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# The Simple Relation Between Certain Dynamical Systems

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**Abstract.** A simple relation between certain dynamical systems is established.

Consider two classical dynamical systems characterized by the Hamiltonians

$$H=\frac{1}{2}\sum_{j=1}^{n} p_j^2+U(q), \qquad q=(q_1,\ldots,q_n) \tag{1}$$

and

$$\tilde{H}=\frac{1}{2}\sum_{j=1}^{n}(p_j^2+\omega^2(t)q_j^2)+\varkappa(t)U(q), \tag{2}$$

where the potential $U(q)$ is a homogeneous function of degree $k$

$$U(\lambda q)=\lambda^k U(q). \tag{3}$$

Let $q(t)$ and $\tilde{q}(t)$ be the solutions of equations of motion for the corresponding systems

$$\ddot{q}_j=F_j(q), \qquad F_j(q)=-\partial U/\partial q_j, \tag{4}$$

$$\ddot{\tilde{q}}_j=\varkappa(t)F_j(\tilde{q})-\omega^2(t)\tilde{q}_j. \tag{5}$$

Let $\alpha_1(t)$ and $\alpha_2(t)$ be two independent solutions of equation

$$\ddot{\alpha}+\omega^2(t)\alpha=0 \tag{6}$$

and the function $\beta(t)$ is defined by the formula

$$\beta(t)=c\alpha_2(t)/\alpha_1(t)=c\int\alpha_1^{-2}(t')dt'. \tag{7}$$

Suppose also that the function $\varkappa(t)$ is of the form

$$\varkappa(t)=c^2\alpha_1^{-(k+2)}(t). \tag{8}$$

**Theorem.** *If $q(t)$ is the solution of Eq.* (4), *then*

$$\tilde{q}(t)=\alpha_1(t)q(\beta(t)) \tag{9}$$

*is the solution of Eq.* (5).

*Proof.* Substitute $\tilde{q}(t)$ from (9) into (5) and take into account the relations (4) and (6)–(8).

Thus, the solution of a more complicated system (5) is reduced to that of a more simple system (4).

From this theorem one can get a number of useful corollaries. Let us give some of them.

**Corollary 1.** *If $U(q)$ is a homogeneous function of degree $k=-2$, then setting $c=1$ we have $\varkappa(t)\equiv 1$ and consequently $\tilde{q}(t)=\alpha_1(t)q(\beta(t))$ is the solution of the system* (5) *with*

$$\tilde{H}=\tfrac{1}{2}\sum_{j=1}^{n}(p_j^2+\omega^2q_j^2)+U(q). \tag{10}$$

**Corollary 2.** *Let $U(q)$ be a homogeneous function of degree $k=-2$ with $H$ of the form* (10) *and $\omega=$*const. *It is known that in a number of cases such a system is completely integrable: $n=3$, $\omega=0$* [1]; *$n=4$* [2]; *$n=5$* [3]; *arbitrary $n$, $\omega=0$* [4] *and we even know explicitely the solution of the system* [5]. *Namely, for $U(q)$ of the form*

$$U(q)=\sum_{j<k}(q_j-q_k)^{-2}. \tag{11}$$

*$q_j(t)$ are the eigenvalues of the matrix*

$$\cos\omega t\cdot q(0)+\frac{1}{\omega}\sin\omega t\cdot L(0), \tag{12}$$

*where*

$$q=\operatorname{diag}(q_1,\dots,q_n),$$

*$L$ is the matrix introduced by Moser* [4]

$$L_{jk}=\delta_{jk}+i(1-\delta_{jk})(q_j-q_k)^{-1}. \tag{13}$$

*Now we have for the system with $H$ of the form* (10) *and $U(q)$ of the form* (11): *$q_j(t)$ are the eigenvalues of the matrix*

$$\alpha_1(t)q(0)+\alpha_2(t)L(0), \tag{14}$$

*where $\alpha_1(t)$ and $\alpha_2(t)$ are the solutions of Eq.* (6) *satisfying the initial conditions*

$$\alpha_1(0)=1,\ \alpha_1'(0)=0\,;\ \alpha_2(0)=0,\ \alpha_2'(0)=1. \tag{15}$$

In particular, for the case of constant frequency the formula

$$\tilde{q}(t)=\cos\omega t\, q\left(\frac{1}{\omega}\operatorname{tg}\omega t\right) \tag{16}$$

is valid.

**Corollary 3.** *Let* $\tilde{q}(t)=q^0$ *be the equilibrium position of the system* (10) *with* $\omega(t)=\text{const}$. *Then*

$$q(t)=\sqrt{a^2+b^2t^2}\cdot q^0,\ b=\omega/a \tag{17}$$

*is the automodel solution of the system with* $H$ *of the form* (1).

**Corollary 4.** *If* $\alpha_1(t)=t$, $\alpha_2=1$ *then* $\omega=0$ *and we get the solution of the system* (5) *with* $\omega=0$ *and* $U(q)$ *of the form*

$$U(q,t)=bt^{-(k+2)}U_k(q),\ U_k(\lambda q)=\lambda^k U_k(q) \tag{18}$$

*in particular, at* $k=-1$ *we have the solution for a "Coulomb" case with*

$$U(q,t)=bt^{-1}U_{-1}(q). \tag{19}$$

It would be interesting to find out whether there exists in the quantum case some analog to the relations obtained here.

## References


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