_1, A_2] \,, \end{aligned} \tag{4.12}$$

where $A_1 A_2$ is the ordinary product of functions and $[A_1, A_2]$ is the Poisson bracket (3.11). Thus the family of the associative algebras $\mathscr{A}_\varkappa$ forms, in the sense of [2] the special quantization of the classical mechanics on M_n described in Section 3.

3. Infinite Number of Degrees of Freedom

Let M stand for any of the two infinite-dimensional manifolds F or B. Denote as E the set of quantum number of the particles forming the system under consideration. Denote the measure on E as dp, $p \in E$ and the Hilbert space of square integrable functions on E with the measure dp as $L^2(E)$. The matrices z in terms of which the original description of the manifold M was given are naturally thought of as the matrices of operators in $L^2(E)$, $z(p,q) = \langle p|z|q\rangle$,

$$(zf)(p) = \int z(p,q) f(q) dq \ .$$

There exists a natural involution in $L^2(E)$ which coinsides with the complex conjugation: $f^*(p) = \overline{f(p)}$. With its aid the operations of complex conjugation and transposition in the space of operators are defined: $(zf^*)^* = \bar{z}f$, $z' = \bar{z}^*$, where the asterisk * stands for Hermitean conjugation when applied to operators. As applied to matrix elements these operations look like usual

$$\langle p|z'|q\rangle = \langle q|z|p\rangle, \qquad \langle p|\bar{z}|q\rangle = \overline{\langle p|z|q\rangle} \ .$$

On the manifold M the group G of the proper canonical transformations acts transitively, although no invariant measure with respect to G exists. The basic formulae (4.1), (4.10), and (4.11) are transferred onto M as follows. A function $f(z)$ on M we call analytical if all the functions $\varphi(v)$ are analytical, where $v = PzP$, $\varphi(v) = f(PzP)$, P is the orthogonal projector onto a finite-dimensional subspace, invariant under the involution. Set

$$(f,f) = \sup c_n(\varkappa) \int |f(PzP)|^2 \det(1 - PzP\bar{z}P)^{-\frac{\varepsilon}{2\varkappa}} d\mu_n(z,\bar{z}) \ . \tag{4.13}$$

Here P is the orthogonal projector onto a finite-dimensional subspace $L \subset L(E)$, invariant under the involution, the term sup. refers to all such subspaces, $n = \dim L$ and $c_n(\varkappa)$, $d\mu_n(z,\bar{z})$ are the same as in Equations (4.2)–(4.4) (z_{ik} are matrix elements of the operator PzP in an orthonormal basis in L). Denote the set of analytical functions, such that $(f,f) < \infty$ as $\mathscr{F}_\varkappa(M)$. It is easy to show that $\mathscr{F}_\varkappa(M)$ is the Hilbert space with the scalar product (4.13).

Let $L_1 \subset L_2 \subset \ldots \subset L^2(E)$ be a sequence of subspaces which are invariant under the involution and $\cup L_k$ be dense in $L^2(E)$. The scalar product in $\mathscr{F}_\varkappa(M)$ may be given as

$$(f_1, f_2) = \lim_{n\to\infty} c_n(\varkappa) \int f_1(z_n)\overline{f_2(z_n)} \det(1 - z_n\bar{z}_n)^{-\frac{\varepsilon}{2\varkappa}} d\mu_n(z,\bar{z}) \ , \tag{4.14}$$

where $z_n = P_n z P_n$, P_n is the orthogonal projector onto L_n.

In the Bose case for $n+1 > \dfrac{1}{\varkappa}$ the integrals in (4.13) and (4.14) are referred to as analytical continuation. Thus, both in the Bose and the Fermi cases the spaces $\mathscr{F}_\varkappa(M)$, $\dim M = \infty$ exist only for the values of $\varkappa$ indicated in Equation (4.6).

Let $\Phi_{\bar{z}}(v) \in \mathscr{F}_\varkappa(M)$ be of the form (4.7). It may be easily shown that the functions of the form

$$f(z) = \sum c_k \Phi_{\bar{z}_k}(z) \ , \tag{4.15}$$

where the sum contains finite number of terms, make up a dense set in $\mathscr{F}_\varkappa(M)$ and that Equation (4.8) remains valid in $\mathscr{F}_\varkappa(M)$. (See the proof of analogous statement in [8].) Equations (4.10) and (4.11) are modified analogously:

$$(\hat{A}f)(z) = \lim_{n\to\infty} c_n(\varkappa) \int A(z_n, \bar{v}_n) f(v_n) \left(\frac{\det(1-z_n\bar{v}_n)}{\det(1-v_n\bar{v}_n)}\right)^{\frac{\varepsilon}{2\varkappa}} d\mu_n(v,\bar{v}) \,,$$

$$(A_1 * A_2)(z,\bar{z}) \tag{4.16}$$

$$= \lim_{n\to\infty} c_n(\varkappa) \int A_1(z_n, \bar{v}_n) A_2(v_n, \bar{z}_n) \left(\frac{\det[(1-z_n\bar{v}_n)(1-v_n\bar{z}_n)]}{\det[(1-z_n\bar{z}_n)(1-v_n\bar{v}_n)]}\right)^{\frac{\varepsilon}{2\varkappa}} d\mu_n(v,\bar{v}) \,.$$

Equations (4.12) remain valid.

Further details concerning the problems touched here may be found in [9].

5. Duplicated Spaces—Statistical Quasiclassics

1. *General Definition*

Consider a fermion or boson system A, whose space of states is $\mathscr{H}_A$ and creation annihilation operators are $\hat{a}^*(p)$ and $\hat{a}(p)$, resp., where $p \in E$, E is a complete set of quantum numbers. We shall refer to system A as a *standard*. Denote as A_N, with N being integer, a system that consists of N subsystems, each being a duplicate of the system A. Denote the space of states of the system A_N as $\mathscr{H}_{A_N}$ and the creation and annihilation operators in $\mathscr{H}_{A_N}$ as $b_k^*(p)$ and $b_k(p)$, resp., where p's are the same quantum numbers as in the system A, $k = 1, 2, \ldots, N$. The commutation relations between $b_k^*(p)$, $b_k(p)$ are of Fermi or Bose type depending on what type of system the system A is:

$$[\hat{b}_k(p), \hat{b}_{k'}^*(p')]_\pm = h\delta_{kk'}\delta_{pp'} \,. \tag{5.1}$$

We shall refer to the system A_N as the *duplicated system A* and to the space $\mathscr{H}_{A_N}$ as the *duplicated space* $\mathscr{H}_A$.

Throughout the present section we do not consider separately the case of finite number of degrees of freedom. In what follows the letter M stands for any of the four manifolds F_n, B_n, F or B. In the case of infinite number of degrees of freedom the quantum numbers are as usually thought of as points of some set E with a measure, $\delta_{pp'}$ in (5.1) standing for the Dirac δ-function with this measure, $\hat{b}_k(p)$, $\hat{b}_k^*(p)$ are, generally, operator distributions and not operators.

2. *Subspaces* $\tilde{\mathscr{H}}_{A_N}$

Set

$$\Psi_{\bar{z}} = e^{-\frac{1}{2}\sum_1^N \hat{b}_k^* \bar{z} \hat{b}_k^*} \Psi_0 \,, \tag{5.2}$$

where Ψ_0 is the vacuum vector in $\mathscr{H}_{A_N}$, z is the operator in $L^2(E)$ (E is the set of quantum number), $\hat{b}_k^* z \hat{b}_k^* = \int \hat{b}_k^*(p)\overline{z(p,q)}\hat{b}_k^*(q)dpdq$. Denote as $\tilde{\mathscr{H}}_{A_N}$ the subspace generated by the vectors of the form

$$\Psi = \sum c_k \Psi_{\bar{z}_k} . \tag{5.3}$$

(The summation is over a finite range.)

Now we show that the space $\tilde{\mathscr{H}}_{A_N}$ is naturally isomorphic to the space $\mathscr{F}_\varkappa(M)$, $\varkappa = \frac{1}{N}$ described in Section 4.

Associate the analytical function $f(z)$ on M

$$f(z) = (\Psi, \Psi_{\bar{z}}) \tag{5.4}$$

to each vector Ψ of the form (5.3). In accordance with [10] one has

$$(\Psi_{\bar{z}}, \Psi_{\bar{v}}) = \int e^{-\frac{1}{2h}\sum(b_k^* \bar{z} b_k^* + b_k v' b_k) - \frac{1}{h}\sum b_k^* b_k} \prod \frac{db_k^* db_k}{h^{-\varepsilon}} ,$$

where ε is the same as in (2.2) and the integration over the anticommuting variables is meant for the Fermi case. (We considered only the $h=I$ case in [10]. The general case is reduced to this in an obvious way.)

By calculating the Gaussian integral in the usual way we find that

$$(\Psi_{\bar{v}}, \Psi_{\bar{z}}) = \det(1 - z\bar{v})^{\varepsilon \frac{N}{2}} = \Phi_{\bar{v}}(z) , \tag{5.5}$$

where $\Phi_{\bar{v}}$ is the same function as in (4.7), $\varkappa = \frac{1}{N}$. Consequently the function

$$f(z) = \sum c_k \Phi_{\bar{z}_k}(z) \tag{5.6}$$

is the image of the element (5.3) under the mapping (5.4). With the use of (5.5) one further concludes that if Ψ has the form (5.3) the following relation takes place

$$(\Psi, \Psi) = \sum c_k \bar{c}_e \Phi_{\bar{z}_k}(z_e) . \tag{5.7}$$

On the other hand Equation (4.8) implies that the right-hand side of (5.7) coincide with the scalar product of f with itself as an element of $\mathscr{F}_\varkappa(M)$, $\varkappa = \frac{1}{N}$. Since the vector (5.3) form a dense set in $\tilde{\mathscr{H}}_{A_N}$, while the vectors (5.6) do this in $\mathscr{F}_\varkappa(M)$ it follows that the mapping (5.4) may be extended up to isomorphism between $\tilde{\mathscr{H}}_{A_N}$ and $\mathscr{F}_\varkappa(M)$.

3. Admissible Operators

Consider the operator distributions in the standard space $\mathscr{H}_A$

$$\hat{A}(p,q) = \hat{a}^*(p)\hat{a}^*(q) , \qquad \hat{B}(p,q) = \hat{a}^*(p)\hat{a}(q) . \tag{5.8}$$

Associate the operator distributions $\hat{A}_N(p,q)$, $\hat{B}_N(p,q)$ in the space $\mathscr{H}_{A_N}$ to them:

$$\begin{aligned} \hat{A}_N(p,q) &= \frac{1}{N}\sum_1^N \hat{b}_k^*(p)\hat{b}_k^*(q) \\ \hat{B}_N(p,q) &= \frac{1}{N}\sum_1^N \hat{b}_k^*(p)\hat{b}_k(q)\,. \end{aligned} \tag{5.9}$$

The space $\tilde{\mathscr{H}}_{A_N}$ is easily seen to be invariant under $\hat{B}_N(p,q)$, $\hat{A}_N(p,q)$, $\hat{A}_N^*(p,q)$. Consequently it is also invariant under the ring of operators created by $\hat{A}_N(p,q)$, $\hat{A}_N^*(p,q)$, $\hat{B}_N(p,q)$. Call this algebra $\mathscr{L}_N$. The elements of the algebra $\mathscr{L}_N$ will be called *admissible operators* below.

Let $\hat{H}$ be an operator in the standard space $\mathscr{H}_A$ which is a function of $\hat{A}(p,q)$, $\hat{A}^*(p,q)$, $\hat{B}(p,q)$,

$$\hat{H} = \sum \int C_{s_1,\dots,s_n}(x_1,\dots,x_n)\hat{K}_{s_1}(x_1)\dots\hat{K}_{s_n}(x_n)d^nx\,, \tag{5.10}$$

where for brevity we put $x_i=(p_i,q_i)$, $s_i=1,2,3$, $\hat{K}_1(x)=\hat{A}(p,q)$, $\hat{K}_2(x)=\hat{A}^*(p,q)$, $\hat{K}_3(x)=\hat{B}(p,q)$.

Due to noncommutativity of $\hat{A}$, $\hat{A}^*$, $\hat{B}$ a given operator is not uniquely represented in the form (5.10). However, once some fixed form is adopted one may associate with H a set of operators H_N in the spaces $\mathscr{H}_{A_N}$ by means of the following formula

$$\hat{H}_N = \sum \int C_{s_1,\dots,s_n}(x_1,\dots,x_n)\hat{K}_{s_1,N}(x_1)\dots\hat{K}_{s_n,N}(x_n)d^nx\,, \tag{5.11}$$

where $\hat{K}_{1,N}=\hat{A}_N$, $\hat{K}_{2,N}=\hat{A}_N^*$, $\hat{K}_{3,N}=\hat{B}_N$.

The operators $\hat{H}_N$ are admissible. Consequently the space $\tilde{\mathscr{H}}_{A_N}$ is invariant under them.

4. Covariant Symbols of Admissible Operators

Associate with each admissible operator $\hat{H}_N$ its expectation value over the vector $\Psi_{\bar{z}}$:

$$H_N(z,\bar{z}) = \frac{(\hat{H}_N\Psi_{\bar{z}},\Psi_{\bar{z}})}{(\Psi_{\bar{z}},\Psi_{\bar{z}})}\,. \tag{5.12}$$

The function $H_N(z,\bar{z})$ is the covariant symbol of the restriction of the operator $\hat{H}_N$ onto $\tilde{\mathscr{H}}_{A_N}$[1]. It is obvious that the symbols of $\hat{A}_N(p,q)$, $\hat{A}_N^*(p,q)$, $\hat{B}_N(p,q)$ do not depend on N and are thus given by Equations (2.10).

It follows, further, from (5.11) and (4.12) that, within the wide range of assumptions about $\hat{H}$ the relation

$$H_N(z,\bar{z}) = H(z,\bar{z}) + \frac{1}{N}\tilde{H}_N(z,\bar{z}) \tag{5.13}$$

holds, where $\tilde{H}_N(z,\bar{z})$ has a limit at $N\to\infty$.

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1 The family of states $\Psi_{\bar{z}}$ is overcomplete (not generalized overcomplete) in the sense of the classic Klauder's definition in the Fermi case if the number of degrees of freedom is finite and in the Bose case if, besides, N is large enough. In all the other cases it is the generalized overcomplete family (see Section 4)

5. *Statistical Quasiclassics*

The Heisenberg equations in the space $\tilde{\mathscr{H}}_{A_N}$ are

$$\frac{h}{iN}\frac{d\hat{A}_N}{dt}=[\hat{H}_N,\hat{A}_N], \qquad \frac{h}{iN}\frac{d\hat{B}_N}{dt}=[\hat{H}_N,\hat{B}_N]\,. \tag{5.14}$$

On passing from the operators to the symbols and with the use of (4.12) we find that these equations are equivalent to the following

$$h\frac{dA}{dt}=[H,A]_P+\frac{1}{N}Q_N, \qquad h\frac{dB}{dt}=[H,B]_P+\frac{1}{N}R_N\,, \tag{5.15}$$

where $[\cdot,\cdot]_P$ is the Poisson bracket (3.11). In the limit $N\to\infty$ Equation (5.15) converge to the equations of the classical mechanics on the manifold M with the Hamiltonian function $h^{-1}H(z,\bar{z})$. One may consider, instead, the equivalent equations in the Lie algebra of the group of the proper canonical transformations. The consideration presented at the end of Section 3 shows that these equations have the form (3.26). The Planck constant h drops out of the equations and survives only in relations (2.11).

Thus, the limiting transition $N\to\infty$ is an ordinary quansiclassical one. The role of the Planck constant is played by the quantity $\varkappa=\frac{1}{N}$. Since the basic quantities (5.9) with whose aid the limiting transition is constructed resemble the analogous quantities known in the statistical physics it is natural to refer to this quasiclassics as *statistical.*

6. *Connection Between the Statistical Quasiclassics and the Usual One*

Set $h=0$. Equations (2.11) for the Bose case admit for $h=0$ the solution $A(p,q)=a^*(p)a^*(q)$, $B(p,q)=a^*(p)a(q)$. For the Fermi case Equations (2.11) have only zero solution. Nevertheless let us in the Fermi case also set $A(p,q)=a^*(p)a^*(q)$, $B(p,q)=a^*(p)a(q)$ and treat $a(p)$, $a^*(p)$ not as ordinary functions but as anticommuting ones. With this convention the symmetry between the Bose and Fermi cases is restored.

Let $H_1(a^*,a)=H(A,\bar{A},B,\bar{B})$, $A(p,q)=a^*(p)a^*(q)$, $\bar{A}(p,q)=a(p)a(q)$, $B(p,q)=a^*(p)a(q)$, $\bar{B}(p,q)=a(p)a^*(q)$. Let, further, $a(p,t)$, $a^*(p,t)$ be a solution of the classical equations:

$$\begin{aligned}\frac{da(p)}{dt}&=[H_1,a(p)]=i\frac{\vec{\partial}}{\partial a^*(p)}H_1\\ \frac{\partial a^*(p)}{\partial t}&=[H_1,a^*(p)]=-iH_1\frac{\overleftarrow{\partial}}{\partial a(p)}\,.\end{aligned} \tag{5.16}$$

The designations $\overleftarrow{\partial}$, $\vec{\partial}$ relate to the Fermi case. They imply the right and left derivatives, respectively. Let us show that the functions $A(p,q|t)=a^*(p,t)a^*(q,t)$,

$B(p,q|t)=a^*(p,t)a(q,t)$ satisfy the classical Equations (3.26). Indeed

$$\frac{d}{dt}A(p,q)=\frac{da^*(p)}{dt}a^*(q)+a^*(p)\frac{da^*(q)}{dt}$$

$$=-i\left[\left(H_1\frac{\overleftarrow{\partial}}{\partial a(p)}\right)a^*(q)+a^*(p)\left(H_1\frac{\overleftarrow{\partial}}{\partial a(q)}\right)\right]$$

$$=-i\int\left[\frac{\partial H}{\partial\bar{A}(p',q')}\left(\bar{A}(p',q')\frac{\overleftarrow{\partial}}{\partial a(p)}\right)+\frac{\partial H}{\partial\bar{B}(p',q')}\left(\bar{B}(p',q')\frac{\overleftarrow{\partial}}{\partial a(p)}\right)\right.$$

$$\left.+\frac{\partial H}{\partial B(p',q')}\left(B(p',q')\frac{\overleftarrow{\partial}}{\partial a(p)}\right)\right]a^*(q)dp'dq'$$

$$-ia^*(p)\int\left[\frac{\partial H}{\partial\bar{A}(p',q')}\left(\bar{A}(p',q')\frac{\overleftarrow{\partial}}{\partial a(q)}\right)+\frac{\partial H}{\partial\bar{B}(p',q')}\left(\bar{B}(p'q')\frac{\overleftarrow{\partial}}{\partial a(q)}\right)\right.$$

$$\left.+\frac{\partial H}{\partial B(p',q')}\left(B(p',q')\frac{\overleftarrow{\partial}}{\partial a(q)}\right)\right]dp'dq'$$

$$=-2i\int\left(\frac{\partial H}{\partial\bar{A}(q',p)}\bar{B}(q',q)+\frac{\partial H}{\partial B(q',p)}A(q',q)\right.$$

$$\left.+B(p,q')\frac{\partial H}{\partial\bar{A}(q,q')}+A(p,q')\frac{\partial H}{\partial\bar{B}(q,q')}\right)dq'\ .$$

We thus get finally

$$\frac{d}{dt}A=-2i\left(\frac{\partial H}{\partial\bar{A}}B+\frac{\partial H}{\partial B}A+B\frac{\partial H}{\partial\bar{A}}+A\frac{\partial H}{\partial\bar{B}}\right). \tag{5.17}$$

Analogously

$$\frac{d}{dt}B=-2i\left(\frac{\partial H}{\partial B}B+\frac{\partial H}{\partial\bar{A}}\bar{A}-B\frac{\partial H}{\partial B}-A\frac{\partial H}{\partial A}\right). \tag{5.18}$$

[In the process of transformations the properties of the functions A, B have been taken into account which result from their special form: $A(p,q)=-\varepsilon A(q,p)$, $B(p,q)=-\varepsilon\bar{B}(q,p)$.]

It is evident that Equations (5.17) and (5.18) are equivalent to (3.26).

6. Concluding Remarks

1. Continual Integral

Let us first consider finite number of degrees of freedom. Let $\hat{H}(t)$ be a family of operators in the space $\mathscr{F}_\varkappa$. Their covariant symbols are $H(z,\bar{z}|t)$. Denote as $\hat{G}(t)$ the evolution operator

$$\frac{\varkappa}{i}\frac{d\hat{G}}{dt}=\hat{H}(t)\hat{G}\,,\qquad \hat{G}(0)=1\ .$$

Following the method suggested in [12] we find the expression for the covariant symbol $G(z,\bar{z}|t)$ of the operator $\hat{G}(t)$ in terms of a continual integral. Set

$$\hat{G}_N=\hat{U}(t_1,t_2)\hat{U}(t_2,t_3)\ldots\hat{U}(t_{N-1},t_N), \qquad t_k=\frac{k}{N}t\,,$$

where $\hat{U}(t,s)$ is the operator in $\mathscr{F}_\varkappa$ whose covariant symbol is

$$U(z,\bar{z}|t,s)=\exp\left(\frac{i}{\varkappa}\int_t^s H(z,\bar{z}|\tau)d\tau\right).$$

The multiple use of Equation (4.11) results in the following expression for the covariant symbol $G_N(z,\bar{z}|t)$ of the operator $\hat{G}_N$:

$$G_N(z,\bar{z}|t)=\int\exp\left(\frac{i}{\varkappa}F_N\right)\prod d\sigma(z_k,\bar{z}_k)\,,$$

$$\begin{aligned}F_N=&\sum_0^{N-1}\int_{t_k}^{t_{k+1}}H(z_k,\bar{z}_{k+1}|\tau)d\tau\\&+\frac{\varepsilon}{i}\sum_0^{N-1}\operatorname{sp}[\ln(1-z_k\bar{z}_{k+1})-\ln(1-z_k\bar{z}_k)]\\&+\frac{\varepsilon}{i}\operatorname{sp}[\ln(1-\bar{z}_0z)-\ln(1-z\bar{z})]\,,\end{aligned}$$

where $z_0=z$, $\bar{z}_N=\bar{z}$, $d\sigma(z,\bar{z})=c_n(\varkappa)d\mu_n(z,\bar{z})$. Let now $z_k=z(t_k)$, where $z(t)$ is a differentiable curve, $t_k=\frac{k}{N}t$. Set $z(t_{k+1})=z(t_k)+\Delta_k$. Up to the first power of Δ_k we have

$$\begin{aligned}&\operatorname{sp}[\ln(1-z_k\bar{z}_{k+1})-\ln(1-z_{k+1}\bar{z}_{k+1})]\\&=\sum\frac{1}{n}\operatorname{sp}[(z_k\bar{z}_k)^n-(z_k(\bar{z}_k+\bar{\Delta}_k))^n]\\&=-\operatorname{sp}[z_k\bar{\Delta}_k(1-z_k\bar{z}_k)^{-1}]\,.\end{aligned}$$

Put $\Delta_k=\dot{z}(t_k)\Delta t_k$, $\bar{\Delta}_k=\dot{\bar{z}}(t_k)\Delta t_k$, $\Delta t_k=\frac{1}{N}t$, and perform the formal limiting transition $N\to\infty$ for the exponent in Equation (6.1). This results in the following final expression for $G(z,\bar{z}|t)$:

$$\begin{aligned}&G(z,\bar{z}|t)=\int\exp\left(\frac{i}{\varkappa}F\right)\prod_{0<\tau<t}d\sigma(z(\tau),\bar{z}(\tau))\\&F=\int_0^t L(z(\tau),\bar{z}(\tau),\dot{z}(\tau),\dot{\bar{z}}(\tau)|\tau)d\tau\\&\quad+\frac{\varepsilon}{i}\operatorname{sp}[\ln(1-\bar{z}(0)z)-\ln(1-z\bar{z})]\,,\end{aligned}\tag{6.2}$$

where

$$L(z, \bar{z}, \dot{z}, \dot{\bar{z}}|\tau) = H(z(\tau), \bar{z}(\tau+0)|\tau) + \frac{\varepsilon}{i}\,\mathrm{sp}[z(\tau)\dot{\bar{z}}(\tau)(1-z(\tau)\bar{z}(\tau))^{-1}], \tag{6.3}$$

$$z(0) = z, \qquad \bar{z}(t) = \bar{z}.$$

The argument of $\bar{z}(\tau+0)$ in (6.3) implies that the extra limiting transition is meant in the continual integral (6.2), namely: one should first calculate $G^{\alpha}(z, \bar{z}|t)$, where $G^{\alpha}(z, \bar{z}|t)$ is given by the formulae to be obtained from (6.2) and (6.3) by the substitution of $H(z(\tau), \bar{z}(\tau)|\tau)$ for $H(z(\tau), \bar{z}(\tau+\alpha)|\tau)$, $\alpha > 0$. After that one should make α tend to zero, $G(z, \bar{z}|t) = \lim_{\alpha\to 0} G^{\alpha}(z, \bar{z}|t)$. We shall show elsewhere that this prócedure implies a certain regularization of the continual integral (6.2).

2. *Dimension of the Space $\tilde{\mathscr{H}}_{A_N}$ in the Fermi Case*

The Hamiltonian of the model A_N is originally defined in the space $\mathscr{H}_{A_N}$ which is wider than $\tilde{\mathscr{H}}_{A_N}$. The question arises on whether we loose any significant information when passing from $\mathscr{H}_{A_N}$ to $\tilde{\mathscr{H}}_{A_N}$. The answer to this question is to some extent given by comparison of the dimensions of these spaces for the Fermi case with finite number of degrees of freedom. Let the model A have $n+1$ degrees of freedom. In such a case

$$\dim \mathscr{H}_A = 2^{n+1}, \qquad \dim \mathscr{H}_{A_N} = 2^{N(n+1)}.$$

Due to the general relation established in [2] $\dim \tilde{\mathscr{H}}_{A_N} = \dfrac{c_n(\varkappa)}{c_n(\infty)}$. Therefore

$$\dim \tilde{\mathscr{H}}_{A_N} = \frac{\Gamma(N+n+1)\dots\Gamma(N+2n)\cdot\Gamma(1)\Gamma(3)\dots\Gamma(2n-1)}{\Gamma(N+1)\dots\Gamma(N+2n-1)\Gamma(n+1)\Gamma(n+2)\dots\Gamma(2n)}. \tag{6.4}$$

Note, that

$$\frac{\Gamma(N+n+1)}{\Gamma(n+1)} = (n+1)\dots(n+N),$$

$$\frac{\Gamma(N+2n-1)}{\Gamma(2n-1)} = (2n-1)(2n)\dots(2n-2+N).$$

Analogous transformations of the other ration in (6.4) along with permutations of the multipliers reduces (6.4) to the form

$$\dim \tilde{\mathscr{H}}_{A_N} = \prod_1^N T_k, \quad T_k = \frac{(n+k)(n+k+1)\dots(2n+k-1)}{k(k+2)\dots(k+2n-2)}. \tag{6.5}$$

Further on

$$T_{2s} = \frac{(n+2s)\dots(2n+2s-1)}{2^n s(s+1)\dots(s+n-1)} = 2^{-n}\frac{(2n+2s-1)!(s-1)!}{(n+2s-1)!(s+n-1)!},$$

$$T_{2s+1} = \frac{(n+2s+1)\dots(2n+2s)\cdot(2s+2)(2s+4)\dots(2s+2n)}{(2s+1)\dots(2s+2n-1)\cdot(2s+2)(2s+4)\dots(2s+2n)}$$

$$= 2^n\frac{(2s)!(s+n)!}{s!(2s+n)!}.$$

Application of the Stirling formula enables us to find the asymptotic value of T_k, when $n \to \infty : T_{2s} \sim 2^{n+2s} n^{-s+1/2}(s-1)!$, $T_{2s+1} \sim 2^n(n+1)^{-s}$. Hence

$$\ln \dim \tilde{\mathscr{H}}_{A_N} = nN \ln 2 - c \ln 2n + O(1) . \tag{6.6}$$

Thus within the leading logarithmic term one concludes that $\dim \mathscr{H}_{A_N} = \dim \tilde{\mathscr{H}}_{A_N}$. This enables one to hope that the passing from the space $\mathscr{H}_{A_N}$ to $\tilde{\mathscr{H}}_{A_N}$ in the case of a large number of degrees of freedom does not yield any significant loss of information.

References

1. Gross-Neveu, D.: Phys. Rev. D **10**, 3235 (1974)
2. Berezin, F.A.: Izv. Akad. Nauk SSR, Ser. Mat. **38**, 1116–1174 (1974); English translation: Math. USSR Izv. **8**, 1109–1165 (1974)
3. Berezin, F.A.: Izv. Akad. Nauk SSR, Ser. Mat. **39**, 363–402 (1975); English translation: Math. USSR Izv. **9**, 341–379 (1975)
4. Klauder, J.R.: J. Math. Phys. **4**, 1055 (1963)
5. Berezin, F.A.: Izv. Akad. Nauk SSR, Ser. Mat. **36**, 1134–1167 (1972); English translation: Math. USSR Izv. **6**, 1117–1151 (1972)
6. Berezin, F.A., Marinov, M.S.: Dynamics of a particle with half-integer spin as the Grassman version of classical mechanics. Preprint ITEP-43 (1976)
7. Hua Loo Keng: Harmonic analysis of the classical domain in the studi of analytic functions of several complex variables. Moskow: Inostrannaja Literatura 1959 (in Russian)
8. Berezin, F. A.: Rep. Math. Phys. **9**, 15–30 (1976)
9. Shereshevskiy, I. A.: Vestn. MGU (1977)
10. Berezin, F. A.: Metod vtorichnogo kvantovaniya. Moscow: Nauka 1965; English translation: Berezin, F. A.: The method of second quantization. New York-London: Academic Press 1966
11. Berezin, F. A.: Funct. Anal. Appl. **1**, 1—14 (1967) (in Russian)
12. Berezin, F. A.: Theor. Math. Phys. **6**, 194–211 (1971) (in Russian)
13. Gindikin, S. G.: Funct. Anal. Appl. **9**, 59—61 (1975) (in Russian)

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