

The $u=0$ Structure Theorem

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Abstract. The previous theorem of the author on the analytic structure of the bubble diagram functions that occur in unitary equations (and are kernels of products of connected scattering operators $S_{m,n}^c$ or $(S^{-1})_{m,n}^c$ and related quantities), is extended to a class of situations, called here in general $u=0$ points, that were not covered by this earlier result.

This new theorem, which is proved on the basis of a refined macrocausality condition, resolves one of the remaining crucial problems in the derivation of discontinuity formulae and related results in S -matrix theory: all points are in fact $u=0$ points for some of the bubble diagram functions, such as $\equiv \oplus \ominus \equiv$ ($\equiv (S^{-1})_{3,3}^c, S_{3,3}^c$), that are encountered even in the simplest cases. In all previous approaches, ad hoc technical assumptions with no a priori physical basis were required for these terms.

The origin of the $u=0$ problem is the absence of information, in general, on a product of distributions that are boundary values of analytic functions from opposite directions, and more generally on the essential support, or singular spectrum, of a product of distributions whose essential supports contain opposite directions. On the other hand, the recent results obtained by Kashiwara-Kawai-Stapp in the framework of hyperfunction theory apply mainly to phase-space factors, whose bubbles are constants times conservation δ -functions rather than actual scattering operators. The present work has basically required the development of new physical and mathematical ideas and methods. In particular, a new general result on the essential support of a product of bounded operators is presented in $u=0$ situations, under a general regularity property on individual terms. The latter follows in the application from the refined macrocausality condition, in the same time as information on the essential support of S -matrix kernels.

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1. Introduction

Discontinuity formulae and related results play an important role in our understanding of the structure of multiparticle scattering functions, and in further applications in S -matrix theory [1]. These formulae were first established in the sixties and the beginning of the seventies, on the basis of a general algebraic analysis of unitary equations [2, 3]. However, their derivation in [2, 3] makes use of several crucial technical assumptions (e.g. the patching assumption and mixed- α cancellation assumption) that have no a priori physical basis¹.

The recent mathematical developments of essential support theory [4], first carried out in connection with the detailed study [5] of the macro-causality condition [6], have made possible a more satisfactory and powerful analysis of these problems, and have led to a number of more refined results [7–9]. The present work, which is a development of [7], is concerned with a basic preliminary part in this domain of research, namely the study of the analytic structure of the “bubble diagram functions”. These functions occur in equations derived from unitarity and the decomposition of the S matrix into its connected components. They are, as will be explained in detail later, integrals over internal on-mass-shell four-momenta of products of connected momentum-space kernels of the S -matrix, or of $S^{-1} = S^\dagger$. These kernels are represented in a usual diagrammatical notation by plus and minus bubbles respectively. Some of the results of the present work apply also to cases, encountered in some applications [8, 9], where the bubbles are more general kernels of bounded operators, or distributions.

¹ These assumptions are not mentioned in [3]. This is due to the fact that, while important aspects of the problems were clearly analysed there, others were ignored. As a consequence, some of the proofs, as they stand, are incomplete and in fact not correct

Early results on the analytic properties of bubble diagram functions were obtained in [2] and [3] from analyticity properties assumed on the S matrix, i.e. on the individual bubbles, by using distortions of contours in the space of complexified variables associated with the internal momentum variables over which there is integration, after elimination of mass-shell and energy-momentum conservation δ -functions. This method presents, however, several difficulties which strongly weaken the value of the results of these works, even in some of the simplest cases. A first difficulty is due to the fact that the scattering functions associated with the individual bubbles (after factorization of their energy-momentum conservation δ -functions) cannot, in the multiparticle case, be expected to be boundary values of analytic functions everywhere. The representation of a scattering function as the boundary value of a single analytic function is expected to fail [5, 6] both at certain points lying on the intersection of several $+\alpha$ -Landau surfaces [10] and, also, at the so-called $\mathcal{M}_0$ points. These latter points are those where several incoming, or alternatively several outgoing, on-mass-shell four-momenta are collinear. Although these special points of both kinds lie in low-dimensional sub-manifolds of the physical region, they may often affect the bubble diagram functions over large portions of their domains of definition. This is because these special points can affect integrals whenever they occur in the domain of integration.

The mathematical framework of essential support theory is adapted to the study of this problem and the analysis carried out in that framework in [7], or the completely similar analysis carried out more recently in [11, 12] in the related framework of hyperfunction theory [13]², completely removed one important technical assumption of the previous proofs, namely the patching assumption, and directly provided in a precise way a theorem that is both more general and more useful for applications [8, 9].

[7] presents general results on products and integrals of distributions. These results yield in turn information on the essential support of any bubble diagram function in terms of the essential supports of its individual bubbles. The latter are known directly [5] from macrocausality in the case of plus bubbles, and with the aid of unitarity, also in the case of minus bubbles³.

The way the analytic structure of the individual bubbles, and of the bubble diagram functions, is characterized by their essential support is explained in [5, 7, 9], and will therefore not be explained again here. Let us only recall that the essential support property associated in [5] with macro-causality can be considered as a general and precise form of basic physical-region analyticity properties of scattering functions (analyticity outside $+\alpha$ -Landau surfaces and plus $i\varepsilon$ rules) which, in earlier approaches to S -matrix theory were derived from the idea of “maximal analyticity”, and which can also be extrapolated from perturbation theory.

² The part of hyperfunction theory which is relevant here is the study of the singular spectrum. Although the notions of essential support and of singular spectrum were introduced independently by very different methods, it turns out that they coincide for distributions [14]

³ Unitarity ensures that the essential support of a minus bubble is opposite to that of the corresponding plus bubble

The structure theorem of [7] (or the similar result of [11, 12]) does not yet provide, however, any information for certain sets $p = \{p_k\}$ of initial and final (on-mass-shell) four-momenta p_k . The excluded points in [7] are those for which there is possible occurrence of $\mathcal{M}_0$ points for some bubbles in integration domains (see above) and are other “ $u=0$ ” points. For reasons that appear later, all these points will be called here $u=0$ points. The basic difficulty in all cases is a general aspect of a second difficulty of the method of [2, 3], which occurs even for values of the internal momenta such that each scattering function is the boundary value of a single analytic function: if, in some part of the integration domain, the boundary values corresponding to the various bubbles cannot be obtained from *common* direction with respect to the appropriate internal variables, then there is a priori no common analyticity domain in which the integration contour can be distorted. This problem is also closely linked with the fact that no information is a priori obtained on the analytic structure of a product of distributions that are boundary values of analytic functions, if these boundary values cannot be obtained from common directions.

A related problem appears also for values of the internal momenta such that some scattering functions are no longer boundary values of single analytic functions, for instance when $\mathcal{M}_0$ points are encountered, and it arises more generally from the absence of general mathematical results on the essential support, or singular spectrum, of a product of distributions, when their essential supports at some point contain opposite directions.

For many bubble diagram functions, the excluded $u=0$ points p are exceptional. However, as emphasized in [11], this is not always the case. In particular, all points p are $u=0$ points for certain bubble diagram functions that occur even in the simplest cases, in the derivation of discontinuity formulae. The most simple example, in a theory with equal mass particles, is the bubble diagram function $\equiv \bigoplus \bigotimes \equiv$ that occurs for instance in the derivativon of the pole-factorization theorem for three-body processes [8]. The absence of information on its essential support at any point p is a first basic problem that completely disrupts the proofs, and an ad hoc technical assumption was still needed in [8] to cope with that problem.

The $u=0$ assumption considered in [8] says essentially, in the cases encountered there, that the same rules that apply at non $u=0$ points should determine the essential support also at $u=0$ points. An assumption of this type had been briefly mentioned in [7], but without justification, and it cannot be expected to be correct, as it stands, in the general case. A detailed analysis of some aspects of the problem was as a matter of fact carried out more recently. It led first [11] to a satisfactory understanding of the essential support of S -matrix kernels at $\mathcal{M}_0$ points, whereas the macrocausality condition used in [7] gave no information at these points. The essential support at $\mathcal{M}_0$ points can be obtained in general by introducing vertices “at infinity” and a certain angular-momentum conservation law, as will be explained in Sect. 4, and can be associated with an extension of the macrocausality ideas to $\mathcal{M}_0$ points. As already mentioned in [18], and as explained here in Sect. 4.3, the methods of [7] then lead to a slightly improved structure theorem that applies to some of the previously excluded $u=0$ points. The most important class of $u=0$ situations is, however, still not covered by this result:

this is because the same problems as before still arise in most cases from the occurrence of $\mathcal{M}_0$ points in integration domains, and because the $u=0$ problem arises also in general from the occurrence of internal momenta that do not correspond to $\mathcal{M}_0$ points for any bubble⁴.

A solution of the $u=0$ problem that occurs in the study of phase-space integrals⁵, was on the other hand obtained in [15], by an explicit analysis of these integrals, on the basis of a mathematical result [16, 17] on the singular spectrum of products of functions of the form $f_j^{s_j}$, where each f_j is analytic. This result led in turn [15] to a corresponding $u=0$ assumption on bubble diagram functions. In that assumption, the previous rules that hold away from $u=0$ points are modified in general at $u=0$ points by the introduction of certain limiting procedures that might enlarge the essential support.

The result of [15] on phase-space integrals is of interest and [15] introduces for the first time the important idea that limiting procedures might have to be considered in the general case at $u=0$ points. However, some aspects of the limiting procedures assumed in [15] are questionable⁶. On the other hand bubble diagram functions are not phase-space integrals, and the methods of [15] do not provide a solution of the fundamental theoretical problem in this domain of research, which is to understand the basic physical and mathematical reasons, in the general structure of the S matrix (i.e. of the individual bubbles), that may lead to a solution of the $u=0$ problem, and hence to derive a $u=0$ structure theorem on actual bubble diagram functions from basic physical properties of the S matrix.

The purpose of this work is to treat this problem and in fact to present a theorem that does cover all $u=0$ points. This new result is based on a refined version of the macrocausality condition. Macrocausality is the assertion that, in a certain asymptotic limit, transition probabilities (or transition amplitudes) fall off exponentially (in a well defined sense) for non causal configurations of displaced particles. The refined version introduced here adds a condition on the way rates of exponential fall off tend to zero when causal directions are approached. This new condition follows, as explained in detail in Sect. 5 on various examples, from the same ideas as the previous macrocausality condition, and it will be assumed to hold in general. It has again a neat mathematical expression, and the results obtained do provide a satisfactory understanding of the $u=0$ properties of bubble diagram functions. The $u=0$ structure theorem proved here introduces, as the $u=0$ assumption of [15], certain limiting procedures that modify in general the

4 For instance, in the case of $\Xi \oplus \ominus \Xi$, all points p are still excluded because of the occurrence of $\mathcal{M}_0$ points, and all points p above the four-particle threshold are also excluded in the same time because of this second problem: see Sect. 4

5 I.e. integrals in which the connected kernels of S or S^{-1} are replaced by constants times energy-momentum conservation δ -functions. The factors associated with internal lines are mass-shell δ -functions rather than Feynman propagators

6 A feature, which seems needed in the approach of [15], is the introduction of complex four-momenta and complex values of the Landau parameters α_i in the course of the limiting procedures (see [17, 15b]). The use made in [15] of macrocausality to get a $u=0$ assumption on bubble diagram functions is then questionable. Even for individual bubbles, the limiting procedures of [15] are not those that can be associated in a natural way with the general macrocausality ideas. (They do not take into account the fact that the intermediate particles cannot in the quantum case be strictly localized along classical trajectories)

rules previously derived at non $u=0$ points in [7]. These limiting procedures are, however, so far different from those of [15]⁷. As the latter, they need not be considered in many usual cases.

The organization of the paper is as follows.

In Sect. 2, the precise definition of bubble diagram functions is given in terms of bounded operators. This definition is both more general and better adapted to our present purposes than the definition of [7] in the framework of distribution theory (or than earlier even less general definitions). In fact, the procedures of [7] do not *a priori* define bubble diagram functions in neighbourhoods of the $u=0$ points⁸. The simple framework set up in this section is more satisfactory than previous ones and has its own autonomous interest. Certain combinatorial aspects are therefore briefly treated for completeness, although they play no role in the remainder of the paper and can be omitted by the non-interested reader.

Section 3, which is independent of Sect. 2, is devoted to the presentation of new mathematical results on the essential support of a product of bounded operators. The mathematical definition of $u=0$ points is given Subsect. 3.1, where a first theorem that applies to non $u=0$ points is presented. This theorem can be easily derived from the results of [4, 7] on products and integrals of distributions. A new, more direct and self-contained proof, which eliminates several unnecessary steps, is given here. It will be directly adapted in Subsect. 3.2, where it leads to a new theorem that covers $u=0$ points, provided the individual operators satisfy a certain regularity property R on the way rates of exponential fall-off of generalized Fourier transforms tend to zero when directions of the essential support are approached. (This property is precisely the condition that arises from refined macro-causality in the physical application). A certain weak conjecture (or a corresponding technical condition) is also used so far in the case of a product of more than two operators. It is introduced in Appendix 2. (It is clearly a minor problem which does not affect the essence of the arguments.)

The general application of the results of Sect. 3 to physical situations is described in Subsect. 4.1 in terms of space-time diagrams. These results apply to cases when the bubbles are not necessarily connected kernels of S or S^{-1} . Geometrical definitions and results are then presented in Subsect. 4.2, and Subsect. 4.3 is the application of the previous results to the usual bubble-diagram functions whose bubbles are connected kernels of S or S^{-1} . The structure theorem of [7] and, as already mentioned, its improved version based on the extension of macro-causality to $\mathcal{M}_0$ points, are first presented. The refined macro-causality condition is then introduced, and the general structure theorem that follows from it is described.

Finally, the physical discussion of the macro-causality and refined macro-causality condition, is given in Sect. 5.

⁷ In contrast to [15] (see previous footnote), only real quantities are involved, but a “doubling” of the internal lines is introduced in the course of the limiting procedures. The origin of this doubling is clear from a physical viewpoint (see Sect. 5.2), and also from a mathematical viewpoint, in view of the results of the present work

⁸ The same comment applies equally to definitions in the framework of hyperfunction theory used in [11, 12]

Appendix 1 establishes the connection between the definition of bubble-diagram functions of [7] and the present one⁹.

In Appendix 2, a mathematical lemma on bounded operators, needed in Subsect. 3.1, is established in the framework of essential support theory. The conjecture needed in Subsect. 3.2, which is a refinement of this lemma, is stated at the end.

We conclude this introduction with some remarks.

i) The precise content of the refined macrocausality condition in terms of analyticity properties has not yet been fully established, the study of this problem being left for further work. Let us only make here the following comment. We consider for simplicity a simple point p of a $+\alpha$ -Landau surface $L^+(G)$ and a system of real analytic local coordinates of the physical region chosen such that $L^+(G)$ is locally represented in this system by $q_1=0$, where q_1 is the first coordinate, and such that the physical side of $L^+(G)$ is represented by $q_1>0$.

The essential support property associated with macrocausality is then equivalent to the assertion that the scattering function f of the process is locally analytic away from the surface $q_1=0$, and is moreover, in a neighborhood of p , the boundary value of an analytic function $\underline{f}$ from the “plus $i\epsilon$ ” directions $\text{Im} q_1>0$, where q_1 is the complexified variable of q_1 . Namely, being given any open cone Γ' with apex at the origin in $\text{Im} q$ -space whose closure is contained (apart from the origin) in the region $\text{Im} q_1>0$, $\underline{f}$ is analytic in a domain of the form $\text{Re} q \in \omega$, $\text{Im} q \in \Gamma'$, $|\text{Im} q|<\epsilon$, $\epsilon>0$, where ω is a real neighborhood of p , and the boundary value f of $\underline{f}$ is obtained in ω from the directions of Γ' . However, ϵ may a priori tend to zero when Γ' expands to the half-space $\text{Im} q_1>0$. Although this is not fully established so far, refined macrocausality is probably closely linked with a slight refinement of this analyticity property, according to which $\underline{f}$ is indeed analytic in a region of the form $\varpi \cap \text{Im} q_1>0$ where ϖ is a complex neighborhood of p .

This is suggested in particular by the remark that concludes the presentation of the regularity property R in Sect. 3.2. This remark cannot be strictly applied here because the regularity property R does not apply to the distribution f , considered as being defined locally in the space of the coordinates q , but applies to the actual connected S -matrix kernel (i.e. the product of f by a global energy-momentum conservation δ -function), defined in the space of all three-momenta variables. There are, however, close links between them.

If this link is established, the use of the refined macrocausality condition would allow one to remove the two assumptions needed in [8] in order to establish the pole-factorization theorem, namely the $u=0$ assumption, and also the “no sprout” assumption.

ii) The refined macrocausality condition contains no specific information on the rates of exponential fall-off and is in fact only, as mentioned above, a general regularity condition on the way these rates tend to zero when causal directions are approached.

On the other hand, just as the previous macrocausality condition, it contains no information for causal configurations of displaced particles. Let us recall that a factorization property of transition amplitudes for causal configurations is also

⁹ Related results have been previously given [19] in more particular situations

considered in the theory and is as a matter of fact essentially equivalent to the discontinuity formulae of scattering functions around $+\alpha$ -Landau surfaces: see [9] and references therein. We do not wish to use it here, since this work is part of the program which aims to prove the discontinuity formulae, and in the same time causal factorization, on the basis essentially of macro-causality and unitarity.

iii) A solution of the $u=0$ problem was given in [11] for a particular class of situations on the basis of specific assumptions on the nature of the singularities of the S -matrix kernels, in particular in the neighborhood of $\mathcal{M}_0$ points. These assumptions go much beyond the analyticity properties associated with macro-causality and can as a matter of fact be derived from the discontinuity formulae, as explained in detail in [11].

However, that result is not general. On the other hand, making specific assumptions on the precise nature of singularities is not a priori satisfactory if one works to establish the discontinuity formulae, since such specific information will as a matter of fact be derived later from these formulae.

The refined macrocausality condition used in the present paper is a much weaker and more general assumption.

iv) The mathematical methods of the present work may in principle lead also to a new derivation of results on phase-space integrals. The expected result is similar to that of [15], with however again some differences in the limiting procedures obtained, similar to those already mentioned earlier (see remark at the end of Sect. 5.2). A complete proof would require checking certain technical points in detail and is not presented here.

2. Bubble Diagram Functions

2.1. General Framework

In the relativistic quantum theory of systems of massive particles with short-range interactions, one is led from basic principles to introduce a unitary operator S from $\mathcal{H}$ to $\mathcal{H}$, where $\mathcal{H}$ is a Hilbert space of free-particle states. The matrix elements $|\langle\psi|S|\phi\rangle|^2$ between unit-norm vectors $|\phi\rangle$ and $|\psi\rangle$ are the transition probabilities from the initial state represented by $|\phi\rangle$ to the final state represented by $|\psi\rangle$.

For simplicity, we consider in this section a theory with only one type of (spinless) particle, a boson of mass $\mu>0$. (The adaptation of the definitions and results to the more general case requires a somewhat more subtle analysis but presents no real difficulty.) The Hilbert space $\mathcal{H}$ is then the direct sum of Hilbert spaces $\mathcal{H}_m$ ($m=1,2,\dots$). The space $\mathcal{H}_m$ is the space of all functions ϕ of m on-mass-shell four-vectors p_i [$i=1,\dots,m$, $p_i^2=(p_i)_0^2-\mathbf{p}_i^2=\mu^2$, $(p_i)_0>0$, $\forall i$] that are square integrable ($\|\phi\|<\infty$, where $\|\phi\|=\langle\phi|\phi\rangle^{1/2}$ is the norm of ϕ ; see definition of scalar product below) and are symmetric under the interchange of any two four-vector variables.

It will be convenient to consider also the space $\hat{\mathcal{H}}_m$ of all (not necessarily symmetric) square integrable functions ϕ . The scalar product in $\hat{\mathcal{H}}_m$ is generally defined by:

$$\langle\phi|\psi\rangle=\int\bar{\phi}(p_1,\dots,p_m)\psi(p_1,\dots,p_m)\frac{1}{m!}\prod_{i=1,\dots,m}d\mu(p_i), \quad (1)$$

where

$$d\mu(p) = \delta(p^2 - \mu^2) \theta(p_0) d^4 p \equiv \frac{d^3 \mathbf{p}}{2(\mathbf{p}^2 + \mu^2)^{3/2}}. \quad (2)$$

The scattering operators $S_{m,n}$ are the linear and bounded operators from $\mathcal{H}_m$ to $\mathcal{H}_n$ whose action on a function ϕ in $\mathcal{H}_m$ is the component $(S|\phi\rangle)_n$ of $S|\phi\rangle$ in $\mathcal{H}_n$. Their extension to all functions ϕ in $\mathcal{H}_m$ will be defined in a natural way by the relation:

$$S_{m,n}(\phi) = S_{m,n} \left(\frac{1}{m!} \mathcal{S}\phi \right), \quad (3)$$

where $\mathcal{S}\phi(p_1, \dots, p_m) = \sum_{\pi} \phi(p_{\pi_1}, \dots, p_{\pi_m})$ and the sum $\sum$ runs over all permutations π of $1, \dots, m$.

The *connected* operators $S_{m,n}^c$ are defined in a precise way [20] by the formula:

$$S_{m,n}^c = S_{m,n} - \sum_{\mathcal{K}: N(\mathcal{K}) > 1} \left(\bigotimes_K S_{m_K, n_K}^c \right)_{\mathcal{K}}, \quad (4)$$

where the sum $\sum$ in the right-hand side runs over all non-trivial [$N(\mathcal{K}) > 1$] partitions $\mathcal{K}$ of the sets $I = 1, \dots, m$ and $J = 1, \dots, n$ of initial and final indices (attributed to each initial, or final, particle) into subsets $I_K, J_K, K = 1, \dots, N(\mathcal{K})$. The order of indices in I_K , or in J_K , or the relative order of the pairs (I_K, J_K) is irrelevant. For a given partition $\mathcal{K}$, the operator $\left(\bigotimes_K S_{m_K, n_K}^c \right)_{\mathcal{K}}$ is defined initially

on functions ϕ of a product form:

$$\phi(p_1, \dots, p_m) = \phi_1(p_1) \times \dots \times \phi_m(p_m) \quad (5)$$

by the formula

$$\left\{ \left(\bigotimes_K S_{m_K, n_K}^c \right)_{\mathcal{K}} (\phi) \right\} (q_1, \dots, q_n) = \frac{1}{m!} \prod_{K=1, \dots, N(\mathcal{K})} m_K! \{ S_{m_K, n_K}^c (\phi_{I_K}) \} (q_{J_K}), \quad (6)$$

where m_K, n_K , are the numbers of indices in I_K and J_K respectively, $q_1, \dots, q_n$ are on-mass-shell four-vector variables, ϕ_{I_K} is the usual (tensorial) product of the one-particle wave functions of the set I_K [for instance if $I_K = (1, 3, 5)$ then $\phi_{I_K}(p_1, p_2, p_3) = \phi_1(p_1) \phi_3(p_2) \phi_5(p_3)$] and q_{J_K} is the subset of $(q_1, \dots, q_n)$ associated with the set J_K [for instance, if $J_K = 2, 3, 5$ then $q_{J_K} = (q_2, q_3, q_5)$].

By induction on the numbers m, n of initial and final particles, the formulae (4) clearly allow one to define each operator $S_{m,n}^c$ for functions ϕ of the product form (5). On the other hand, it is also easily seen [21] that they provide moreover a (unique) definition of the operators $S_{m,n}^c$ as linear and bounded operators from $\mathcal{H}_m$ to $\mathcal{H}_n$ (or from $\mathcal{H}_m$ to $\mathcal{H}_n$: the operators $S_{m,n}^c$ have the same symmetry properties as the non-connected operators $S_{m,n}$). We give below an argument that is slightly different from that used in [21] and is better adapted to our later purposes.

The result is known for $S_{1,1}^c = S_{1,1} = \mathbb{1}_{1,1}$. If it is known for all operators $S_{m',n'}^c$, where $m' \leq m, n' < n$, or $m' < m, n' \leq n$, then the condition $N(\mathcal{K}) > 1$ in Eq. (4),

together with mathematical results on tensorial products of bounded operators¹⁰, ensures that each operator $\left(\bigotimes_K S_{m_K, n_K}^c\right)_{\mathcal{K}}$ in Eq. (4) is well defined (in a unique way) as a linear and bounded operator from $\hat{\mathcal{H}}_m$ to $\hat{\mathcal{H}}_n$. Since $S_{m, n}$ is already known to be a well-defined linear and bounded operator from $\hat{\mathcal{H}}_m$ to $\hat{\mathcal{H}}_n$, the result follows for $S_{m, n}^c$. [The symmetry properties of $S_{m, n}^c$ follow from the fact that the sums $\sum$ in Eq. (4) run over all (non-trivial) partitions $\mathcal{K}$.] Q.E.D.

At the same time, the above argument shows (by induction) that each operator $\left(\bigotimes_K S_{m_K, n_K}^c\right)_{\mathcal{K}}$ is a well-defined linear and bounded operator from $\hat{\mathcal{H}}_m$ to $\hat{\mathcal{H}}_n$. The same conclusions hold similarly for tensorial products $\left[\bigotimes_K (S^{-1})_{m_K, n_K}^c\right]_{\mathcal{K}}$ of connected operators associated with the operator $S^{-1} = S^\dagger$.

The *bubble diagram operators* that arise in equations derived from unitary $(SS^{-1} = S^{-1}S = \mathbb{1}, SS^{-1}S = S, \text{ etc.}, \text{ where } S^{-1} = S^\dagger)$, from the “cluster decomposition” of the S -matrix:

$$S_{m, n} = \sum_{\mathcal{K}; N(\mathcal{K}) \geq 1} \left(\bigotimes_K S_{m_K, n_K}^c\right)_{\mathcal{K}} \quad (7)$$

and from the analogous cluster decomposition of S^{-1} , are by definition (see below Proposition 1) sums of equal operators of the form:

$$A = \left(\bigotimes_K S_{m_K, n_K}^{(-1)^c}\right)_{\mathcal{K}[I_{(q)}, J]} \cdots \left(\bigotimes_K S_{m_K, n_K}^{(-1)^c}\right)_{\mathcal{K}[I_{(1)}, I_{(2)}]} \left(\bigotimes_K S_{m_K, n_K}^{(-1)^c}\right)_{\mathcal{K}[I, I_{(1)}]} \quad (8)$$

where each $\mathcal{K}[I_{(r)}, I_{(r+1)}]$, $r=0, \dots, q$ [$I_{(0)} = I, I_{(q+1)} = J$] is a given partition of $I_{(r)} = [1, \dots, m_{(r)}]$ and $I_{(r+1)} = [1, \dots, m_{(r+1)}]$ into subsets $I_{(r)K}^r$ and $I_{(r+1)K}^r$, and where $S_{m_K, n_K}^{(-1)^c}$ stands either for S_{m_K, n_K}^c or for $(S^{-1})_{m_K, n_K}^c$.

In view of the previous analysis, each term $\left[\bigotimes_K S_{m_K, n_K}^{(-1)^c}\right]_{\mathcal{K}[I_{(r)}, I_{(r+1)}]}$ is a well-defined linear and bounded operator from $\hat{\mathcal{H}}_{m_{(r)}}$ to $\hat{\mathcal{H}}_{m_{(r+1)}}$, and consequently one has¹¹:

10 The mathematical results needed are the same as those proved in [21]. We learn from Prof. V. Glaser that they are completely standard in mathematics. They ensure here that there is a unique, well-defined, linear and bounded operator $\left(\bigotimes_K S_{m_K, n_K}^c\right)_{\mathcal{K}}$ from $\hat{\mathcal{H}}_m$ to $\hat{\mathcal{H}}_n$, whose restriction to functions ϕ of the form $\phi(p_1, \dots, p_m) = \prod_K \phi_{I_K}(p_{I_K})$ is given by:

$$\left\{ \left(\bigotimes_K S_{m_K, n_K}^c\right)_{\mathcal{K}} (\phi) \right\} (q_1, \dots, q_n) = \frac{1}{m!} \prod_K m_K! \{S_{m_K, n_K}^c(\phi_{I_K})\} (q_{J_K})$$

11 The extension of each operator $\left[\bigotimes_K S_{m_K, n_K}^{(-1)^c}\right]_{\mathcal{K}[I_{(r)}, I_{(r+1)}]}$ to all functions of $\hat{\mathcal{H}}_{m_{(r)}}$ is crucial here, since each one of these operators transforms a function of a product form into a function that is no longer, in general, at the next step, of a product form with respect to the appropriate variables

Proposition 1. *Bubble diagram operators are well-defined linear and bounded operators from $\hat{\mathcal{H}}_m$ to $\hat{\mathcal{H}}_n$, where m and n are the numbers of indices of the external (initial and final) sets I, J .*

2.2. Bubble Diagram Operators and Diagrammatical Notation

We now complete the definition of bubble diagram operators. The analysis presented below provides in a precise and general way results analogous to those given in Appendix A of [22] in terms of momentum-space functions. It is not crucial for the main purposes of the present work and can therefore be omitted by non-interested readers.

In order to write equations derived from unitarity (and the cluster decompositions of S and S^{-1}) in a simple way, it is convenient to group together all operators that correspond to different sets of partitions but are trivially equal, namely all operators that can be obtained from a given operator A of the form (8) by permutations π_r inside each set $I_{(r)}$ of intermediate indices ($r=1, \dots, q$). More precisely, each π_r is here a given permutation of $I_{(r)}$, i.e. is the same, whether $I_{(r)}$ is considered as the set of final indices for the operator $\left[\bigotimes_K S_{m_K, n_K}^{(-1)^c} \right]_{\mathcal{K}[I_{(r-1)}, I_{(r)}]}$ or as the set of initial indices for the operator $\left[\bigotimes_K S_{m_K, n_K}^{(-1)^c} \right]_{\mathcal{K}[I_{(r)}, I_{(r+1)}]}$. Given a set π of permutations π_r ($r=1, \dots, q$), the operator A_π derived from A is defined by replacing in (8) the partitions $\mathcal{K}[I_{(r)}, I_{(r+1)}]$ ($r=0, \dots, q$) by new partitions $\mathcal{K}^\pi[I_{(r)}, I_{(r+1)}]$. The new subsets of $I_{(r)}$ and $I_{(r+1)}$, respectively, are obtained from the previous ones by the respective transformations π_r and π_{r+1} ¹² (π_0 and π_{q+1} are here the identity).

The *bubble diagram operator* associated with A is the sum of all equal operators that can be obtained in that way and correspond to *different sets of partitions* $\mathcal{K}^\pi$, i.e. all sets π of permutations that give the same set of new partitions are identified and counted only once. The reason for defining bubble diagram operators in that way is that there is only *one* term for each partition in the definition of the connected operators: see [20]. The sets π that give the same partitions are those that differ only by changes of orderings of indices *inside* each subset of each partition or by changes of orderings of the pairs of subsets of each partition: these changes are irrelevant and do not define new partitions (see definition of the partitions above).

The number of equal operators thus obtained is $(1/N) \prod_{r=1}^q m_{(r)}!$, where N is the number of sets π that give the same set of partitions. For a given set of partitions in (8), N is clearly independent of the set of new partitions considered. The calculation of N will be explained below in the diagrammatical notation now introduced.

In this notation, each operator $S_{m,n}^c$ or $(S^{-1})_{m,n}^c$ is represented by a bubble



¹² For instance, if $I_{(r)} = (1, \dots, 5)$, $\pi_r = (1 \rightarrow 3, 2 \rightarrow 4, 3 \rightarrow 1, 4 \rightarrow 5, 5 \rightarrow 2)$, then the subset $(1, 3, 4)$ of $I_{(r)}$ is transformed into $(3, 1, 5) \equiv (1, 3, 5)$

respectively, with m initial and n final lines¹³. Each operator $\left(\bigotimes_K S_{m_K, n_K}^c\right)_{\mathcal{K}(I, J)}$ is represented by a column

$$\begin{array}{c} I_1 \left(\begin{array}{c} \text{---} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \text{---} \end{array} \oplus \right) J_1 \\ \text{---} \text{---} \text{---} \text{---} \\ I_K \left(\begin{array}{c} \text{---} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \text{---} \end{array} \oplus \right) J_K \\ \text{---} \text{---} \text{---} \text{---} \end{array}$$

(where the ordering of bubbles is irrelevant), and each operator $\left[\bigotimes_K (S^{-1})_{m_K, n_K}^c\right]_{\mathcal{K}(I, J)}$ is represented by a similar column with minus bubbles. An operator A of the form (8) is represented by drawing, from left to right, the various columns associated with each operator $\left[\bigotimes_K S_{m_K, n_K}^{(-1)c}\right]_{\mathcal{K}[I_{(r)}, I_{(r+1)}]} (r=0, \dots, q)$. Finally, the bubble diagram operator associated with A is represented by a bubble diagram obtained as follows: two lines that belong to bubbles in two successive columns are joined whenever there is a common index in the corresponding subsets $I_{(r)K}^{(r-1)}$ and $I_{(r)K}^{(r)}$, of $I_{(r)}\{I_{(r)K}^{(r-1)}\}$ and $I_{(r)}^{(r)}\{I_{(r)K}^{(r)}\}$ refer respectively to subsets of $I_{(r)}$ in the partitions $\mathcal{K}[I_{(r-1)}, I_{(r)}]$ and $\mathcal{K}[I_{(r)}, I_{(r+1)}]$. Each line on the right of a column is joined in that way to one, and only one, line on the left of the next column. Finally, all mentions of indices on the internal lines that run between successive columns are removed. In the case of a bubble

$$\text{---} \text{---} \text{---} \text{---} \oplus \text{---} \text{---} \text{---} \text{---} = \text{---} \text{---} \text{---} \text{---} \ominus \text{---} \text{---} \text{---} \text{---} = \mathbb{1}_{1,1}$$

with only one incoming and one outgoing line, the bubble $\oplus$ or $\ominus$ can also be removed.

Example. If A is the operator

$$A = \left(\begin{array}{c} 1,2 \text{---} \text{---} \text{---} \text{---} \oplus \text{---} \text{---} \text{---} \text{---} \\ 3 \text{---} \text{---} \text{---} \text{---} \oplus \text{---} \text{---} \text{---} \text{---} \end{array} \right) \left(\begin{array}{c} 1,3 \text{---} \text{---} \text{---} \text{---} \ominus \text{---} \text{---} \text{---} \text{---} \\ 2 \text{---} \text{---} \text{---} \text{---} \oplus \text{---} \text{---} \text{---} \text{---} \end{array} \right) \left(\begin{array}{c} 1,3 \text{---} \text{---} \text{---} \text{---} \oplus \text{---} \text{---} \text{---} \text{---} \\ 2 \text{---} \text{---} \text{---} \text{---} \oplus \text{---} \text{---} \text{---} \text{---} \end{array} \right) \begin{array}{c} 1,2 \\ 3 \end{array} \quad (9)$$

where the product, according to the conventions mentioned above, is written from left to right, the corresponding bubble diagram operator is the sum of all equal operators of the form

$$\left(\begin{array}{c} 1,2 \text{---} \text{---} \text{---} \text{---} \oplus \text{---} \text{---} \text{---} \text{---} \\ 3 \text{---} \text{---} \text{---} \text{---} \oplus \text{---} \text{---} \text{---} \text{---} \end{array} \right) \left(\begin{array}{c} i,k \text{---} \text{---} \text{---} \text{---} \ominus \text{---} \text{---} \text{---} \text{---} \\ j \text{---} \text{---} \text{---} \text{---} \oplus \text{---} \text{---} \text{---} \text{---} \end{array} \right) \left(\begin{array}{c} i',k' \text{---} \text{---} \text{---} \text{---} \oplus \text{---} \text{---} \text{---} \text{---} \\ j' \text{---} \text{---} \text{---} \text{---} \oplus \text{---} \text{---} \text{---} \text{---} \end{array} \right) \begin{array}{c} 1,2 \\ 3 \end{array}$$

obtained by permutations $\pi_{(1)}=(1 \rightarrow i, 2 \rightarrow j, 3 \rightarrow k)$ and $\pi_{(2)}=(1 \rightarrow i', 2 \rightarrow j', 3 \rightarrow k')$ of $(1, 2, 3)$. It is represented by

$$B = \begin{array}{c} 1,2 \text{---} \text{---} \text{---} \text{---} \oplus \text{---} \text{---} \text{---} \text{---} \oplus \text{---} \text{---} \text{---} \text{---} \\ 3 \text{---} \text{---} \text{---} \text{---} \oplus \text{---} \text{---} \text{---} \text{---} \ominus \text{---} \text{---} \text{---} \text{---} \oplus \text{---} \text{---} \text{---} \text{---} \end{array} \begin{array}{c} 1,2 \\ 3 \end{array} \quad (10)$$

¹³ In accordance with some conventions, $\text{---} \text{---} \text{---} \text{---} \ominus \text{---} \text{---} \text{---} \text{---}$ would represent the operator $-(S^{-1})_{m,n}^c$

It is easily seen that there is a well-defined 1–1 correspondence between bubble diagram operators and bubble diagrams B of the type that has been described. Given B , it defines a sum of equal operators obtained by drawing the bubbles $\rightarrow \bigcirc \leftarrow$ on the appropriate single lines, and by attributing indices $1, \dots, m_{(r)}$ to the $m_{(r)}$ lines that run between two successive columns ($r=1, \dots, q$). These sets of indices clearly define corresponding partitions $\mathcal{K}[I_{(r)}, I_{(r+1)}]$, $r=0, \dots, q$. Operators that correspond to the same set of partitions are identified and counted only once.

In the example of Eq. (10), the number N of sets π that give the same set of partitions is one: all different ways of attributing indices to the internal lines correspond to different sets of partitions. More generally, this number, also denoted by $N(B)$ since it depends only on the structure of B , is the product of two terms. The first one is the product, over all pairs (b_1, b_2) of bubbles in successive columns of the factors $\alpha(b_1, b_2)!$, where $\alpha(b_1, b_2)$ is the number of common lines joining b_1, b_2 : this term clearly accounts for possible changes of orderings of indices *inside* each subset of *each* partition. The second one is the product, over all sets of *identical* bubbles $b_1, \dots, b_p$ inside a common column (i.e. bubbles with the same numbers of incoming *and* outgoing lines) of corresponding factors $p!$: this term clearly accounts for possible changes of orderings of the *pairs* of subsets of *each* partition.

For instance, if

$$B = 1,2,3,4 \left(\begin{array}{c} \text{Diagram with two columns of bubbles. The first column has a bubble with a '+' sign and four incoming lines from the left. The second column has two bubbles, one with a '-' sign and one with a '+' sign, both with four incoming lines from the first column and four outgoing lines to the right. The lines between columns are connected in a way that suggests a specific partitioning of indices.} \end{array} \right) 1,2,3 \quad (11)$$

then $N(B)=2! \times 2! \times 3! \times 3! \times 2!$

2.3. Momentum-Space Kernels and Bubble Diagram Functions

The action of any (linear, bounded) operator $F_{m,n}$ from $\hat{\mathcal{H}}_m$ to $\hat{\mathcal{H}}_n$ on a function $\phi_{(m)}$ in $\hat{\mathcal{H}}_m$ can be written formally as:

$$\{F_{m,n}(\phi_{(m)})\}(q_1, \dots, q_n) = \int F_{m,n}(p_1, \dots, p_n; q_1, \dots, q_n) \phi_{(m)}(p_1, \dots, p_m) \frac{1}{m!} d\mu(p_1) \dots d\mu(p_m), \quad (12)$$

where $d\mu(p)$ is defined in Eq. (2).

With the normalization conventions (1) and (12), one checks from Eqs. (6) and (7) that

$$S_{m,n}(p_1, \dots, p_m; q_1, \dots, q_n) = \sum_{\mathcal{K}[I,J]} \prod_{K \in \mathcal{K}} S_{m_K, n_K}^c(p_{I_K}; q_{J_K}). \quad (13)$$

The kernel $F_B(p_1, \dots, p_m; q_1, \dots, q_n)$ of a bubble diagram operator F_B (associated with a bubble diagram B) can be correspondingly written (formally) as an integral, over all on-mass-shell values of internal four-momenta k_l attributed to each internal line l of B , of momentum-space kernels associated with each bubble of B . The integration measure is, in view of the normalization conventions

$$[1/N(B)] \prod_l d\mu(k_l) \equiv [1/N(B)] \prod_l [2(\mathbf{k}_l^2 + \mu^2)]^{-1/2} d^3\mathbf{k}_l.$$

For instance, if B is the bubble diagram of Eq. (10), then

$$F_B(p_1, p_2, p_3; q_1, q_2, q_3) = \int S_{2,2}^c(p_1, p_2; k_1, k_2) \times (S^{-1})_{2,2}^c(k_2, p_3; k_3, q_3) \times S_{2,2}^c(k_1, k_3; q_1, q_2) \prod_{l=1,2,3} d^3 \mathbf{k}_l / 2(\mathbf{k}_l^2 + \mu^2)^{1/2}. \quad (14)$$

Expressions such as (14) are formal: bubble diagram operators, and their kernels, are well defined by the procedures described earlier. We recall, on the other hand, that kernels of linear bounded operators [such as $S_{m,n}^{(-1)}$, $S_{m,n}^{(-1)c}$, F_B] are always in particular well-defined *tempered distributions*¹⁴. The distribution thus associated with a bubble diagram operator F_B will be called a bubble-diagram function. It is still denoted by F_B in Sect. 4.

We finally recall that, as the operators $S_{m,n}^{(-1)}$ or $S_{m,n}^{(-1)c}$, the operators F_B satisfy energy-momentum conservation, as easily checked from their definition (i.e. the support of $(F_B|\phi_{(m)})_n$ is at most that derived from the support of $\phi_{(m)}$ by energy-momentum conservation). Just as in the case of the S -matrix kernels, the distribution F_B can correspondingly be written (if the initial and final four-momenta are not all collinear) in the form:

$$F_B = f_B \times \delta^4\left(\sum_{i \in I} p_i - \sum_{j \in J} p_j\right), \quad (15)$$

where the sums $\sum$ in the right-hand side run over the initial and final variables respectively, and where f_B is a distribution defined on the physical region of the process $I \rightarrow J$ (i.e. the manifold whose points $p = \{p_k\}$ are sets of initial and final on-mass-shell four-momenta satisfying energy-momentum conservation).

3. Essential Support of Products of Bounded Operators: Mathematical Results

The mathematical notations used for convenience in the present section are somewhat different from those used in the physical situation. In accordance with the present notations, the words: “ $u=0$ points” should be replaced in the titles of Subsects. 1 and 2, by: “ $(u, v)=0$ points” (this is the notation that appears in the main text). They become $u=0$ points when different notations are used, as in the physical application, namely notations in which the variable u denotes the set of all “initial” and “final” variables presently denoted by u and v respectively.

¹⁴ I.e. the functional $F_{m,n}$ whose action on pairs $\phi_{(m)}, \psi_{(n)}$ of functions in $\hat{\mathcal{H}}_m$ and $\hat{\mathcal{H}}_n$ respectively, is given by:

$$\begin{aligned} F_{m,n}(\phi_{(m)}, \psi_{(n)}) &= \langle \bar{\psi}_{(n)} | F_{m,n} | \phi_{(m)} \rangle \\ &= \int F_{m,n}(p_1, \dots, p_m; q_1, \dots, q_n) \phi_{(m)}(p_1, \dots, p_m) \psi_{(n)}(q_1, \dots, q_n) \\ &\quad \cdot \frac{1}{m!} \frac{1}{n!} \prod_{i=1, \dots, m} d\mu(p_i) \prod_{j=1, \dots, n} d\mu(q_j) \end{aligned}$$

can be extended, as a linear and continuous functional in the Schwartz topology, to all functions $\phi_{(m+n)}$ of $m+n$ on-mass-shell four-momenta variables $p_1, \dots, p_m, q_1, \dots, q_n$, that are infinitely differentiable and have a rapid decrease at infinity, as well as their derivatives, and are no longer necessarily of a product form in the variables $p_1, \dots, p_m$ and $q_1, \dots, q_n$ separately

On the other hand, the analogue of the distribution a introduced below and associated with an operator A is, in the physical application, the distribution still denoted by F_B [and *not* the distribution denoted f_B of Eq. (15)].

3.1. Results away from $u=0$ Situations

We first consider below two linear bounded operators A', A'' from $\mathcal{H}$ to $\mathcal{H}'$ and $\mathcal{H}'$ to $\mathcal{H}''$ respectively, where $\mathcal{H}$, $\mathcal{H}'$, and $\mathcal{H}''$ are here (non-symmetrized) Hilbert spaces of square integrable functions of real variables $x=x_1, \dots, x_m$, $t=t_1, \dots, t_p$, and $y=y_1, \dots, y_n$ respectively¹⁵. Let $A=A''A'$ and let a', a'', a denote the distributions associated with A', A'', A , respectively, as in the footnote at the end of Sect. 2. For instance, the distribution a is well-defined by its action on Schwartz test functions χ of a product form $[\chi(x, y)=\phi(x)\psi(y)]$, in which case one has

$$a(\chi)=\langle \bar{\psi}|A''A'|\phi\rangle. \quad (16)$$

It is finally assumed that A' and A'' satisfy a certain support property. Stated for instance on A' , it says this: $A'|\phi\rangle$ has a given compact support whenever ϕ has a given compact support, the support of $A'|\phi\rangle$ depending only on the support of ϕ (this property will be satisfied in the physical problem in view of energy-momentum conservation).

The dual variables of the variables x, y, t will be denoted respectively by u, v, w and scalar products will be defined for convenience by the formula:

$$(u, v) \cdot (x, y) = u \cdot x - v \cdot y, \quad (17)$$

where $u \cdot x = \sum_{i=1}^m u_i x_i$, $v \cdot y = \sum_{j=1}^n v_j y_j$, the scalar products $(u, w) \cdot (x, t)$ and $(w, v) \cdot (t, y)$ being defined similarly.

Definition 1. A point (X, Y) in $R_{(x)}^m \times R_{(y)}^n$ ¹⁶ is called a $(u, v)=0$ point relative to the product $A''A'$ if there exist a point T in $R_{(t)}^p$ and a point $W \neq 0$ in $R_{(w)}^p$ such that

$$(O, W) \in ES_{X, T}(a') \quad (18)$$

and

$$(W, O) \in ES_{T, Y}(a''). \quad (19)$$

Here $ES_{X, T}(a')$ and $ES_{T, Y}(a'')$ denote respectively the essential supports of a' at the point (X, T) and of a'' at the point (T, Y) . (See definitions in [4, 7]. Throughout this paper, essential supports are considered as closed cones with apex at the origin in the appropriate spaces, rather than closed sets of directions. For instance $ES_{X, T}(a')$ is a cone with its apex at the origin in the dual space $R_{(u)}^m \times R_{(v)}^p$ of $R_{(x)}^m \times R_{(t)}^p$.)

¹⁵ Each variable $x_i (i=1, \dots, m)$, $y_j (j=1, \dots, n)$ or $t_k (k=1, \dots, p)$ is here one-dimensional. The variables x, y or t will be in the physical problem, sets of components of three-momenta variables (see Sect. 4)
¹⁶ The index $x, y, \dots$, in $R_{(x)}^m, R_{(y)}^n, \dots$, serves only to recall the name of the variables in the space $R^m, R^n, \dots$, considered

The following theorem then holds away from $(u, v) = 0$ points:

Theorem 1. *Let (X, Y) be a point $R_{(x)}^m \times R_{(y)}^n$ and let (U, V) be a point of the dual space $R_{(u)}^m \times R_{(v)}^n$. If the following conditions are satisfied:*

- i) (X, Y) is not a $(u, v) = 0$ point relative to the product $A''A'$;
- ii) *there exists no pair of points $T, W, T \in R_{(t)}^p, W \in R_{(w)}^p$ such that $(U, W) \in ES_{X, T}(a')$ and $(W, V) \in ES_{T, Y}(a'')$, then (U, V) does not belong to the essential support $ES_{X, Y}(a)$ of a at the point (X, Y) .*

Remark. The conditions i) and ii) of Theorem 1 can be equivalently replaced by the unique condition:

- i') There exists no point T and no sequence of points W_k such that the directions defined by the points (U, W_k) and (W_k, V) in $R_{(u)}^m \times R_{(w)}^p$ and $R_{(w)}^p \times R_{(v)}^n$, both become arbitrarily close to directions that belong to $ES_{X, T}(a')$ and $ES_{T, Y}(a'')$ respectively.

Proof of Theorem 1. Theorem 1 can be easily derived from the results of [7] (together with those of Appendix 1 of the present paper). In fact, Theorem 4 of [7] allows one to define the product $a(x, t)a(t, y)$, in the sense of distributions, away from $(u, v) = 0$ points, and to show that its essential support at a point X, Y, T is contained in the set of points (U, V, W) of the form $(U, V, W_1 + W_2)$ where $(U, W_1) \in ES_{X, T}(a')$ and $(W_2, V) \in ES_{T, Y}(a'')$. Then Theorem 5 of [7] allows one to study the essential support of the integral $\int a'(x, t)a''(t, y)dt$ and to obtain the announced result.

As mentioned in the Introduction, we present below a more direct, self-contained proof.

Let g and h be functions of x and y , respectively, with compact support, chosen for instance C^∞ (infinitely differentiable) and equal to one in neighbourhoods of X and Y , and let F be the generalized Fourier transform at X, Y of $a(x, y)g(x)h(y)$:

$$F(u, v; v_0) = \int a(x, y)g(x)h(y)e^{-i(u \cdot x - v \cdot y)} \times e^{-v_0[(x-X)^2 + (y-Y)^2]} dx dy. \quad (20)$$

In view of the definition of a , F can be written [see Eq. (16) with $\phi(x) = g(x)e^{-i u \cdot x - v_0(x-X)^2}$, $\psi(y) = h(y)e^{-i v \cdot y - v_0(y-Y)^2}$], as

$$F(u, v; v_0) = \int dt A'(t; u, v_0) A''(t; v, v_0), \quad (21)$$

where

$$\begin{aligned} A'(t; u, v_0) &= \int a'(x, t) e^{-i u \cdot x - v_0(x-X)^2} g(x) dx \\ &\equiv (A'|g(x)e^{-i u \cdot x - v_0(x-X)^2})(t) \end{aligned}$$

and $A''(t; v, v_0)$ is defined similarly. These two functions are clearly square integrable functions of t for any given u, v, v_0 . Moreover, $A'(t; u, v_0)$ has a given compact support in $\mathbb{R}_{(t)}^p$ as a consequence of the support property assumed on the operator A' .

By standard arguments on integrals of functions, Eq. (21) can alternatively be written, for $v_0 > 0$, in the form

$$\begin{aligned} F(u, v; v_0) &= \pi^{-p/2} \int (2v_0)^{p/2} dT \int A'(t; u, v_0) e^{-v_0(t-T)^2} \\ &\quad \times A''(t; v, v_0) e^{-v_0(t-T)^2} dt. \end{aligned} \quad (22)$$

Finally, by standard arguments on the Fourier transformation of square integrable functions, one has

$$F(u, v; v_0) = \pi^{-p/2} \int (2v_0)^{p/2} dT \int dw F'(u, w; v_0, X, T) F''(w, v; v_0, T, Y), \quad (23)$$

where F' and F'' are, respectively, the generalized Fourier transforms of $a'(x, t)g(x)$ and $a''(t, y)h(y)$ at the points (X, T) and (T, Y) . For instance:

$$\begin{aligned} F'(u, w; v_0, X, T) &= \int A'(t; u, v_0) e^{-v_0(t-T)^2} e^{iwt} dt \\ &= \int a'(x, t) g(x) e^{-i(ux-wt)} e^{-v_0[(t-T)^2 + (x-X)^2]} dx dt. \end{aligned} \quad (24)$$

Equation (23) is the basis of the proof of Theorem 1. In view of the very definition of the essential support, conditions i) and ii) of Theorem 1 will in fact ensure exponential fall-off properties of F' or F'' , which in turn directly yield the needed exponential fall-off properties of F , if Lemmas 2 and 3 of Appendix 2 are used. The details of the proof are easy and we only give below some brief, but sufficient, indications.

First, let D be the given compact region in $\mathbb{R}_{(t)}^p$ (which depends only by assumption on the support of g) where the product $A'(t; u, v_0)A''(t; v, v_0)$ has its support, and let D_{α_1} , $\alpha_1 > 0$, be the set of points whose distance to D is less than (or equal to) $1/\sqrt{\alpha_1}$. Then the part of the integral (23) corresponding to T outside D_{α_1} clearly satisfies a bound of the form $Ce^{-\alpha v_0}$, $\alpha > 0$, in the whole region $v_0 > 0^{17}$. We are therefore left with the study of the contribution of the compact region $T \in D_{\alpha_1}$.

Given U, V , the integration regions over T and w are then divided into a finite number of sufficiently small neighbourhoods of given points T_i, W_j , including points W_j "at infinity": in this latter case W_j is defined by a certain direction in $\mathbb{R}_{(w)}^p$ and a (sufficiently small) neighbourhood of W_j is the part of a (sufficiently small) cone in $\mathbb{R}_{(w)}^p$ with its apex at the origin, around this direction, that lies outside a sphere with (sufficiently large) radius $\varrho_j > 0$. Given any point u, v , $(u^2 + v^2)^{1/2} = \tau(U^2 + V^2)^{1/2}$, the integration regions over T and w are divided into corresponding neighbourhoods of the points T_i and τW_j : the neighbourhoods of the points T_i are unchanged, while those of the points W_j are obtained from the previous ones by the dilation $\tau(w \rightarrow \tau w)$.

Given any set of indices i, j , the conditions i) and ii) of Theorem 1, applied respectively to cases when W_j is at finite and "infinite" distances, and Lemma 2 of Appendix 2 allow one to show that either F' or F'' satisfies (uniform) exponential fall-off bounds of the form given in Lemma 2, when u or v lies in a sufficiently small cone with its apex at the origin around U , or around V , respectively, and when T, w lie in the corresponding neighbourhoods of T_i and τW_j introduced above. The remaining function F'' or F' , still satisfies the trivial bounds of Lemma 3 of Appendix 2.

17 This can be seen by using Eq. (22) together with the inequalities

$$(i) \quad \int dt |A'(t; u, v_0)| \cdot |A''(t; v, v_0)| \leq \|A'\| \cdot \|A''\| \cdot \|g\| \cdot \|h\|,$$

where $\|A'\|$, $\|A''\|$, $\|g\|$, $\|h\|$ are the norms of the operators A' , A'' and of the functions g, h , and

$$(ii) \quad \int_{|T| > 1/\sqrt{\alpha_1}} (2v_0)^{p/2} dT e^{-2v_0 T^2} < \text{const } e^{-\alpha v_0}, \quad \alpha > 0$$

Since the number of regions considered is finite, it is easily seen that one may extract a common exponential fall-off factor $e^{-\alpha v_0}$ ($\alpha > 0$) in a common cone with its apex at the origin in $\mathbb{R}_{(u)}^m \times \mathbb{R}_{(v)}^n$ around (U, V) and in a common region $0 < v_0 < \gamma_0 \tau$, $\gamma_0 > 0$. The remaining integrals over w are bounded by a constant independent of u, v, v_0 by virtue of the norm properties of the square integrable functions involved in the bounds of Lemmas 2 and 3 of Appendix 2. Theorem 1 is therefore proved.

Theorem 1 is easily extended to the case of a product of $q > 2$ linear bounded operators $A_1, A_2, \dots, A_q$ from $\mathcal{H}^{(0)}$ to $\mathcal{H}^{(1)}$, $\mathcal{H}^{(1)}$ to $\mathcal{H}^{(2)}$, ..., $\mathcal{H}^{(q-1)}$ to $\mathcal{H}^{(q)}$, respectively, where $\mathcal{H}^{(0)}, \mathcal{H}^{(1)}, \dots, \mathcal{H}^{(q)}$ are Hilbert spaces of square integrable functions of real (possibly multi-dimensional) variables $t^{(0)} \equiv x, t^{(1)}, \dots, t^{(q)} \equiv y$, respectively. If we denote here by $w^{(0)} \equiv u, w^{(1)}, \dots, w^{(q)} \equiv v$ the dual variables of the variables $x, t^{(1)}, \dots, y$, respectively, Definition 1 is first extended as follows.

Definition 2. A point (X, Y) in $R_{(x)}^m \times R_{(y)}^n$ is called a $(u, v) = 0$ point relative to the product $A_q \dots A_1$, if there exist points $T^{(1)}, \dots, T^{(q-1)}$ and $W^{(1)}, \dots, W^{(q-1)}, W^{(r)} \neq 0$ for at least one index r , such that:

$$\begin{aligned} (O, W^{(1)}) &\in ES_{X, T^{(1)}}(a_1) \\ (W^{(r)}, W^{(r+1)}) &\in ES_{T^{(r)}, T^{(r+1)}}(a_{r+1}), \quad r = 1, \dots, q-2 \\ (W^{(q-1)}, 0) &\in ES_{T^{(q-1)}, Y}(a_q). \end{aligned}$$

The following extension of Theorem 1 then holds:

Theorem 2. If X, Y is not a $(u, v) = 0$ point relative to the product $A_q \dots A_1$ and if there exist no set of points $T^{(1)}, \dots, T^{(q-1)}, W^{(1)}, \dots, W^{(q-1)}$ such that

$$\begin{aligned} (U, W^{(1)}) &\in ES_{X, T^{(1)}}(a_1) \\ (W^{(r)}, W^{(r+1)}) &\in ES_{T^{(r)}, T^{(r+1)}}(a_{r+1}), \quad r = 1, \dots, q-2 \\ (W^{(q-1)}, V) &\in ES_{T^{(q-1)}, Y}(a_q), \end{aligned}$$

then $(U, V) \notin ES_{X, Y}(a)$.

The remark that follows Theorem 1 can be readily adapted to the present case.

As Theorem 1, Theorem 2 can be easily derived from the results of [7] (and those of the present Appendix 1). Alternatively, it is easily derived by induction from Theorem 1. It is sufficient to notice, as easily checked, that if (X, Y) is not a $(u, v) = 0$ point of the product $A_q \dots A_1$, then for any point $T^{(q-1)}$, the point $[X, T^{(q-1)}]$ cannot be a $[u, w^{(q-1)}] = 0$ point relative to the product $A_{q-1} \dots A_1$.

3.2. $U = 0$ Results

The exclusion of $(u, v) = 0$ points in Theorem 1 ensures, as we have seen, the absence of problems when the points w in the integration domain tend to infinity: uniform exponential fall-off bounds were obtained in each region, including those associated with points W_j "at infinity". The situation is a priori different if (X, Y) is a $(u, v) = 0$ point, since there exist by definition points T_i and points $\tilde{W}_j$, which can be chosen for instance on the unit sphere in $R_{(w)}^p$, such that $(O, \tilde{W}_j) \in ES_{X, T_i}(a')$ and $(\tilde{W}_j, O) \in ES_{T_i, Y}(a'')$. Therefore, being given any point U, V , the directions determined by the points $(U, \lambda \tilde{W}_j)$ and $(\lambda \tilde{W}_j, V)$ both tend to directions that belong to

$ES_{X,T_i}(a')$ and $ES_{T_i,Y}(a'')$ respectively when $\lambda \rightarrow \infty$, and the previous proof does not allow one to extract uniform exponential fall-off bounds.

A more detailed analysis shows, however, that this is still possible under a certain condition. Let us as a matter of fact consider linear bounded operators A', A'' which, besides the support property already mentioned in Subsect. 1, satisfy a regularity property on the way rates of exponential fall-off tend to zero when directions of the essential support are approached. We state it below on a general linear bounded operator A and then denote by x, y, u, v the initial and final variables and their dual variables. The actual mathematical significance and content of this property will be analysed below.

Regularity Property R. The operator A is said to satisfy the regularity property R if, being given any point (X, Y) , any real neighbourhood $\mathcal{N}'$ of (X, Y) in $R_{(x)}^m \times R_{(y)}^n$ and any open cone $\mathcal{C}$ with apex at the origin in $R_{(u)}^m \times R_{(v)}^n$ such that:

$$(u, v) \notin ES_{x,y}(a), \quad \forall (u, v) \in \mathcal{C} \quad (25)$$

$$\forall (x, y) \in \mathcal{N}'$$

there exist $\alpha > 0$, $\gamma_0 > 0$, a neighbourhood $\mathcal{N}'$ of (X, Y) contained in $\mathcal{N}$, and functions d_1, d_2 of the variables u, v, v_0 , which are square integrable with respect to u and v respectively and whose norms $[\int d_1^2(u, v, v_0) du]^{1/2}$, $[\int d_2^2(u, v, v_0) dv]^{1/2}$ are independent of v, v_0 and of u, v_0 respectively, such that:

$$|F(u, v; v_0, x, y)| < d_i(u, v, v_0) e^{-\alpha v_0} \quad (26)$$

$i = 1, 2$ in the region $(x, y) \in \mathcal{N}'$, $(u, v) \in \mathcal{C}$ and:

$$0 \leq v_0 \leq \gamma_0 (\widehat{u, v}; \partial \mathcal{C}) (|u| + |v|). \quad (27)$$

In Eq. (26), F is as before the generalized Fourier transform of $a(x', y')g(x')h(y')$ at the point (x, y) and $(\widehat{u, v}; \partial \mathcal{C})$ is in Eq. (27) the angle of the direction determined by the point (u, v) with the boundary $\partial \mathcal{C}$ of $\mathcal{C}$. We note on the other hand that the bound (27) can be equivalently replaced by:

$$0 \leq v_0 \leq \gamma'_0 \text{dist}((u, v); \partial \mathcal{C}), \gamma'_0 > 0, \quad (27')$$

where γ'_0 is possibly different from γ_0 but is again independent of (u, v) and where $\text{dist}((u, v); \partial \mathcal{C})$ is the distance of the point (u, v) to $\partial \mathcal{C}$.

If the intersection of the closure of $\mathcal{C}$ with $ES_{X,Y}$ is empty (apart from the origin), the regularity property R holds automatically, as a simple consequence of Lemma 2 of Appendix 2. [The factor $(\widehat{u, v}; \partial \mathcal{C})$ can as a matter of fact be removed in this case from the inequality (27).] The regularity property R gives however further information when some directions of the boundary of $\mathcal{C}$ lie in $ES_{X,Y}(a)$.

The fact that there is a uniform constant $\alpha > 0$ in the exponential fall-off factor of the right-hand side of Eq. (26) is natural and expected in view of the results of essential support theory. It comes from the fact that there is a *common* uniform neighbourhood $\mathcal{N}'$ of (X, Y) , such that all directions of $\mathcal{C}$ lie outside the essential support of a at any point (x, y) of this neighbourhood. Being given a distribution f defined on a real vector space¹⁸ $R_{(x)}^n$, the best rate α of exponential fall-off

18 The notations used here are those used in [4, 7] for general distributions. In the application, the variables x and u have to be replaced by (x, y) and (u, v) respectively

obtained when x varies in a neighbourhood $\mathcal{N}'$ of a point X and for a given direction $\hat{u}$ in the dual space of $R_{(x)}^n$, depends as a matter of fact on the size of the neighbourhood $\mathcal{N}$ of X such that $\hat{u}$ lies outside $ES_x(f)$, $\forall x \in \mathcal{N}$ (see [4]). It is strictly positive if $\mathcal{N}$ contains the closure of $\mathcal{N}'$ in its interior. The constant α obtained is thus independent of the direction considered, if $\mathcal{N}'$ and $\mathcal{N}$ are independent of it.

For any given direction $\widehat{u, v}$ in $\mathcal{C}$, one therefore expects an exponential fall-off factor $e^{-\alpha v_0}$, with a common α , in a region of the form :

$$0 \leq v_0 < \gamma_{\max}(\widehat{u, v}) \times (|u| + |v|),$$

where $\gamma_{\max}(\widehat{u, v})$ is strictly positive, but depends on the direction $\widehat{u, v}$ considered, and tends to zero when one approaches any direction of $\partial\mathcal{C}$ that lies in $ES_{X, Y}(a)$. The crucial content of the regularity property R lies in the fact that $\gamma_{\max}(\widehat{u, v})$ is assumed to decrease not faster than linearly with respect to the angle, when the direction $\widehat{u, v}$ tends to a direction of $ES_{X, Y}(a)$.

The inclusion of the function d_i in the bounds (26) is natural in view of the results (and conjecture) of Appendix 2, when the distribution a is the kernel of a bounded operator.

Remark. The regularity property R can also be stated for distributions which are not necessarily kernels of bounded operators, with only minor modifications: the inclusion in the bounds of the function d_i is then no longer justified. Instead, polynomial factors of the variables u, v are expected in general.

In order to give an idea of the content of the regularity property R in terms of analyticity properties, let us consider the simple example (which is not that encountered in physical situations in the present work) of a distribution f whose essential support $ES_x(f)$, at all points x in a given neighbourhood $\mathcal{N}$ of a point X , is contained in a given closed convex salient cone C . (As previously, f is here defined in $R_{(x)}^n$, the variables x, y and u, v being replaced by x and u respectively.) In view of the results of essential support theory, f is equivalently (independently of the regularity property R) the boundary value in $\mathcal{N}$ of an analytic function $\underline{f}$ from the directions of the open dual cone Γ of C : namely, being given any real neighborhood ω of X whose closure is contained in $\mathcal{N}$, and any open cone Γ' with apex at the origin whose closure is contained (apart from the origin) in Γ , there exists $\varepsilon > 0$ such that $\underline{f}$ is analytic in the domain $x = \operatorname{Re} z \in \omega$, $\operatorname{Im} z \in \Gamma'$, $|\operatorname{Im} z| < \varepsilon$, where z is the complexified variable of x . However, for any given ω , ε may tend to zero when Γ' expands to Γ .

In view of the results of essential support theory, the regularity property R , applied here to the case when $\mathcal{C}$ is the complement of C in $R_{(u)}^n$ implies that this is not the case, and it is as a matter of fact essentially equivalent to that result: namely $\underline{f}$ is analytic in $\mathcal{N} \cap \{\operatorname{Im} z \in \Gamma\}$, where $\mathcal{N}$ is a complex neighborhood of X .

The general regularity property R can be considered as an extension of this analyticity property to more general situations.

After the above presentation and discussion of the regularity property R , we now state:

Theorem 3. *Let A', A'' be two bounded operators satisfying the same properties as in Subsect. 1 and the further regularity property R , and let (X, Y) and (U, V) be given points in $R_{(x)}^m \times R_{(y)}^n$ and $R_{(u)}^m \times R_{(v)}^n$ respectively.*

If there exists $\varepsilon > 0$ such that, being given any set of points $x, y, u, v, t_1, t_2, w_1, w_2$ satisfying the relations

$$|x - X| < \varepsilon, \quad |y - Y| < \varepsilon, \quad |u - U| < \varepsilon, \quad |v - V| < \varepsilon, \quad |t_1 - t_2| < \varepsilon, \quad |w_1 - w_2| < \varepsilon.$$

One of the following conditions at least is satisfied:

$$(u, w_1) \notin ES_{x, t_1}(a') \quad (28)$$

or

$$(w_2, v) \notin ES_{t_2, y}(a'') \quad (29)$$

then $(U, V) \notin ES_{X, Y}(a)$.

Remarks. i) Theorem 3 can be equivalently stated as follows: if $(U, V) \in ES_{X, Y}(a)$, then there exists a sequence, when $\varepsilon \rightarrow 0$, of points $x_\varepsilon, y_\varepsilon, u_\varepsilon, v_\varepsilon, t_{1, \varepsilon}, t_{2, \varepsilon}, w_{1, \varepsilon}, w_{2, \varepsilon}$ satisfying the relations

$$|x_\varepsilon - X| < \varepsilon, \quad |y_\varepsilon - Y| < \varepsilon, \quad |u_\varepsilon - U| < \varepsilon, \quad |v_\varepsilon - V| < \varepsilon, \\ |t_{1, \varepsilon} - t_{2, \varepsilon}| < \varepsilon, \quad |w_{1, \varepsilon} - w_{2, \varepsilon}| < \varepsilon$$

such that:

$$(u_\varepsilon, w_{1, \varepsilon}) \in ES_{x_\varepsilon, t_{1, \varepsilon}}(a')$$

and

$$(w_{2, \varepsilon}, v_\varepsilon) \in ES_{t_{2, \varepsilon}, y_\varepsilon}(a'')$$

for all values of ε .

In these sequences, $w_{1, \varepsilon}$ and $w_{2, \varepsilon}$ are points of $R_{(w)}^p$ and are allowed to tend to infinity when $\varepsilon \rightarrow 0$.

ii) Theorem 3 is interesting only if (X, Y) is a $(u, v) = 0$ point, since otherwise a stronger result (namely Theorem 1), in which the regularity property R is not required and in which the introduction of sequences is not needed, is obtained.

The theorem does provide information at $(u, v) = 0$ points. In fact, the existence of T and W such that $(0, W) \in ES_{X, T}(a')$ and $(W, 0) \in ES_{T, Y}(a'')$ does not prevent the conditions of the theorem to be satisfied, even though the directions determined by the points $(U, \lambda W)$ and $(\lambda W, V)$ both tend, as already mentioned, to directions of $ES_{X, T}(a')$ and $ES_{T, Y}(a'')$ respectively when $\lambda \rightarrow \infty$.

iii) The regularity property R is needed only in certain situations, and the hypotheses of the theorem may be correspondingly weakened, as will appear in the proof of the theorem.

Proof of Theorem 3. The proof of the theorem, given below, is very close to that of Theorem 1. Although one has here to consider situations in which the maximal constants $\gamma_{\max}(\bar{u}, \bar{w})$ or $\gamma_{\max}(\bar{w}, \bar{v})$, tend to zero when $|\bar{w}|$ tends to infinity, the conditions of the theorem and the fact that the decrease is not faster than linear with respect to the angle to the essential support, will ensure the existence of uniform exponential fall-off factors $e^{-\alpha v_0}$ in uniform regions of the form

$$0 \leq v_0 \leq \delta_0(|u| + |v|), \quad \delta_0 > 0$$

The beginning of the proof is the same as in Theorem 1. Namely Eq. (23) still holds and the integration domain over T can be restricted similarly to a compact region D_{α_1} .

Being given a point T_0 in $R_{(t)}^p$, let us now denote by $\mathcal{V}'_\varepsilon$ the set of points w such that :

$$(u, w') \notin ES_{x,t}(a') \quad (30)$$

whenever x, t, u, w' satisfy the relations $|x - X| < \varepsilon/2$, $|t - T_0| < \varepsilon/2$, $|u - U| < \varepsilon/2$, $|w' - w| < \varepsilon/2$.

A set $\mathcal{V}''_\varepsilon$ is defined similarly as the set of points w such that :

$$(w'', v) \notin ES_{t,y}(a'') \quad (31)$$

whenever t, y, w'', v satisfy the relations $|y - Y| < \varepsilon/2$, $|t - T_0| < \varepsilon/2$, $|v - V| < \varepsilon/2$, $|w'' - w| < \varepsilon/2$.

It is easily seen from the conditions of Theorem 3 that

$$\mathcal{V}'_\varepsilon \cup \mathcal{V}''_\varepsilon = \mathbb{R}_{(w)}^p. \quad (32)$$

The regularity property R applied to F' and Lemma 3 of Appendix 2, applied to F'' , ensure the existence of $\alpha' > 0$, $\delta'_0 > 0$, of a constant C' , of a (sufficiently small) neighbourhood $\mathcal{N}'$ of T_0 in $R_{(t)}^p$ and of a (sufficiently small) open cone $\mathcal{V}'$ with apex at the origin in $R_{(u)}^m$ containing U such that :

$$\int_{\substack{T \in \mathcal{N}'(T_0) \\ w \in \tau \mathcal{V}'_\varepsilon}} dT dw |F'(u, w; v_0, X, T)| \times |F''(w, v; v_0, T, Y)| < C' e^{-\alpha v_0} \quad (33)$$

in the region $u \in \mathcal{V}'$, $0 \leq v_0 \leq \delta'_0 \tau$; τ is defined as previously by the relation $(u^2 + v^2)^{1/2} = \tau(U^2 + V^2)^{1/2}$.

To see this, it is sufficient to remark that, in view of the conditions of the theorem, $\text{Min}(\widehat{u, w}; \partial \mathcal{C}') \times (|u| + |w|)$ is strictly positive when u belongs to a sufficiently small neighbourhood of U and when w belongs to $\mathcal{V}'_\varepsilon$, $\mathcal{C}'$ being here the (open) cone obtained by drawing all lines issued from the origin and joining the points (u, w) just mentioned. The distance of (u, w) to $\partial \mathcal{C}'$ is in fact always larger than $\varepsilon/2$.

An analogous result clearly holds when the roles of a' and a'' are exchanged, and in view of (32), one concludes easily that there exists a (sufficiently small) neighbourhood $\mathcal{N}$ of T_0 , a (sufficiently small) open cone $\mathcal{V}$ with apex at the origin in $R_{(u)}^m \times R_{(v)}^n$, containing (U, V) , $\alpha > 0$, $\delta_0 > 0$, and C such that :

$$\int_{T \in \mathcal{N}} dT dw |F'(u, w; v_0, X, T)| \times |F''(w, v; v_0, T, Y)| < C e^{-\alpha v_0}, \quad (34)$$

in the region $(u, v) \in \mathcal{V}$, $0 \leq v_0 \leq \delta_0 \tau$.

Since the domain of integration D_{α_1} with respect to T is compact, the same conclusions hold also for the integral over D_{α_1} and the theorem is therefore proved.

The methods used in the proof of Theorem 3 yield also the following slightly stronger result, which will be useful below :

Lemma. *Let A' , A'' be two linear bounded operators satisfying the same conditions as in Theorem 3 and let X , Y and $U \neq 0$ be given points in $R_{(x)}^m \times R_{(y)}^n$ and $R_{(u)}^m$ respectively.*

Being given $\varepsilon > 0$, let $\mathcal{D}_\varepsilon$ be the set of points in $R_{(v)}^n$ such that whenever $x, y, u, v', t_1, t_2, w_1, w_2$, satisfy the relations

$$|x - X| < \varepsilon, \quad |y - Y| < \varepsilon, \quad |u - U| < \varepsilon, \quad |v' - v| < \varepsilon, \quad |t_1 - t_2| < \varepsilon, \quad |w_1 - w_2| < \varepsilon$$

one of the following two conditions at least is satisfied

$$(u, w_1) \notin ES_{x, t_1}(a')$$

or

$$(w_2, v') \notin ES_{t_2, y}(a'').$$

Then, there exist a neighborhood $\mathcal{N}$ of X, Y in $R_{(x)}^m \times R_{(y)}^n$, a (sufficiently small) open cone $\mathcal{V}$ with apex at the origin in $R_{(u)}^m$ containing U , $\alpha > 0$, $\delta_0 > 0$ and a polynomial $\mathcal{P}$ of the variable v_0 such that :

$$|F(u, v; v_0, x, y)| < \mathcal{P}(v_0) e^{-\alpha v_0} \quad (35)$$

in the region $(x, y) \in \mathcal{N}$, $u \in \mathcal{V}$, $v \in \tau \mathcal{D}_\varepsilon$, $0 \leq v_0 \leq \delta_0 \tau$, where $\tau = |u|/|U|$.

If the conjecture stated at the end of Appendix 2 is accepted, the polynomial factor $\mathcal{P}(v_0)$ in the bounds (35) can moreover be replaced by a function d of the variables u, v, v_0 which is square integrable with respect to v and whose norm with respect to v is independent of u, v_0 .

The analogous result holds if $U = 0$ but $V \neq 0$, and the results can be extended by induction to a product of more than two bounded operators A_r ($r = 1, \dots, q$). The following extension of Theorem 3 is then obtained.

Theorem 4. Let $A_1, \dots, A_q$ be linear bounded operators satisfying the same properties as in Subsect. 1, and the further regularity property R , and let (X, Y) and (U, V) be given points in $R_{(x)}^m \times R_{(y)}^n$ and $R_{(u)}^m \times R_{(v)}^n$ respectively.

If there exists $\varepsilon > 0$ such that, being given any set of points $x, y, u, v, t_1^{(1)}, t_2^{(1)}, w_1^{(1)}, w_2^{(1)}, \dots, t_1^{(q-1)}, t_2^{(q-1)}, w_1^{(q-1)}, w_2^{(q-1)}$ satisfying the relations $|x - X| < \varepsilon, |y - Y| < \varepsilon, |u - U| < \varepsilon, |v - V| < \varepsilon, |t_1^{(r)} - t_2^{(r)}| < \varepsilon, |w_1^{(r)} - w_2^{(r)}| < \varepsilon, r = 1, \dots, q-1$, the following condition cannot be all satisfied :

$$\begin{aligned} (u, w_1^{(1)}) &\in ES_{x, t_1^{(1)}}(a_1) \\ (w_2^{(r-1)}, w_1^{(r)}) &\in ES_{t_2^{(r-1)}, t_1^{(r)}}(a_r), \quad r = 2, \dots, q-1 \\ (w_2^{(q-1)}, v) &\in ES_{t_2^{(q-1)}, y}(a_q), \end{aligned}$$

then $(U, V) \notin ES_{X, Y}(a)$.

The Remarks i)–iii) that follow Theorem 3 can be likewise extended to the present situation.

As appears above, the present proof of Theorem 4 makes use (when $q > 2$) of the conjecture stated at the end of Appendix 2. If this conjecture turned out not to be always satisfied, a corresponding technical condition would be needed so far on all operators $A^{(r)} = A_r \dots A_1$ ($r = 2, \dots, q$).

4. Refined Macro-Causality Condition and $u=0$ Structure Theorem

4.1. General Results in Physical Situations

The considerations of this section apply to bubble diagrams B whose external and internal lines are not necessarily associated with the same type of particle. (As already mentioned, the results of Sect. 2 can be adapted to this more general case.) It will be convenient to label all external and internal lines in a given order; k and l will then denote indices referring to the external and internal lines, respectively, and p_k and p_l will denote on-mass-shell four-momenta variables associated with the external and internal lines respectively.

For simplicity, the bubble diagram functions F_B , and the distributions F_b associated with each bubble b of B , will be considered, as in [7], as being defined on the space of all initial and final three-momenta variables $\mathbf{p}_k$, or of all three-momenta variables $\{\mathbf{p}_k, \mathbf{p}_l\}_b$ involved at b , respectively¹⁹. The essential support of F_B , or F_b , at a given point $\{\mathbf{P}_k\}$, or $\{\mathbf{P}_k, \mathbf{P}_l\}_b$, respectively, is then a cone with its apex at the origin in the space of the dual variables $\mathbf{v}_k$ of the variables $\mathbf{p}_k$, or of the variables of the set $\{\mathbf{v}_k, \mathbf{v}_l\}_b$, respectively. The definition of scalar products is the same as in [7].

It is useful to associate with each point $\{\mathbf{p}_k\}$, or $\{\mathbf{p}_k, \mathbf{p}_l\}_b$, and each point $\{\mathbf{v}_k\}$, or $\{\mathbf{v}_k, \mathbf{v}_l\}_b$, in the space of dual variables a certain configuration of trajectories in four-dimensional space-time. The trajectory of line k , or line l , is the *full line* in space-time that is parallel to the on-mass-shell four-vector $p_k[(p_k)_0 = (\mathbf{p}_k^2 + m_k^2)^{1/2}]$, or p_l , respectively, and passes through the space-time point $v_k = (0, \mathbf{v}_k)$ or $v_l = (0, \mathbf{v}_l)$, respectively. Diagrams $\mathcal{E}_b$, $\mathcal{E}_B$, and $(\mathcal{E}_B)_\varepsilon$, where ε is a given positive number ($\varepsilon > 0$), are then defined as follows:

Definition 3. A diagram $\mathcal{E}_b$ associated with a bubble b is a configuration of space-time trajectories associated with each incoming and outgoing line of b such that $\{\mathbf{v}_k, \mathbf{v}_l\}_b$ lies in the essential support of F_b at the point $\{\mathbf{p}_k, \mathbf{p}_l\}_b$.

A diagram $\mathcal{E}_B$, resp. $(\mathcal{E}_B)_\varepsilon$, associated with a bubble diagram B is a collection of diagrams $\mathcal{E}_b$ (associated with each bubble b of B) that fit together, resp. fit together up to ε : if b_1 and b_2 are the two bubbles of B at which an internal line l of B is respectively outgoing and incoming, then the corresponding lines associated with l in $\mathcal{E}_{b_1}$ and $\mathcal{E}_{b_2}$ must coincide in space-time (and can be identified as a unique internal line of $\mathcal{E}_B$), respectively correspond to points $(\mathbf{p}_l)_1, (\mathbf{v}_l)_1$ and $(\mathbf{p}_l)_2, (\mathbf{v}_l)_2$, such that

$$|(\mathbf{p}_l)_1 - (\mathbf{p}_l)_2| < \varepsilon \quad (36)$$

$$|(\mathbf{v}_l)_1 - (\mathbf{v}_l)_2| < \varepsilon. \quad (37)$$

In the case of diagrams $(\mathcal{E}_B)_\varepsilon$, the two lines associated with l in $\mathcal{E}_{b_1}$ and $\mathcal{E}_{b_2}$ respectively are thus allowed to be slightly displaced and twisted with respect to each other, in the sense of Eqs. (36) and (37).

¹⁹ This is fully legitimate since the energy variables can be expressed in terms of the three-momenta variables in view of the mass-shell conditions. One can alternatively translate everything that follows by considering F_B and F_b as being defined in the space of on-mass-shell four-momenta. Both ways are equivalent (see [9])

Definition 4. A point $\{\mathbf{p}_k\}$ is called a $u=0$ point of B in the sense of the diagrams $\mathcal{E}_B$ if there exists at least one $\mathcal{E}_B$ such that all its external trajectories pass through the origin and have the on-mass-shell four-momenta $p_k[(p_k)_0 = (\mathbf{p}_k^2 + m_k^2)^{1/2}]$, while at least one internal trajectory does not pass through the origin.

The following theorem then holds:

Theorem 5. *The only possible points $\{\mathbf{v}_k\}$ in the essential support of F_B at a point $\{\mathbf{p}_k\}$ that is not a $u=0$ point of B (in the sense of Definition 4), are those corresponding to configurations of external trajectories of some $\mathcal{E}_B$.*

The proof of this result is part of the proof of the structure theorem of [7], as can be seen by withdrawing from it all specific informations on the essential supports of each F_b . It applies to cases when each F_b is not necessarily a connected kernel of S or S^{-1} but may be a more general kernel of bounded operator or also a more general distribution satisfying energy momentum conservation.

When each F_b is the kernel of a bounded operator, Theorem 5 is alternatively a direct consequence of Theorem 2 of Sect. 3.1. Each operator A_r is, in the application, a tensorial product of operators F_b , and the essential support of its kernel is easily determined in terms of the essential supports of the kernels of each F_b , since these kernels do not depend on the same variables. As a matter of fact, an elementary result of essential support theory, which directly follows from the definition of the essential support, says this: the essential support at a point $X = [X_{(1)}, \dots, X_{(s)}]$ of the (tensorial) product $f_1[x_{(1)}]f_2[x_{(2)}] \dots f_s[x_{(s)}]$ of s distributions $f_1, \dots, f_s$ that do not depend on the same, possibly multi-dimensional, variables $x_{(1)}, \dots, x_{(s)}$, is the set of points $u = [u_{(1)}, \dots, u_{(s)}]$ such that

$$u_{(i)} \in ES_{X_{(i)}}(f_i). \quad (38)$$

In the application, a point in the essential support of the kernel of the tensorial product A_r of operators F_b is therefore characterized by a set of independent configurations of trajectories, each one of these configurations being here associated with one of the operators F_b .

Theorem 5 applies only away from $u=0$ points. If each F_b is the kernel of a bounded operator satisfying energy-momentum conservation and the regularity property R stated in Sect. 3, the following result is derived from Theorems 3 and 4 at $u=0$ points²⁰.

Theorem 6. *The only possible points $\{\mathbf{v}_k\}$ in the essential support of F_B at a $u=0$ point $\{\mathbf{p}_k\}$ are those for which there exists a sequence, when $\varepsilon \rightarrow 0$, of diagrams $(\mathcal{E}_B)_\varepsilon$ whose external trajectories correspond, for each k , to points $(\mathbf{p}_k)_\varepsilon, (\mathbf{v}_k)_\varepsilon$ such that*

$$|(\mathbf{p}_k)_\varepsilon - \mathbf{p}_k| < \varepsilon \quad (38)$$

$$|(\mathbf{v}_k)_\varepsilon - \mathbf{v}_k| < \varepsilon. \quad (39)$$

In other words, in each diagram of the sequence, the two lines associated with an internal line of B are as before allowed to be displaced and twisted with respect to each other up to ε [in the sense of Eqs. (36) and (37)], the external trajectories are

²⁰ In the case of a product of more than two operators A_r , the conjecture or technical condition used in Sect. 3 in the proof of Theorem 4 is also needed so far

allowed to be displaced and twisted up to ε [in the sense of Eqs. (38) and (39)] with respect to the given trajectories defined (for each k) by the points $\mathbf{p}_k, \mathbf{v}_k$ and finally $\varepsilon \rightarrow 0$ in the limit.

We conclude this subsection by recalling the connection between the essential support of F_B and the essential support of the distribution f_B defined in Eq. (15) (see [5, 7] for details). The essential support of a distribution f defined on a real analytic manifold $\mathcal{M}$, at a point p of $\mathcal{M}$, is a cone with apex at the origin in the cotangent space $T_p^* \mathcal{M}$ at p to $\mathcal{M}$. When $\mathcal{M}$ is the physical region of a given process, a point in $T_p^* \mathcal{M}$, where $p = \{p_k\}$, is a collection $u = \{u_k\}$ of four-vectors u_k , defined modulo addition, for each k , of vectors of the form $\lambda_k p_k + a$, where a is independent of k . Equivalently, a point in $T_p^* \mathcal{M}$ is characterized by a relative configuration, defined modulo global space-time translations, of space-time trajectories associated with each initial and final particle and parallel to the respective four-momenta p_k . If a configuration of external trajectories corresponds to a point in the essential support of F_B , the class of configurations obtained by global space-time translations defines a point in the essential support of f_B . (Conversely, if a configuration, defined modulo global space-time translations, corresponds to a point in the essential support of f_B , then all configurations derived from it correspond to points in the essential support of F_B).

4.2. Geometrical Definitions

In this subsection, we present preliminary definitions that will be needed in Subsect. 3.

Definition 5 below of graphs G_b is the usual definition of multiple scattering graphs. The definition of the graphs G_B is also the usual definition, given for instance in [7], of topological graphs associated with a bubble diagram B . Definition 6 of diagrams $\mathcal{D}_b$ is, when b is a plus bubble, the usual definition of classical multiple scattering space-time diagrams ($+\alpha$ -Landau diagrams), except that vertices “at infinity” are also introduced when all incoming and outgoing trajectories are parallel. In this case, these trajectories are not required to coincide, but must satisfy an angular-momentum conservation law [11, 15]²¹. Definition 7 of the diagram $\mathcal{D}_B$ that can be associated with a bubble diagram B is again analogous to that given in [7], but vertices at infinity are also introduced. Finally, the diagrams $(\mathcal{D}_b)_\varepsilon$ and $(\mathcal{D}_B)_\varepsilon$ introduced in Definitions 6 and 7 are defined similarly, but some of the constraints of the diagrams $\mathcal{D}_b$ or $\mathcal{D}_B$ are satisfied only “up to ε ”. For a physical discussion of the introduction of these diagrams, see Sect. 5.

Definition 5. A connected multiple scattering graph G_b associated with a bubble b is a connected topological graph characterized by a set of vertices and of oriented external and internal lines. There is one external line for each incoming or outgoing line of b . It is associated with a given vertex and is incoming or outgoing, at that vertex, if it corresponds to an incoming or outgoing line of b , respectively. Each internal line of G_b is associated with two vertices. It is outgoing from one of

²¹ In the terminology of [15], diagrams $\mathcal{D}_b$ would be called “generalized $+\alpha$ -Landau diagrams” if they contain vertices at infinity

them and is incoming to the second one. A particle, with a given mass, is moreover associated with each internal line. Finally, there are at least two incoming and two outgoing lines at each vertex (or one incoming and one outgoing line of identical particles).

A graph G_B associated with a bubble diagram B is a collection of graphs G_b associated with each bubble b of B ; being given any internal line of B , which runs between two bubbles b_1 and b_2 where it is respectively outgoing and incoming, the two corresponding lines in G_{b_1} and G_{b_2} are moreover identified as a common internal line of G_B . It is outgoing and incoming at the respective vertices of G_{b_1} and G_{b_2} .

Definition 6. An elementary space-time diagram $\mathcal{D}_v$ [resp. $(\mathcal{D}_v)_\varepsilon$] associated with a vertex v of a graph G_b or G_B , is a space-time representation of that vertex and of the incoming and outgoing lines at that vertex. In this representation, each line is represented by a full line in space-time, which must be parallel to a given on-mass-shell four-momentum. Energy-momentum must be conserved :

$$\sum_{i \in I_v} p_i = \sum_{j \in J_v} p_j, \quad (40)$$

where I_v and J_v are the sets of incoming and outgoing lines respectively.

Finally, the incoming and outgoing lines of $\mathcal{D}_v$ [resp. $(\mathcal{D}_v)_\varepsilon$] must pass through a common space-time vertex V that represents v , (resp. must pass at a distance less than ε of that vertex) except for the following possibility: if all incoming and outgoing four-momenta are colinear, V is called a parallel vertex and is possibly "at infinity" in some direction. Then the incoming and outgoing lines of $\mathcal{D}_v$, [resp. of $(\mathcal{D}_v)_\varepsilon$], which are parallel, are not required to coincide, but must satisfy angular-momentum conservation²²:

$$\sum_{i \in I_v} ((u_i)_\mu (p_i)_\nu - (u_i)_\nu (p_i)_\mu) = \sum_{j \in J_v} ((u_j)_\mu (p_j)_\nu - (u_j)_\nu (p_j)_\mu), \quad (41)$$

$\mu, \nu = 0, 1, 2, 3$ where u_i , or u_j , is an arbitrary point on line i , or j , respectively, and μ, ν denote components of the four-vectors considered, (resp. must satisfy angular-momentum conservation up to ε : if $u_k = (0, \mathbf{u}_k)$ denotes the point of line k with zero time component, then the quantity

$$\left| \sum_{i \in I_v} (\mathbf{u}_i)(p_i)_0 - \sum_{j \in J_v} (\mathbf{u}_j)(p_j)_0 \right| \text{ must be less than } \varepsilon).$$

A connected diagram $\mathcal{D}_b$, resp. $(\mathcal{D}_b)_\varepsilon$, associated with a connected graph G_b is a collection of diagrams $\mathcal{D}_v$ that fit together, resp. is a collection of diagrams $(\mathcal{D}_v)_\varepsilon$ that fit together up to ε : if v_1, v_2 are the vertices of G_b at which an internal line l of G_b is outgoing and incoming respectively, then the corresponding lines in $\mathcal{D}_{v_1}, \mathcal{D}_{v_2}$, resp. in $(\mathcal{D}_{v_1})_\varepsilon$ and $(\mathcal{D}_{v_2})_\varepsilon$, must coincide and are identified as a common internal

²² One sees easily that this condition is independent of the choice of u_i , or u_j . Condition (41) is automatically satisfied by virtue of energy-momentum conservation if v is not at infinity (to see this, it is sufficient to choose all points u_i or u_j at v). When v is at infinity, it is on the other hand sufficient to state it for the components $\mu=0, \nu=1, 2, 3$. Since all four-momenta involved at v are parallel, Eq. (41) is then also satisfied for the values $\mu, \nu=1, 2, 3$

line of $\mathcal{D}_b$, resp. must coincide up to ε in the sense of Eqs. (36) and (37). Moreover, the following time ordering condition must be satisfied: the space-time vertex that represents v_2 must be strictly later in time²³ than the space-time vertex that represents v_1 if b is a plus bubble, or strictly earlier in time if v is a minus bubble.

Definition 7. A diagram $\mathcal{D}_B$, resp. $(\mathcal{D}_B)_\varepsilon$, associated with a bubble diagram B is a collection of diagrams $\mathcal{D}_b$, resp. $(\mathcal{D}_b)_\varepsilon$, that are associated with each bubble b of B and fit together, resp. fit together up to ε . Equivalently, it is a collection of diagrams $\mathcal{D}_v$, resp. $(\mathcal{D}_v)_\varepsilon$, that are associated with each vertex of a graph G_B , fit together, resp. fit together up to ε , and satisfy the time ordering conditions introduced in Definition 6 for pairs of space-time vertices corresponding to subgraphs G_b of G_B .

There is no time ordering condition on pairs of space-time vertices corresponding to vertices of an internal line of G_B associated with an original internal line of B .

The following further definitions will be useful. In these definitions, a point $\{\mathbf{p}_k\}$ is a set of external, initial and final, three-momenta associated with each external line of a bubble diagram B . The definitions cover the case when B is composed of a single bubble b , or the case of non-trivial bubble diagrams B .

Definition 8. $\mathcal{C}_B(\{\mathbf{p}_k\})$ is the set of points $\{\mathbf{v}_k\}$ such that the trajectories defined for each k , by the points $\mathbf{p}_k, \mathbf{v}_k$ are the external trajectories of at least one diagram $\mathcal{D}_B$.

$\hat{\mathcal{C}}_B(\{\mathbf{p}_k\})$ is the set of points $\{\mathbf{v}_k\}$ such that the trajectories defined, for each k , by the points $\mathbf{p}_k, \mathbf{v}_k$ can be obtained as limits, when $\varepsilon \rightarrow 0$, of the external trajectories of a sequence of diagrams $(\mathcal{D}_B)_\varepsilon$.

We next introduce a geometrical definition of $u=0$ points in the sense of space-time diagrams. They are closely related to the $u=0$ points introduced in Sect. 3, in the case when the bubbles are connected kernels of S or S^{-1} , i.e. when their essential supports will be those provided by macrocausality and unitarity (see Sect. 4.3).

Definition 9. A point $\{\mathbf{p}_k\}$ is called a $u=0$ point of B in the sense of diagrams $\mathcal{D}_B$ if there exists a $\mathcal{D}_B$ such that all its external lines pass through the origin and have the respective on-mass-shell four-momenta p_k , while at least one internal line does not pass through the origin.

The internal lines of $\mathcal{D}_B$ considered here can either be internal lines of subdiagrams $\mathcal{D}_b$ or internal lines associated with original internal lines of B .

Definition 10. A point $\{\mathbf{p}_k\}$ is called a restricted $u=0$ point of B (in the sense of diagrams $\mathcal{D}_B$) if there exists a $\mathcal{D}_B$ such that all its external lines pass through the origin and have the (on-mass-shell) four-momenta p_k and such that at least one of the two following conditions is satisfied:

- i) One internal line of $\mathcal{D}_B$ associated with an original internal line of B does not pass through the origin.
- ii) One internal line of a subdiagram $\mathcal{D}_b$ of $\mathcal{D}_B$ has specified vertices that are not both parallel vertices²⁴ and does not pass through the origin.

²³ An appropriate extension of this notion is introduced to cover cases in which two space-time vertices would be at infinity in the same direction

²⁴ The specified vertices of a line are the vertices where it is outgoing and incoming. A vertex is a parallel vertex if all incoming and outgoing four-momenta at that vertex are colinear (see Definition 6)

Conditions i) and ii) can equivalently be replaced in Definition 10 by the following Conditions i') and ii'):

i') One internal line associated with an original internal line of B has specified vertices that are both parallel vertices, and does not pass through the origin.

ii') One internal line of $\mathcal{D}_B$ has specified vertices that are not both parallel vertices, and does not pass through the origin.

Condition ii') can itself be equivalently replaced in Definition 10 by:

ii'') Non parallel vertices do not all lie at the origin.

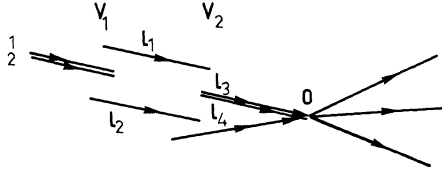
In fact Condition ii') obviously entails Condition ii''). Conversely, let v be a non-parallel vertex that does not lie at the origin. In view of the properties of the diagrams $\mathcal{D}_B$ considered in Definition 9, all external lines of $\mathcal{D}_B$ pass through the origin. Since v is not a parallel vertex, at least one internal line whose v is a specified vertex does not pass through the origin.

Restricted $u=0$ points will be in turn divided for some purposes.

*Definition 11*²⁵. A restricted $u=0$ point of B is called a $u=0$ point of the second kind if there exists a $\mathcal{D}_B$ satisfying the conditions of Definition 10 and such that Condition ii') or equivalently Condition ii'') is satisfied.

We give below some examples of $u=0$ points.

Examples. a) First consider a *single* bubble b . Then all points $\{\mathbf{p}_k\}$ of the set $\mathcal{M}_0$ (i.e. some initial, or some final, on-mass-shell four-momenta p_k are collinear) are $u=0$ points of b in the sense of Definition 9. If for instance two initial four-vectors p_1, p_2 are parallel, one can always consider diagrams $\mathcal{D}_b$ of the form



where the vertices v_1, v_2 are both at infinity. All external lines, as well as the internal lines l_3, l_4 pass through the origin, but l_1, l_2 do not.

However, the points of the set $\mathcal{M}_0$ are not in general *restricted* $u=0$ points of b . The only restricted $u=0$ points of a single bubble b are (by definition) of the second kind. The only possible examples known to us occur in very exceptional situations (special values of masses, etc.) at very special $\mathcal{M}_0$ points.

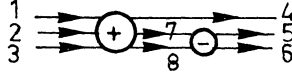
b) Let B be more generally a bubble diagram with more than one bubble, and let $\{\mathbf{p}_k\}$ be a set of external three-momenta. If there exists a set of internal three-momenta $\mathbf{p}_l$ in the allowed integration domains such that $\{\mathbf{p}_k, \mathbf{p}_l\}_b$ is a $\mathcal{M}_0$ point of at least one bubble b , then $\{\mathbf{p}_k\}$ is always a $u=0$ point of B in the sense of Definition 9. To see this, it is sufficient to consider a diagram $\mathcal{D}_b$ of the type introduced in the above example a) and to construct a corresponding diagram $\mathcal{D}_B$ by choosing all other subdiagrams $\mathcal{D}_{b'}$ of $\mathcal{D}_B$ with only one vertex put at the origin (and no internal line).

Such diagrams $\mathcal{D}_B$ are excluded from the Definition 10 of *restricted* $u=0$ points, and $\{\mathbf{p}_k\}$ is therefore not a restricted $u=0$ point, unless the conditions of Definition

²⁵ Second kind $u=0$ points seem to coincide with the “generalized $u=0$ points” of [15]

10 are satisfied by other diagrams $\mathcal{D}_B$. In general, $\{\mathbf{p}_k\}$ will not be a restricted $u=0$ point of B if the only origin of the $u=0$ problem is the occurrence of internal momenta (for the internal lines of B) that correspond to $\mathcal{M}_0$ points for some bubbles, and if the internal lines of B that may have parallel four-momenta do not run between two common bubbles.

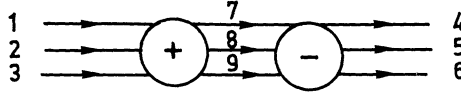
An example, which occurs in [7], is the bubble diagram



in a theory with equal-mass particles. Let us consider a point $\{\mathbf{p}_k\} = \{\mathbf{p}_1, \dots, \mathbf{p}_6\}$ such that $\mathbf{p}_i \neq \mathbf{p}_j$ when $i \neq j$, $i, j = 1, 2, 3$ or $i, j = 4, 5, 6$. The condition $\mathbf{p}_5 \neq \mathbf{p}_6$ ensures that $\mathbf{p}_7 \neq \mathbf{p}_8$, where the indices 7, 8 refer to the two internal lines of B . However, there may exist in certain cases internal on-mass-shell four-momenta p_7, p_8 (satisfying $p_7 + p_8 = p_5 + p_6$) such that for instance $\mathbf{p}_4 = \mathbf{p}_7$. The point $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4, \mathbf{p}_7, \mathbf{p}_8\}$ is then an $\mathcal{M}_0$ point of the plus bubble and correspondingly the point $\{\mathbf{p}_k\}$ is a $u=0$ point of B in the sense of Definition 9. However, it is not a restricted $u=0$ point.

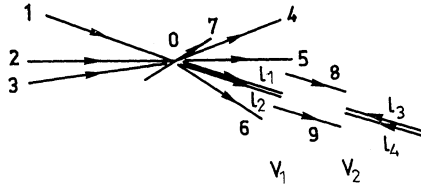
If the lines that may have parallel four-momenta run between two common bubbles, the point $\{\mathbf{p}_k\}$ will on the other hand be also a restricted $u=0$ point of B : see next example.

c) For some bubble diagram functions, all points $\{\mathbf{p}_k\}$ are restricted $u=0$ points. An example, which occurs in [7], is the bubble diagram



with equal-mass particles.

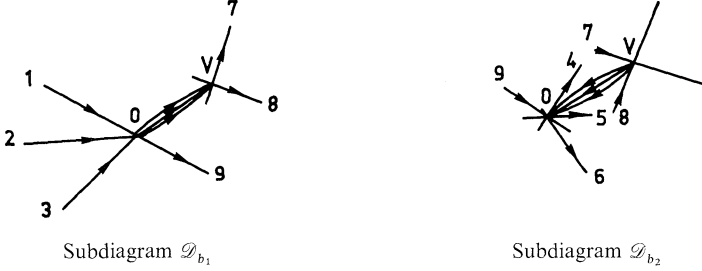
Given any point $\{\mathbf{p}_k\}$, there always exist, in fact, on-mass-shell four-momenta p_l ($l = 7, 8, 9$) such that $p_7 = p_8$, $\sum_{l=7,8,9} p_l = \sum_{i=1,2,3} p_i = \sum_{j=4,5,6} p_j$. The diagram $\mathcal{B}_B$ considered is then of the form:



where the vertices v_1 and v_2 are both at infinity (the lines l_1, l_2 , and l_3, l_4 , are internal lines of the diagrams $\mathcal{D}_{b_1}$ and $\mathcal{D}_{b_2}$, associated with the plus and minus bubbles, respectively).

For any point $\{\mathbf{p}_k\}$, one source of the $u=0$ problem is thus the occurrence of $\mathcal{M}_0$ points in integration domains. When the point $\{\mathbf{p}_k\}$ is above the four-particle threshold $((p_1 + p_2 + p_3)^2 > 16\mu^2)$, the $u=0$ problem arises also from other possibilities, which entail again that $\{\mathbf{p}_k\}$ is a restricted $u=0$ point of B . Another diagram $\mathcal{D}_B$ whose all external lines pass through the origin, but such that the internal lines associated with the original internal lines of B do not all pass through the origin is

for instance the diagram $\mathcal{D}_B$ obtained as the collection of the following diagrams $\mathcal{D}_{b_1}, \mathcal{D}_{b_2}$ associated with the plus and minus bubble respectively, after identification of the internal lines 7, 8, 9 associated with the original lines of B :



With this example, we end our presentation of $u=0$ points and we conclude this subsection with the following result:

Lemma. *If $\{\mathbf{p}_k\}$ is not a second kind $u=0$ point of B , then $\hat{\mathcal{C}}_B(\{\mathbf{p}_k\}) = \mathcal{C}_B(\{\mathbf{p}_k\})$.*

This result is analogous to a result announced independently in [15]. We therefore only outline the proof below. Since $\mathcal{C}_B(\{\mathbf{p}_k\}) \subset \hat{\mathcal{C}}_B(\{\mathbf{p}_k\})$, it is sufficient to show that conversely $\hat{\mathcal{C}}_B(\{\mathbf{p}_k\}) \subset \mathcal{C}_B(\{\mathbf{p}_k\})$ if $\{\mathbf{p}_k\}$ is not a second kind $u=0$ point.

We may restrict our attention to sequences of diagrams $(\mathcal{D}_B)_\varepsilon$ that have a common topological structure (i.e. are space-time representations of given graphs G_B). In general, points $\{\mathbf{v}_k\}$ that belong to $\hat{\mathcal{C}}_B(\{\mathbf{p}_k\})$, but not to $\mathcal{C}_B(\{\mathbf{p}_k\})$, may be obtained in the following situation. In the course of the limiting procedure ($\varepsilon \rightarrow 0$), certain vertices whose incoming and outgoing four-momenta do not tend to be all parallel may be sent to infinity, although the points $(\mathbf{v}_k)_\varepsilon$ that are associated with the external trajectories tend to actual points $\mathbf{v}_k$ (that remain at finite distances). Hence a limiting point $\{\mathbf{v}_k\}$ and a corresponding limit configuration of external trajectories are obtained, although there is no limiting diagram $\mathcal{D}_B$.

Let us now fix a scale in the limiting procedure by keeping constant the sum of all distances between the vertices whose incoming and outgoing four-momenta do not tend to be all parallel. In this scale, these vertices are therefore not sent to infinity and a limiting diagram $\mathcal{D}_B$ exists²⁶. If in the limit the external lines of $\mathcal{D}_B$ do not all pass through the origin, then the point $\{\mathbf{v}_k\}$ obtained also belongs to $\mathcal{C}_B(\{\mathbf{p}_k\})$. On the other hand, if in the limit, all external lines pass through the origin, then by definition [see Condition ii'')] $\{\mathbf{p}_k\}$ is a $u=0$ point of the second kind (non-parallel vertices do not all lie at the origin). The lemma makes no claim at these points.

4.3. $U=0$ Results on Usual Bubble Diagram Functions

As explained in [5, 7], the macro-causality condition (and unitarity if b is a minus bubble) lead to the following property: the essential support of any bubble

²⁶ The lines of $(\mathcal{D}_B)_\varepsilon$ that are involved at vertices whose incoming and outgoing lines tend to be all parallel might be sent to infinity in the limit. However, by using the fact that the two lines associated in $(\mathcal{D}_B)_\varepsilon$ with any internal line of B must become close up to ε , it can be checked that they can be replaced in the limit by lines that remain at finite distances (even when the corresponding parallel vertices are “at infinity”)

function F_b at any non $\mathcal{M}_0$ point²⁷ $\{\mathbf{p}_k\}$ is contained in the set $\mathcal{C}_b(\{\mathbf{p}_k\})$ introduced in Definition 8 (in the case when B is a single bubble b)²⁸.

In view of this essential support property, Theorem 5 of Subsect. 1 then leads to the following theorem, which coincides with the structure theorem of [7].

Theorem 7. *If $\{\mathbf{p}_k\}$ is not a $u=0$ point of B in the sense of diagrams $\mathcal{D}_B$ (Definition 9), then the essential support of F_B at $\{\mathbf{p}_k\}$ is contained in $\mathcal{C}_B(\{\mathbf{p}_k\})$.*

As a matter of fact, the assumption that $\{\mathbf{p}_k\}$ is not a $u=0$ point of B excludes in particular the occurrence of $\mathcal{M}_0$ points for any bubble in integration domains [see remark in the Example b) of $u=0$ points in Subsect. 2]. Hence the above mentioned essential support property of each F_b covers all points encountered in integration domains and the result follows.

Remark. Theorem 7 is not the best result that follows from Theorem 5 and from the above mentioned essential support property of each F_b . In fact, some information on the essential support of F_B at some $u=0$ points can be obtained²⁹. However, it is not of much interest and is omitted here.

The arguments upon which macrocausality is based lead on the other hand to admit in general certain limiting procedures in the statement of macrocausality that might enlarge the essential support. This is discussed here in Sect. 5. The macrocausality condition then leads in general to the following postulate if b is a plus bubble: the essential support of F_b at any point $\{\mathbf{p}_k\}$ is contained in the set $\mathcal{C}_b(\{\mathbf{p}_k\})$. As previously, the same essential support property is derived for minus bubbles from unitarity.

As a consequence of the lemma of Subsect. 2, applied in the case when B has a single bubble b , $\mathcal{C}_b(\{\mathbf{p}_k\})$ can be replaced in the statement by $\mathcal{C}_B(\{\mathbf{p}_k\})$ whenever $\{\mathbf{p}_k\}$ is not a second kind $u=0$ point of b . This includes the cases when $\{\mathbf{p}_k\}$ is not an $\mathcal{M}_0$ point, but it also includes most of the $\mathcal{M}_0$ points (see discussion in Subsect. 2). Theorem 5 of Subsect. 1 and the above mentioned essential support property of each F_b then yield the following extension of Theorem 7 to $u=0$ points of B that are not restricted $u=0$ points.

Theorem 8. *If $\{\mathbf{p}_k\}$ is not a restricted $u=0$ point of B in the sense of Definition 10, then the essential support of F_B at $\{\mathbf{p}_k\}$ is contained in $\mathcal{C}_B(\{\mathbf{p}_k\})$.*

In other words, Theorem 7 can be extended without change to all points that are not restricted $u=0$ points of B , if the definition of the set $\mathcal{C}_B$ given in Subsect. 2 is adopted. Theorem 8 applies for instance to the case described in the Example b) of $u=0$ points in Subsect. 2.

The proof of Theorem 8 follows in the same way as before. It is sufficient to note that if $\{\mathbf{p}_k\}$ is not a restricted $u=0$ point of B , then given any bubble b of B and

²⁷ $\{\mathbf{p}_k\}$ denotes here the set of the incoming and outgoing momenta of b . In the application to the study of bubble diagram functions, it has to be replaced by $\{\mathbf{p}_k, \mathbf{p}_i\}_b$

²⁸ The set $\mathcal{C}_b$ introduced in [5, 7] is defined in terms of ordinary $+\alpha$ -Landau diagrams. However, the two definitions coincide if $\{\mathbf{p}_k\}$ is not an $\mathcal{M}_0$ point. This is because a diagram $\mathcal{D}_b$ whose set of external momenta is not an $\mathcal{M}_0$ point cannot have vertices "at infinity"

²⁹ The best result would be obtained by using Theorem 5 with the essential support property given at non $\mathcal{M}_0$ points, the essential support at $\mathcal{M}_0$ points being allowed to be the whole space of the variables $\{\mathbf{v}_k\}$. The result can be stated by introducing diagrams that differ from the diagrams $\mathcal{D}_B$ when $\mathcal{M}_0$ points are involved

any point $\{\mathbf{p}_l\}_b$ in the allowed integration domains, $\{\mathbf{p}_k, \mathbf{p}_l\}_b$ cannot be a second kind $u=0$ point of b . Otherwise, there would exist a $\mathcal{D}_b$ such that all its external lines pass through the origin (and have the four-momenta of the set $\{\mathbf{p}_k, \mathbf{p}_l\}_b$), while at least one internal line would not have its specified vertices both parallel and would not pass through the origin. But then there would also exist a $\mathcal{D}_B$ with the analogous property, defined by choosing all other subdiagrams $\mathcal{D}_{b'}$ with only one vertex put at the origin (and no internal line).

The point $\{\mathbf{p}_k\}$ would therefore be a restricted $u=0$ point of B , a situation for which Theorem 8 makes no claim.

Theorem 8 does not cover the restricted $u=0$ points, such as the points described in the example c) of $u=0$ points in Subsect. 2. As mentioned there, all points $\{\mathbf{p}_k\}$ are as a matter of fact restricted $u=0$ points of the bubble diagram



In order to cover these points, we shall use the following refined macro-causality condition which, as explained in Sect. 5.3, follows from the same physical ideas as the previous macro-causality condition.

Refined Macro-Causality Condition. Being given any plus bubble b , the essential support of F_b at any point $\{\mathbf{p}_k\}$ is contained in the set $\hat{\mathcal{C}}_b(\{\mathbf{p}_k\})$. Moreover, F_b satisfies the regularity property R of Sect. 3³⁰ when directions of $\hat{\mathcal{C}}_b(\{\mathbf{p}_k\})$ are approached.

As previously the same properties are derived for minus bubbles from unitarity.

Refined macro-causality (so far, together with the mathematical conjecture, or technical condition, needed in Sect. 3 in the case of a product of more than two bounded operators) then leads, in view of Theorem 6, to the following general structure theorem.

$U=0$ Structure Theorem. The essential support of F_B at any point $\{\mathbf{p}_k\}$ is contained in $\hat{\mathcal{C}}_B(\{\mathbf{p}_k\})$.

The proof of the theorem follows from a straightforward application of Theorem 6. As a matter of fact, Theorem 6 provides a slightly more refined result: namely, the essential support of F_B at $\{\mathbf{p}_k\}$ is composed at most of the points $\{\mathbf{v}_k\}$ such that the trajectories $(\mathbf{p}_k, \mathbf{v}_k)$ can be obtained as limits when $\varepsilon \rightarrow 0$ of the external trajectories of diagrams $(\mathcal{E}_B)_\varepsilon$; $(\mathcal{E}_B)_\varepsilon$ is here a collection of diagrams $\mathcal{E}_b$ that fit together up to ε [in the sense of Eqs. (36) and (37)]. For each bubble b , $\mathcal{E}_b$ is here, in view of the essential support property of F_b , a configuration of incoming and outgoing trajectories that can be obtained as limits, when $\varepsilon' \rightarrow 0$, of the incoming and outgoing trajectories of a sequence of diagrams $(\mathcal{D}_b)_{\varepsilon'}$.

If one does not encounter, in integration domains, points $\{\mathbf{p}_k, \mathbf{p}_l\}_b$ that are second kind $u=0$ points of some bubble b , $\mathcal{E}_b$ is simply, in view of the lemma of Subsect. 2, the configuration of external trajectories of a diagram $\mathcal{D}_b$ and the set $\hat{\mathcal{C}}_B(\{\mathbf{p}_k\})$ can correspondingly be replaced in the theorem by a somewhat more simple set. To our present knowledge, this should be a very general situation. On

30 We recall that the analogues of the variable x, y of Sect. 3 are here the sets of components of the incoming and outgoing momenta, and that the analogues of the variables u, v are correspondingly the sets of components of the space three-vectors $\mathbf{v}_k$ associated with the incoming and outgoing lines of b

the other hand, if moreover $\{\mathbf{p}_k\}$ is not a second kind $u=0$ point of B , the set $\hat{C}_B(\{\mathbf{p}_k\})$ can be replaced in the theorem by the set $\mathcal{C}_B(\{\mathbf{p}_k\})$, again by virtue of the lemma of Subsect. 2. However, second kind $u=0$ points of B are not always exceptional and this result is thus not sufficient to eliminate limiting procedures in general. It is our present belief that the set $\hat{C}_B$ can be replaced by $\mathcal{C}_B$, i.e. that limiting procedures do not enlarge the essential support, in more general situations, but we have so far no general result of this type.

The connection between the essential support of F_B and the essential support of the distribution f_B of Eq. (15) has already been outlined at the end of Subsect. 1.

5. Macro-Causality and Refined Macro-Causality: Physical Discussion

The purpose of this section is to describe the physical ideas upon which macrocausality is based, in order to explain how they lead in the same time to the refined macrocausality condition stated in Sect. 4.3. These ideas are those already essentially contained in [6]. A somewhat more detailed discussion will however be needed. We shall omit from it all technical aspects which can be directly derived from [5] or [6].

Macro-causality is an expression of a certain classical limit of quantum theory in terms of particles, and more precisely of the idea that all transfers of energy-momentum over large distances that cannot be attributed to stable real particles in accordance with classical ideas, give effects that are damped exponentially with distance³¹.

This idea is expressed in the form of exponential fall-off properties of transition probabilities. To that purpose, it is useful to consider a special class of initial and final displaced wave functions. These wave functions correspond to particles that have sharp localization properties both in momentum space and space-time and behave in fact like classical particles, in a certain asymptotic limit, up to exponential fall-off properties. They are introduced in Subsect. 1, where the main results needed later on their localization properties will be described.

A certain semi-classical model is then introduced in Subsect. 2, where we explain how it leads to the macro-causality condition of [5, 6], and to its extension at $\mathcal{M}_0$ points. Finally, the way it leads at the same time to the refined macro-causality condition is described and explained on simple examples in Subsect. 3.

Before giving details, we emphasize that there is no attempt to “prove” macrocausality or refined macro-causality, or even to determine precise rates of exponential fall-off. The aim is only to determine the type of exponential fall-off properties that can be reasonably expected. The semi-classical model of Subsect. 2 is not the only one that might be considered. However, it seems to us so far the most satisfactory from a physical viewpoint and is supported by the mathematical methods of the present work, applied for instance to the study of Feynman or phase-space integrals. An alternative model, which can be elaborated on the basis of the results of [15, 17] on Feynman and phase-space integrals, is proposed in [15]. It will be briefly presented and discussed at the end of Subsect. 3.

31 We recall that we consider only systems of massive particles with short-range interactions

5.1. Localization Properties of Free Particles

We consider initial and final displaced wave functions of the form :

$$\phi_{k,\tau}^{u_k}(p'_k) = \chi_k(p'_k) e^{-\gamma\tau\Phi(p'_k;p_k)} e^{-ip'_k \cdot \tau u_k}, \quad (42)$$

where p_k is a given on-mass-shell four-momentum, χ_k is locally analytic in the neighbourhood of p_k , Φ is for instance, of the form $\Phi(p'_k;p_k) = (\mathbf{p}'_k - \mathbf{p}_k)^2$, γ is a positive constant ($\gamma \geq 0$), τ is a space-time dilation parameter and u_k is a space-time translation four-vector which will be taken in the following of the form $(0, \mathbf{u}_k)$, i.e. $(u_k)_0 = 0$. The index k labels as previously the initial and final particles.

In a space-time coordinate system scaled to τ (i.e. in $\hat{x}$ -space, where $x = \hat{x}\tau$), the free particle whose wave function is given by (42) behaves like a classical particle with on-mass-shell four-momentum p_k and with a well defined space-time trajectory that is parallel to p_k and passes through the point u_k , up to exponential fall-off properties in the $\tau \rightarrow \infty$ limit. More precisely, the following results are obtained (for simplicity, we leave below the index k implicit) :

i) The probability density in momentum space, which is equal to $|\phi_\tau^{u_k}(p')|^2$ up to normalization factors, decreases like $e^{-\alpha\gamma\tau}$ when p' is different from p ; $\alpha > 0$ increases with the distance to p .

This probability density vanishes, as $\phi_\tau^{u_k}$ itself, if p' is not on-mass-shell.

ii) The fall-off properties of the probability densities in macroscopic space-time³² are extrapolated by assumption from those of space-time wave functions. These functions fall-off exponentially like $e^{-\alpha\gamma\tau}$ for all sufficiently small values of γ , away from the trajectory defined by (p, u) in the space-time coordinate system scaled to τ .

The value of α is constant (and strictly positive) on any given line issued from the point u and not parallel to p . It is an increasing function of the angle of this line with the trajectory (p, u) (and it tends to zero when this angle tends to zero). The maximal value γ_0 of γ for which the fall-off factor $e^{-\alpha\gamma\tau}$ is obtained on a given line issued from u , is proportional to the distance to u . The proportionality coefficient is again an increasing function of the angle of this line with the trajectory (p, u) and is strictly positive when this angle is non-zero.

iii) More generally, the fall-off properties of joint probability densities with respect to both momentum-space and space-time variables³³ are extrapolated by assumption from those of joint density functions [23].

Being given an (on-mass-shell) four-momentum p' , these functions remain constant when the point x in space-time varies along any given line parallel to p' . In other words, they are functions of space-time trajectories.

32 In the relativistic quantum case, space-time probability densities make sense only at the macroscopic level. This level is obtained in the $\tau \rightarrow \infty$ limit: any arbitrarily small region in the space-time coordinate system scaled to τ becomes macroscopic in actual space-time in this limit. For a more detailed analysis, see [5, 6]

33 Joint probability densities make sense only, in the relativistic quantum case, at the macroscopic space-time level and over regions large compared to $\hbar^3$. These conditions are again satisfied for large τ : in particular, any arbitrarily small region centered around a given momentum and a given point in space, at time zero, in the space-time coordinate system scaled to τ , becomes large compared to $\hbar^3$ in the $\tau \rightarrow \infty$ limit

For a trajectory that is parallel to an (on-mass-shell) four-momentum p' and passes through a point $u'=(0, \mathbf{u}')$ in the space-time coordinate scaled to τ , the exponential fall-off factor obtained when $p' \neq p$ or $\mathbf{u}' \neq \mathbf{u}$ is approximately a product of two terms. A factor $e^{-\alpha\gamma\tau}$ is obtained in a way similar to that of Paragraph i) if p' is different from p . A further factor $e^{-\alpha'\gamma\tau}$ is obtained for sufficiently small values of γ if $\mathbf{u}' \neq \mathbf{u}$. Here $\alpha' > 0$ is a given constant and the maximal value γ_0 of γ is proportional to the distance of $\mathbf{u}'$ to $\mathbf{u}$, with a certain strictly positive proportionality coefficient.

Remark. The properties mentioned in Paragraphs i)–iii) can be likewise extended to integrals of the previous functions over appropriate regions (see [5]).

5.2. Semi-Classical Model and Macro-Causality Condition

Being given a set of initial and final displaced wave functions of the form (42), the transition probability of the process is not expected to fall-off exponentially with τ if there exists a diagram $\mathcal{D}_b$ (see Sect. 4.2, Definition 6) whose external trajectories are the trajectories (p_k, u_k) in the space-time coordinate system scaled to τ : energy-momentum can then be transferred from the initial to the final particles, via stable physical particles, in accordance with classical ideas. (The bubble b is always here a plus bubble.)

Macro-causality gives information when such a diagram does not exist. Before going on, we recall on the other hand that, for reasons explained elsewhere [24], the *connected* diagrams are the only ones relevant in the study of the exponential fall-off properties of the connected amplitudes.

It is assumed that the type of exponential fall-off properties obtained is that suggested by semi-classical arguments. More precisely, we shall consider, in the space-time coordinate system scaled to τ , diagrams of the form $(\mathcal{D}_b)_\varepsilon$ (see Definition 6 in Sect. 4.2) whose external trajectories need not coincide with the trajectories (p_k, u_k) . We shall below denote by p'_k the on-mass-shell four-momentum of line k in such a diagram and by $u'_k=(0, \mathbf{u}'_k)$ the point that lies on this line at time zero.

The diagrams $(\mathcal{D}_b)_\varepsilon$ are similar to the diagrams $\mathcal{D}_b$, except that there are on the one hand possible violations of the locality conditions at each vertex (or of angular-momentum conservation if the vertex is at infinity) and that there are on the other hand possibly two different lines in $(\mathcal{D}_b)_\varepsilon$ for each internal line of G_b . The introduction in the model of this latter possibility is due to the fact that, just as the external particles, an internal particle cannot in the quantum case be strictly localized, even asymptotically, along a given classical trajectory. The consideration of all possible trajectories for an internal particle must be replaced in the quantum case by the consideration of all possible gaussian-type wave functions of a form similar to (42) (see below) with mean trajectories $(\mathbf{K}, \mathbf{w})$ corresponding to all possible classical trajectories. This interpretation is linked with the fact³⁴ that a sum over all possible intermediate states of a particle is an integral, over all possible classical trajectories, of projectors on corresponding gaussian-type wave functions:

$$\int d\mathbf{K} d\mathbf{w} |\phi_{\mathbf{K}, \mathbf{w}}\rangle \langle \phi_{\mathbf{K}, \mathbf{w}}| = 1,$$

34 Note that this fact is essentially that used in the derivation of Eq. (23) in Sect. 3

where

$$\phi_{\mathbf{k}, \mathbf{w}}(k) = (2\pi v_0)^{3/4} (2\omega(\mathbf{k}))^{1/2} e^{-v_0(\mathbf{k}-\mathbf{K})^2} e^{-i\mathbf{k} \cdot \mathbf{w}},$$

$$\omega(\mathbf{k}) = (\mathbf{k}^2 + m^2)^{1/2}, \quad v_0 \equiv \gamma\tau.$$

With any diagram $(\mathcal{D}_b)_\varepsilon$ is then associated an exponential fall-off which is the product of the following factors:

i) If the incoming and outgoing trajectories at a given vertex do not pass through this vertex, a factor of the form $e^{-\beta\tau}$, $\beta > 0$, is obtained. The value of β is assumed to be proportional to the violation of locality, which is characterized (in the space-time coordinate system scaled to τ) by the sum of the distances of this vertex to the incoming and outgoing trajectories at that vertex.

The proportionality coefficient depends physically on the various mechanisms that may give rise to the non-locality effect (unstable particles, etc. ...). It will be assumed that it has a fixed, strictly positive, lower bound.

In the case of a vertex at infinity, it will be similarly assumed that β is proportional to the violation of angular-momentum conservation, i.e. to the quantity $\left| \sum_{i \in J_v} (\mathbf{u}_i)(p_i)_0 - \sum_{j \in J_v} (\mathbf{u}_j)(p_j)_0 \right|$ introduced in Definition 6.

ii) If the trajectory of an external line k is different from (p_k, u_k) , an exponential fall-off factor arises in the way described in Subsect. 1.

If $\mathbf{p}'_k \neq \mathbf{p}_k$, it can be chosen to be that associated with the fall-off of momentum-space probabilities. If the approximate interaction region that involves line k as an incoming, or outgoing, line does not intersect the trajectory (p_k, u_k) , it can be chosen to be that associated with the fall-off of space-time probabilities in that region (in the $\tau \rightarrow \infty$ limit). More generally, it is the factor associated with the fall-off properties of the density functions away from the trajectory (p_k, u_k) .

iii) If the trajectories of the two internal lines associated in $(\mathcal{D}_b)_\varepsilon$ with a common internal line of G_b do not coincide, a similar exponential fall-off factor is considered, in accordance with the previous analysis, the pairs of points $\mathbf{p}_k, \mathbf{v}_k$ and $\mathbf{p}'_k, \mathbf{v}'_k$ in the case of an external line being here replaced by the pairs of points $(\mathbf{p}_l)_1, (\mathbf{v}_l)_1$ and $(\mathbf{p}_l)_2, (\mathbf{v}_l)_2$ corresponding to these two lines.

The rate of exponential fall-off of the connected amplitude between initial and final wave functions of the form (42) is then determined, in the model, as the minimal one obtained by considering all possible diagrams.

We first note that, if there exists a sequence of diagrams $(\mathcal{D}_b)_\varepsilon$ whose external trajectories coincide, in the $\varepsilon \rightarrow 0$ limit, with the trajectories (p_k, u_k) , the rates of exponential fall-off that arise from the above analysis are arbitrarily small, if ε is chosen sufficiently small. Hence exponential fall-off cannot be expected in these situations. According to Definition 8 of Sect. 4.2, they correspond to the cases when $\{\mathbf{u}_k\} \in \hat{\mathcal{C}}_b(\{\mathbf{p}_k\})$.

If on the other hand $\{\mathbf{u}_k\} \notin \hat{\mathcal{C}}_b(\{\mathbf{p}_k\})$, then an exponential fall-off factor $e^{-\alpha\gamma\tau}$ is obtained (see below) for some $\alpha > 0$ and for sufficiently small values of $\gamma (0 \leq \gamma \leq \gamma_0, \gamma_0 > 0)$. It is directly at the origin of the essential support property of F_b stated in Sect. 4.3, in view of the definition of the essential support (see details in [5]).

The above result can be obtained for instance as follows. When $\{\mathbf{u}_k\} \notin \hat{\mathcal{C}}_b(\{\mathbf{p}_k\})$, there exists by definition $\varepsilon > 0$ such that in any diagram $(\mathcal{D}_b)_\varepsilon$ either one of the

violations of the locality conditions at each vertex (or of angular-momentum conservation) is larger than ε_0 , or one of the quantities $|\mathbf{p}'_k - \mathbf{p}_k|$, $|\mathbf{u}'_k - \mathbf{u}_k|$ or $|(\mathbf{p}_l)_1 - (\mathbf{p}_l)_2|$, $|(\mathbf{v}_l)_1 - (\mathbf{v}_l)_2|$ is larger than ε_0 . In the second case, an exponential fall-off factor $e^{-\alpha\gamma\tau}$ is obtained for sufficiently small values of γ : see Paragraphs ii) and iii) of the present subsection. Moreover, it is easily seen, in view of the discussion of Subsect. 1, that $\alpha > 0$ and the maximal value $\gamma'_0 > 0$ of γ are independent of the diagram considered. In the first case, an exponential fall-off factor of the form $e^{-\beta\tau}$, $\beta > 0$, is obtained [see Paragraph i) of the present subsection], and β is again independent of the diagram considered (it depends only on ε_0). This fall-off factor implies a fall-off factor of the form $e^{-\alpha\gamma\tau}$ for all γ satisfying $0 \leq \gamma \leq \gamma'_0 = \beta/\alpha$.

The minimal fall-off factor obtained for any diagram $(\mathcal{D}_b)_\varepsilon$ is therefore $e^{-\alpha\gamma\tau}$ when $0 \leq \gamma \leq \gamma_0 = \text{Min}(\gamma'_0, \gamma''_0)$ and the announced result is proved. Q.E.D.

Remark. If the S matrix is replaced by the Feynman or phase-space integral associated with a given graph G_b , the type of exponential fall-off properties of the corresponding amplitudes can probably be determined similarly, but without any need of introducing violations of the locality conditions (or of angular-momentum conservation) at each vertex. The latter are introduced for the actual physical transition amplitudes. They correspond to complex singularities of the S matrix different from those associated with Feynman or phase-space integrals (poles corresponding to unstable particles, ...). In the case of phase-space integrals, the time ordering conditions of the diagrams $(\mathcal{D}_b)_\varepsilon$ should on the other hand be removed.

5.3. Refined Macro-Causality

A detailed analysis of the way the fall-off factor $e^{-\alpha\gamma\tau}$ is obtained when $\{\mathbf{u}_k\} \notin \hat{\mathcal{C}}_b(\{\mathbf{p}_k\})$ in Subsect. 2 shows that α depends only on the distance of $\{\mathbf{p}_k\}$ to the nearest point $\{\mathbf{p}''_k\}$ such that $\{\mathbf{u}_k\} \in \hat{\mathcal{C}}_b(\{\mathbf{p}''_k\})$, while γ_0 depends on the distance of $\{\mathbf{p}_k, \mathbf{u}_k\}$ to the nearest point $\{\mathbf{p}''_k, \mathbf{u}''_k\}$ such that $\{\mathbf{u}''_k\} \in \hat{\mathcal{C}}_b(\{\mathbf{p}''_k\})$.

If we consider an open cone $\mathcal{C}$ (with apex at the origin) of points $\{\mathbf{u}_k\}$ that all lie outside $\hat{\mathcal{C}}_b(\{\mathbf{p}_k\})$ at all points $\{\mathbf{p}'_k\}$ of a neighbourhood $\mathcal{N}$ of a point $\{\mathbf{p}_k\}$, then there is more precisely a common $\alpha > 0$ for all points of any given neighbourhood $\mathcal{N}'$ of $\{\mathbf{p}_k\}$ whose closure is contained in $\mathcal{N}$ and all points $\{\mathbf{u}_k\}$ in $\mathcal{C}$. The maximal value of γ may depend on the point $\{\mathbf{u}_k\}$ but is at least equal to $\gamma_0|\{\mathbf{u}_k\}; \partial\mathcal{C}|$ or $\gamma_0|\{\mathbf{u}_k\} \times \{\overline{\{\mathbf{u}_k\}}; \partial\mathcal{C}\}|$, where $\gamma_0 > 0$ is a constant independent of $\{\mathbf{u}_k\}$, $|\{\mathbf{u}_k\}; \partial\mathcal{C}|$ is the distance of $\{\mathbf{u}_k\}$ to $\partial\mathcal{C}$, and $\{\overline{\{\mathbf{u}_k\}}; \partial\mathcal{C}\}$ is the angle of the direction of $\{\mathbf{u}_k\}$ with $\partial\mathcal{C}$.

To obtain these results, one may here divide, for each point $\{\mathbf{u}_k\}$ in $\mathcal{C}$, the diagrams $(\mathcal{D}_b)_\varepsilon$ into two sub-classes that include respectively the diagrams such that the sum of the quantities associated with the violations of locality and angular-momentum conservation is less than, or larger than, a given percentage (to be determined) of $|\{\mathbf{u}_k\}; \partial\mathcal{C}|$. These two classes are then treated as in the argument given at the end of Subsect. 2. To complete the proof, one must show here that in any diagram of the first class at least one of the quantities $|\mathbf{p}''_k - \mathbf{p}'_k|$, $\forall \{\mathbf{p}'_k\} \in \mathcal{N}'$, or $|\mathbf{u}'_k - \mathbf{u}_k|$, or $|(\mathbf{p}_l)_1 - (\mathbf{p}_l)_2|$, or $|(\mathbf{v}_l)_1 - (\mathbf{v}_l)_2|$ is larger than a certain percentage of $|\{\mathbf{u}_k\}; \partial\mathcal{C}|$.

The way these results arise is easily seen on the two simple examples described below. The connection between these informations and the behaviour of the maximal value of γ in the regularity property R of Sect. 3 is established by noting

that the analogues of the variables v_0 and (u, v) of Sect. 3 are in the physical application the variables $\gamma\tau$ and $\{\tau\mathbf{u}_k\}$ respectively: see the expression of the connected amplitude between initial and final displaced wave functions of the form (42).

The examples that we shall consider occur in the study of the bubble $\equiv\oplus\equiv$, in a theory with only one mass, when one wishes to determine the essential support of the bubble diagram function $\equiv\oplus\ominus\equiv$ according to the method of Sect. 3.2. We recall that all points are $u=0$ points for this bubble diagram function (see Sect. 4.2).

We shall denote, as in Subsect. 2, by 1,2,3 and 4,5,6 the initial and final particles of the bubble $\equiv\oplus\equiv$ and by $\mathbf{p}_1, \dots, \mathbf{p}_6$, and $\mathbf{u}_1, \dots, \mathbf{u}_6$ corresponding momenta and space displacements. In the two cases that we shall consider, $\mathbf{p}_1, \dots, \mathbf{p}_6$ vary in (sufficiently small) neighbourhoods of points $\mathbf{P}_1, \dots, \mathbf{P}_6$ such that $\mathbf{P}_i \neq \mathbf{P}_j$, $i, j = 1, 2, 3$ and $\mathbf{P}_5 = \mathbf{P}_6$, and $\mathbf{u}_1, \dots, \mathbf{u}_4$ vary in (sufficiently small) neighbourhoods of given points $\mathbf{U}_1, \dots, \mathbf{U}_4$. On the other hand, $\mathbf{u}_5$ and $\mathbf{u}_6$ are chosen sufficiently large and may tend to infinity. The analysis extends easily to the cone $\mathcal{C}$ of lines issued from the origin and passing through these points $\{\mathbf{u}_k\}$.

First Case. We first consider the case when the initial trajectories defined by the points $(\mathbf{P}_k, \mathbf{U}_k)$, $k = 1, 2, 3$ do not meet.

Then, the trajectories defined by the points $\mathbf{p}_k, \mathbf{u}_k$, $k = 1, \dots, 4$ still do not meet when $\mathbf{p}_k$ and $\mathbf{u}_k$ lie in sufficiently small neighbourhoods of $\mathbf{P}_k$ and $\mathbf{U}_k$ respectively. Because of this fact, it is easily seen that, independently of the (sufficiently large) values of $\mathbf{u}_5$ and $\mathbf{u}_6$, $\{\mathbf{u}_k\}_{k=1, \dots, 6} \notin \hat{\mathcal{C}}_b(\{\mathbf{p}_k\})$. Moreover, a common fall-off factor $e^{-\alpha\gamma\tau}$, $\alpha > 0$, is obtained for $0 \leq \gamma \leq \gamma_0$, $\gamma_0 > 0$, where α and γ_0 are given constants independent of $\mathbf{u}_5, \mathbf{u}_6$, as a consequence of the informations of Subsect. 1 and 2 applied to the trajectories 1,2,3,4.

This conclusion holds in particular if the region over which $\mathbf{u}_5$ and $\mathbf{u}_6$ vary includes sets of points of the form $\mathbf{u}_5 = \varrho\mathbf{U}_5$, $\mathbf{u}_6 = \varrho\mathbf{U}_6$, where $\mathbf{U}_5$ and $\mathbf{U}_6$ are given points such that $\mathbf{U}_6 = -\mathbf{U}_5$ and ϱ is sufficiently large. In this case, the direction determined by the point $\{\mathbf{u}_k\}_{k=1, \dots, 6}$ tends, when $\varrho \rightarrow \infty$, to the direction determined by the point $(0, 0, 0, 0, \mathbf{U}_5, \mathbf{U}_6)$ which does belong to $\hat{\mathcal{C}}_b(\{\mathbf{P}_k\})$:

In the application to the study of the essential support of $\equiv\oplus\ominus\equiv$, the points $\mathbf{p}_k, \mathbf{u}_k$, $k = 4, 5, 6$ are sets of possible momenta and space displacements of the intermediate particles, over which there is integration. The refined macro-causality condition in this case is relevant to solve the $u=0$ problem when the initial momenta and displacements are $\mathbf{P}_k, \mathbf{U}_k$, $k = 1, 2, 3$ independently of the given momenta and space displacements of the final particles; as we have seen, a uniform exponential fall-off factor $e^{-\alpha\gamma\tau}$, $0 \leq \gamma \leq \gamma_0$, can in fact be extracted, for all sufficiently large values of $\mathbf{u}_5$ and $\mathbf{u}_6$, from the generalized Fourier transform F' associated with the bubble $\equiv\oplus\equiv$.

Second Case. We next consider the case when the trajectories $(\mathbf{P}_k, \mathbf{U}_k)$, $k = 1, \dots, 4$ meet at the origin ($\mathbf{U}_k = 0$, $k = 1, \dots, 4$) and when $\mathbf{u}_5, \mathbf{u}_6$ vary in a region such that $|\mathbf{u}_5 + \mathbf{u}_6| \geq \delta$, where δ is a strictly positive given number; $|\mathbf{u}_5 + \mathbf{u}_6|$ is allowed to be arbitrarily close to δ even when $\mathbf{u}_5$ and $\mathbf{u}_6$ tend to infinity.

The condition $|\mathbf{u}_5 + \mathbf{u}_6| \geq \delta > 0$ does not allow angular-momentum conservation to be satisfied and hence it is again easily seen that $\{\mathbf{u}_k\}_{k=1, \dots, 6} \notin \hat{\mathcal{C}}_b(\{\mathbf{p}_k\})$,

although the direction determined by the point $\{\mathbf{u}_k\}$ tends to a direction of $\mathcal{C}_b(\{\mathbf{P}_k\})$ when $\mathbf{u}_5$ and $\mathbf{u}_6$ tend to infinity in a way such that $|\mathbf{u}_5 + \mathbf{u}_6|$ remains close to δ . This direction is again that determined by the point $(0,0,0,0, \mathbf{U}_5, \mathbf{U}_6 = -\mathbf{U}_5)$ if for instance $\mathbf{u}_5 = \varrho \mathbf{U}_5$, $\mathbf{u}_6 = -\varrho \mathbf{U}_5 + \delta$, $|\delta| = \delta$, $\varrho \rightarrow \infty$. Moreover, a common exponential fall-off factor $e^{-\alpha\gamma\tau}$, $\alpha > 0$, is again obtained for $0 \leq \gamma \leq \gamma_0$, $\gamma_0 > 0$, where α and γ_0 are constants independent of $\mathbf{u}_5$ and $\mathbf{u}_6$. To see this, one applies again the rules given in Subsect. 2.

In the application to the study of the essential support of $\equiv \oplus \ominus \equiv$, the refined macro-causality condition in this second case is relevant in the solution of the $u=0$ problem when the initial trajectories meet at the origin, the final trajectories meet at a point V , and when there exist moreover intermediate on-mass-shell four-momenta P_4, P_5, P_6 such that $P_5 = P_6$ and $V = \lambda P_4$, $\lambda > 0$: see Fig. 1. (Other cases are treated more easily.)

In fact, let δ be the point in space that lies at time zero on the line passing through V and parallel to $P_5 = P_6$ and let us consider points $\mathbf{u}_4, \mathbf{u}_5, \mathbf{u}_6$ such that $\mathbf{u}_4$ is close to zero and $\mathbf{u}_5 + \mathbf{u}_6$ is close to 2δ . We note that $\delta \neq 0$ in view of the assumption $\mathbf{P}_i \neq \mathbf{P}_j$, $i, j = 1, 2, 3$ (it implies that $\mathbf{P}_4 \neq \mathbf{P}_5$).

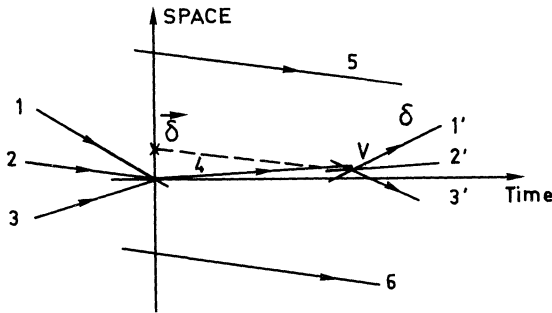
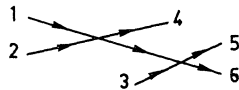


Fig. 1

Then the configuration of incoming and outgoing trajectories 4, 5, 6 and 1', 2', 3' (where 1', 2', 3' label here the final particles) may correspond to a point in the essential support of the bubble $\equiv \ominus \equiv$, and hence no exponential fall-off factor can be extracted from the generalized Fourier transform F'' associated with it. But in these situations a common exponential fall-off factor $e^{-\alpha\gamma\tau}$, $0 \leq \gamma \leq \gamma_0$, can be extracted as we have seen from the generalized Fourier transform F' associated with the bubble $\equiv \oplus \equiv$.

We have in this subsection considered examples in which the point $(\mathbf{P}_1, \dots, \mathbf{P}_6)$ was an $\mathcal{M}_0$ point. On the other hand, one may consider other situations in which $\{\mathbf{P}_k\}$ is not an $\mathcal{M}_0$ point and is, for instance, a simple point of a $+\alpha$ -Landau surface $L_0^+(G)$. In this case the scattering function of the process is known from macro-causality to be locally the plus $i\varepsilon$ boundary value of an analytic function.

In such a case, refined macro-causality still gives informations and follows from the same physical ideas. Let G_b be for instance the graph



and let us start from a corresponding diagram $\mathcal{D}_b$ of the form shown in Fig. 2a, in which the trajectories 1,2,4 and 3,5,6 meet respectively at the origin and at a point V , with $V = \lambda K$, $\lambda > 0$, $K = P_1 + P_2 - P_4$. A typical situation in which refined macrocausality gives information is obtained by displacing the trajectories 1,2,4 in a way such that they do not meet (Fig. 2b) and by letting λ tend to infinity.

The situation is then analogous to that of the first example described above. In fact, let $\mathbf{U}_1, \mathbf{U}_2, \mathbf{U}_4$ be the given space displacements of the trajectories 1,2,4. If $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_4$ lie in sufficiently small neighbourhood of $\mathbf{U}_1, \mathbf{U}_2, \mathbf{U}_4$, and if $\mathbf{p}_1, \dots, \mathbf{p}_6$ lie in sufficiently small neighbourhoods of $\mathbf{P}_1, \dots, \mathbf{P}_6$, then $\{\mathbf{u}_k\}_{k=1,\dots,6} \notin \mathcal{C}_b(\{\mathbf{p}_k\})$ even though $\lambda \rightarrow \infty$, but the direction of $\{\mathbf{u}_k\}$ tends in this limit to the causal direction in $\mathcal{C}_b(\{\mathbf{P}_k\})$ determined by the configuration of external trajectories of the diagram $\mathcal{D}_b$ of Fig. 2a.

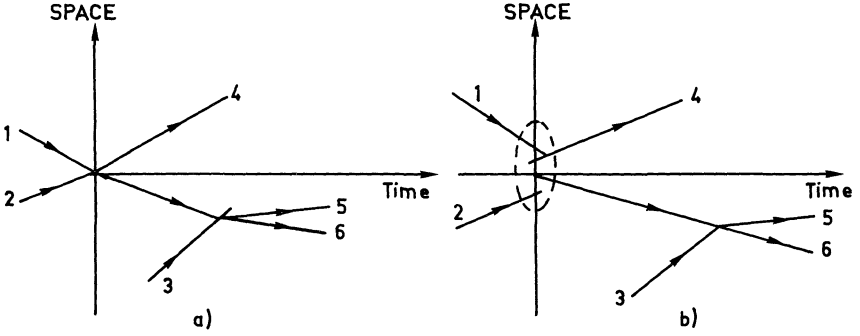


Fig. 2

The refined macrocausality condition then follows again physically from the fact that the displaced trajectories 1, 2, 4 do not meet.

Although this is not yet fully established, there is probably a link between the refined macrocausality condition in such situations and the “no sprout” property which is introduced in [8] and is a slight refinement of the plus $i\epsilon$ rule: see Remark i) at the end of Sect. 1.

We conclude this section by a brief presentation and discussion of the alternative semi-classical model proposed in [15]. In contrast to the model presented in Subsect. 2, each internal line of a graph G_b is always represented in this model by a *unique* internal line in the corresponding space-time diagrams, and vertices at infinity are not a priori introduced. On the other hand, besides the violations of locality at each vertex, violations of the mass-shell conditions for the internal and external lines are introduced. The set $\hat{\mathcal{C}}_b(\{\mathbf{p}_k\})$ obtained by considering sequences of modified diagrams of this type is not a priori identical to the set $\mathcal{C}_b(\{\mathbf{p}_k\})$ defined in Sect. 4.2. On the other hand, it is contained in the set $\mathcal{C}_b(\{\mathbf{p}_k\})$, if $\{\mathbf{p}_k\}$ is not a second kind $u=0$ point³⁵.

Most of the configurations of external trajectories of diagrams $\mathcal{D}_b$, including possibly vertices at infinity, can be obtained as limits of configurations of external

³⁵ We recall that the sets $\hat{\mathcal{C}}_b(\{\mathbf{p}_k\})$ and $\mathcal{C}_b(\{\mathbf{p}_k\})$ of Sect. 4.2 coincide if $\{\mathbf{p}_k\}$ is not a second kind $u=0$ point, and that second kind $u=0$ points of b are exceptional

trajectories of the above modified diagrams, as easily seen on simple examples³⁶, the angular-momentum conservation law being then derived as a consequence of the properties of the model (see [15]).

In connection with the present work, the refined macrocausality condition, in particular at $\mathcal{M}_0$ points, can still be derived in such a model, if the rates of exponential fall-off associated with a mass-shell violation are assumed to be proportional to $(p_k^2 - m_k^2)d_k$ or $(p_l^2 - m_l^2)d_l$, where p_k , resp. p_l , is the (off-shell) four-momentum of the external line k , or of the internal line l , and d_k , resp. d_l , is the distance of the vertex where line k is incoming or outgoing to the region of intersection of this line with some fixed temporal region (for instance the time zero hyperplane), resp. is the distance between the vertices where line l is respectively outgoing and incoming. These conditions are suggested by the results of [15, 17] on Feynman and phase-space integrals.

While this model presents certain interesting features, one has, however, to consider, in accordance with the discussion of [17], complex (and not only real) values of the four-momenta in the course of the limiting procedures as also possibly complex values of the Landau parameters α_l .

Remark. In some situations, the set $\hat{\mathcal{C}}_b(\{\mathbf{p}_k\})$ obtained seems to be *strictly contained* in the set $\mathcal{C}_b(\{\mathbf{p}_k\})$ [25]. Points of $\mathcal{C}_b(\{\mathbf{p}_k\})$ that do not belong to $\hat{\mathcal{C}}_b(\{\mathbf{p}_k\})$ would therefore be outside the essential support of F_b at $\{\mathbf{p}_k\}$. This is not impossible, but would be surprising.

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Appendix 1. Comparison Between Different Definitions of Bubble Diagram Functions

In this appendix, we check that the definition of bubble diagram functions given in [7] in the framework of distribution theory away from $u=0$ points, coincides with that given here in Sects. 2 and 3 when the bubbles are kernels of bounded operators.

We first consider a product $A=A''A'$ of two linear bounded operators that satisfy the properties stated at the beginning of Sect. 3.1. The distribution a associated with A is then well defined by its action on Schwartz test functions of a product form according to Eq. (16). We then wish to show that $a(\chi)$ is equivalently defined by the procedures of Sect. 2b of [7] on products and integrals of distributions.

It is sufficient to consider functions ϕ and ψ with sufficiently small supports. (Otherwise partitions of unity can be used.) On the other hand, by virtue of the assumption on the support of $A'|\phi\rangle$, the distribution a' vanishes outside a certain compact set K in the space of the variables t , when x lies in the support of ϕ . Let h

³⁶ Some vertices of these modified diagrams tend to infinity in the limit, all four-momenta involved at these vertices tending to be colinear. (These four-momenta are possibly off-shell, but tend to on-shell values)

be an infinitely differentiable function of t , with compact support, equal to one in K , and let $h(t) = \sum_i h_i(t)$ be a decomposition of h into a (finite) sum of infinitely differentiable functions h_i with sufficiently small supports. Then, according to the procedures described in [7], $a(\chi)$ is defined, under certain conditions on the essential supports of a' and a' ³⁷, by

$$a(\chi) = \sum_i \int \tilde{a}'_{\chi_{1,i}}(u+u', v+v', w+w') \tilde{a}''_{\chi_{2,i}}(u', v', w') \cdot \tilde{\phi}(u) \tilde{\psi}(v) \tilde{h}_i(w) du dv dw du' dv' dw', \quad (43)$$

where $\tilde{a}'_{\chi_{1,i}}$ and $\tilde{a}''_{\chi_{2,i}}$ are, respectively, the Fourier transforms of $a'(x, t)\chi_{1,i}(x, t, y)$ and $a''(t, y)\chi_{2,i}(x, t, y)$ in relation to the variables x, y, t , and $\chi_{1,i}, \chi_{2,i}$ are infinitely differentiable functions with sufficiently small support, equal to one when $\phi(x)\psi(y)h_i(t) \neq 0$. Finally, $\tilde{\phi}, \tilde{\psi}, \tilde{h}_i$ are the respective Fourier transforms of ϕ, ψ, h_i . (The dual variables of the variables x, y, t are denoted, respectively, by u, v, w .)

The definition (43) does not depend on the choice of $\chi_{1,i}$ and $\chi_{2,i}$ with the above properties (see [4, 7]). By choosing functions $\chi_{1,i}$ and $\chi_{2,i}$ of a product form:

$$\begin{aligned} \chi_{1,i}(x, t, y) &= \chi_{1,i}^{(1)}(x) \chi_{1,i}^{(2)}(t) \chi_{1,i}^{(3)}(y) \\ \chi_{2,i}(x, t, y) &= \chi_{2,i}^{(1)}(x) \chi_{2,i}^{(2)}(t) \chi_{2,i}^{(3)}(y) \end{aligned}$$

it is easily seen that Eq. (43) can alternatively be written in the form

$$a(\chi) = \sum_i \int \tilde{a}'_{\chi_{1,i}^{(2)}}(\phi, w+w') \tilde{a}''_{\chi_{2,i}^{(2)}}(w', \psi) \tilde{h}_i(w) dw dw', \quad (44)$$

where $\tilde{a}'_{\chi_{1,i}^{(2)}}(\phi, w)$ and $\tilde{a}''_{\chi_{2,i}^{(2)}}(w, \psi)$ are the Fourier transforms in relation to t of $a'(\phi, t)\chi_{1,i}^{(2)}(t)$ and $a''(t, \psi)\chi_{2,i}^{(2)}(t)$, respectively; $a'(\phi, t)$ and $a''(t, \psi)$ are the distributions in the variable t obtained by taking the partial action of a' on ϕ , and of a'' on ψ , respectively.

Since a' and a'' are the respective kernels of A' and A'' , $a'(\phi, t)$ and $a''(t, \psi)$ are square integrable functions of t . For instance

$$a'(\phi, t) = (A'|\phi\rangle)(t) \quad (45)$$

since its action on any Schwartz test function r of t is $\langle \bar{r}|A'|\phi\rangle = \int (A'|\phi\rangle)(t)r(t)dt$.

Hence, it follows from Eq. (44) and standard results on Fourier transforms of products of square integrable functions that

$$\begin{aligned} a(\chi) &= \sum_i \int a'(\phi, t) a''(t, \psi) h_i(t) dt \\ &= \int a'(\phi, t) a''(t, \psi) dt \end{aligned} \quad (46)$$

a result which clearly coincides [see Eq. (45)] with Eq. (16). Q.E.D.

³⁷ These conditions ensure absolute convergence properties of the integrals of Eq. (43) (as ordinary integrals of functions) for general distributions a', a'' . We shall see below that the Definition (43) coincides with (16) and hence defines $a(\chi)$ independently of these conditions, when a' and a'' are kernels of bounded operators

In order to extend this result to general bubble diagram functions, we first indicate the following results which can be checked in the framework of Sect. 2b of [7] (and are as a matter of fact, partly needed to complete the proofs given there):

i) The product of r distributions ($r > 2$) is well defined under the conditions stated in Theorem 4' of [7]. While this is directly seen by induction from the previous result of Theorem 4 on the product of two distributions, it can moreover be checked³⁸ that the definition is independent of the order in which the product is made, and that one can equivalently first make partial products, in any order, of subgroups of distributions.

ii) The following property holds on integrals of distributions that satisfy adequate support properties:

$$\int [f_1(x, t', t) f_2(t, y)] dt dt' = \int [\int f_1(x, t', t) dt'] f_2(t, y) dt.$$

We then first consider a product of two operators

$$A' = \left[\bigotimes_K S_{m_K, n_K}^{(-1)c} \right]_{\mathcal{K}[I, I_{(1)}]} \quad \text{and} \quad A'' = \left[\bigotimes_K S_{m_K, n_K}^{(-1)c} \right]_{\mathcal{K}[I_{(1)}, J]}.$$

The product of the distributions associated with the operators $S_{m_K, n_K}^{(-1)c}$ involved in A' (or A'') is clearly the distribution a' associated with A' ³⁹ (or the distribution a'' associated with A''). Hence, the previous analysis can be directly applied, as a consequence of Remark i) above.

In the case of a product of more than two operators ($A = A_q \dots A_2 A_1$), the result follows by induction on q , and by applying on the one hand Remark i) and on the other hand Remark ii) with $f_1(x, t', t) = a_1[x, t_{(1)}] \dots a_{q-1}[t_{(q-2)}, t]$, $t' = [t_{(1)}, \dots, t_{(q-2)}]$, $f_2(t, y) = a_q(t, y)$.

Appendix 2. Essential Support Theory and Bounded Operators

In this appendix, we present some simple results, established in the framework of essential support theory on kernels of bounded operators. Lemmas 2 and 3 are directly used in Sect. 3.1. Lemma 1 is an intermediate step in the proof of Lemma 2. It may have other applications and is therefore presented separately.

The mathematical conjecture used in the proof of Theorem 4 in Sect. 3.2 is described at the end. It is based on the one hand on Lemma 3, and on the other hand on Lemma 4, which is an improved version of Lemma 2.

The notations are the same as in Sect. 3.

Lemma 1. *Let f be a square integrable function of a real variable $x = (x_1, \dots, x_n)$ and let F be its generalized Fourier transform:*

$$F(v; v_0; X) = \int f(x) e^{-iv \cdot x - v_0(x - X)^2} dx.$$

³⁸ To see this, one may for instance define the product by multiple convolutions that generalize Eq. (5) of [7] when $r > 2$

³⁹ This is a straightforward consequence of the fact that these distributions do not depend on the same variables [see Eq. (13): for each K , p_{i_K} and q_{j_K} are different sets of on-mass-shell four-momenta]

If $V=(V_1, \dots, V_n) \in ES_X(f)$, then there exists an open cone $\mathcal{V}$ with its apex at the origin in $R_{(v)}^n$ containing V , $\alpha > 0$, $\gamma_0 > 0$, such that :

$$|F(v; v_0, X)| < d(v, v_0) e^{-\alpha v_0} \quad (47)$$

in the region $v \in \mathcal{V}$, $0 \leq v_0 \leq \gamma_0 |v|$, where d is, for any $v_0 \geq 0$, a square integrable function of v whose norm $[\int d^2(v, v_0) dv]^{1/2}$ is bounded by a constant independent of v_0 .

Proof. By definition of the essential support, the condition $V \notin ES_X(f)$ implies bounds of the form (47), where d is replaced by a polynomial $\mathcal{P}$ of v, v_0 (times an inverse power of v_0). However, it is also known [4] that the generalized Fourier transform F_χ of χf at X , where χ is a C^∞ function with a sufficiently small compact support, equal for instance to one in a neighbourhood of X , satisfies analogous bounds in which d is replaced by a rapid fall-off factor $C_N/(1+|v|)^N$, $\forall N \geq 0$, where C_N is independent of v_0 .

On the other hand, the generalized Fourier transform $F_{(1-\chi)}$ of $(1-\chi)f$ is trivially bounded in the whole region $v_0 \geq 0$ by

$$|F_{(1-\chi)}(v, v_0; X)| < e^{-v_0 \delta(X, 1-\chi)} \left| \int f(x) (1-\chi)(x) \cdot e^{-iv \cdot x} e^{-v_0 [(x-X)^2 + \delta(X, 1-\chi)]} dx \right|, \quad (48)$$

where $\delta(X, 1-\chi) > 0$ is the distance of X to the support of $1-\chi$. The integral on the right-hand side of Eq. (48) is the Fourier transform with respect to x of a function that is square integrable for any given $v_0 \geq 0$ and whose norm is bounded by a constant independent of v_0 .

Since $F = F_\chi + F_{(1-\chi)}$, Lemma 1 is therefore proved. Q.E.D.

Lemma 2. Let $a(x, y)$ [$x=(x_1, \dots, x_m)$, $y=(y_1, \dots, y_n)$] be the kernel of a linear bounded operator A and let F be its generalized Fourier transform :

$$F(u, v; v_0, X, Y) = \int a(x, y) e^{-iu \cdot x - iv \cdot y - v_0 [(x-X)^2 + (y-Y)^2]} \times dx dy. \quad (49)$$

If $(U, V) \notin ES_{X, Y}(a)$, then there exists an open cone $\mathcal{V}$ with its apex at the origin in $\mathbb{R}_{(u)}^m \times \mathbb{R}_{(v)}^n$ containing (U, V) , a neighbourhood $\mathcal{N}$ of X, Y in $\mathbb{R}_{(x)}^m \times \mathbb{R}_{(y)}^n$, $\alpha > 0$, $\gamma_0 > 0$ such that :

$$|F(u, v; v_0, X', Y')| < d(u, v, v_0, X', Y') e^{-\alpha v_0} \quad (50)$$

in the region $(X', Y') \in \mathcal{N}(X, Y)$, $(u, v) \in \mathcal{V}(U, V)$, $0 < v_0 \leq \gamma_0 (u^2 + v^2)^{1/2}$; d is, for any given u, v_0, X', Y' a square integrable function of v whose norm is bounded by $C v_0^{-q/2}$ where C and q are constants independent of u, v_0, X', Y' :

$$[\int d^2(u, v, v_0, X', Y') dv]^{1/2} < C v_0^{-q/2}. \quad (51)$$

If a is replaced by $a(x, y)g(x)$, where g is a C^∞ function of x with compact support, equal to one in some neighbourhood of X , then the same result holds with moreover $q=0$ in the bounds (51).

Proof. If χ and χ' are C^∞ functions of x and y , respectively, with sufficiently small supports, equal to one in respective neighbourhoods of X, Y , then it is known (as in the proof of Lemma 1 above) that the generalized Fourier transform at X, Y of

$a(x, y)\chi(x)\chi'(y)$ satisfies bounds of the form (50), in which d is as a matter of fact replaced by a rapid fall-off factor $C_N/(1+|u|+|v|)^N$ (C_N is independent of u, v, v_0).

The fact that one can also obtain uniform bounds of the same type for all X', Y' in a sufficiently small neighbourhood of X, Y is analogous to Remark 2 of [7] (p. 41) and is proved by methods similar to those of [4].

Finally, the treatment of the generalized Fourier transforms of

$$\begin{aligned} a(x, y)\chi(x) \times (1 - \chi')(y), \quad a(x, y) \times (1 - \chi)(x) \times \chi'(y), \\ a(x, y) \times (1 - \chi)(x) \times (1 - \chi')(y), \end{aligned}$$

presents no difficulty and is analogous to that given at the end of the proof of Lemma 1. It is sufficient to notice that the integral

$$\int a(x, y)e^{-iux - v_0(x-X)^2}\chi(x)dx,$$

or

$$\int a(x, y)e^{-iux - v_0(x-X)^2 + v_0\delta(X, 1-\chi)} \times (1 - \chi)(x)dx,$$

is a square integrable function of y whose norm is bounded by $\|A\| \times \|e^{-v_0(x-X)^2}\chi(x)\|$ or

$$\|A\| \times \|e^{-v_0(x-X)^2 + v_0\delta(X, 1-\chi)} \times (1 - \chi)(x)\|.$$

The same analysis provides the announced results when a is replaced by $a(x, y)g(x)$. Q.E.D.

The following trivial lemma is presented for completeness, in view of the applications to Sect. 3. The proof follows from an argument analogous to that given at the end of the proof of Lemma 2.

Lemma 3. *Let a and g be defined as in Lemma 2. Then there exists a function d of v, u, v_0 which is square integrable with respect to v and whose norm satisfies the properties described in Lemma 2, such that :*

$$|F(u, v, v_0, X', Y')| < d(u, v, v_0, X', Y') \quad (52)$$

in the region $v_0 \geq 0$, and at all points X', Y', u, v .

Results analogous to Lemmas 2 and 3 clearly holds also when the roles of u and v are exchanged.

The following improved version of Lemma 2 holds and can be proved by the same methods:

Lemma 4. *Let a and g be defined as in Lemma 2, let F be the generalized Fourier transform of $a(x, y)g(x)$, and let us assume it satisfies the bounds :*

$$|F(u, v; v_0, X', Y')| < \mathcal{P}(u, v, v_0)e^{-\alpha v_0}, \quad \alpha > 0 \quad (53)$$

in the region $u, v \in \mathcal{C}$, $(X', Y') \in \mathcal{N}(X, Y)$, $0 \leq v_0 \leq \gamma_0(\widehat{u, v})(u^2 + v^2)^{1/2}$ where $\alpha > 0$ is a given constant, $\mathcal{N}$ is a neighbourhood of a point (X, Y) , $\mathcal{C}$ is an open cone with apex at the origin in $R_{(u)}^m \times R_{(v)}^n$ and γ_0 is continuous function of the direction $\widehat{u, v}$ which has

a strictly positive lower bound over any closed subset of directions contained in $\mathcal{C}^{40}$. Finally, $\mathcal{P}$ is a given polynomial of the variables u, v, v_0 .

Then, being given any open cone $\mathcal{C}'$ with apex at the origin whose closure is contained in $\mathcal{C}$ (apart from the origin), any $\alpha' < \alpha$, any neighbourhood $\mathcal{N}'$ of (X, Y) whose closure is contained in $\mathcal{N}$, and any continuous function γ'_0 such that $\text{Min}_{u, v \in \mathcal{C}'} [\gamma_0(\widehat{u, v}) - \gamma'_0(\widehat{u, v})] > 0$, there exists a function d of u, v, v_0, X', Y' , which is square integrable with respect to v and whose norm is independent of u, v_0, X', Y' , such that

$$|F(u, v; v_0, X', Y')| < d(u, v, v_0, X', Y') e^{-\alpha' v_0} \quad (54)$$

in the region $(X', Y') \in \mathcal{N}', (u, v) \in \mathcal{C}', 0 \leq v_0 \leq \gamma'_0(\widehat{u, v})(u^2 + v^2)^{1/2}$.

We next state:

Conjecture. Let the assumptions of Lemma 4 be satisfied. Then being given any $\alpha' < \alpha$, any $\mathcal{N}'$ whose closure is contained in $\mathcal{N}$ and any continuous function γ'_0 of $\widehat{u, v}$ such $\gamma_0 - \gamma'_0$ has a strictly positive lower bound over any closed subset of directions contained in $\mathcal{C}$, there exists a function d with the same properties as in Lemma 4, such that the bounds (54) are satisfied in the region $(X', Y') \in \mathcal{N}', (u, v) \in \mathcal{C}, 0 \leq v_0 \leq \gamma'_0(\widehat{u, v})(u^2 + v^2)^{1/2}$.

Compared to Lemma 4, this conjecture asserts that the function d of Lemma 4 can be chosen independent of $\mathcal{C}'$, i.e. uniform when one comes close to the boundary $\partial\mathcal{C}$ of $\mathcal{C}$. The methods used so far in the proof of Lemma 4 are not sufficient to ensure this uniformity property. It is, however, suggested by Lemma 4 on the one hand, and by the fact that F always satisfies on the other hand the uniform bounds of Lemma 3, at all points $u, v, v_0 \geq 0, X', Y'$.

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40 For certain reasons that appear in [4], the functions γ_0 considered here must also satisfy a further geometrical condition, which plays a completely unessential role for our purposes. Namely, let R be the region in (u, v, v_0) -space defined by the inequalities $0 \leq v_0 \leq \gamma_0(\widehat{u, v})(u^2 + v^2)^{1/2}$ and let C_α be the cone with apex at the origin in (u, v, v_0) -space whose basis in the hyperplane $v_0 = 1$ is the sphere of radius $2\sqrt{\alpha}$ with centre at $u = v = 0$. Then $R \cap C_\alpha$ must be empty (apart from the origin). Moreover, if a point (u, v, v_0) belongs to R , then all points (u', v', v'_0) , $v'_0 \geq 0$, of the cone $(u, v, v_0) - C_\alpha$ must also belong to R

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Note Added in Proof: The validity of the regularity property R for the S -matrix (refined macrocausality) has not been fully established in the general case in Sect. 5. Progress towards a more complete justification of this assumption from various viewpoints has been made recently in a collaboration with Professor H. P. Stapp.