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The Navier-Stokes Equations in Space Dimension Four*

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Abstract. Solutions to the Navier-Stokes equations in four space dimensions are continuous except for a closed set whose three dimensional Hausdorff measure is finite.

Section 1. Introduction and Notation

In this paper we will prove Theorem 1.1 below. The terminology and notation in the statement of this theorem are explained in the remainder of this section.

Theorem 1.1. *Let* $v:R^4\to R^4$ *be a measurable function such that* $\int|v(x)|^2dx<\infty$ *and* $\operatorname{div}(v)=0$. *Then there exists a measurable function* $u:R^4\times(0,\infty)\to R^4$ *such that*

$$\int_0^\infty\int|u(x,t)|^3dxdt<\infty\,,$$

u is a weak solution to the Navier-Stokes equations of incompressible fluid flow with initial condition v, and the following property holds: There exists a set $A\subset R^4\times(0,\infty)$ *such that*

a) $A\cap(R^4\times[\varepsilon,\infty))$ *is compact for every* $\varepsilon>0$,

b) *the* 3-*dimensional Hausdorff measure of A is finite, and*

c) *the restriction of u to the complement of A is a continuous function.*

Notation. Hausdorff measure is defined in Section 2 (Definition 2.8). If X and Y are euclidean spaces then $C^\infty(X,Y)$ is the set of all infinitely differentiable functions from X into Y, and $C_0^\infty(X,Y)$ is the set of all functions in $C^\infty(X,Y)$ with compact support. If f is a C^∞ function defined on an open subset of $R^4\times R$ (R^4 should be thought of as space and R as time) then D_if, $D_{ij}f$, and $D_{ijk}f$ are the partial derivatives $(\partial/\partial x_i)f$, $(\partial^2/\partial x_i\partial x_j)f$, and $(\partial^3/\partial x_i\partial x_j\partial x_k)f$ with respect to the variables x_1,x_2,x_3,x_4 of R^4. The partial derivative of f with respect to the R variable of

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$R^4 \times R$ is denoted by $D_t f$ (the time derivative). We also set $\Delta f = \sum_{i=1}^{4} D_{ii} f = D_{ii} f$ (repeated indices are always summed), $\operatorname{div}(f) = \sum_{i=1}^{4} D_i f_i = D_i f_i$ if the range of f is R^4, and $\nabla(f) = (D_1 f, D_2 f, D_3 f, D_4 f)$. Note that the time derivative is not included in the definitions of Δ, div, and ∇. Analogous definitions are made for functions with domain R^4. If $v: R^4 \to R^4$ is square integrable then $\operatorname{div}(v) = 0$ is interpreted in the distribution sense. We also set $[a, b) = \{t : a \leqq t < b\}$. Closed and open intervals are denoted $[a, b]$ and (a, b), respectively.

Suppose $v: R^4 \to R^4$ is square integrable with $\operatorname{div}(v) = 0$. A measurable function $u: R^4 \times (0, \infty) \to R^4$ satisfying $\int_0^\infty \int |u(x, t)|^3 dx dt < \infty$ is called a weak solution to the Navier-Stokes equations of incompressible fluid flow with initial condition v when (1.1) and (1.2) are satisfied:

$$\int_0^\infty \int u_i(x, t) D_i \phi(x, t) dx dt = 0 \quad \text{if} \quad \phi \in C_0^\infty(R^4 \times R, R), \tag{1.1}$$

$$\int v_i(x) \phi_i(x, 0) dx + \int_0^\infty \int u_i(x, t)(D_t \phi_i + \Delta \phi_i)(x, t) dx dt$$
$$+ \int_0^\infty \int u_j(x, t) u_i(x, t) D_j \phi_i(x, t) dx dt = 0 \quad \text{if} \quad \phi \in C_0^\infty(R^4 \times R, R^4), \operatorname{div}(\phi) = 0. \tag{1.2}$$

Observe that Hölder's inequality, the integrability of $|u|^3$, and the integrability of $|v|^2$ imply that the integrals in (1.1) and (1.2) make sense.

If $a \in R^4$ and $0 < r < \infty$ we set $B(a, r) = \{x \in R^4 : |x - a| \leqq r\}$. If X is a measure space and Y is a euclidean space then $L^p(X, Y)$ is the Lebesgue L^p space of functions with domain X and range Y. The L^p norm is denoted $\| \ \|_p$. The norm $| \ |$ is always euclidean norm. If f is a C^∞ function defined on an open subset of $R^4 \times R$ or R^4 then ∇f (see above) will also be denoted Df. In addition, $D^2 f$ and $D^3 f$ will be the vector valued functions with components $D_{ij} f$ and $D_{ijk} f$, respectively ($i, j, k \in \{1, 2, 3, 4\}$). If $\phi \in C_0^\infty(X, Y)$ (see above) then $\operatorname{spt}(\phi)$ is the closure of $\{x : \phi(x) \neq 0\}$. If f is a function defined on a subset of $R^4 \times R$ and g, k are functions defined on R^4 then we set

$$(f * k)(x, t) = \int f(y, t) k(x - y) dy,$$
$$(g * k)(x) = \int g(y) k(x - y) dy$$

whenever the integrals make sense. If $0 < t < \infty$ we define $H_t : R^4 \to R$ by

$$H_t(x) = (4\pi t)^{-2} \exp(-|x|^2/4t). \tag{1.3}$$

If t has a complicated form we will write

$$H[t] = H_t. \tag{1.4}$$

We also define $K: R^4 - \{0\} \to R$ by

$$K(x) = -(4\pi^2 |x|^2)^{-1}. \tag{1.5}$$

An *absolute constant* is a finite positive constant that does not depend on any of the parameters in this paper. The symbol C will always denote an absolute constant, and the value of C may change from one line to the next (e.g. $2C \leqq C$). The symbols $C_1, C_2, C_3, \ldots$ will also denote absolute constants, but their values will not change in the course of the paper.

We fix v satisfying the hypothesis of Theorem 1.1 and set

$$L = \int |v(x)|^2 dx. \tag{1.6}$$

See [4], [5] and [6] for theorems on the Navier-Stokes equations in 3-dimensional space. This program was inspired by the work of Mandelbrot [3] and Almgren [1].

Section 2. Preliminary Results Involving Hausdorff Measure

Throughout this section we fix a positive real number D and functions f and f_n in $L^3(R^4 \times [0,\infty), R^4)$ for $n \in \{1,2,3,\ldots\}$ such that $\|f_n\|_3^3 \leqq D$ and the sequence f_n converges to f weakly in L^3.

Definition 2.1. If $a \in R^4$, $0 < b < \infty$, m is an integer satisfying $2^{-2m} \leqq b$, and $n \in \{1,2,3,\ldots\}$ then we set

$$A(a,b,m,n) = 2^m \left(\int_{b-2^{-2m}}^{b} \int_{R^4} |f_n(x,t)|^3 (|x-a| + 2^{-m})^{-5} dx dt \right), \tag{2.1}$$

$$B(a,b,m,n) = 2^{6m} \left(\int_{b-2^{-2m}}^{b} \int_{B(a,2^{-m})} |f_n(x,t)|^3 dx dt \right). \tag{2.2}$$

Definition 2.2. Whenever $0 < M < \infty$ the following statement will be known as property $P(M)$: If $n_1, n_2, n_3, \ldots$ is an increasing sequence of positive integers, p and q are integers, $p < q$, $a \in R^4$, $0 < b < \infty$, and $2^{-2p} \leqq b$ then

$$\liminf_{k\to\infty} B(a,b,q,n_k)$$
$$\leqq M \left(\liminf_{k\to\infty} \left(A(a,b,p,n_k) + \left(2^{-p} A(a,b,p,n_k) + \sum_{m=p}^{q-1} 2^{-m} B(a,b,m,n_k) \right)^{3/2} \right) \right).$$

From (2.1) and (2.2) we conclude that there exists an absolute constant C_1 such that

$$B(a,b,m,n) \leqq C_1 (A(a,b,m,n)). \tag{2.3}$$

We use C_1 in the following lemma:

Lemma 2.3. *For every positive real number M there exists $\varepsilon > 0$ such that the following holds: If property $P(M)$ is satisfied, $a \in R^4$, $0 < b < \infty$, p is an integer, $2^{-2p} \leqq b$, and*

$$\liminf_{n\to\infty} A(a,b,p,n) \leqq \varepsilon 2^{3p} \tag{2.4}$$

then

$$\liminf_{n\to\infty} B(a,b,q,n) \leqq \varepsilon(M+C_1+1)2^{3p} \tag{2.5}$$

for all integers q satisfying $q \geqq p$.

Proof. We choose $\varepsilon > 0$ such that

$$M(1+2(M+C_1+1))^{3/2}\varepsilon^{3/2} \leqq \varepsilon. \tag{2.6}$$

There is an increasing sequence $n_1, n_2, n_3, \ldots$ of positive integers such that

$$\liminf_{n\to\infty} A(a,b,p,n) = \lim_{k\to\infty} A(a,b,p,n_k). \tag{2.7}$$

Using the Cantor diagonal process and passing to a subsequence, we may assume

$$\lim_{k\to\infty} B(a,b,m,n_k) \quad \text{exists if} \quad m \geqq p. \tag{2.8}$$

It suffices to prove

$$\lim_{k\to\infty} B(a,b,m,n_k) \leqq \varepsilon(M+C_1+1)2^{3p} \tag{2.9}$$

if $m \geqq p$. We proceed by induction on m. If $m=p$ then (2.9) follows from (2.7), (2.8), (2.3), and (2.4). Now suppose that (2.9) holds for $m=p, p+1, \ldots, q-1$. Then from (2.8), property $P(M)$, Definition 2.2, (2.7), (2.8), (2.4), and (2.6) we obtain

$$\begin{aligned}
&\lim_{k\to\infty} B(a,b,q,n_k)\\
&\leqq M\left(\liminf_{k\to\infty}\left(A(a,b,p,n_k) + \left(2^{-p}A(a,b,p,n_k) + \sum_{m=p}^{q-1} 2^{-m}B(a,b,m.\,n_k)\right)^{3/2}\right)\right)\\
&= M\left(\lim_{k\to\infty} A(a,b,p,n_k)\right)\\
&\quad + M\left(2^{-p}\left(\lim_{k\to\infty} A(a,b,p,n_k)\right) + \sum_{m=p}^{q-1} 2^{-m}\left(\lim_{k\to\infty} B(a,b,m,n_k)\right)\right)^{3/2}\\
&\leqq M\varepsilon 2^{3p} + M\left(2^{-p}\varepsilon 2^{3p} + \sum_{m=p}^{q-1} 2^{-m}\varepsilon(M+C_1+1)2^{3p}\right)^{3/2}\\
&\quad < M\varepsilon 2^{3p} + M(\varepsilon 2^{2p} + 2^{-p+1}\varepsilon(M+C_1+1)2^{3p})^{3/2}\\
&= M\varepsilon 2^{3p} + M\varepsilon^{3/2}2^{3p}(1+2(M+C_1+1))^{3/2} \leqq M\varepsilon 2^{3p} + \varepsilon 2^{3p} < \varepsilon(M+C_1+1)2^{3p}.
\end{aligned}$$

The lemma has been proved.

Lemma 2.4. *There exists an absolute constant C_2 with the following property: If $a \in R^4$, $0 < b < \infty$, p is an integer, and $2^{-2p+2} \leqq b$ then*

$$A(c,d,p,n) \leqq C_2(A(a,b,p-1,n))$$

whenever $c \in R^4$, $|a_i - c_i| \leqq 2^{-p-1}$ for $i \in \{1,2,3,4\}$, and $b - 2^{-2p} \leqq d \leqq b$.

Proof. This follows easily from (2.1).

Definition 2.5. Let p be an integer. We let $Z(p)$ be the collection of all points $(a,b) \in R^4 \times [2^{-2p+2}, \infty)$ which satisfy the following: $a = (i_1 2^{-p}, i_2 2^{-p}, i_3 2^{-p}, i_4 2^{-p})$ where i_1, i_2, i_3, i_4 are integers, and $b = j2^{-2p}$ where j is an integer satisfying $j \geqq 4$.

In the following lemma we use the function f given at the start of this section.

Lemma 2.6. *Suppose M is a positive real number, property $P(M)$ holds, ε is as in Lemma 2.3, p is an integer, $p \geqq 0$, $(a,b) \in Z(p)$,*

$$\liminf_{n\to\infty} A(a,b,p-1,n) \leqq C_2^{-1} \varepsilon 2^{3p}$$

(see Lemma 2.4), and

$$Q = \{(c,d) \in R^4 \times R : |a_i - c_i| \leqq 2^{-p-1} \quad \text{for} \quad i \in \{1,2,3,4\}, \tag{2.10}$$

and $b - 2^{-2p} \leqq d \leqq b\}$.

Then $|f(c,d)|^3 \leqq C\varepsilon(M + C_1 + 1)2^{3p}$ for almost every $(c,d) \in Q$.

Proof. From Definition 2.5, Lemma 2.4, and the hypotheses we obtain

$$\liminf_{n\to\infty} A(c,d,p,n) \leqq C_2 \left(\liminf_{n\to\infty} A(a,b,p-1,n) \right) \leqq \varepsilon 2^{3p}$$

if $(c,d) \in Q$. Hence Lemma 2.3 yields

$$\liminf_{n\to\infty} B(c,d,q,n) \leqq \varepsilon(M + C_1 + 1)2^{3p}$$

if $q \geqq p$ and $(c,d) \in Q$. Hence (2.2) and the assumption that f_n converges to f weakly in L^3 imply

$$2^{6q} \left(\int_{d-2^{-2q}}^{d} \int_{B(c,2^{-q})} |f(x,t)|^3 dx dt \right) \leqq \varepsilon(M + C_1 + 1)2^{3p} \tag{2.11}$$

if q is an integer, $q \geqq p$, and $(c,d) \in Q$. We set

$$Q' = \{(y,s) \in R^4 \times R : |a_i - y_i| < 2^{-p-1} \quad \text{for} \quad i \in \{1,2,3,4\}, \tag{2.12}$$

and $b - 2^{-2p} < s < b\}$.

Let $(y,s) \in Q'$. Choose an integer q' such that $q' \geqq p$, $|a_i - y_i| + 2^{-q'} \leqq 2^{-p-1}$ for $i \in \{1,2,3,4\}$, $b - 2^{-2p} \leqq s - 2^{-q'}$, and $s + 2^{-q'} \leqq b$. Let q be an integer such that $q \geqq q'$. We set

$$Q_j = \{(x,t) \in R^4 \times R : |x-y| \leqq 2^{-q}, (s + j2^{-2q}) - 2^{-2q} \leqq t \leqq s + j2^{-2q}\} \tag{2.13}$$

whenever j is an integer and $1 - 2^q \leqq j \leqq 2^q$. We also set

$$Q'' = \{(x,t) \in R^4 \times R : |x-y| \leqq 2^{-q}, |s-t| \leqq 2^{-q}\}. \tag{2.14}$$

From $q \geqq q' \geqq p \geqq 0$ we conclude that $1 - 2^q$ and 2^q are integers and satisfy $1 - 2^q < 2^q$. Hence (2.13) and (2.14) yield

$$\int_{Q''} |f|^3 = \sum_j \left(\int_{Q_j} |f|^3 \right) \quad \text{where the sum is taken over } 1 - 2^q \leqq j \leqq 2^q. \tag{2.15}$$

Using (2.15), (2.13), (2.11) with $(c,d) = (y, s + j2^{-2q})$, the properties of q', and (2.10) we obtain

$$2^{6q} \left(\int_{Q''} |f|^3 \right) \leqq 2^{q+1}(\varepsilon)(M + C_1 + 1)2^{3p}. \tag{2.16}$$

For every $r>0$ we set

$$B(y,s,r)=\{(x,t)\in R^4\times R: |y-x|^2+|s-t|^2\leqq r^2\} \tag{2.17}$$

and we let m be the Lebesgue measure on $R^4\times R$. From (2.17) and (2.14) we conclude $B(y,s,2^{-q})\subset Q''$. Hence (2.16) and (2.17) yield

$$(m(B(y,s,2^{-q})))^{-1}\left(\int_{B(y,s,2^{-q})}|f|^3\right)\leqq C\varepsilon(M+C_1+1)2^{3p}. \tag{2.18}$$

Since $|f|^3$ is an integrable function, (2.17) yields

$$\lim_{q\to\infty}(m(B(y,s,2^{-q})))^{-1}\left(\int_{B(y,s,2^{-q})}|f|^3\right)=|f(y,s)|^3 \tag{2.19}$$

for almost every $(y,s)\in Q'$. Now (2.18), (2.19), and the fact that Q' is almost all of Q yield the conclusion of the lemma.

In the lemma below we use Definition 2.5 and the number D that was fixed at the start of this section.

Lemma 2.7. *There exists an absolute constant C_3 which satisfies the following: If p is an integer then*

$$\sum_{(a,b)\in Z(p)}\liminf_{n\to\infty} A(a,b,p-1,n)\leqq C_3 2^{6p}D.$$

Proof. For each $(a,b)\in Z(p)$ (see Definition 2.5) we define $\phi_{a,b}:R^4\times[0,\infty)\to R$ by

$$\phi_{a,b}(x,t)=(|x-a|+2^{-p+1})^{-5} \quad \text{if} \quad b-2^{-2p+2}\leqq t\leqq b,$$

$$\phi_{a,b}(x,t)=0 \quad \text{otherwise}.$$

We have

$$\left\|\sum_{(a,b)\in Z(p)}\phi_{a,b}\right\|_\infty \leqq C2^{5p}.$$

From (2.1) we obtain

$$A(a,b,p-1,n)=2^{p-1}\left(\int|f_n|^3\phi_{a,b}\right). \text{Hence}$$

$$\sum_{(a,b)\in Z(p)}A(a,b,p-1,n)\leqq C2^{6p}\|f_n\|_3^3\leqq C2^{6p}D.$$

The conclusion follows from the inequality $\liminf_{n\to\infty}(a_n)+\liminf_{n\to\infty}(b_n)\leqq\liminf_{n\to\infty}(a_n+b_n)$.

Definition 2.8. For any nonempty subset B of $R^4\times R$ we define

$$\operatorname{diam}(B)=\sup\{(|a-c|^2+|b-d|^2)^{1/2}:(a,b)\in B \quad \text{and} \quad (c,d)\in B\}.$$

Let A be a subset of $R^4\times R$. For every $\delta>0$ we define $\phi_\delta(A)$ to be the infimum of all numbers of the form

$$\sum_{i=1}^{\infty}(4/3)\pi(2^{-1}\operatorname{diam}(A_i))^3,$$

where A_i is a nonempty subset of $R^4\times R$, $A\subset\bigcup_{i=1}^{\infty}A_i$, and $\operatorname{diam}(A_i)\leqq\delta$. Observe that

$\phi_\delta(A) \geqq \phi_\eta(A)$ if $\delta \leqq \eta$. This allows us to define $H^3(A) = \lim_{\delta \to 0} \phi_\delta(A)$. The number $H^3(A)$ is called the 3-dimensional Hausdorff measure of A. There is an extensive treatment of Hausdorff measure in [2].

Lemma 2.9. *Suppose p is an integer, $p \geqq 0$, M is a positive real number, property $P(M)$ holds, and ε is as in Lemma 2.3. Then there exists a set A_p such that*

1) *$A_p \subset R^4 \times [3(2^{-2p}), \infty)$ and A_p is compact,*
2) *$\phi_\delta(A_p) \leqq C\varepsilon^{-1}D$ if $\delta \geqq 5^{1/2}2^{-p}$,*
3) *$|f(x,t)|^3 \leqq C\varepsilon(M + C_1 + 1)2^{3p}$ for almost every (x,t) that satisfies the conditions $(x,t) \in R^4 \times [3(2^{-2p}), \infty)$ and $(x,t) \notin A_p$.*

Proof. For each point $(a,b) \in Z(p)$ (see Definition 2.5) we set

$$Q(a,b) = \{(c,d) \in R^4 \times R : |a_i - c_i| \leqq 2^{-p-1} \quad \text{for} \quad i \in \{1,2,3,4\}, \tag{2.20}$$

and $b - 2^{-2p} \leqq d \leqq b\}$.

From Definition 2.5 we obtain

$$\cup \{Q(a,b) : (a,b) \in Z(p)\} = R^4 \times [3(2^{-2p}), \infty). \tag{2.21}$$

We set

$$Y(p) = \{(a,b) \in Z(p) : \liminf_{n \to \infty} A(a,b,p-1,n) \geqq C_2^{-1}\varepsilon 2^{3p}\}, \tag{2.22}$$

$$A_p = \cup \{Q(a,b) : (a,b) \in Y(p)\}. \tag{2.23}$$

From (2.22) and Lemma 2.7 we obtain

$$\begin{aligned} (\text{cardinality}(Y(p)))C_2^{-1}\varepsilon 2^{3p} &= \sum_{(a,b) \in Y(p)} C_2^{-1}\varepsilon 2^{3p} \\ &\leqq \sum_{(a,b) \in Y(p)} \liminf_{n \to \infty} A(a,b,p-1,n) \\ &\leqq \sum_{(a,b) \in Z(p)} \liminf_{n \to \infty} A(a,b,p-1,n) \leqq C_3 2^{6p} D < \infty. \end{aligned} \tag{2.24}$$

We conclude from (2.24) that $Y(p)$ is a finite set. Combining this with (2.20) and (2.23) we obtain that A_p is compact. This fact and (2.21) yield part 1). From (2.20) and $p \geqq 0$ we conclude (see Definition 2.8)

$$\operatorname{diam}(Q(a,b)) = (4(2^{-2p}) + 2^{-4p})^{1/2} \leqq 5^{1/2}2^{-p}. \tag{2.25}$$

Combining (2.24) and (2.25) we obtain

$$\sum_{(a,b) \in Y(p)} (4/3)\pi(2^{-1}\operatorname{diam}(Q(a,b)))^3 \leqq C\varepsilon^{-1}D. \tag{2.26}$$

Now the countability of $Z(p)$, (2.23), (2.25), and (2.26) yield part 2) of the lemma. From (2.21) and (2.23) we obtain

$$(R^4 \times [3(2^{-2p}), \infty)) - A_p \subset \cup \{Q(a,b) : (a,b) \in Z(p) - Y(p)\}. \tag{2.27}$$

Take $(a,b) \in Z(p)$ such that $(a,b) \notin Y(p)$. Then (2.22) yields

$$\liminf_{n \to \infty} A(a,b,p-1,n) < C_2^{-1}\varepsilon 2^{3p}.$$

Hence Lemma 2.6 and (2.20) yield

$$|f(x,t)|^3 \leqq C\varepsilon(M+C_1+1)2^{3p} \quad \text{for almost every} \quad (x,t)\in Q(a,b) \tag{2.28}$$

if $(a,b)\in Z(p)-Y(p)$. Finally, (2.27), (2.28), and the countability of $Z(p)$ yield part 3) of the lemma.

Section 3. Estimates on Vector Fields

Throughout this section we fix a positive real number ζ and a C^∞ function $w:R^4\to R^4$ such that $\operatorname{div}(w)=0$, $\int|w|^2<\infty$, and $\int|Dw|^2<\infty$.

Lemma 3.1. $\int|w|^3 \leqq C(\int|Dw|^2)(\int|w|^2)^{1/2}$.

Proof. The Schwarz inequality and the case $n=4$, $p=2$, $q=4$ of [7, Line 9, p. 127] yield

$$\int|w|^3=\int|w|^2|w| \leqq (\int|w|^4)^{1/2}(\int|w|^2)^{1/2} \leqq C(\int|Dw|^2)(\int|w|^2)^{1/2}.$$

Lemma 3.2. *If* $a\in R^4$, $t>0$, *and* $t^{1/2}\leqq r<\infty$ *then (see* (1.3)*)*

$$\int|(w*H_t)(x)|^3(|x-a|+r)^{-5}dx \leqq C(\int|w(x)|^3(|x-a|+r)^{-5}dx).$$

Proof. For every $i\in\{0,1,2,\ldots\}$ we define $h_i:R^4\to R$ as follows: If $i>0$ then $h_i(x)=1$ for every $x\in B(0,2^i r)-B(0,2^{i-1}r)$ and $h_i(x)=0$ otherwise. We set $h_0(x)=1$ if $x\in B(0,r)$ and $h_0(x)=0$ otherwise. For every i we use Young's inequality to obtain

$$\begin{aligned}
&\int|(w*H_t h_i)(x)|^3(|x-a|+r)^{-5}dx\\
&\leqq C\left(\sum_{j=0}^{\infty}\int_{B(a,2^j r)}|w*H_t h_i|^3(2^j r)^{-5}\right)\\
&\leqq C\left(\sum_{j=0}^{\infty}\left(\int_{B(a,2^j r+2^i r)}|w|^3(2^j r)^{-5}\right)\|H_t h_i\|_1^3\right)\\
&\leqq C\left(\sum_{j=0}^{i}\left(\int_{B(a,2^{i+1}r)}|w|^3(2^j r)^{-5}\right)\|H_t h_i\|_1^3\right)\\
&\quad+C\left(\sum_{j=i+1}^{\infty}\left(\int_{B(a,2^{j+1}r)}|w|^3(2^j r)^{-5}\right)\|H_t h_i\|_1^3\right)\\
&\leqq C\left(\int_{B(a,2^{i+1}r)}|w|^3(2^i r)^{-5}2^{5i}\right)\|H_t h_i\|_1^3\\
&\quad+C\left(\sum_{j=i+1}^{\infty}\left(\int_{B(a,2^{j+1}r)}|w|^3(2^j r)^{-5}2^{5i}\right)\|H_t h_i\|_1^3\right)\\
&\leqq C\left(\sum_{j=0}^{\infty}\left(\int_{B(a,2^{j+1}r)}|w|^3(2^j r)^{-5}\right)2^{5i}\|H_t h_i\|_1^3\right)\\
&\leqq C(\int|w(x)|^3(|x-a|+r)^{-5}dx)2^{5i}\|H_t h_i\|_1^3.
\end{aligned} \tag{3.1}$$

We define a measure m on R^4 by $m(E)=\int_E (|x-a|+r)^{-5}dx$. Minkowski's inequality and (3.1) yield

$$\begin{aligned}(\int|(w*H_t)(x)|^3(|x-a|+r)^{-5}dx)^{1/3}&=(\int|w*H_t|^3dm)^{1/3} \\ &\leqq \sum_{i=0}^{\infty}(\int|w*H_th_i|^3dm)^{1/3} \\ &= \sum_{i=0}^{\infty}(\int|(w*H_th_i)(x)|^3(|x-a|+r)^{-5}dx)^{1/3} \\ &\leqq C(\int|w(x)|^3(|x-a|+r)^{-5}dx)^{1/3}\left(\sum_{i=0}^{\infty}2^{5i/3}\|H_th_i\|_1\right).\end{aligned} \tag{3.2}$$

From $t^{1/2}\leqq r$ we obtain $\sum_{i=0}^{\infty}2^{5i/3}\|H_th_i\|_1\leqq C$. This inequality and (3.2) complete the proof of the lemma.

Definition 3.3. We define $f:R^4\to R^4$ by $f_i(x)=(w_j*H_\zeta)(x)D_jw_i(x)$. We obtain

$$\|f\|_2\leqq\|w*H_\zeta\|_\infty\|Dw\|_2\leqq\|w\|_2\|H_\zeta\|_2\|Dw\|_2<\infty. \tag{3.3}$$

It is elementary that every $\phi\in C_0^\infty(R^4,R^4)$ has an orthogonal decomposition (consisting of a divergence free vector field and a gradient vector field) $\phi=\phi'+\phi''$ in the Hilbert space $L^2(R^4,R^4)$ where [see (1.5)] $\phi''=(\nabla\operatorname{div}(\phi))*K=\operatorname{div}(\phi)*\nabla K$ [so that $\Delta\phi''=\nabla\operatorname{div}(\phi)$]. We conclude

$$\|\phi-((\nabla\operatorname{div}(\phi))*K)\|_2=\|\phi'\|_2\leqq\|\phi\|_2.$$

Hence we can use (3.3) to construct $g\in L^2(R^4,R^4)$ such that (3.4) and (3.5) hold:

$$\|g\|_2\leqq\|f\|_2, \tag{3.4}$$

$$\int g\cdot\phi=\int f\cdot(\phi-((\nabla\operatorname{div}(\phi))*K)) \quad \text{if} \quad \phi\in C_0^\infty(R^4,R^4). \tag{3.5}$$

Lemma 3.4. *If* $a\in R^4$, $\zeta^{1/2}\leqq r<\infty$, $\phi\in C_0^\infty(R^4,R)$, *and* $\operatorname{spt}(\phi)\subset B(a,r)$ *then*

$$\begin{aligned}&|\int g_i(x)w_i(x)\phi(x)dx| \\ &\leqq C(\|D\phi\|_\infty+r^{-1}\|\phi\|_\infty)r^5(\int|w(x)|^3(|x-a|+r)^{-5}dx).\end{aligned}$$

Proof. We define $J:R^4\to R$ by

$$J(x)=(|x-a|+r)^{-5}. \tag{3.6}$$

Let $\varepsilon>0$ such that $4\varepsilon<r$ and $4\varepsilon<r^{-1}$. We construct C^∞ functions α', β', γ' with domain R^4 and range $[0,1]$ such that the following conditions are satisfied: $\alpha'(x)=1$ if $x\in B(0,\varepsilon)$, $\alpha'(x)=0$ if $x\notin B(0,2\varepsilon)$, $\varepsilon\|D\alpha'\|_\infty+\varepsilon^2\|D^2\alpha'\|_\infty\leqq C$, $\beta'(x)=1$ if $x\in B(0,r)$, $\beta'(x)=0$ if $x\notin B(0,2r)$, $r\|D\beta'\|_\infty+r^2\|D^2\beta'\|_\infty+r^3\|D^3\beta'\|_\infty\leqq C$, $\gamma'(x)=1$ if $x\in B(0,\varepsilon^{-1})$, $\gamma'(x)=0$ if $x\notin B(0,2\varepsilon^{-1})$, $\varepsilon^{-1}\|D\gamma'\|_\infty+\varepsilon^{-2}\|D^2\gamma'\|_\infty+\varepsilon^{-3}\|D^3\gamma'\|_\infty\leqq C$. We define functions α, β, γ, δ with domain R^4 and range $[0,1]$ as follows: $\alpha=\alpha'$, $\beta=\beta'-\alpha'$, $\gamma=\gamma'-\beta'$, $\delta=1-\gamma'$. We have

$$\alpha+\beta+\gamma+\delta=1, \tag{3.7}$$

$$\|\alpha K\|_1\leqq C\varepsilon^2,\ |D^3(\gamma K)(x)|\leqq C(|x|+r)^{-5},\ \|D(\delta K)\|_2\leqq C\varepsilon. \tag{3.8}$$

Since $\Delta K(x)=0$ for every $x\neq 0$, we have $\Delta(\beta K)(x)=0$ for every $x\in B(0,r)-B(0,2\varepsilon)$. We conclude

$$\|\Delta(\beta K)\|_1\leqq C. \tag{3.9}$$

We use Definition 3.3, $\operatorname{div}(w)=0$, the Schwarz inequality, Young's inequality, and (3.8) to obtain

$$\begin{aligned}
&|\textstyle\int f_i D_i(\operatorname{div}(w\phi)*\alpha K)|\\
&=|\textstyle\int f_i D_i((w_j D_j\phi)*\alpha K)|\\
&=|\textstyle\int f_i (D_i(w_j D_j\phi)*\alpha K)|\\
&\leqq \|f\|_2\,\|D(w_j D_j\phi)*\alpha K\|_2\\
&\leqq \|f\|_2\,\|D(w_j D_j\phi)\|_2\,\|\alpha K\|_1\\
&\leqq C\|f\|_2(\|Dw\|_2\,\|D\phi\|_\infty+\|w\|_2\,\|D^2\phi\|_\infty)\varepsilon^2.
\end{aligned} \tag{3.10}$$

We use Definition 3.3, $\operatorname{div}(w)=0$, $\operatorname{spt}(\phi)\subset B(a,r)$, $\operatorname{spt}(\beta)\subset B(0,2r)$, and the generalized Hölder inequality to estimate

$$\begin{aligned}
&|\textstyle\int f_i D_i(\operatorname{div}(w\phi)*\beta K)|\\
&=|\textstyle\int (w_j*H_\zeta)D_j w_i D_i((w_k D_k\phi)*\beta K)|\\
&=|\textstyle\int (w_j*H_\zeta)w_i D_{ij}((w_k D_k\phi)*\beta K)|\\
&=\Big|\int_{B(a,3r)} (w_j*H_\zeta)w_i D_{ij}((w_k D_k\phi)*\beta K)\Big|\\
&\leqq \sum_{i,j}\Big(\int_{B(a,3r)}|w*H_\zeta|^3\Big)^{1/3}\Big(\int_{B(a,3r)}|w|^3\Big)^{1/3}\|D_{ij}((w_k D_k\phi)*\beta K)\|_3.
\end{aligned} \tag{3.11}$$

From Lemma 3.2 we obtain

$$\begin{aligned}
\int_{B(a,3r)}|w*H_\zeta|^3 &\leqq Cr^5(\textstyle\int |(w*H_\zeta)(x)|^3(|x-a|+r)^{-5}dx)\\
&\leqq Cr^5(\textstyle\int |w(x)|^3(|x-a|+r)^{-5}dx).
\end{aligned} \tag{3.12}$$

We also have

$$\int_{B(a,3r)}|w|^3\leqq Cr^5(\textstyle\int |w(x)|^3(|x-a|+r)^{-5}dx). \tag{3.13}$$

Now [7, Proposition 3, p. 59], $\operatorname{spt}(\phi)\subset B(a,r)$, $\operatorname{spt}(\beta)\subset B(0,2r)$, Young's inequality, and (3.9) yield

$$\begin{aligned}
&\|D_{ij}((w_k D_k\phi)*\beta K)\|_3\leqq C\|\Delta((w_k D_k\phi)*\beta K)\|_3=C\|w_k D_k\phi*\Delta(\beta K)\|_3\\
&\leqq C\|w_k D_k\phi\|_3\,\|\Delta(\beta K)\|_1\leqq C\|w_k D_k\phi\|_3\leqq C\Big(\int_{B(a,r)}|w|^3\Big)^{1/3}\|D\phi\|_\infty\\
&\leqq Cr^{5/3}\|D\phi\|_\infty(\textstyle\int |w(x)|^3(|x-a|+r)^{-5}dx)^{1/3}.
\end{aligned} \tag{3.14}$$

Combining (3.11), (3.12), (3.13), and (3.14) we obtain

$$|\textstyle\int f_i D_i(\operatorname{div}(w\phi)*\beta K)|\leqq Cr^5\|D\phi\|_\infty(\textstyle\int |w(x)|^3(|x-a|+r)^{-5}dx). \tag{3.15}$$

From Definition 3.3 and the hypothesis $\operatorname{div}(w)=0$ we obtain

$$\begin{aligned}
&|\textstyle\int f_i D_i(\operatorname{div}(w\phi)*\gamma K)|\\
&=|\textstyle\int (w_j*H_\zeta)D_j w_i D_i(D_k(w_k\phi)*\gamma K)|\\
&=|\textstyle\int (w_j*H_\zeta)w_i D_{ijk}(w_k\phi*\gamma K)|\\
&=|\textstyle\int (w_j*H_\zeta)w_i(w_k\phi*D_{ijk}(\gamma K))|\,.
\end{aligned} \tag{3.16}$$

Now (3.8), $\operatorname{spt}(\phi)\subset B(a,r)$, and Hölder's inequality yield

$$\begin{aligned}
&|(w_k\phi*D_{ijk}(\gamma K))(x)|\\
&=|\textstyle\int (w_k\phi)(y)(D_{ijk}(\gamma K))(x-y)dy|\\
&\leqq C(\textstyle\int |(w\phi)(y)|(|x-y|+r)^{-5}dy)\\
&\leqq C(\textstyle\int |(w\phi)(y)|dy)(\sup\{(|x-y|+r)^{-5}: y\in B(a,r)\})\\
&\leqq C\|\phi\|_\infty\Big(\int_{B(a,r)} |w(y)|dy\Big)(|x-a|+r)^{-5}\\
&\leqq C\|\phi\|_\infty\Big(\int_{B(a,r)} |w(y)|^3dy\Big)^{1/3}\Big(\int_{B(a,r)} (1)^{3/2}\Big)^{2/3}(|x-a|+r)^{-5}\\
&\leqq C\|\phi\|_\infty\Big(\int_{B(a,r)} |w(y)|^3dy\Big)^{1/3}r^{8/3}(|x-a|+r)^{-5}\\
&\leqq C\|\phi\|_\infty r^{13/3}(\textstyle\int |w(y)|^3(|y-a|+r)^{-5}dy)^{1/3}(|x-a|+r)^{-5}\,.
\end{aligned} \tag{3.17}$$

Combining (3.16), (3.17), and (3.6) we obtain

$$\begin{aligned}
&|\textstyle\int f_i D_i(\operatorname{div}(w\phi)*\gamma K)|\\
&\leqq C\|\phi\|_\infty r^{13/3}(\textstyle\int |w(y)|^3 J(y)dy)^{1/3}(\int |(w*H_\zeta)(x)|\,|w(x)|J(x)dx)\,.
\end{aligned} \tag{3.18}$$

Furthermore, the generalized Hölder inequality, Lemma 3.2, and (3.6) yield

$$\begin{aligned}
&\textstyle\int |(w*H_\zeta)(x)|\,|w(x)|J(x)dx\\
&=\textstyle\int (|(w*H_\zeta)(x)|(J(x))^{1/3})(|w(x)|(J(x))^{1/3})((J(x))^{1/3})dx\\
&\leqq(\textstyle\int |(w*H_\zeta)(x)|^3 J(x)dx)^{1/3}(\int |w(x)|^3 J(x)dx)^{1/3}(\int J(x)dx)^{1/3}\\
&\leqq C(\textstyle\int |w(x)|^3 J(x)dx)^{2/3}(\int J(x)dx)^{1/3}\\
&\leqq Cr^{-1/3}(\textstyle\int |w(x)|^3 J(x)dx)^{2/3}\,.
\end{aligned} \tag{3.19}$$

Combining (3.18), (3.19), and (3.6) we obtain

$$|\textstyle\int f_i D_i(\operatorname{div}(w\phi)*\gamma K)|\leqq C\|\phi\|_\infty r^4(\int |w(x)|^3(|x-a|+r)^{-5}dx)\,. \tag{3.20}$$

We use $\operatorname{div}(w)=0$, the Schwarz inequality, Definition 3.3, (3.8), and Young's inequality to estimate

$$\begin{aligned}
&|\textstyle\int f_i D_i(\operatorname{div}(w\phi)*\delta K)|\\
&=|\textstyle\int f_i(w_k D_k\phi*D_i(\delta K))|\\
&\leqq\|f\|_1\,\|w_k D_k\phi*D(\delta K)\|_\infty\\
&\leqq\|f\|_1\,\|w_k D_k\phi\|_2\,\|D(\delta K)\|_2\\
&\leqq C\|w*H_\zeta\|_2\,\|Dw\|_2\,\|w\|_2\,\|D\phi\|_\infty\varepsilon\\
&\leqq C\|w\|_2\,\|H_\zeta\|_1\,\|Dw\|_2\,\|w\|_2\,\|D\phi\|_\infty\varepsilon\,.
\end{aligned} \tag{3.21}$$

Now we use (3.7), (3.10), (3.15), (3.20), (3.21), (3.3), $\|w\|_2<\infty$, $\|Dw\|_2<\infty$, $\|H_\zeta\|_1<\infty$, $\phi\in C_0^\infty(R^4,R)$, and the fact that ε can be made arbitrarily small to conclude

$$\begin{aligned}&|\textstyle\int f_i D_i(\operatorname{div}(w\phi)*K)|\\&\leqq C(\|D\phi\|_\infty+r^{-1}\|\phi\|_\infty)r^5(\textstyle\int|w(x)|^3(|x-a|+r)^{-5}dx).\end{aligned}\tag{3.22}$$

Using Definition 3.3, div.$(w)=0$, spt$(\phi)\subset B(a,r)$, Hölder's inequality, (3.12), and (3.6) we find

$$\begin{aligned}&|\textstyle\int f_i w_i\phi|=|\int(w_j*H_\zeta)(D_jw_i)w_i\phi|=(1/2)|\int(w_j*H_\zeta)D_j(|w|^2)\phi|\\&=(1/2)|\textstyle\int(w_j*H_\zeta)|w|^2D_j\phi|\leqq(1/2)\Big(\int_{B(a,r)}|w*H_\zeta||w|^2\Big)\|D\phi\|_\infty\\&\leqq(1/2)\|D\phi\|_\infty\Big(\textstyle\int_{B(a,r)}|w*H_\zeta|^3\Big)^{1/3}\Big(\int_{B(a,r)}|w|^3\Big)^{2/3}\\&\leqq C\|D\phi\|_\infty r^{5/3}(\textstyle\int|w(x)|^3J(x)dx)^{1/3}r^{10/3}(\int|w(x)|^3J(x)dx)^{2/3}\\&=C\|D\phi\|_\infty r^5(\textstyle\int|w(x)|^3(|x-a|+r)^{-5}dx).\end{aligned}\tag{3.23}$$

Finally (3.5) (with ϕ replaced by $w\phi$), (3.22), and (3.23) imply the conclusion of Lemma 3.4.

For every $n\in\{1,2,3,\ldots\}$ we set $r_n=n\zeta^{1/2}$ and construct a C^∞ function $\phi_n:R^4\to[0,1]$ such that $\phi_n(x)=1$ if $x\in B(0,r_n/2)$, $\phi_n(x)=0$ if $x\notin B(0,r_n)$, and $\|D\phi_n\|_\infty\leqq Cr_n^{-1}$. Lemma 3.4 (with $a=0$) yields

$$|\textstyle\int g_iw_i\phi_n|\leqq Cr_n^{-1}\|w\|_3^3.\tag{3.24}$$

From Lemma 3.1, $\|w\|_2<\infty$, and $\|Dw\|_2<\infty$ we conclude $\|w\|_3<\infty$. We also have $g\in L^2(R^4,R^4)$ (see Definition 3.3) and $w\in L^2(R^4,R^4)$. Hence we can take the limit as $n\to\infty$ in (3.24) and conclude

$$\textstyle\int g_iw_i=0.\tag{3.25}$$

Lemma 3.5. *Suppose $a\in R^4$, p is an integer, t is a real number, $\zeta\leqq t\leqq 2^{-2p+1}$, $\alpha:R^4\to[0,1]$ is a C^∞ function, $\alpha(x)=1$ if $x\in B(a,2^{-p-1})$, $\alpha(x)=0$ if $x\notin B(a,2^{-p})$, and $\|D\alpha\|_\infty\leqq 2^{p+2}$. Then*

$$|\textstyle\int g_i(x)w_i(x)H_t(x-a)\alpha(x)dx|\leqq C(\int|w(x)|^3(|x-a|+t^{1/2})^{-5}dx).$$

Proof. We define k by the properties

$$2^{-2k-1}<t\leqq 2^{-2k+1},\ k\text{ is an integer}.\tag{3.26}$$

The hypotheses imply $p\leqq k$. For every integer j satisfying $p\leqq j\leqq k$ we construct α_j as follows: We set $\alpha_p=\alpha$; if $p<j\leqq k$ we choose a C^∞ function $\alpha_j:R^4\to[0,1]$ such that $\alpha_j(x)=1$ if $x\in B(a,2^{-j-1})$, $\alpha_j(x)=0$ if $x\notin B(a,2^{-j})$, $\|D\alpha_j\|_\infty\leqq 2^{j+2}$. We define $\phi_j:R^4\to R$ as follows:

$$\begin{aligned}&\phi_j(x)=H_t(x-a)(\alpha_j(x)-\alpha_{j+1}(x))\quad\text{if}\quad p\leqq j<k\\&\phi_k(x)=H_t(x-a)\alpha_k(x).\end{aligned}$$

From (3.26) and the hypotheses we conclude $2^{-j+1} \geqq 2^{-k+1} \geqq (2t)^{1/2} > \zeta^{1/2}$ if $p \leqq j \leqq k$. We also have $\operatorname{spt}(\phi_j) \subset B(a, 2^{-j+1})$. The last two statements and Lemma 3.4 yield

$$\begin{aligned}
&|\int g_i(x) w_i(x) H_t(x-a) \alpha(x) dx| \\
&\leqq \sum_{j=p}^{k} |\int g_i(x) w_i(x) \phi_j(x) dx| \\
&\leqq \sum_{j=p}^{k} C(\|D\phi_j\|_\infty + 2^{j-1}\|\phi_j\|_\infty) 2^{-5j+5} (\int |w(x)|^3 (|x-a| + 2^{-j+1})^{-5} dx).
\end{aligned} \tag{3.27}$$

We have

$$0 < H_t(x) \leqq C t^{1/2} (|x| + t^{1/2})^{-5}, \ |D(H_t)(x)| \leqq C t^{1/2} (|x| + t^{1/2})^{-6}. \tag{3.28}$$

We also have

$$\begin{aligned}
&\operatorname{spt}(\alpha_j - \alpha_{j+1}) \subset B(a, 2^{-j}) - B(a, 2^{-j-2}) \quad \text{if} \quad j < k, \\
&\operatorname{spt}(\alpha_k) \subset B(a, 2^{-k}).
\end{aligned} \tag{3.29}$$

From (3.26), (3.28), and (3.29) we conclude

$$\begin{aligned}
&0 < H_t(x-a) \leqq C 2^{5j-k} \quad \text{if} \quad x \in \operatorname{spt}(\alpha_j - \alpha_{j+1}) \quad \text{and} \quad j < k, \\
&0 < H_t(x-a) \leqq C 2^{4k} \quad \text{if} \quad x \in \operatorname{spt}(\alpha_k), \\
&|D(H_t)(x-a)| \leqq C 2^{6j-k} \quad \text{if} \quad x \in \operatorname{spt}(\alpha_j - \alpha_{j+1}) \quad \text{and} \quad j < k, \\
&|D(H_t)(x-a)| \leqq C 2^{5k} \quad \text{if} \quad x \in \operatorname{spt}(\alpha_k).
\end{aligned} \tag{3.30}$$

From (3.30) and the properties of α_j we obtain

$$\|D\phi_j\|_\infty + 2^{j-1}\|\phi_j\|_\infty \leqq C 2^{6j-k} \quad \text{if} \quad p \leqq j \leqq k. \tag{3.31}$$

Now (3.27), (3.31), and (3.26) yield

$$\begin{aligned}
&|\int g_i(x) w_i(x) H_t(x-a) \alpha(x) dx| \\
&\leqq \sum_{j=p}^{k} C 2^{j-k} (\int |w(x)|^3 (|x-a| + 2^{-j+1})^{-5} dx) \\
&\leqq \sum_{j=p}^{k} C 2^{j-k} (\int |w(x)|^3 (|x-a| + 2^{-k})^{-5} dx) \\
&\leqq C (\int |w(x)|^3 (|x-a| + 2^{-k})^{-5} dx) \\
&\leqq C (\int |w(x)|^3 (|x-a| + t^{1/2})^{-5} dx).
\end{aligned}$$

The lemma has been proved.

Lemma 3.6. *If* $0 < t < \infty$, $f \in C^\infty(R^4, R)$, $\int |f|^2 < \infty$, *and* $\int |Df|^2 < \infty$ *then*

$$\|f - (f * H_t)\|_2 \leqq C t^{1/2} \|Df\|_2 .$$

Proof. Define $g: R^4 \times (0, \infty) \to R$ by $g(x, s) = (f * H_s)(x)$. The relation $D_t g = \Delta g$ yields

$$\begin{aligned}
(f * H_t)(x) - f(x) &= \int_0^t D_t g(x, s) ds = \int_0^t \Delta g(x, s) ds \\
&= \int_0^t D_{ii}(f * H_s)(x) ds = \int_0^t (D_i f * D_i H_s)(x) ds .
\end{aligned}$$

Hence Minkowski's integral inequality and Young's inequality yield

$$(\int |(f*H_t)(x)-f(x)|^2 dx)^{1/2}$$
$$=\left(\int\left|\int_0^t (D_i f*D_i H_s)(x)ds\right|^2 dx\right)^{1/2}$$
$$\leqq \int_0^t (\int |(D_i f*D_i H_s)(x)|^2 dx)^{1/2} ds$$
$$\leqq \int_0^t \|Df\|_2 \|DH_s\|_1 ds = \|Df\|_2 \left(\int_0^t (Cs^{-1/2})ds\right) = C\|Df\|_2 t^{1/2}.$$

Section 4. Estimates on Approximate Solutions

Throughout this section we fix positive real numbers ζ and d such that (see Section 1)

$$d\leqq\zeta,\ dL\|H_\zeta\|_2^2\leqq 1. \tag{4.1}$$

Definition 4.1. We use induction to define functions v^{-1}, v^0, v^1, $v^2,\ldots$ such that $v^k\in L^2(R^4,R^4)$ and

$$\|v^k\|_2^2\leqq L,\ \operatorname{div}(v^k)=0. \tag{4.2}$$

We set $v^{-1}=v$ (see Section 1). Suppose that $k\geqq 0$ and $v^{k-1}\in L^2(R^4,R^4)$ has been defined so that (4.3) holds:

$$\|v^{k-1}\|_2^2\leqq L,\ \operatorname{div}(v^{k-1})=0. \tag{4.3}$$

We will define v^k with the aid of several auxiliary functions. Let $u^k: R^4\times[kd,\infty)\to R^4$ be given by $u^k(x,kd)=v^{k-1}(x)$, $u^k(x,kd+t)=(v^{k-1}*H_t)(x)$ if $t>0$. Let $w^k: R^4\to R^4$ be given by $w^k(x)=u^k(x,kd+d)$. The relationship $D_t(u^k)=\Delta(u^k)$ implies

$$\|w^k\|_2^2=\|v^{k-1}\|_2^2-2\left(\int_{kd}^{kd+d}\int |Du^k(x,t)|^2 dx\,dt\right). \tag{4.4}$$

Similarly, the relationship $D_t(D_i u^k)=\Delta(D_i u^k)$ implies

$$\|D_i w^k\|_2^2=\int |D_i u^k(x,t)|^2 dx-2\left(\int_t^{kd+d}\int |D(D_i u^k)(x,s)|^2 dx\,ds\right) \tag{4.5}$$

for all $t\in(kd,kd+d)$. Averaging (4.5) over t and summing over i we obtain

$$\|Dw^k\|_2^2\leqq d^{-1}\left(\int_{kd}^{kd+d}\int |Du^k(x,t)|^2 dx\,dt\right). \tag{4.6}$$

From (4.3), (4.4), (4.6), and the definition of w^k we conclude that w^k satisfies

$$\|w^k\|_2^2\leqq\|v^{k-1}\|_2^2\leqq L, \tag{4.7}$$

$$\|Dw^k\|_2^2\leqq(2d)^{-1}\|v^{k-1}\|_2^2\leqq(2d)^{-1}L, \tag{4.8}$$

$$w^k\in C^\infty(R^4,R^4)\quad\text{and}\quad\operatorname{div}(w^k)=0. \tag{4.9}$$

Hence we can replace w by w^k in Section 3 and construct functions f^k and g^k corresponding to f and g in Definition 3.3. We set

$$v^k = w^k - (d)(g^k). \tag{4.10}$$

In order to complete the inductive definition, we must show that (4.2) holds. Using (3.4), (3.3), (4.7), and (4.6) we obtain

$$\begin{aligned} d^2\|g^k\|_2^2 &\leqq d^2\|f^k\|_2^2 \leqq d^2\|w^k\|_2^2\|H_\zeta\|_2^2\|Dw^k\|_2^2 \leqq d^2L\|H_\zeta\|_2^2\|Dw^k\|_2^2 \\ &\leqq dL\|H_\zeta\|_2^2\left(\int_{kd}^{kd+d}\int|Du^k(x,t)|^2dxdt\right). \end{aligned} \tag{4.11}$$

From (4.10), (3.25), (4.4), (4.11), and (4.1) we obtain

$$\begin{aligned} \|v^k\|_2^2 &= \|w^k\|_2^2 + d^2\|g^k\|_2^2 \\ &= \|v^{k-1}\|_2^2 - 2\left(\int_{kd}^{kd+d}\int|Du^k(x,t)|^2dxdt\right) + d^2\|g^k\|_2^2 \\ &\leqq \|v^{k-1}\|_2^2 - \int_{kd}^{kd+d}\int|Du^k(x,t)|^2dxdt. \end{aligned} \tag{4.12}$$

From (4.12) and (4.3) we conclude

$$\|v^k\|_2^2 \leqq \|v^{k-1}\|_2^2 \leqq L. \tag{4.13}$$

If $\phi\in C_0^\infty(R^4, R)$ then $(\nabla\operatorname{div}(\nabla\phi))*K = \nabla(\Delta\phi * K) = \nabla\phi$. Hence (3.5) yields $\int g^k\cdot\nabla\phi = 0$. We conclude $\operatorname{div}(g^k)=0$. Combining this with (4.9), (4.10), and (4.13) we obtain that (4.2) holds. The definition of the v^k is complete.

We define $u: R^4\times[0,\infty)\to R^4$ (this is not the u in Theorem 1.1) by

$$u(x,t) = u^k(x,t) \quad \text{if} \quad kd \leqq t < kd + d. \tag{4.14}$$

From (4.2) and (4.12) we conclude

$$\int_0^\infty\int|Du(x,t)|^2dxdt \leqq L. \tag{4.15}$$

If $t\geqq kd$ then the property $\int|u^k(x,t)|^2dx \leqq \|v^{k-1}\|_2^2$ follows in the same way as (4.4). Hence (4.2) yields

$$\int|u(x,t)|^2dx \leqq L \quad \text{for all} \quad t\geqq 0. \tag{4.16}$$

The argument in the proof of Lemma 3.1, (4.15), and (4.16) yield

$$\begin{aligned} &\int_0^\infty\int|u(x,t)|^3dxdt \\ &\quad\leqq C\left(\int_0^\infty\left(\int|Du(x,t)|^2dx\right)\left(\int|u(x,t)|^2dx\right)^{1/2}dt\right) \leqq CL^{3/2}. \end{aligned} \tag{4.17}$$

Definition 4.2. In Lemmas 4.3 and 4.4 we fix $a\in R^4$, $0<b<\infty$, and integers p and q satisfying $p<q$, $2^{-2p}\leqq b$, and $d\leqq 2^{-2p-2}$. We construct C^∞ functions $\alpha: R^4\to[0,1]$

and $\beta: R \to [0,1]$ such that $\alpha(x)=1$ if $x \in B(a, 2^{-p-1})$, $\alpha(x)=0$ if $x \notin B(a, 2^{-p})$, $\|D\alpha\|_\infty \leqq 2^{p+2}$, $\|D^2\alpha\|_\infty \leqq C2^{2p}$, $\beta(t)=1$ if $t \geqq b-2^{-2p-2}$, $\beta(t)=0$ if $t \leqq b-2^{-2p}+d$, and $\|(\partial/\partial t)\beta\|_\infty \leqq C2^{2p}$. We define $\phi: R^4 \times [0,b] \to R$ by [see (1.4)]

$$\phi(x,t) = (H[b-t+2^{-2q}])(x-a)\alpha(x)\beta(t). \tag{4.18}$$

We have $(D_t\phi + \Delta\phi)(x,t)=0$ if $|x-a| < 2^{-p-1}$ and $b-2^{-2p-2} < t < b$. From this we conclude

$$\|D_t\phi + \Delta\phi\|_\infty \leqq C2^{6p}. \tag{4.19}$$

We also have

$$\|\phi\|_\infty \leqq C2^{4q}. \tag{4.20}$$

Lemma 4.3. *If* $k \in \{0,1,2,\ldots\}$, $b-2^{-2p} \leqq kd+d \leqq b$ *and* $\zeta \leqq 2^{-2q}$ *(see Definition 4.2) then*

$$\begin{aligned}&\int |u^{k+1}(x,kd+d)|^2\phi(x,kd+d)dx \\ &\quad -\int |u^k(x,kd+d)|^2\phi(x,kd+d)dx \\ &\quad \leqq C\left(\int_{kd}^{kd+d}\int |u(x,t)|^3(|x-a|+(b-t+2^{-2q})^{1/2})^{-5}dxdt\right) \\ &\quad + CdL\|H_\zeta\|_2^2(2^{4q})\left(\int_{kd}^{kd+d}\int |Du(x,t)|^2 dxdt\right).\end{aligned}$$

Proof. From Definition 4.1 and (4.10) we obtain

$$\begin{aligned}&\int |u^{k+1}(x,kd+d)|^2\phi(x,kd+d)dx \\ &\qquad -\int |u^k(x,kd+d)|^2\phi(x,kd+d)dx \\ &\quad = \int |v^k(x)|^2\phi(x,kd+d)dx - \int |w^k(x)|^2\phi(x,kd+d)dx \\ &\quad = -2d\left(\int g_i^k(x)w_i^k(x)\phi(x,kd+d)dx\right) + d^2\left(\int |g^k(x)|^2\phi(x,kd+d)dx\right).\end{aligned} \tag{4.21}$$

Using (4.20), (4.11), and (4.14) we find

$$\begin{aligned}&d^2\left(\int |g^k(x)|^2\phi(x,kd+d)dx\right) \leqq Cd^2\|g^k\|_2^2(2^{4q}) \\ &\quad \leqq CdL\|H_\zeta\|_2^2(2^{4q})\left(\int_{kd}^{kd+d}\int |Du(x,t)|^2dxdt\right).\end{aligned} \tag{4.22}$$

The hypotheses on p, q, and k imply

$$b-(kd+d)+2^{-2q} \leqq 2^{-2p}+2^{-2q} < 2^{-2p+1}. \tag{4.23}$$

The hypothesis of Lemma 4.3 implies

$$\zeta \leqq 2^{-2q} \leqq b-(kd+d)+2^{-2q}. \tag{4.24}$$

Now (4.18), Definition 4.2, (4.23), (4.24), and Lemma 3.5 yield [see (1.4)]

$$\begin{aligned}&\left|\int g_i^k(x)w_i^k(x)\phi(x,kd+d)dx\right| \\ &\quad = \beta(kd+d)\left|\int g_i^k(x)w_i^k(x)(H[b-(kd+d)+2^{-2q}])(x-a)\alpha(x)dx\right| \\ &\quad \leqq C\left(\int |w^k(x)|^3(|x-a|+(b-(kd+d)+2^{-2q})^{1/2})^{-5}dx\right).\end{aligned} \tag{4.25}$$

For every $t \in (kd, kd+d)$ we define $h^t: R^4 \to R^4$ by

$$h^t(x) = u^k(x,t) = u(x,t). \tag{4.26}$$

From Definition 4.1 and the semigroup property $H_a * H_b = H_{a+b}$ (see [8, Corollary 1.28, p. 16]) we obtain

$$\begin{aligned} w^k(x) &= u^k(x, kd+d) = (v^{k-1} * H_d)(x) \\ &= (v^{k-1} * H[t-kd] * H[kd+d-t])(x) \\ &= (h^t * H[kd+d-t])(x). \end{aligned} \tag{4.27}$$

The hypotheses of Lemma 4.3 and (4.1) yield the following for $kd < t < kd+d$:

$$(kd+d-t)^{1/2} < d^{1/2} \leqq \zeta^{1/2} \leqq (2^{-2q})^{1/2} \leqq (b-(kd+d)+2^{-2q})^{1/2}. \tag{4.28}$$

Now (4.27), (4.28), and the proof of Lemma 3.2 yield

$$\begin{aligned} &\int |w^k(x)|^3 (|x-a| + (b-(kd+d)+2^{-2q})^{1/2})^{-5} dx \\ &\quad \leqq C\left(\int |h^t(x)|^3 (|x-a| + (b-(kd+d)+2^{-2q})^{1/2})^{-5} dx\right) \end{aligned} \tag{4.29}$$

if $kd < t < kd+d$. From (4.1) and the hypotheses $\zeta \leqq 2^{-2q}$, $kd+d \leqq b$ we obtain $d \leqq \zeta \leqq 2^{-2q}$ and $d \leqq b-kd$. This implies $d \leqq (1/2)(b-kd+2^{-2q})$, which in turn implies

$$\begin{aligned} b-(kd+d)+2^{-2q} &= (b-kd+2^{-2q}) - d \geqq (1/2)(b-kd+2^{-2q}) \\ &\geqq (1/2)(b-t+2^{-2q}) \end{aligned}$$

whenever $kd < t < kd+d$. We conclude

$$(b-t+2^{-2q})^{1/2} > (b-(kd+d)+2^{-2q})^{1/2} \geqq (1/2)^{1/2}(b-t+2^{-2q})^{1/2} \tag{4.30}$$

if $kd < t < kd+d$. Now (4.30) implies

$$\begin{aligned} &\int |h^t(x)|^3 (|x-a| + (b-(kd+d)+2^{-2q})^{1/2})^{-5} dx \\ &\leqq C\left(\int |h^t(x)|^3 (|x-a| + (b-t+2^{-2q})^{1/2})^{-5} dx\right) \end{aligned} \tag{4.31}$$

if $kd < t < kd+d$. Using (4.29) and (4.31), averaging over t, and using (4.26) we obtain

$$\begin{aligned} &\int |w^k(x)|^3 (|x-a| + (b-(kd+d)+2^{-2q})^{1/2})^{-5} dx \\ &\leqq Cd^{-1}\left(\int_{kd}^{kd+d} \int |u(x,t)|^3 (|x-a| + (b-t+2^{-2q})^{1/2})^{-5} dx\, dt\right). \end{aligned} \tag{4.32}$$

Finally, (4.21), (4.22), (4.25), and (4.32) yield the conclusion of the lemma.

Lemma 4.4. *If $b-2^{-2q} \leqq s \leqq b$ and $\zeta \leqq 2^{-2q}$ (see Definition 4.2) then*

$$\begin{aligned} &(1/2)\left(\int |u(x,s)|^2 \phi(x,s)\,dx\right) + \int_{b-2^{-2p}}^{s} \int |Du(x,t)|^2 \phi(x,t)\,dx\,dt \\ &\leqq C\left(\int_{b-2^{-2p}}^{b} \int |u(x,t)|^3 (|x-a| + (b-t+2^{-2q})^{1/2})^{-5} dx\,dt\right) \\ &\quad + CdL^2 \|H_\zeta\|_2^2 (2^{4q}) + C(2^{6p})\left(\int_{b-2^{-2p}}^{b} \int_{B(a,2^{-p})} |u(x,t)|^2 dx\,dt\right). \end{aligned}$$

Proof. We define integers k' and k'' by the relations

$$(k'-1)d<b-2^{-2p}\leqq k'd,\, k''d+d\leqq s<(k''+1)d+d. \tag{4.33}$$

From (4.33) and $b-2^{-2p}\geqq 0$ we obtain $k'\geqq 0$. From Definition 4.2 we obtain $2^{-2p}-2^{-2q}\geqq(3/4)2^{-2p}\geqq 3d$. Hence (4.33) and the hypotheses yield

$$k'd<d+b-2^{-2p}\leqq -2d+b-2^{-2q}\leqq -2d+s\leqq k''d.$$

Since $k'\geqq 0$ we conclude $0\leqq k'<k''$. From (4.14) we obtain

$$\begin{aligned}
&(1/2)\left(\int |u(x,k''d+d)|^2\phi(x,k''d+d)dx\right)\\
&-(1/2)\left(\int |u(x,k'd)|^2\phi(x,k'd)dx\right)\\
&=(1/2)\left(\sum_{k=k'}^{k''}\left(\int |u(x,kd+d)|^2\phi(x,kd+d)dx-\int |u(x,kd)|^2\phi(x,kd)dx\right)\right)\\
&=(1/2)\left(\sum_{k=k'}^{k''}\left(\int |u^{k+1}(x,kd+d)|^2\phi(x,kd+d)dx-\int |u^k(x,kd)|^2\phi(x,kd)dx\right)\right)\\
&=(1/2)\left(\sum_{k=k'}^{k''}\left(\int |u^{k+1}(x,kd+d)|^2\phi(x,kd+d)dx\right.\right.\\
&\qquad\left.\left.-\int |u^k(x,kd+d)|^2\phi(x,kd+d)dx\right)\right)\\
&+(1/2)\left(\sum_{k=k'}^{k''}\left(\int |u^k(x,kd+d)|^2\phi(x,kd+d)dx\right.\right.\\
&\qquad\left.\left.-\int |u^k(x,kd)|^2\phi(x,kd)dx\right)\right).
\end{aligned}\tag{4.34}$$

Taking the inner product of the relation $D_t(u^k)=\Delta(u^k)$ with $u^k\phi$ (see Definition 4.1) and using (4.14), Definition 4.2, and (4.19) we obtain the following whenever $k'\leqq k\leqq k''$:

$$\begin{aligned}
&(1/2)\left(\int |u^k(x,kd+d)|^2\phi(x,kd+d)dx\right)\\
&-(1/2)\left(\int |u^k(x,kd)|^2\phi(x,kd)dx\right)\\
&=-\int_{kd}^{kd+d}\int |Du^k(x,t)|^2\phi(x,t)dx\,dt\\
&+(1/2)\left(\int_{kd}^{kd+d}\int |u^k(x,t)|^2(D_t\phi+\Delta\phi)(x,t)dx\,dt\right)\\
&\leqq -\int_{kd}^{kd+d}\int |Du(x,t)|^2\phi(x,t)dx\,dt\\
&+C(2^{6p})\left(\int_{kd}^{kd+d}\int_{B(a,2^{-p})}|u(x,t)|^2dx\,dt\right).
\end{aligned}\tag{4.35}$$

From (4.34), (4.35), Lemma 4.3, the hypothesis of Lemma 4.4, and (4.33) we conclude

$$\begin{aligned}
&(1/2)\left(\int |u(x,k''d+d)|^2\phi(x,k''d+d)dx\right)\\
&-(1/2)\left(\int |u(x,k'd)|^2\phi(x,k'd)dx\right)
\end{aligned}$$

$$\leq C\left(\int_{b-2^{-2p}}^{b} \int |u(x,t)|^3(|x-a|+(b-t+2^{-2q})^{1/2})^{-5}dx\,dt\right)$$

$$+CdL\|H_\zeta\|_2^2(2^{4q})\left(\int_{b-2^{-2p}}^{b} \int |Du(x,t)|^2 dx\,dt\right)$$

$$-\int_{k'd}^{k''d+d} \int |Du(x,t)|^2\phi(x,t)dx\,dt$$

$$+C(2^{6p})\left(\int_{b-2^{-2p}}^{k''d+d} \int_{B(a,2^{-p})} |u(x,t)|^2 dx\,dt\right). \tag{4.36}$$

Using (4.33), (4.14), Definition 4.1, and the argument that produced (4.35) we obtain

$$(1/2)(\textstyle\int |u(x,s)|^2\phi(x,s)dx)$$

$$-(1/2)(\textstyle\int |u(x,k''d+d)|^2\phi(x,k''d+d)dx)$$

$$=-\int_{k''d+d}^{s} \int |Du(x,t)|^2\phi(x,t)dx\,dt$$

$$+(1/2)\left(\int_{k''d+d}^{s} \int |u(x,t)|^2(D_t\phi+\Delta\phi)(x,t)dx\,dt\right)$$

$$\leq -\int_{k''d+d}^{s} \int |Du(x,t)|^2\phi(x,t)dx\,dt$$

$$+C(2^{6p})\left(\int_{k''d+d}^{s} \int_{B(a,2^{-p})} |u(x,t)|^2 dx\,dt\right). \tag{4.37}$$

From Definition 4.2 and (4.33) we obtain $\beta(t)=0$ if $t\leq k'd$. Hence (4.18) yields

$$\phi(x,t)=0 \quad \text{if} \quad t\leq k'd. \tag{4.38}$$

Combining (4.36), (4.15), (4.37), (4.38), (4.33), and the hypothesis $s\leq b$ we obtain the conclusion of the lemma.

Lemma 4.5. *If $f\in C^\infty(R^4,R^4)$, $a\in R^4$, and $0<r<\infty$ then*

$$\int_{B(a,r)} |f|^3$$

$$\leq Cr^{-2}\left(\int_{B(a,2r)} |f|^2\right)^{3/2}+C\left(\int_{B(a,2r)} |Df|^2\right)\left(\int_{B(a,2r)} |f|^2\right)^{1/2}.$$

Proof. Let $\psi:R^4\to[0,1]$ be a C^∞ function such that $\psi(x)=1$ if $x\in B(a,r)$, $\psi(x)=0$ if $x\notin B(a,2r)$, and $\|D\psi\|_\infty\leq Cr^{-1}$. Applying the argument in Lemma 3.1 to ψf we find

$$\int_{B(a,r)} |f|^3\leq \int |\psi f|^3\leq C(\textstyle\int |D(\psi f)|^2)(\int |\psi f|^2)^{1/2}$$

$$\leq C\left(\|D\psi\|_\infty^2\left(\int_{B(a,2r)} |f|^2\right)+\|\psi\|_\infty^2\left(\int_{B(a,2r)} |Df|^2\right)\right)\left(\|\psi\|_\infty^2\left(\int_{B(a,2r)} |f|^2\right)\right)^{1/2}$$

$$\leq Cr^{-2}\left(\int_{B(a,2r)} |f|^2\right)^{3/2}+C\left(\int_{B(a,2r)} |Df|^2\right)\left(\int_{B(a,2r)} |f|^2\right)^{1/2}.$$

Definition 4.6. If $a \in R^4$, $0 < b < \infty$, m is an integer, and $2^{-2m} \leqq b$ we set

$$A(a,b,m) = 2^m\left(\int_{b-2^{-2m}}^{b} \int |u(x,t)|^3(|x-a| + 2^{-m})^{-5}\,dx\,dt\right),$$

$$B(a,b,m) = 2^{6m}\left(\int_{b-2^{-2m}}^{b} \int_{B(a,2^{-m})} |u(x,t)|^3\,dx\,dt\right).$$

Theorem 4.7. *If $a \in R^4$, $0 < b < \infty$, and p, q are integers satisfying $p < q$, $2^{-2p} \leqq b$, and $\zeta \leqq 2^{-2q}$ then*

$$B(a,b,q) \leqq C(A(a,b,p)) + Cd^{3/2}L^3\|H_\zeta\|_2^3(2^{6q})$$
$$+ C\left(2^{-p}A(a,b,p) + \sum_{m=p}^{q-1} 2^{-m}B(a,b,m)\right)^{3/2}.$$

Proof. It is easy to see that Definition 4.6 yields

$$\int_{b-2^{-2p}}^{b} \int |u(x,t)|^3(|x-a| + (b-t+2^{-2q})^{1/2})^{-5}\,dx\,dt$$
$$\leqq C\left(2^{-p}A(a,b,p) + \sum_{m=p}^{q-1} 2^{-m}B(a,b,m)\right). \tag{4.39}$$

From Hölder's inequality and Definition 4.6 we obtain

$$2^{6p}\left(\int_{b-2^{-2p}}^{b} \int_{B(a,2^{-p})} |u(x,t)|^2\,dx\,dt\right)$$
$$\leqq 2^{6p}(2^{-6p}B(a,b,p))^{2/3}\left(\int_{b-2^{-2p}}^{b} \int_{B(a,2^{-p})} (1)^3\,dx\,dt\right)^{1/3}$$
$$\leqq C(B(a,b,p))^{2/3} \leqq C(A(a,b,p))^{2/3}. \tag{4.40}$$

From (4.1), $p < q$, and $\zeta \leqq 2^{-2q}$ we obtain $d \leqq 2^{-2p-2}$. Hence a, b, p, q satisfy the properties required in Definition 4.2. Therefore we can define ϕ as in Definition 4.2 and use $\zeta \leqq 2^{-2q}$, Lemma 4.4, (4.39), and (4.40) to conclude

$$(1/2)\left(\int |u(x,s)|^2\phi(x,s)\,dx\right) + \int_{b-2^{-2p}}^{s} \int |Du(x,t)|^2\phi(x,t)\,dx\,dt$$
$$\leqq C\left(2^{-p}A(a,b,p) + \sum_{m=p}^{q-1} 2^{-m}B(a,b,m)\right) + CdL^2\|H_\zeta\|_2^2(2^{4q}) + C(A(a,b,p))^{2/3} \tag{4.41}$$

if $b - 2^{-2q} \leqq s \leqq b$. We set

$$Z = 2^{-p}A(a,b,p) + \left(\sum_{m=p}^{q-1} 2^{-m}B(a,b,m)\right) + dL^2\|H_\zeta\|_2^2(2^{4q}) + (A(a,b,p))^{2/3}. \tag{4.42}$$

We consider two cases: $q > p+1$ and $q = p+1$. Assume that $q > p+1$ holds. Definition 4.2 yields

$$2^{4q} \leqq C\phi(x,t) \quad \text{if} \quad x \in B(a, 2^{-q+1}) \quad \text{and} \quad b - 2^{-2q} \leqq t \leqq b. \tag{4.43}$$

From (4.41), (4.42), (4.43), and $\phi \geqq 0$ we obtain

$$\int_{B(a,2^{-q+1})} |u(x,s)|^2 dx \leqq C(2^{-4q})Z \quad \text{if} \quad b-2^{-2q} \leqq s \leqq b, \tag{4.44}$$

$$\int_{b-2^{-2q}}^{b} \int_{B(a,2^{-q+1})} |Du(x,t)|^2 dx\,dt \leqq C(2^{-4q})Z. \tag{4.45}$$

Since u is C^∞ except at the points (x, kd) for $k=0,1,2,\ldots$, Lemma 4.5 yields

$$\begin{aligned} &\int_{B(a,2^{-q})} |u(x,s)|^3 dx \\ &\leqq C(2^{2q})\left(\int_{B(a,2^{-q+1})} |u(x,s)|^2 dx\right)^{3/2} \\ &\quad + C\left(\int_{B(a,2^{-q+1})} |Du(x,s)|^2 dx\right)\left(\int_{B(a,2^{-q+1})} |u(x,s)|^2 dx\right)^{1/2} \end{aligned} \tag{4.46}$$

for almost every $s \in (b-2^{-2q}, b)$. Now Definition 4.6, (4.44), (4.45), and (4.46) yield

$$2^{-6q} B(a,b,q) \leqq C(2^{-6q}) Z^{3/2}. \tag{4.47}$$

From (4.42) we obtain

$$\begin{aligned} Z^{3/2} &\leqq C(2^{-p}A(a,b,p) + \sum_{m=p}^{q-1} 2^{-m} B(a,b,m))^{3/2} + C d^{3/2} L^3 \|H_\zeta\|_2^3 (2^{6q}) \\ &\quad + C(A(a,b,p)). \end{aligned} \tag{4.48}$$

Then (4.47) and (4.48) imply the conclusion of Theorem 4.7 in the case $q > p+1$. Now we assume $q = p+1$. We have

$$B(a,b,q) = B(a,b,p+1) \leqq C(B(a,b,p)) \leqq C(A(a,b,p)).$$

The proof of Theorem 4.7 is complete.

Theorem 4.8. *Let $\phi \in C_0^\infty(R^4 \times R, R^4)$ such that* $\operatorname{div}(\phi) = 0$. *Let N be a positive real number such that $|D\phi(x,t) - D\phi(x,s)| \leqq N|t-s|$ for all $x \in R^4$ and $s, t \in R$. Let T be a positive real number such that $\phi(x,t) = 0$ whenever $x \in R^4$ and $t \geqq T$. Then*

$$\begin{aligned} &\left| \int v_i(x)\phi_i(x,0)dx + \int_0^\infty \int u_i(x,t)(D_t\phi_i + \Delta\phi_i)(x,t)dx\,dt \right. \\ &\quad \left. + \int_0^\infty \int u_j(x,t)u_i(x,t) D_j\phi_i(x,t) dx\,dt \right| \\ &\leqq C(T+d)LNd + C(T+d)(d+\zeta)^{1/2}\|D\phi\|_\infty + C(d+\zeta)^{1/2} L^2 \|D\phi\|_\infty. \end{aligned}$$

Proof. We define k' by the properties

$$k'd \leqq T < k'd + d, \ k' \text{ is an integer}. \tag{4.49}$$

Observe that we have $k' \geqq 0$. Now let $k \in \{0,1,2,\ldots,k'\}$. We use Definition 4.1, the semigroup property $H_a * H_b = H_{a+b}$ (see [8, Corollary 1.28, p. 16]) and (4.14) to write the following whenever $kd<t<kd+d$:

$$
\begin{aligned}
w_i^k(x) &= u_i^k(x, kd+d) = (v_i^{k-1} * H_d)(x) \\
&= ((v_i^{k-1} * (H[t-kd])) * (H[kd+d-t]))(x) \\
&= (u_i^k * (H[kd+d-t]))(x,t) = (u_i * (H[kd+d-t]))(x,t) .
\end{aligned} \tag{4.50}
$$

From (4.50) and $H_a * H_b = H_{a+b}$ we conclude

$$
\begin{aligned}
(w_j^k * H_\zeta)(x) &= ((u_j * (H[kd+d-t])) * H_\zeta)(x,t) \\
&= (u_j * (H[kd+d-t+\zeta]))(x,t)
\end{aligned} \tag{4.51}
$$

if $kd<t<kd+d$. We set

$$H_t' = H[kd+d-t+\zeta] \quad \text{if} \quad kd<t<kd+d , \tag{4.52}$$

$$H_t'' = H[kd+d-t] \quad \text{if} \quad kd<t<kd+d . \tag{4.53}$$

Using (4.50), (4.51), (4.52), and (4.53) we obtain

$$
\begin{aligned}
&\int (w_j^k * H_\zeta)(x) w_i^k(x) D_j \phi_i(x, kd+d) dx \\
&= \int (u_j * H_t')(x,t)(u_i * H_t'')(x,t) D_j \phi_i(x, kd+d) dx
\end{aligned} \tag{4.54}
$$

if $kd<t<kd+d$. Averaging (4.54) over t we obtain

$$
\begin{aligned}
&\int (w_j^k * H_\zeta)(x) w_i^k(x) D_j \phi_i(x, kd+d) dx \\
&= d^{-1}\left(\int_{kd}^{kd+d} \int (u_j * H_t')(x,t)(u_i * H_t'')(x,t) D_j \phi_i(x, kd+d) dx\, dt \right) .
\end{aligned} \tag{4.55}
$$

From Definition 4.1, (4.10), (3.5), the assumption $\operatorname{div}(\phi)=0$, Definition 3.3, integration by parts, and (4.9) we obtain

$$
\begin{aligned}
&\int u_i^{k+1}(x, kd+d) \phi_i(x, kd+d) dx \\
&= \int v_i^k(x) \phi_i(x, kd+d) dx \\
&= \int w_i^k(x) \phi_i(x, kd+d) dx - d\left(\int g_i^k(x) \phi_i(x, kd+d) dx\right) \\
&= \int u_i^k(x, kd+d) \phi_i(x, kd+d) dx - d\left(\int f_i^k(x) \phi_i(x, kd+d) dx\right) \\
&= \int u_i^k(x, kd+d) \phi_i(x, kd+d) dx \\
&\quad - d\left(\int (w_j^k * H_\zeta)(x) D_j w_i^k(x) \phi_i(x, kd+d) dx\right) \\
&= \int u_i^k(x, kd+d) \phi_i(x, kd+d) dx \\
&\quad + d\left(\int (w_j^k * H_\zeta)(x) w_i^k(x) D_j \phi_i(x, kd+d) dx\right) .
\end{aligned} \tag{4.56}
$$

From (4.55) and (4.56) we obtain

$$
\begin{aligned}
&\int u_i^{k+1}(x, kd+d) \phi_i(x, kd+d) dx \\
&- \int u_i^k(x, kd+d) \phi_i(x, kd+d) dx \\
&= \int_{kd}^{kd+d} \int (u_j * H_t')(x,t)(u_i * H_t'')(x,t) D_j \phi_i(x, kd+d) dx\, dt .
\end{aligned} \tag{4.57}
$$

Taking the inner product of the relation $D_t(u^k)=\Delta(u^k)$ with ϕ and using (4.14) we obtain

$$\begin{aligned}&\int u_i^k(x,kd+d)\phi_i(x,kd+d)dx-\int u_i^k(x,kd)\phi_i(x,kd)dx\\&=\int_{kd}^{kd+d}\int u_i^k(x,t)(D_t\phi_i+\Delta\phi_i)(x,t)dx\,dt\\&=\int_{kd}^{kd+d}\int u_i(x,t)(D_t\phi_i+\Delta\phi_i)(x,t)dx\,dt.\end{aligned}\tag{4.58}$$

Summing (4.57) and (4.58) over all $k\in\{0,1,2,\ldots,k'\}$ and using $u^0(x,0)=v(x)$, (4.49), and the assumption on T we obtain

$$\begin{aligned}&-\int v_i(x)\phi_i(x,0)dx\\&=\left(\sum_{k=0}^{k'}\int_{kd}^{kd+d}\int(u_j*H_t')(x,t)(u_i*H_t'')(x,t)D_j\phi_i(x,kd+d)dx\,dt\right)\\&\quad+\int_0^\infty\int u_i(x,t)(D_t\phi_i+\Delta\phi_i)(x,t)dx\,dt.\end{aligned}\tag{4.59}$$

The Schwarz inequality, Young's inequality, (4.52), (4.53), and (4.16) yield the following if $kd<t<kd+d$:

$$\begin{aligned}&\int|(u*H_t')(x,t)|\,|(u*H_t'')(x,t)|dx\\&\leqq\left(\int|(u*H_t')(x,t)|^2dx\right)^{1/2}\left(\int|(u*H_t'')(x,t)|^2dx\right)^{1/2}\\&\leqq\left(\int|u(x,t)|^2dx\right)^{1/2}\|H_t'\|_1\left(\int|u(x,t)|^2dx\right)^{1/2}\|H_t''\|_1\\&=\int|u(x,t)|^2dx\leqq L.\end{aligned}\tag{4.60}$$

From (4.60) and the assumption on N we obtain

$$\begin{aligned}&\Bigg|\int_{kd}^{kd+d}\int(u_j*H_t')(x,t)(u_i*H_t'')(x,t)D_j\phi_i(x,kd+d)dxdt\\&\quad-\int_{kd}^{kd+d}\int(u_j*H_t')(x,t)(u_i*H_t'')(x,t)D_j\phi_i(x,t)dxdt\Bigg|\leqq CLNd^2.\end{aligned}\tag{4.61}$$

Suppose $kd<t<kd+d$. The Schwarz inequality, the argument in (4.60), Lemma 3.6, (4.52), (4.53), the estimates $kd+d-t<d$, $kd+d-t+\zeta<d+\zeta$, and (4.16) yield

$$\begin{aligned}&\int|(u_j*H_t')(x,t)(u_i*H_t'')(x,t)-u_j(x,t)u_i(x,t)|dx\\&\quad\leqq\int|(u_j*H_t')(x,t)((u_i*H_t'')(x,t)-u_i(x,t))|dx\\&\qquad+\int|((u_j*H_t')(x,t)-u_j(x,t))u_i(x,t)|dx\\&\quad\leqq\left(\int|(u*H_t')(x,t)|^2dx\right)^{1/2}\left(\int|(u*H_t'')(x,t)-u(x,t)|^2dx\right)^{1/2}\\&\qquad+\left(\int|(u*H_t')(x,t)-u(x,t)|^2dx\right)^{1/2}\left(\int|u(x,t)|^2dx\right)^{1/2}\\&\quad\leqq CL^{1/2}d^{1/2}\left(\int|Du(x,t)|^2dx\right)^{1/2}+C(d+\zeta)^{1/2}\left(\int|Du(x,t)|^2dx\right)^{1/2}L^{1/2}\\&\quad\leqq C(d+\zeta)^{1/2}L^{1/2}\left(\int|Du(x,t)|^2dx\right)^{1/2}\\&\quad=C((d+\zeta)^{1/4})((d+\zeta)^{1/4}L^{1/2}(\int|Du(x,t)|^2dx)^{1/2})\\&\quad\leqq C(d+\zeta)^{1/2}+C(d+\zeta)^{1/2}L\left(\int|Du(x,t)|^2dx\right).\end{aligned}\tag{4.62}$$

From (4.62) we obtain

$$\left|\int_{kd}^{kd+d}\int (u_j * H_t')(x,t)(u_i * H_t'')(x,t)D_j\phi_i(x,t)\,dx\,dt - \int_{kd}^{kd+d}\int u_j(x,t)u_i(x,t)D_j\phi_i(x,t)\,dx\,dt\right|$$
$$\leqq C(d+\zeta)^{1/2}\|D\phi\|_\infty d + C(d+\zeta)^{1/2}L\|D\phi\|_\infty\left(\int_{kd}^{kd+d}\int |Du(x,t)|^2\,dx\,dt\right). \tag{4.63}$$

Combining (4.61) and (4.63), summing over $k\in\{0,1,2,\ldots,k'\}$, and using (4.49), the assumption on T, and (4.15) we obtain

$$\left|\left(\sum_{k=0}^{k'}\int_{kd}^{kd+d}\int (u_j * H_t')(x,t)(u_i * H_t'')(x,t)D_j\phi_i(x,kd+d)\,dx\,dt\right) - \int_0^\infty\int u_j(x,t)u_i(x,t)D_j\phi_i(x,t)\,dx\,dt\right|$$
$$\leqq C(k'+1)LNd^2 + C(k'+1)(d+\zeta)^{1/2}\|D\phi\|_\infty d$$
$$+ C(d+\zeta)^{1/2}L\|D\phi\|_\infty\left(\int_0^\infty\int |Du(x,t)|^2\,dx\,dt\right)$$
$$\leqq C(k'd+d)LNd + C(k'd+d)(d+\zeta)^{1/2}\|D\phi\|_\infty + C(d+\zeta)^{1/2}L^2\|D\phi\|_\infty$$
$$\leqq C(T+d)LNd + C(T+d)(d+\zeta)^{1/2}\|D\phi\|_\infty + C(d+\zeta)^{1/2}L^2\|D\phi\|_\infty. \tag{4.64}$$

Finally, (4.59) and (4.64) yield the conclusion of Theorem 4.8.

Lemma 4.9. *If* $\phi\in C_0^\infty(R^4\times R,R)$ *then* $\int_0^\infty\int u_i(x,t)D_i\phi(x,t)\,dx\,dt=0$.

Proof. This follows from (4.2), the definition of u^k, and (4.14).

Lemma 4.10. *If* $0<\eta<\infty$, $a\in R^4$, *and* $0\leqq t'\leqq t''<\infty$ *then*

$$|(u*H_\eta)(a,t'')-(u*H_\eta)(a,t')|\leqq C(d+(t''-t'))(L^{1/2}\eta^{-2}+L\eta^{-5/2}).$$

Proof. Define k' and k'' by

$$k'd\leqq t'<k'd+d,\ k''d\leqq t''<k''d+d,\ k' \quad\text{and}\quad k'' \quad\text{are integers}. \tag{4.65}$$

We clearly have $0\leqq k'\leqq k''$. From (4.14) and Definition 4.1 we obtain

$$(u*H_\eta)(a,k'd)=(u^{k'}*H_\eta)(a,k'd)=(v^{k'-1}*H_\eta)(a). \tag{4.66}$$

If $k'd<t'$ then (4.65), (4.14), Definition 4.1, and the semigroup property $H_{a+b} = H_a * H_b$ yield

$$(u*H_\eta)(a,t')=(u^{k'}*H_\eta)(a,t')=((v^{k'-1}*H[t'-k'd])*H_\eta)(a)$$
$$=(v^{k'-1}*H[t'-k'd+\eta])(a) \quad\text{if}\quad k'd<t'. \tag{4.67}$$

Now (4.66), (4.67), the Schwarz inequality, (4.2), and (4.65) yield

$$\begin{aligned}&|(u*H_\eta)(a,t')-(u*H_\eta)(a,k'd)|\\&\quad=|(v^{k'-1}*(H[t'-k'd+\eta]-H[\eta]))(a)|\\&\quad\leqq\|v^{k'-1}\|_2\|H[t'-k'd+\eta]-H[\eta]\|_2\\&\quad\leqq CL^{1/2}(t'-k'd)\eta^{-2}\leqq CL^{1/2}d\eta^{-2}.\end{aligned}\tag{4.68}$$

A similar computation yields

$$|(u*H_\eta)(a,t'')-(u*H_\eta)(a,k''d)|\leqq CL^{1/2}d\eta^{-2}.\tag{4.69}$$

Now we fix an integer k satisfying $k\geqq 0$. The arguments that yielded (4.66), (4.67), and (4.68) also yield

$$\begin{aligned}&|(u^k*H_\eta)(a,kd)-(u^k*H_\eta)(a,kd+d)|\\&\quad=|(v^{k-1}*H_\eta)(a)-(v^{k-1}*H[d+\eta])(a)|\leqq CL^{1/2}d\eta^{-2}.\end{aligned}\tag{4.70}$$

From Definition 4.1 [in particular (4.10)] we obtain

$$\begin{aligned}&(u^{k+1}*H_\eta)(a,kd+d)-(u^k*H_\eta)(a,kd+d)\\&\quad=(v^k*H_\eta)(a)-(w^k*H_\eta)(a)=-d(g^k*H_\eta)(a).\end{aligned}\tag{4.71}$$

We fix $n\in\{1,2,3,4\}$ and define $\phi:R^4\to R^4$ by $\phi_n(x)=H_\eta(a-x)$, $\phi_i(x)=0$ if $i\neq n$. Then approximation of ϕ by functions with compact support, (3.5), Definition 3.3, (4.7), (4.8), integration by parts, (4.9), (1.5), Young's inequality, (1.3), and (4.7) yield

$$\begin{aligned}&|(g_n^k*H_\eta)(a)|=|\textstyle\int g_n^k(x)H_\eta(a-x)dx|\\&\quad=|\textstyle\int g_i^k(x)\phi_i(x)dx|=|\textstyle\int f_i^k(x)(\phi_i-((D_i\operatorname{div}(\phi))*K))(x)dx|\\&\quad=|\textstyle\int(w_j^k*H_\zeta)(x)D_jw_i^k(x)(\phi_i-((D_i\operatorname{div}(\phi))*K))(x)dx|\\&\quad=|\textstyle\int(w_j^k*H_\zeta)(x)w_i^k(x)(D_j\phi_i-((D_{ij}\operatorname{div}(\phi))*K))(x)dx|\\&\quad\leqq C\|w^k*H_\zeta\|_2\|w^k\|_2\eta^{-5/2}\leqq C\|w^k\|_2\|H_\zeta\|_1\|w^k\|_2\eta^{-5/2}\leqq CL\eta^{-5/2}.\end{aligned}\tag{4.72}$$

Now (4.71) and (4.72) yield

$$|(u^{k+1}*H_\eta)(a,kd+d)-(u^k*H_\eta)(a,kd+d)|\leqq CLd\eta^{-5/2}.\tag{4.73}$$

From (4.70), (4.73), and (4.14) we obtain

$$|(u*H_\eta)(a,kd)-(u*H_\eta)(a,(k+1)d)|\leqq CL^{1/2}d\eta^{-2}+CLd\eta^{-5/2}.\tag{4.74}$$

From (4.68), (4.69), and (4.74) we obtain

$$\begin{aligned}&|(u*H_\eta)(a,t'')-(u*H_\eta)(a,t')|\\&\quad\leqq CL^{1/2}d\eta^{-2}+(k''-k')(CL^{1/2}d\eta^{-2}+CLd\eta^{-5/2}).\end{aligned}\tag{4.75}$$

The conclusion of Lemma 4.10 follows from (4.75) and the inequality $(k''-k')d\leqq(t''-t')+d$.

Lemma 4.11. *If* $0<\eta<\infty$, $a\in R^4$, $0\leqq t<\infty$, *and* $\delta>0$ *then*

$$\left|(u*H_\eta)(a,t)-(1/\delta)\left(\int_t^{t+\delta}(u*H_\eta)(a,s)ds\right)\right|$$
$$\leqq C(d+\delta)(L^{1/2}\eta^{-2}+L\eta^{-5/2}).$$

Proof. This follows immediately from Lemma 4.10.

Section 5. Passage to the Limit

Definition 5.1. We choose infinite sequences $\zeta_1,\zeta_2,\zeta_3,\ldots$ and $d_1,d_2,d_3,\ldots$ of positive real numbers satisfying $d_n\leqq\zeta_n$, $d_nL\|H_{\zeta_n}\|_2^2\leqq 1$, $\lim_{n\to\infty}\zeta_n=0$, and $\lim_{n\to\infty}d_n\|H_{\zeta_n}\|_2^2=0$. For each n we define the function $(u,n):R^4\times[0,\infty)\to R^4$ by $(u,n)=u$ where u is the function obtained as in Section 4 using $\zeta=\zeta_n$ and $d=d_n$ [Definition 4.1 and (4.14)]. From (4.17) we obtain

$$\int_0^\infty\int|(u,n)(x,t)|^3dxdt\leqq CL^{3/2}. \tag{5.1}$$

Hence, by passing to a subsequence, we may assume that there is a function $u\in L^3(R^4\times[0,\infty),R^4)$ such that

$$u \text{ is the weak limit of } (u,n) \text{ in } L^3. \tag{5.2}$$

Lemma 5.2. *Let* $0<T<\infty$. *Then*

$$\lim_{n\to\infty}\int_0^T\int_{B(0,T)}|(u,n)(x,t)-u(x,t)|^2dxdt=0.$$

Proof. Suppose $\varepsilon>0$ is given. Let η be a positive real number such that $\eta^{1/2}L^{1/2}\leqq\varepsilon$. Let δ be a positive real number such that $\delta(L^{1/2}\eta^{-2}+L\eta^{-5/2})T^{5/2}\leqq\varepsilon$. Define $f_n:R^4\times[0,\infty)\to R^4$ and $f:R^4\times[0,\infty)\to R^4$ by

$$f_n(a,t)=(1/\delta)\left(\int_t^{t+\delta}((u,n)*H_\eta)(a,s)ds\right),$$

$$f(a,t)=(1/\delta)\left(\int_t^{t+\delta}(u*H_\eta)(a,s)ds\right).$$

Then (5.1) implies that the sequence f_n is equicontinuous. From (5.2) we conclude $\lim_{n\to\infty}f_n(a,t)=f(a,t)$ for all (a,t). Therefore f_n converges to f uniformly on the compact set $B(0,T)\times[0,T]$, and hence

$$\lim_{n\to\infty}\int_0^T\int_{B(0,T)}|f_n(x,t)-f(x,t)|^2dxdt=0. \tag{5.3}$$

If n is a positive integer then Minkowski's inequality, Lemmas 3.6, 4.11, (4.15), and the choice of η and δ yield

$$\begin{aligned}
&\left(\int_0^T \int_{B(0,T)} |(u,n)(x,t)-f_n(x,t)|^2 dxdt\right)^{1/2} \\
&\leqq \left(\int_0^\infty \int |(u,n)(x,t)-((u,n)*H_\eta)(x,t)|^2 dxdt\right)^{1/2} \\
&\quad + \left(\int_0^T \int_{B(0,T)} |((u,n)*H_\eta)(x,t)-f_n(x,t)|^2 dxdt\right)^{1/2} \\
&\leqq \left(\int_0^\infty C\eta\left(\int |D(u,n)(x,t)|^2 dx\right)dt\right)^{1/2} \\
&\quad + \left(\int_0^T \int_{B(0,T)} (C(d_n+\delta)(L^{1/2}\eta^{-2}+L\eta^{-5/2}))^2 dxdt\right)^{1/2} \\
&\leqq C\eta^{1/2}L^{1/2}+C(d_n+\delta)(L^{1/2}\eta^{-2}+L\eta^{-5/2})T^{5/2} \\
&\leqq C\varepsilon + Cd_n(L^{1/2}\eta^{-2}+L\eta^{-5/2})T^{5/2}\,. \qquad (5.4)
\end{aligned}$$

Since $\lim_{n\to\infty} d_n = 0$, we can choose a positive integer N such that $d_n(L^{1/2}\eta^{-2} + L\eta^{-5/2})T^{5/2} \leqq \varepsilon$ if $n \geqq N$. We may assume [using (5.3)] that

$$\left(\int_0^T \int_{B(0,T)} |f_n(x,t)-f_N(x,t)|^2 dxdt\right)^{1/2} \leqq \varepsilon \quad \text{if} \quad n \geqq N\,.$$

Hence (5.4) implies

$$\left(\int_0^T \int_{B(0,T)} |(u,n)(x,t)-(u,N)(x,t)|^2 dxdt\right)^{1/2} \leqq C\varepsilon \quad \text{if} \quad n \geqq N\,. \qquad (5.5)$$

From (5.2) we conclude that the restriction of u to $B(0,T)\times(0,T)$ is the weak limit in L^2 of the restrictions of the (u,n) to $B(0,T)\times(0,T)$. Hence (5.5) and the fact that the unit ball of L^2 is weakly closed imply

$$\left(\int_0^T \int_{B(0,T)} |u(x,t)-(u,N)(x,t)|^2 dxdt\right)^{1/2} \leqq C\varepsilon\,. \qquad (5.6)$$

Now (5.5) and (5.6) imply the conclusion of the lemma.

Theorem 5.3. *The function u is a weak solution to the Navier-Stokes equations of incompressible fluid flow with initial condition v.*

Proof. From Definition 5.1 we obtain $u \in L^3(R^4 \times [0,\infty), R^4)$. Let $\phi \in C_0^\infty(R^4 \times R, R)$. Then Lemma 4.9 and (5.2) yield (1.1). Now let $\phi \in C_0^\infty(R^4 \times R, R^4)$ with $\operatorname{div}(\phi)=0$. Let N be as in Theorem 4.8 and let T be a positive real number such that $\operatorname{spt}(\phi) \cap (R^4 \times [0,\infty)) \subset B(0,T) \times [0,T]$. Then Lemma 5.2, Theorem 4.8, $\lim_{n\to\infty} d_n = 0$, and $\lim_{n\to\infty} \zeta_n = 0$ imply (1.2).

We can now finish the proof of Theorem 1.1. We set $f_n = (u,n)$ and $f = u$ (Definition 5.1). Then (5.1) and (5.2) imply that there exists a positive real number

D such that the conditions in the first paragraph of Section 2 are satisfied. From Definitions 5.1, 4.6, and Theorem 4.7 we conclude that there exists $0 < M < \infty$ such that property $P(M)$ holds (see Definitions 2.2 and 2.1). Let $\varepsilon > 0$ correspond to M as in Lemma 2.3. Then Lemma 2.9 implies that there exist sets A_p satisfying 1), 2), and 3) of Lemma 2.9. From parts 1), 3) of that lemma, the fact $f = u$, Theorem 5.3, and the argument at the end of Section 2 in [5], we conclude that the restriction of u to $(R^4 \times (3(2^{-2p}), \infty)) - A_p$ is equal almost everywhere to a continuous function. Hence, by modifying u on a set of Lebesgue measure zero and setting

$$A = \bigcap_{p=0}^{\infty} (A_p \cup (R^4 \times (0, 3(2^{-2p})])),$$

we can conclude that the restriction of u to $(R^4 \times (0, \infty)) - A$ is continuous. This proves part c) of Theorem 1.1. From part 1) of Lemma 2.9 we conclude that part a) of Theorem 1.1 holds. Let $p \geqq 0$ be an integer and let $\delta > 0$ be given. There exists an integer q such that $p \leqq q$ and $5^{1/2} 2^{-q} \leqq \delta$. From

$$\begin{aligned} &A \cap (R^4 \times (3(2^{-2p}), \infty)) \\ &\quad \subset (A_q \cup (R^4 \times (0, 3(2^{-2q})])) \cap (R^4 \times (3(2^{-2p}), \infty)) \subset A_q, \end{aligned}$$

Definition 2.8, and part 2) of Lemma 2.9 we conclude

$$\phi_\delta(A \cap (R^4 \times (3(2^{-2p}), \infty))) \leqq \phi_\delta(A_q) \leqq C\varepsilon^{-1} D.$$

Hence Definition 2.8 yields $H^3(A \cap (R^4 \times (3(2^{-2p}), \infty))) \leqq C\varepsilon^{-1} D$. Since H^3 is a Borel measure, we can use $A \subset R^4 \times (0, \infty)$ to conclude $H^3(A) \leqq C\varepsilon^{-1} D$. This proves part b) of Theorem 1.1. From Theorem 5.3 and the fact $u \in L^3(R^4 \times [0, \infty), R^4)$ (see Definition 5.1) we obtain the remaining conclusions of Theorem 1.1.

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