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Analyticity and Clustering Properties of Unbounded Spin Systems*

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Abstract. Transforming any lattice system in a polymer model, we use known analytic and cluster properties of the latter to derive similar ones for general lattice models with two-body interactions. These properties of the lattice model hold when the temperature is high enough.

Introduction

Our purpose here is to study various lattice models in some weak coupling regime. In particular we will prove that, at high enough temperatures, the free energy and the correlation functions of these models are analytic functions of any parameter on which the hamiltonian depends analytically. Moreover, under the same conditions and for finite range interactions, the two-point functions will be proved to decay exponentially.

These models contain as special cases, lattice gases with two-body interactions, classical Heisenberg models and lattice approximations of field theoretical models. They describe also some anharmonic crystals, of interest for ferroelectricity. Results of this kind were already obtained for some of these models [1, 2]. However, the technique used relied heavily on further properties of the model in question, such as the boundedness of the values taken by the spin variables. They could not therefore be generalized to lattice approximations of field models for example.

Our strategy here is the following: we transform any lattice model in a so-called polymer model, which can be seen as a generalized lattice gas, with hard core interactions. These polymer models were studied previously [3, 4] and various analytic and clustering properties were established for their gaseous phase (i.e. in the weak coupling region). The remaining task is therefore to estimate the parameters of the polymer model in terms of those of the corresponding lattice models. This is done in the case of two-body interactions only in order to simplify as much as possible the analysis. Our expansion is very much related to the old

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Mayer expansion for continuous systems and this suggest that there should exist a simple unifying approach to all these systems. It seems also that there exist some connection with the cluster expansion of Glimm-Jaffe-Spencer.

The idea to transform a lattice model into a polymer one is not new. It appeared in the physical litterature mainly in connection with the so called cell-cluster theory of liquids [10], and has been called by Hurst and Green the general association problem [11]. The systems discussed however were lattice models with descrete spin variables.

In the meanwhile, some of the problems we discuss in this paper have been attacked by other workers [5, 6] using different techniques. Their results, however, are similar to ours.

1. General Lattice Systems and Polymer Models

Our purpose here is to show how quite generally any lattice system can be recast into an associated polymer model. Let us recall first what we mean by a general lattice system. To each point $x \in \mathbb{Z}^\nu$ is associated a given subsystem whose "states" are numbered by the variables $s_x \in \mathbb{R}^d$. A configuration s_Λ in a finite box $\Lambda \subset \mathbb{Z}^\nu$ is given by the $|\Lambda|$-tuple: $s_\Lambda = \{s_x | x \in \Lambda\}$. The potential energy of a configuration s_Λ, in the box Λ, is a real function: $(\mathbb{R}^{d|\Lambda|}, \Lambda) \to \mathbb{R}$, denoted by $U_\Lambda(s_\Lambda)$. It is choosen to be such that $U_x(s_x) = 0$ $\forall x \in \mathbb{Z}^\nu$. The Gibbs probability distribution of this system is given by

$$Q_\Lambda^{-1} \exp - \beta U_\Lambda(s_\Lambda) \qquad w_\beta(ds_\Lambda) \tag{1}$$

where

$$w_\beta(ds_\Lambda) = \prod_{x \in \Lambda} w_\beta(ds_x) , \tag{2}$$

$w_\beta(ds)$ being a measure on $\mathbb{R}^d$ normalised to 1, i.e.

$$\int_{\mathbb{R}^d} w_\beta(ds) = 1 , \tag{3}$$

and

$$Q_\Lambda = \int w_\beta(ds_\Lambda) e^{-\beta U_\Lambda(s_\Lambda)} \tag{4}$$

in the partition function. In physical applications one takes

$$w_\beta(ds) = \frac{e^{-\beta V(s)} \mu(ds)}{\int_{\mathbb{R}^d} e^{-\beta V(s)} \mu(ds)} , \tag{5}$$

$\mu(ds)$ being a measure independent of β, and $V(s)$ a real continuous function on $\mathbb{R}^d$.

The correlation functions of such a system are defined as

$$\varrho_{\Lambda, X}(s_X) = Q_\Lambda^{-1} \int w_\beta(ds_{\Lambda \setminus X}) e^{-\beta U_\Lambda(s_\Lambda)} \qquad \forall X \subset \Lambda . \tag{6}$$

The formula we have written make sense if $0 < Q_\Lambda < \infty$, $\forall \Lambda \subset \mathbb{Z}^\nu$ is finite, a property we will suppose to hold from now on.

Let us now define more precisely a polymer model and its associated partition function and correlation functions.

We consider a finite set $\Lambda \subset \mathbb{Z}^\nu$ consisting of $|\Lambda|$ points called sites, which are denoted by small letters, $x, y \ldots$

With $X = \{x_1, \ldots, x_n\}$ a finite subset of Λ, a "polymer X" is a rigid system of n particles which can be placed on Λ in such a way as to cover X. The polymers consisting of one particle will be called "monomers", of two particles "dimers", ... of particles "n-mers".

Polymers are placed on Λ and we assume that each site is covered by one and only one particle.

A configuration of the polymer system is therefore defined as a partition $\{X_1, X_2, \ldots, X_k\}$ of the set Λ; we recall that by definition of a partition, we have $\Lambda = \bigcup_{i=1}^{k} X_i$, $X_i \neq \emptyset$, $X_i \cap X_j = \emptyset$ if $i \neq j$, we will denote from now on this partition by the symbol

$$\Lambda = \sum_{i=1}^{k} X_i .$$

The state of the system is defined as usual by a probability measure $\nu_\Lambda(\{X_1, \ldots, X_k\})$ on the configuration space; for polymer systems this measure is caracterised by a positive, bounded function $\Phi(X)$ defined on subsets $X \subset \Lambda$, which is interpreted as the "activity of the polymer X", and

$$\nu_\Lambda(\{X_1, \ldots, X_k\}) = P_\Lambda^{-1} \prod_{i=1}^{k} \Phi(X_i) \tag{7}$$

where

$$P_\Lambda[\Phi] = \sum_k \sum_{\Lambda = \sum_{i=1}^{k} X_i} \prod_{i=1}^{k} \Phi(X_i) \tag{8}$$

is the partition function.

The correlation functions $\varrho_\Lambda(X_1; \ldots; X_p)$ are defined as the probability of finding polymers $X_1, \ldots, X_p$. With the above probability measure we have

$$\begin{aligned} \varrho_\Lambda(X_1, \ldots, X_p) &= 0 \quad \text{if} X_i \cap X_j \neq \emptyset \quad \text{for some} \quad i \neq j \\ &\qquad \text{or} \quad X_i \not\subset \Lambda \end{aligned} \tag{9}$$

$$\varrho_\Lambda(X_1, \ldots, X_p) = P_\Lambda[\Phi]^{-1} \prod_{i=1}^{p} \Phi(X_i) \sum_{\Lambda \setminus \bigcup_{i=1}^{p} X_i = \sum_j Y_j} \prod_j \Phi(Y_j)$$

otherwise.

From this follows that all the correlation functions can be expressed in terms of the various ratios of partition functions:

Namely

$$\varrho_\Lambda(X_1; \ldots; X_p) = \prod_{i=1}^{p} \Phi(X_i) \bar{\varrho}_\Lambda\left(\bigcup_{i=1}^{p} X_i\right) \tag{10}$$

where

$$\bar{\varrho}_\Lambda(X) = \chi_\Lambda(X) \frac{P_{\Lambda \setminus X}[\Phi]}{P_\Lambda[\Phi]} \tag{11}$$

and

$$\chi_A(X)=\begin{cases}1 & X\subset A\\ 0 & X\not\subset A\,.\end{cases} \tag{12}$$

In order to map a lattice system into a polymer model, we need to introduce some well known algebraic formalism.

Let M_A be the complex vector space of functions $F_X(s_X)\in \mathbb{L}_1(\mathbb{R}^{d|A|},\omega_\beta(\cdot))$ defined on all the subsets $X\subset A$. This vector space becomes an algebra with unit element $\mathbb{1}$, when we introduce the following * product.

$$(F*G)_X(s_X)=\sum_{Y\subset X} F_Y(s_Y)G_{X\setminus Y}(s_{X\setminus Y}),\qquad F,G\in M_A \tag{13}$$

$\mathbb{1}$ being the vector defined by

$$\mathbb{1}_X(s_X)=\begin{cases}1 & X=\emptyset\\ 0 & X\neq\emptyset\,.\end{cases} \tag{14}$$

If M_A^+ denotes the subspace of M_A formed by the functions F such that $F_\emptyset(s_\emptyset)=0$, we define as usual on M_A^+ an exponential Γ.

$$\Gamma F=\sum_{n=0}^{\infty}\frac{1}{n!}F^{*n}\,. \tag{15}$$

It appears also useful to introduce the following mapping on this algebra

$$(D_{s_X}F)_Y(s_Y)=F_{X\cup Y}(s_{X\cup Y})\delta_{X\cap Y,\emptyset}\,. \tag{16}$$

This has the basic property

$$[D_{s_x}(F*G)]_Y(s_Y)=\{(D_{s_x}F*G)_Y(s_Y)+(F*D_{s_x}G)_Y(s_Y)\}\delta_{x\cap Y,\emptyset}\,, \tag{17}$$

from which follows that

$$[D_{s_x}(\Gamma G)]_Y(s_Y)=[D_{s_x}G*\Gamma G]_Y(s_Y)\delta_{x\cap Y,\emptyset} \tag{18}$$

and

$$[D_{s_X}(\Gamma G)]_Y(s_Y)=\sum_{X=\sum_{i=1}^{r}X_i}[D_{s_{X_1}}G*\ldots*D_{s_{X_r}}G*\Gamma G]_Y(s_Y)\delta_{X\cap Y,\emptyset}\,. \tag{19}$$

This formulation allows us to define in a simple way the Ursell functions $\Psi\in M_A$ of our lattice system. They are defined by

$$\Psi_x(s_x)=1\quad\text{and}\quad e^{-\beta U_X(s_X)}=(\Gamma\Psi)_X(s_X)\quad\text{when}\quad |X|>1\,. \tag{20}$$

We can now state precisely the correspondance between a lattice and a polymer model.

Theorem 1.

1) $Q_A=P_A[\Phi]$ (21)

where

$$\Phi(X)=\int w_\beta(ds_X)\Psi_X(s_X) \tag{22}$$

$P_\Lambda[\Phi]$ *being the polymer partition function of a system of activities* $\{\Phi(X)\}$.

2) $$\varrho_{\Lambda,X}(s_X)=\sum_Y \int w_\beta(ds_Y)F_{s_X}(s_Y)\bar{\varrho}_\Lambda(X\cup Y) \tag{23}$$

where

$$\bar{\varrho}_\Lambda(X)=\chi_\Lambda(X)\frac{P_{\Lambda\setminus X}[\Phi]}{P_\Lambda[\Phi]} \tag{24}$$

and

$$F_{s_X}(s_Y)=[(\Gamma\Psi)^{-1}*D_{s_X}(\Gamma\Psi)]_Y(s_Y)\delta_{X\cap Y,\emptyset} \tag{25}$$

$(\Gamma\Psi)^{-1}$ *being the* $*$ *inverse of* $\Gamma\Psi$.

N.B. when $Y=\emptyset$ *in the formula we do not integrate.*

Proof. 1) $$Q_\Lambda=\int\omega_\beta(ds_\Lambda)(\Gamma\Psi)_\Lambda(s_Y)=\sum_{k=1}^{|\Lambda|}\int\omega_\beta(ds_\Lambda)\sum_{\substack{\Lambda_1\dots\Lambda_k\\ \Lambda_i\cap\Lambda_j=\emptyset\\ \bigcup_{i=1}^k\Lambda_i=\Lambda}}\prod_{i=1}^k\Psi_{\Lambda_i}(s_{\Lambda_i})$$
$$=\sum_{k=1}^{|\Lambda|}\sum_{\substack{\Lambda_1\dots\Lambda_k\\ \Lambda_i\cap\Lambda_j=\emptyset\\ \bigcup_{i=1}^k\Lambda_i=\Lambda}}\prod_{i=1}^k\Phi(\Lambda_i)=P_\Lambda[\Phi]\,.$$

2) $$\varrho_{\Lambda,X}(s_X)=Q_\Lambda^{-1}\int\omega_\beta(ds_{\Lambda\setminus X})(\Gamma\Psi)_\Lambda(s_\Lambda)=Q_\Lambda^{-1}\int\omega_\beta(ds_{\Lambda\setminus X})[D_{s_X}(\Gamma\Psi)](s_{\Lambda\setminus X})$$
$$=Q_\Lambda^{-1}\int\omega_\beta(ds_{\Lambda\setminus X})\sum_{X=\sum_{i=1}^r X_i}\ \sum_{Y\subset\Lambda\setminus X}(D_{s_{X_1}}\Psi*\dots*D_{s_{X_r}}\Psi)(s_Y)(\Gamma\Psi)(s_{\Lambda\setminus X\cup Y})$$
$$=\sum_{X=\sum_{i=1}^r X_i}\ \sum_{Y\cap X=\emptyset}\int\omega_\beta(ds_Y)(D_{s_{X_1}}\Psi*\dots*D_{s_{X_r}}\Psi)(s_Y)\bar{\varrho}_\Lambda(X\cup Y)\,.$$

But since

$$[D_{s_X}(\Gamma\Psi)](s_Y)=\sum_{X=\sum_{i=1}^r X_i}[D_{s_{X_1}}\Psi*\dots*D_{s_{X_r}}\Psi*\Gamma\Psi](s_Y)\delta_{X\cap Y,\emptyset}$$

from the definition of $F_{s_X}(s_Y)$ we get

$$F_{s_X}(s_Y)=\delta_{X\cap Y,\emptyset}\sum_{Z\subset Y}(\Gamma\Psi)^{-1}(s_{Y\setminus Z})[D_{s_X}(\Gamma\Psi)](s_Z)$$
$$=\delta_{X\cap Y,\emptyset}\sum_{Z\subset Y}(\Gamma\Psi)^{-1}(s_{Y\setminus Z})\sum_{X=\sum_{i=1}^r X_i}(D_{s_{X_1}}\Psi*\dots*D_{s_{X_r}}\Psi*\Gamma\Psi)(s_Z)$$

since

$$Z\cap X=\emptyset$$

and

$$F_{s_X}(s_Y)=\delta_{X\cap Y,\emptyset}\sum_{X=\sum_{i=1}^r X_i}(D_{s_{X_1}}\Psi*\dots*D_{s_{X_r}}\Psi)(s_Y)$$

which concludes the proof.

2. Analyticity and Clustering Properties of the Polymer Model

The following result has been obtained about the analyticity properties of the polymer model [4]. This is the analogue of the results obtained by Ruelle and Penrose about the convergence of the Mayer expansion for imperfect classical gases.

Theorem 2. *Let $\Delta(\xi)$ be the set of complex activities Φ such that*

$$|||\Phi|||_\xi = \sup_x \sum_{X \ni x} |\Phi(X)| \xi^{|X|} < \infty \quad \text{for some} \quad \xi \in \mathbb{R}^+ \tag{26}$$

and

$$|\Phi(x)| > \sup_x \xi^{-1} \left[1 + \sum_{\substack{X \in x \\ |X| \geqq 2}} |\Phi(X)| \xi^{|X|} \right] = R(\xi) \tag{27}$$

then for any $\Phi \in \Delta(\xi)$.

1) $P_\Lambda[\Phi] \neq 0$.

Moreover, if one of the two conditions are satistied

a) *$\Phi(X)$ is finite ranged (i.e. $\Phi(X) = 0$ when* $\operatorname{diam} X > d$*),*

b) *$\Phi(X)$ is translation invariant*

then.

2) *There exists a positive, decreasing function $\varepsilon(\lambda)$ such that*

$$\lim_{\lambda \to \infty} \varepsilon(\lambda) = 0$$

and a function $\bar{\varrho}(X)$ such that

$$|\bar{\varrho}_\Lambda(X) - \bar{\varrho}(X)| \leqq \xi^{|X|} \varepsilon(\lambda)$$

where λ is the minimum distance from $x_i \in X$ to the boundary of Λ.

3) *$\bar{\varrho}_\Lambda$ and $\bar{\varrho}$ defined on the Banach space of activities $\{\Phi(X)\}$ with the norm $|||\Phi|||_\xi$ is norm analytic in $\Delta(\xi)$ the norm of $\bar{\varrho}_\Lambda$ and $\bar{\varrho}$ being defined as*

$$\|\bar{\varrho}_\Lambda\| = \sup_{X \subset \mathbb{Z}_\nu} |\bar{\varrho}_\Lambda(X)| \xi^{-|X|} \tag{28}$$

4) $\|\bar{\varrho}_\Lambda\|_\xi \leqq M(\xi)$, (29)

$M(\xi)$ being some constant function of $R(\xi)$ and ξ.

It is also known that the correlation functions have good clustering properties, when the monomer activity is sufficiently large. More precisely, we have the following general result [3, 7].

Theorem 3. *Let Δ' be the set of complex activities such that $z = \inf_x |\Phi(x)| > z_0$, where z_0 is the positive root of the equation*

$$\frac{1}{2} = \sum_{n=2}^{\infty} z_0^{-n} \sup_x \sum_{\substack{X \ni x \\ |X| = n}} |\Phi(X)| \, . \tag{30}$$

Then if $\Phi\in\Delta'\cap\Delta(\xi)$ and if one of the two conditions is satisfied

a) *$\Phi(X)$ is finite ranged,*

b) *$\Phi(X)$ is translation invariant.*

Then

$$\sup_x \sum_{\substack{X\ni x\\|X|=n}} |\bar{\varrho}^T(X)|\leqq z^{-1}+\frac{z^{-n}\left(\frac{z_0}{z}\right)^n}{\left(1-\frac{z_0}{z}\right)^n} \tag{31}$$

$\bar{\varrho}^T(X)$ being the truncated correlation functions associated to $\bar{\varrho}(X)$.

In fact, using standard techniques, it is also possible to prove in some cases, exponential weak and strong clustering [7, 8].

Theorem 4. *If $\Phi\in\Delta'\cap\Delta(\xi)$*

$$(\Gamma\varphi)(X\cup Y)=(\Gamma\varphi)(X)(\Gamma\varphi)(Y)\quad \text{when}\quad d(X,Y)\geqq\delta \tag{32}$$

where $d(X,Y)$ is the distance between the sets X and Y and $\varphi(X)=\Phi(X)-\delta_{|X|,1}$ then the following properties hold

$$1)\quad |\bar{\varrho}(X\cup Y)-\bar{\varrho}(X)\bar{\varrho}(Y)|\leqq Lc^{|X|+|Y|}e^{-kd(X,Y)}, \tag{33}$$

$$2)\quad |\bar{\varrho}^T(X)|\leqq Lc'^{|X|}e^{-\frac{k}{2}\hat{L}(X)}\quad \text{when}\quad |X|>1 \tag{34}$$

where $\hat{L}(X)$ is the length of the minimal tree built on X, and

$$c=\frac{\alpha z_0}{(\alpha-1)z^2},\qquad c'=\frac{1}{(\alpha-1)z},\qquad k=\ln\frac{z}{\alpha z_0},\qquad L=\frac{1}{1-\alpha\frac{z_0}{z}} \tag{35}$$

α being any number such that

$$1<\alpha<\frac{z}{z_0}\,. \tag{36}$$

3. Analyticity Properties of the Lattice Model with 2-Body Potentials

The general strategy would be to exploit known results about the polymer model, such as those described above, in order to get similar ones for a general lattice system. To achieve this, we need to estimate the polymer activities, as well as the functions $F_{s_X}(s_Y)$, in terms of the lattice model potential. This analysis, although possible in the general case, can be made simple enough only when we restrict ourselves to the case of two-body potentials. This is what we will do from now on. It is defined by the condition that

$$U_\Lambda(s_\Lambda)=\sum_{(x,y)\subset\Lambda}\varphi_{xy}(s_x,s_y)\qquad \forall\Lambda\subset\mathbb{Z}^\nu \tag{37}$$

$\varphi_{x,y}(\cdot,\cdot)$ being the two-body potential.

We will suppose that the two-body potential satisfies the following conditions:

$$|\varphi_{xy}(s_x,s_y)|\leqq J(x,y)v(s_x)v(s_y) \tag{38}$$

where $v(s):\mathbb{R}^d\to\mathbb{R}^+$ is a function such that

$$\int |w_\beta(ds)| e^{xv^2(s)} < \infty \qquad \forall x\in\mathbb{R} \tag{39}$$

and

$$J(x,y)\geqq 0\,, \quad \text{with} \quad J=\sup_x \sum_{y\in\mathbb{Z}_v} J(x,y)<\infty\,. \tag{40}$$

These conditions ensure that our system is superstable and upper-lower regular in the sense of [9], when the potential is translation invariant.

Let us define now the following quantity, which will play an essential role in the analysis.

$$\begin{aligned} A^Y_{s_X} &= \int w_\beta(ds_Y) e^{-\frac{\beta J}{2}\gamma \sum_{x\in X} v^2(s_x)} F_{s_X}(s_Y) \qquad Y\neq\emptyset \\ A^\emptyset_{s_X} &= F_{s_X}(s_\emptyset) e^{-\frac{\beta J}{2}\gamma \sum_{x\in X} v^2(s_x)} \end{aligned} \tag{41}$$

γ being for the moment an arbitrary number.

It can be estimated in terms of the potential as follows:

Lemma 2. *Let*

$$B_\gamma = |\beta|^{\frac{1}{2}} \int |w_\beta(ds)| v(s) e^{\frac{Jv^2(s)}{2}(|\beta|+\gamma\,\mathrm{Re}\,\beta)} \inf\left[\bar{J}, |\beta|^{\frac{1}{2}} J \sup_{s\in\Omega} v(s)\right] \tag{42}$$

with

$$\mathrm{Re}\,\beta>0 \quad \text{and} \quad \gamma>2+\frac{|\beta|}{\mathrm{Re}\,\beta}$$

Ω denoting the support of the measure $\omega_\beta(ds)$ and

$$\bar{J}=\sup_x \sum_{t\in\mathbb{Z}^v} J^{\frac{1}{2}}_{x,t}\,. \tag{43}$$

If

$$I(m,n)=\sup_{\substack{X,s_X\\ |X|=n}} \sum_{\substack{Y\\ |Y|=m}} |A^Y_{s_X}| \tag{44}$$

then we have

$$I(m,n)\leqq a^n \left(\frac{e^{B_\gamma a}}{a}\right)^{m+n} \tag{45}$$

a being any positive number.

Proof. In the appendix we show that the A's satisfy the following recursion formula in the case of two-body interactions.

$$\begin{aligned} A^Y_{s_X} = \delta_{x\cap Y,\emptyset}\, e^{-\beta U^{(x)}(s_X)-\frac{\beta J}{2}\gamma v^2(s_x)} \Bigg[A^Y_{s_{X'}} + \sum_{\emptyset\neq T\subset Y} \int w_\beta(ds_T) \\ \cdot \prod_{t\in T} \left\{ (e^{-\beta\varphi_{xt}(s_x,s_t)}-1) e^{\frac{\beta J}{2}\gamma v^2(s_t)} \right\} A^{Y\setminus T}_{s_{X'\cup T}} \Bigg]\,. \end{aligned} \tag{46}$$

Here x is any point of X and $X'=X\backslash x$

$$U^{(x)}(s_X)=\sum_{y\in X'}\varphi_{xy}(s_x,s_y)\quad\text{when}\quad |X|\geqq 2$$
$$=0\qquad\qquad\text{when}\quad |X|=1\,.$$

Condition (38) on the potential implies that

$$\sum_{(x,y)\subset\Lambda}\varphi_{xy}(s_x,s_y)\geqq -\frac{J}{2}\sum_{x\in\Lambda}v^2(s_x)$$

and consequently there is at least one $x\in X$, noted $w(X)$ such that

$$U^{w(X)}(s_X)\geqq -Jv^2(s_{w(X)})\,.$$

in Formula (46), we take then for x, always $w(X)$, therefore

$$\left|e^{-\beta U^{\omega(X)}(s_X)-\frac{\beta J}{2}\gamma v^2(s_{\omega(X)})}\right|\leqq 1$$

when $\operatorname{Re}\beta>0$ and $\gamma>2$.

The recursion formula gives

$$I(m,n)\leqq I(m,n-1)+\sup_{x,s_x}\sum_{k=0}^{m-1}\sum_{T:|T|=m-k}\prod_{t\in T}b_{s_x,x}(t)I(k;n-1+m-k)$$

where

$$b_{s_x,x}(t)=e^{-\frac{J_{xt}}{2}\operatorname{Re}\beta(\gamma-2)v^2(s_x)}\int|w_\beta(ds')e^{\frac{\operatorname{Re}\beta\gamma}{2}Jv^2(s')}|e^{-\beta\varphi_{xt}(s_x,s')}-1|$$

hence if

$$b=\sup_{x,s_x}\sum_{t\in\mathbb{Z}^\nu}b_{s_x,x}(t)$$

we get

$$I(m,n)\leqq I(m,n-1)+\sum_{l=1}^{m}\frac{b^l}{l!}I(m-l,n-1+l)\tag{47}$$

since $I(1,0)=0$ and $I(0,1)=1$, (45) follows simply from (47) by induction on $m+n$ if $b\leqq B_\gamma$. This is therefore what we have to prove now. Using the inequality $|e^{xy}-1|\leqq(e^{y^2}-1)^{\frac{1}{2}}(e^{y^2}-1)^{\frac{1}{2}}$ and property (38) of the potential, we get

$$b\leqq\sup_{x,s}\int|w_\beta(ds')|e^{\frac{\operatorname{Re}\beta J\gamma}{2}v^2(s')}\sum_t(1-e^{-J_{xt}|\beta|v^2(s)})^{\frac{1}{2}}(e^{|\beta|J_{xt}v^2(s')}-1)^{\frac{1}{2}}$$

since

$$\operatorname{Re}\beta>0\quad\text{and}\quad\gamma>2+\frac{|\beta|}{\operatorname{Re}\beta}$$

when $w_\beta(ds)$ has compact support, we can take $s\in\Omega$ and Schwartz inequality gives us

$$b\leqq|\beta|J\left(\sup_{s\in\Omega}v(s)\right)\int|w_\beta(ds')v(s')e^{\frac{Jv^2(s')}{2}(|\beta|+\gamma\operatorname{Re}\beta)}$$

whereas in the general case we get

$$b \leqq |\beta|^{\frac{1}{2}} \bar{J} \int |w_\beta(ds')| v(s') e^{\frac{J v^2(s')}{2}(|\beta| + \gamma \operatorname{Re}\beta)} .$$

This crucial estimate allow us to use Theorem 2 to get the desired analytic properties of our class of lattice models.

Theorem 5. *If the potential energy is given by*

$$U_\Lambda(s_\Lambda) = \sum_{(x,y) \subset \Lambda} \varphi_{xy}(s_x, s_y)$$

where the two-body potential φ_{xy} and the measure $w_\beta(ds)$ satisfy Conditions (38) *and* (39). *If moreover the two-body potential is finite ranged i.e. $\varphi_{xy}(s,s')=0$ when $|x-y| \geqq \delta$ or translation invariant, i.e. $\varphi_{xy}(s,s')=\varphi_{x-y}(s,s')$ then in the domain defined by*

$$\operatorname{Re}\beta > 0 \qquad B_\gamma + 2(B_\gamma D_\gamma)^{\frac{1}{2}} \leqq e^{-1} \tag{48}$$

where

$$D_\gamma = \int |w_\beta(ds)| e^{\frac{J}{2}\gamma \operatorname{Re}\beta v^2(s)} \tag{49}$$

γ being any number larger than $2 + \dfrac{|\beta|}{\operatorname{Re}\beta}$.

The following properties hold

1) $Q_\Lambda \neq 0$,

2) $\lim_{\Lambda \nearrow \mathbb{Z}^\nu} \varrho_{\Lambda,X}(s_X) = \varrho_X(s_X)$,

exist and extends to an analytic function of β and of any parameter on which $w_\beta(\cdot)$ or φ depends analytically. If the potential is translation invariant, then so are the correlation functions $\varrho_X(s_X)$.

$$3) \quad |\varrho_{\Lambda,X}(s_X)| \leqq \bar{M} e^{\operatorname{Re}\beta \frac{J\gamma}{2} \sum_{x \in X} v^2(s_x)} \bar{\xi}^{|X|} \tag{50}$$

where $\bar{\xi} = \dfrac{1}{B_\gamma + (B_\gamma D_\gamma)^{\frac{1}{2}}}$ and $\bar{M}$ is some constant depending on B_γ.

Proof. In inequality (45), we choose $a = B_\gamma^{-1}$ and we get

$$I(m,n) \leqq (eB_\gamma)^m e^n . \tag{51}$$

We can now estimate the polymer activities $\Phi(X)$, since $\Phi(x)=1$ and

$$\Phi(X) = \int w_\beta(ds_x) e^{\frac{\beta J\gamma}{2} v^2(s_x)} A^{X'}_{s_x} \quad \text{when} \quad X' = X \backslash x \tag{52}$$

because $F_{s_x}(s_{X'}) = \Psi(s_{x \cup X'})$, as can be seen from (22) and (25).

The correlation functions $\varrho_{\Lambda,X}(s_X)$ are given by:

$$\varrho_{\Lambda,X}(s_X) = e^{\frac{\beta J\gamma}{2} \sum_{x \in X} v^2(s_x)} \sum_Y A^Y_{s_X} \bar{\varrho}_\Lambda(X \cup Y)$$

according to (23).

Inserting (51) in (52), we get that $R(\xi)$ defined in Theorem 2 is bounded above by

$$\xi^{-1}+D_\gamma e\frac{(B_\gamma e\xi)}{1-(B_\gamma e\xi)} \quad \text{choosing} \quad \xi=\frac{e^{-1}}{B_\gamma+(B_\gamma D_\gamma)^{\frac{1}{2}}},$$

we can check that Condition (27) of Theorem 2 is satisfied if (48) holds. Therefore $\Phi\in\Delta(\xi)$.

Theorem 2, Part 4, tells us then that

$$|\bar{\varrho}_\Lambda(X)|\leqq M(\xi)\xi^{|X|}.$$

Using the estimate (51) for $I(m,n)$, we obtain

$$|\varrho_{\Lambda,X}(s_X)|\leqq e^{\frac{\mathrm{Re}\,\beta J}{2}\gamma\sum_{x\in X}v^2(s_x)}\bar{\xi}^{|X|}\bar{M}$$

with

$$\bar{\xi}=\xi e \quad \text{and} \quad \bar{M}=\frac{M(\xi)}{1-\xi e B_\gamma}$$

which is (50).

Part 1 of our theorem follows from Part 1 of Theorem 2.

For the same reasons $\bar{\varrho}(X)$ depends analytically on β or any parameter on which φ depends analytically by Theorem 2, Part 3. The same is true of $\varrho_{\Lambda,X}(s_X)$ and of $\varrho_X(s_X)$ by (50), and this proves Part 2 of our theorem.

It remains to see under which physical Conditions (48) is satisfied. We want to discuss here the range of temperatures, i.e. β for which (48) holds, by looking at three important special cases defined by various conditions on the measure $w_\beta(\cdot)$.

Notice first that if we write γ as $\gamma=2+\lambda\frac{|\beta|}{\mathrm{Re}\beta}$ with $\lambda>1$ then we have

$$B_\gamma\leqq|\beta|^{\frac{1}{2}}\int|w_\beta(ds)|v(s)e^{\frac{|\beta|}{2}(2+\lambda)v^2(s)}\inf\left[\bar{J},|\beta|^{\frac{1}{2}}J\sup_{s\in\Omega}v(s)\right] \tag{53}$$

therefore if we fix $\lambda>1$ independent of β, then

$$\lim_{\beta\to 0}B_\gamma(\beta)=0$$

if

$$\lim_{\beta\to 0}|\beta|^{\frac{1}{2}}\int|w_\beta(ds)|v(s)=0. \tag{54}$$

And since

$$D_\gamma\leqq\int|w_\beta(ds)|e^{|\beta|(1+\frac{\lambda}{2})v^2(s)} \tag{55}$$

we see that

$$\lim_{\beta\to 0}B_\gamma(\beta)+2(B_\gamma(\beta)D_\gamma(\beta))^{\frac{1}{2}}=0$$

if

$$\lim_{\beta\to 0}|\beta|^{\frac{1}{2}}(\int|w_\beta(ds)|(\int|w_\beta(ds)|v(s))=0 \tag{56}$$

and (54) hold. Therefore (48) will always be satisfied at sufficiently high temperatures if $w_\beta(\cdot)$ fulfills the condition (54) and (56).

Let us consider now three special cases:

a) $w_\beta(ds)=w(ds)$ with $w(ds)$ satisfying (39), then clearly (54) and (56) hold

b) $$w_\beta(ds)=\frac{e^{-\beta V(s)}\mu(ds)}{\int e^{-\beta V(s)}\mu(ds)}$$

with $\mu(ds)\geqq 0$ and $\int\mu(ds)>0$, $\int\mu(ds)e^{|\beta|(V(s)+\alpha v^2(s))}<\infty$ when $0\leqq|\beta|\leqq\beta_0$, $\forall\alpha>0$.

Then, there exists a constant d independent of β such that

$$\left|\int\mu(ds)e^{-\beta V(s)}\right|\geqq d>0 \quad \text{when} \quad 0\leqq|\beta|\leqq\beta_1\leqq\beta_0$$

and $w_\beta(ds)$ will be analytic in β inside this circle and (54) and (56) will be satisfied.

c) If

$$w_\beta(ds)=\frac{e^{-\beta V(s)-\beta h\cdot s}d^d s}{\int e^{-\beta V(s)-\beta h\cdot s}d^d s}$$

where $V(s)=V(|s|)$ is a polynomial of degree $2n$ in $|s|$ with a positive coefficient for the term of highest degree, then taking

$$v(s)=|s|^m \quad \text{with} \quad m<n$$

we will have

$$\left|\int d^d s e^{-\beta V(s)-\beta h\cdot s}\right|\geqq d>0$$

when $\mathrm{Re}\,\beta>0$ and β in some domain D in the complex plane containing the origin. $w_\beta(ds)$ will be analytic in this domain and since when β is real

$$\beta^{\frac{1}{2}}\int w_\beta(ds)v(s)\sim\beta^{\overline{\frac{n-m}{2n}}}$$

we see that (54) and (56) will be satisfied when β is in the domain D.

We can summarise these results in the following

Theorem 6. *If $w_\beta(\cdot)$ satisfies one of the three Conditions a, b, c, then there exists a domain D in the complex β plane containing a segment $(0,\beta_0)$ of the positive real axis, such that if $w_\beta(\cdot)$, and the potential satisfies the conditions of Theorem 5.*

1) $Q_\Lambda \neq 0$,

2) $\lim\limits_{\Lambda\nearrow\mathbb{Z}^\nu}\varrho_{\Lambda,X}(s_X)=\varrho_X(s_X)$

exist and extends to an analytic function of β.

3) $|\varrho_{\Lambda,X}(s_X)|\leqq\bar{M}e^{\frac{\mathrm{Re}\,J\gamma}{2}\sum_{x\in X}v^2(s_x)}\bar{\xi}^{|X|}$.

4. Clustering Properties of the Two-Point Function of the Lattice Model

We will not in this section discuss the clustering properties of general n-point functions. Moreover, we will restrict our attention to finite range interactions. The main reason for this, as will appear clearly in the course of the proof, is that the connection between correlation functions of the lattice model and the polymer model being quite complicated in the general case of n-point function, more precise estimates on the functions $A^Y_{s_X}$ would be needed than those we have obtained so far.

Our result is the following:

Theorem 7. *Suppose that the two-body potential and the measure* $w_\beta(ds)$ *satisfy conditions* (38) *and* (39). *If the potential is finite ranged, then in the domain defined by*

$$(1+2B_\gamma e)[B_\gamma+(2B_\gamma D_\gamma e)^{\frac{1}{2}}]<e^{-1} \tag{57}$$

we have

$$|\varrho_{xy}(s,s')-\varrho_x(s)\varrho_y(s')|\leqq A(s)A(s')e^{-m|x-y|} \tag{58}$$

where m is a function of B_γ and D_γ, and $A(s)=Ae^{\frac{\mathrm{Re}\,\beta\gamma}{2}Jv^2(s)}$, A being some function of B_γ.

Proof. The idea is of course to use Theorem 4, and various estimates established before for the $A^Y_{s_X}$.

Φ belongs to Δ' when (57) holds because $z_0<B_\gamma e+(2B_\gamma D_\gamma e^2)^{\frac{1}{2}}$ as can be seen easily by using the estimate

$$I(m,1)\leqq e(eB_\gamma)^m \tag{59}$$

obtained from (51).

Moreover in the course of the proof of Theorem 5, we have shown that

$$\Phi\in\Delta\left[\frac{e^{-1}}{B_\gamma+(B_\gamma D_\gamma)^{\frac{1}{2}}}\right]. \tag{60}$$

On the other hand, since $(\Gamma\varphi)(X)=\sum\limits_{Y\subset X}(\Gamma\Phi)(Y)(-1)^{|X|-|Y|}$ and $(\Gamma\Phi)(Y)=Q_Y$ from Theorem 1, with $Q_{Y_1\cup Y_2}=Q_{Y_1}Q_{Y_2}$ when $d(Y_1,Y_2)\geqq\delta$, we see that $(\Gamma\varphi)(X_1\cup X_2)=(\Gamma\varphi)(X_1)(\Gamma\varphi)(X_2)$ when $d(X_1,X_2)\geqq\delta$ and all the conditions necessary to apply Theorem 4 are fulfilled.

We will need moreover the following estimates:

$$\Psi_X(s_X)=0 \quad \text{if} \quad \operatorname{diam}X\geqq|X|\delta \tag{61}$$

since Ψ_X is the Ursell function of a system with a two-body potential of range δ [8].

On the other hand

$$\sum_{Y:|Y|=m}\left|\int w_\beta(ds_Y)\Psi_{xY}(s_x,s_Y)\right|\leqq e^{\frac{\mathrm{Re}\,\beta J\gamma}{2}v^2(s_x)}e(eB_\gamma)^m \tag{62}$$

by (59) and similarly

$$\sum_{Y:|Y|=m}\left|\int w_\beta(ds_Y)\Psi_{x_1x_2Y}(s_{x_1\cup x_2\cup Y})\right|\leqq e^{\frac{\mathrm{Re}\,\beta J}{2}\gamma(v^2(s_{x_1})+v^2(s_{x_2}))}e^2(eB_\gamma)^m(m+1) \tag{63}$$

since

$$e^{-\frac{\beta J}{2}\gamma(v^2(s_{x_1})+v^2(s_{x_2}))}\int w_\beta(ds_Y)\Psi_{x_1x_2Y}(s_{x_1\cup x_2\cup Y})=A^Y_{x_1x_2}-\sum_{Y=Y_1+Y_2}A^{Y_1}_{x_1}A^{Y_2}_{x_2}.$$

Theorem 2, Part 4, gives us, since $\Phi\in\Delta\left[\frac{e^{-1}}{B_\gamma+(D_\gamma B_\gamma)^{\frac{1}{2}}}\right]$

$$|\bar\varrho(X)|\leqq M\left(\frac{e^{-1}}{B_\gamma+(D_\gamma B_\gamma)^{\frac{1}{2}}}\right)^{|X|} \tag{64}$$

and by Theorem 4

$$|\bar\varrho(X\cup Y)-\bar\varrho(X)\bar\varrho(Y)|\leqq L\left[2\frac{eB_\gamma+(2B_\gamma D_\gamma e^2)^{\frac{1}{2}}}{1-eB_\gamma-(2B_\gamma De)^{\frac{1}{2}}}\right]^{|X|+|Y|}e^{-kd(X,Y)} \tag{65}$$

if we choose for the constant α in this theorem

$$\alpha=\frac{2}{1+B_\gamma e+(2B_\gamma D_\gamma e^2)^{\frac{1}{2}}}\,.$$

Using Theorem 1, Part 2, we see that we can write the two-point truncated correlation function as follows:

$$\begin{aligned}&\varrho_{x_1x_2}(s,s')-\varrho_{x_1}(s)\varrho_{x_2}(s')\\&\quad=\bar\varrho(x_1x_2)-\bar\varrho(x_1)\bar\varrho(x_2)+D_{x_1x_2}(s,s')+E_{x_1x_2}(s)+E_{x_2x_1}(s')\\&\qquad+F_{x_1x_2}(s,s')-G_{x_1x_2}(s,s')\end{aligned}$$

where

$$\begin{aligned}D_{x_1x_2}(s,s')&=\sum_{Y\not\ni(x_1x_2)}\int w_\beta(ds_Y)\Psi_{x_1x_2Y}(s,s',s_Y)\bar\varrho(x_1x_2Y)\,,\\E_{x_1x_2}(s)&=\sum_{Y\not\ni x_1}\int w_\beta(ds_Y)\Psi_{x_1Y}(s,s_Y)[\bar\varrho(x_1x_2Y)-\bar\varrho(x_2)\bar\varrho(x_1Y)]\,,\\F_{x_1x_2}(s,s')&=\sum_{\substack{Y_1\neq\emptyset\\Y_2\neq\emptyset\\Y_1\cap Y_2=\emptyset}}\int w_\beta(ds_{Y_1})\Psi_{x_1Y_1}(s,s_{Y_1})\int w_\beta(ds_{Y_2})\Psi_{x_2Y_2}(s',s_{Y_2})\\&\quad\cdot[\bar\varrho(x_1Y_1x_2Y_2)-\bar\varrho(x_1Y_1)\bar\varrho(x_2Y_2)]\,,\\G_{x_1x_2}(s,s')&=\sum_{Y_1\cap Y_2\neq\emptyset}\int w_\beta(ds_{Y_1})\Psi_{x_1Y_1}(s,s_{Y_1})\int w_\beta(ds_{Y_2})\\&\quad\cdot\Psi_{x_1Y_2}(s',s_{Y_2})\bar\varrho(x_1Y_1)\bar\varrho(x_2Y_2)\,.\end{aligned}$$

Let us analyse now each of these terms.
From (61), (63), (64), we get

$$\begin{aligned}|D_{x_1x_2}(s,s')|&\leqq g(s)g(s')Me^2\sum_{m\geqq\frac{|x_1-x_2|}{\delta}}\left(\frac{B_\gamma}{B_\gamma+(B_\gamma D_\gamma)^{\frac{1}{2}}}\right)^{m+2}(m+1)\\&\leqq M_1g(s)g(s')e^{-k_1|x_1-x_2|}\end{aligned}$$

where $g(s)=e^{\frac{\mathrm{Re}\,\beta}{2}J_{\gamma}v^2(s)}$ and M_1, K_1 are some constants, $K_1>0$. Since $d(x_1;x_2Y) \geqq |x_1-x_2|-\operatorname{diam} x_1Y$ and $\operatorname{diam} x_1Y \leqq (|Y|+1)\delta$ in the Y_s' appearing in the sum defining $E_{x_1x_2}(s,s')$ we get

$$|E_{x_1x_2}(s)| \leqq g(s)Le\left(2\frac{eB_\gamma+(2B_\gamma D_\gamma e^2)^{\frac{1}{2}}}{1-eB_\gamma-(2B_\gamma D_\gamma e^2)^{\frac{1}{2}}}\right)$$
$$\cdot\left\{\sum_{m=0}^{[\delta^{-1}|x_1-x_2|]}\left(eB_\gamma\frac{eB_\gamma+(2B_\gamma D_\gamma e^2)^{\frac{1}{2}}}{1-eB_\gamma-(2B_\gamma D_\gamma e^2)^{\frac{1}{2}}}\right)^m\right.$$
$$\left.+\sum_{m=[\delta^{-1}|x_1-x_2|]+1}^{\infty}\left(eB_\gamma\frac{eB_\gamma+(2B_\gamma D_\gamma e^2)^{\frac{1}{2}}}{1-eB_\gamma-(2B_\gamma D_\gamma e^2)^{\frac{1}{2}}}\right)^m e^{-k[|x_1-x_2|-(m+1)\delta]}\right\}$$

using (65) and (59).

From this it follows that

$$|E_{x_1x_2}(s)| \leqq M_2 g(s) e^{-k_2|x_1-x_2|}$$

when (57) holds.

$F_{x_1x_2}$ is treated in the same way, but this time we use the inequality

$$d(x_1Y_1;x_2Y_2) \geqq |x_1-x_2|-\operatorname{diam} x_1Y_1-\operatorname{diam} x_2Y_2$$

and note that

$$\operatorname{diam} x_1Y_1 \leqq (|Y_1|+1)\delta \qquad \operatorname{diam} x_2Y_2 \leqq (|Y_2|+1)\delta$$

for the $Y's$ appearing in the sum defining $F_{x_1x_2}(s,s')$ we get

$$|F_{x_1x_2}(s,s')| \leqq M_3 g(s)g(s')e^{-k_3|x_1-x_2|}\,.$$

It remains to discuss $G_{x_1x_2}$

$$|G_{x_1x_2}(s,s')| \leqq M^2 \sum_{Y_1\cap Y_2\neq\emptyset}\left(\frac{e^{-1}}{B_\gamma+\sqrt{B_\gamma D_\gamma}}\right)^{|Y_1|+|Y_2|+2} \left|\int w_\beta(ds_{Y_1})\Psi_{x_1Y_1}(s,s_{Y_1})\right|$$
$$\cdot\left|\int w_\beta(ds_{Y_2})\Psi_{x_2Y_2}(s',s_{Y_2})\right|$$

by (55). But if $Y_1\cap Y_2\neq\emptyset$, $|x_1-x_2|\leqq \operatorname{diam} x_1Y_1+\operatorname{diam} x_2Y_2$ and since $\operatorname{diam} x_1Y_1 \leqq (|Y_1|+1)\delta$, $\operatorname{diam} x_2Y_2 \leqq (|Y_2|+1)\delta$ in the sum, we see that

$$|G_{x_1x_2}(s,s')| \leqq g(s)g(s')M^2B_\gamma^{-2} \sum_{\substack{m_1m_2\\ m_1+m_2\geqq\delta^{-1}|x_1-x_2|}}\left(\frac{B_\gamma}{B_\gamma+\sqrt{B_\gamma D_\gamma}}\right)^{m_1+m_2+2}$$
$$\leqq M_4 e^{-k_4|x_1-x_2|}\,.$$

Collecting all these estimates, we see that we have proven the desired cluster property of the two-point functions.

It is clear from the discussion following Theorem 5, that condition (49) will be satisfied if the measure w_β belongs to the three classes a, b, and c discussed, and if the temperature is high enough. We can therefore conclude that at high enough temperatures, the two-point function will cluster exponentially for finite range potentials.

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Appendix

We want to derive here the basic recursion formula. For notational simplicity, from now on $F_X(s_X) \equiv F(s_X)\ \forall F$

$$F_{s_X}(s_Y) = (W^{-1} * D_{s_X} W)(s_Y)\delta_{X\cap Y,\emptyset}$$

where

$$W = (\Gamma\Psi) = e^{-\beta U}\ .$$

If

$$X = x \cup X'$$

then

$$F_{s_X}(s_Y) = \delta_{x\cap Y,\emptyset} \sum_{Z\subset Y} W^{-1}(s_{Y\setminus Z}) W(s_{x\cup X'\cup Z})\delta_{X'\cap Y,\emptyset}$$

but if $X' \neq \emptyset$,

$$W(s_{x\cup X'\cup Z}) = e^{-\beta \sum\limits_{y\in Z} \varphi_{xy}(s_x,s_y) - \beta \sum\limits_{y\in X'} \varphi_{xy}(s_x,s_y) - \beta U_{X'\cup Z}(s_{X'\cup Z})}$$

$$= e^{-\beta \sum\limits_{y\in X'} \varphi_{xy}(s_x,s_y)} W(s_{X'\cup Z}) \left(1 + \sum_{T\subset Z} K(s_x, s_T)\right)$$

where

$$K(s_x, s_T) = \prod_{t\in T} (e^{-\beta\varphi_{xt}(s_x,s_t)} - 1) \quad \text{when} \quad T \neq \emptyset$$

$$K(s_x, s_T) = 0 \quad \text{when} \quad T = \emptyset\ .$$

Then we have

$$F_{s_X}(s_Y) = \delta_{x\cap Y,\emptyset} \sum_{Z\subset Y} W^{-1}(s_{Y\setminus Z}) e^{-\beta \sum\limits_{y\in X'} \varphi_{xy}(s_x,s_y)}$$

$$\cdot \left(1 + \sum_{T\subset Z} K(s_x, s_T)\right) W(s_{X'\cup Z})\delta_{X'\cap Y,\emptyset}$$

$$= e^{-\beta U^{(x)}(s_X)} \delta_{x\cap Y,\emptyset} \sum_{Z\subset Y} W^{-1}(s_{Y\setminus Z})(D_{s_{X'}} W)(s_Z)\delta_{X'\cap Y,\emptyset}$$

$$+ e^{-\beta U^{(x)}(s_X)} \delta_{x\cap Y,\emptyset} \sum_{\substack{T\subset Y\\ T\neq\emptyset}} K(s_x, s_T) \sum_{T\subset Z\subset Y} W^{-1}(s_{Y\setminus Z})$$

$$\cdot W(s_{X'\cup Z})\delta_{X'\cap Y,\emptyset}$$

with

$$U^{(x)}(s_X) = \sum_{y\in X'} \varphi_{xy}(s_x, s_y) \quad \text{when} \quad |X| > 1$$

and we have therefore the desired equation

$$F_{s_X}(s_Y)=\delta_{x\cap Y,\emptyset}\, e^{-\beta U^{(x)}(s_X)}\, F_{s_{X'}}(s_Y)+\sum_{\substack{T\subset Y\\ T\neq\emptyset\\ T\cap X'=\emptyset}} K(s_x, s_T)F_{s_{X'\cup T}}(s_{Y\setminus T})$$

when $X=x$ (i.e. $X'=\emptyset$) the same equation is valid but now we have $U^{(x)}(s_x)=0$.

From this equation, we easily derive the equation for A

$$A^Y_{s_X}=\int w_\beta(ds_Y)e^{\beta\sum_{x\in X}\bar{v}(s_x)}F_{s_X}(s_Y)\,,$$

$$A^Y_{s_X}=\delta_{x\cap Y,\emptyset}\, e^{-\beta U^{(x)}(s_X)+\beta\bar{v}(s_x)}$$

$$\cdot\left[A^Y_{s_{X'}}+\sum_{\substack{T\subset Y\\ T\neq\emptyset\\ T\cap X'=\emptyset}}\int w_\beta(ds_T)K(s_x, s_T)\, e^{-\beta\sum_{t\in T}\bar{v}(s_t)}A^{Y\setminus T}_{s_{X'\cup T}}\right]$$

where $\bar{v}(s_x)=-\dfrac{J}{2}\gamma v^2(s_x)$.

References

1. Gallavotti, G., Miracle-Sole, S.: Commun. math. Phys. **5**, 317 (1967)
2. Israël, R.B.: Commun. math. Phys. **50**, 245 (1976)
3. Kunz, H.: Statistical mechanical treatment of the polymer model. Thesis, Lausanne (1971)
4. Gruber, C., Kunz, H.: Commun. math. Phys. **22**, 133 (1971)
5. Sylvester, C.S.: Weakly coupled Gibbs measures. Rockfeller preprint (1977)
6. Presutti, E., Olivieri, E.: Private communication
7. Kunz, H., Souillard, B.: In preparation
8. Duneau, M., Iagolnitzer, D., Souillard, B.: Commun. math. Phys. **31**, 191 (1973); **35**, 307 (1974)
9. Ruelle, D.: Commun. math. Phys. **50**, 189 (1976)
10. Levelt, J.M.H., Cohen, E.G.D.: Studies in statistical mechanics, Vol. II. Amsterdam: North Holland Publ. 1964
11. Green, H.S., Hurst, C.A.: Order-Disorder Phenomena. Monograph in statistical physics, Vol. 5, pp. 29–34. New York: Interscience 1964

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