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High-Temperature Analyticity in Classical Lattice Systems

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Abstract. We prove analyticity of the correlation functions for classical lattice systems, including "continuous-spin" systems, at high temperatures and in strong external fields. For systems whose configuration spaces are homogeneous spaces for compact groups (e.g. Ising, plane rotator and classical Heisenberg models), improved estimates on the region of analyticity are obtained by generalizing an integral equation of Gruber and Merlini. Exponential cluster properties are also obtained for such systems with a finite-range interaction.

I. Introduction

Using an equation of the "Kirkwood-Salsburg" type, Gallavotti and Miracle [2] have shown that the correlation functions and thermodynamic pressure of a classical lattice gas are analytic in the interaction parameters at high temperature and low activity. A similar equation can be used to deal with models where the "spin" at each site can take on any finite number of values, as opposed to the two values 0 and 1 for the lattice gas. However, this method does not extend to "continuous-spin" models.

In this paper, we present two models of proving analyticity for more general classical lattice systems. The first (Section II) works for very general systems, and is based on a theorem of Dobrushin [1] on uniqueness of measures with given conditional probabilities. The second method (Section III) requires some additional structure on the configuration space at each site: it must admit a transitive action by a compact group of measure-preserving homeomorphisms. We can then define an integral equation, generalizing one considered by Gruber and Merlini [3] for the spin $-\frac{1}{2}$ model. This involves a "Fourier series" expansion of the correlation functions; for a classical Heisenberg model, it is an expansion in spherical harmonics. In Section IV we obtain exponential cluster properties for finite-range interactions from this equation.

II. Analyticity for Random Fields

Our "lattice" will be a countable set S of "sites". At each site j there is a "configuration space" Ω_j, which we assume, for the sake of simplicity, to be a compact Hausdorff space. For each subset Λ of S, the Cartesian product $\Omega_\Lambda = \prod_{j\in\Lambda} \Omega_j$ (with the product topology) is the configuration space for Λ. We abbreviate Ω_S as Ω. All "measures" will be complex Borel measures. $\mathcal{M}(\Omega_\Lambda)$ and $C(\Omega_\Lambda)$ respectively will denote the spaces of measures and complex-valued continuous functions on Ω_Λ, with norms denoted by $\|\cdot\|$. The number of sites in $X \subset S$ is denoted by $|X|$.

A probability measure ϱ on Ω is said to have the *conditional probabilities* $\mu_j(ds|x)$ for $j\in S$, $x\in\Omega_{\{j\}^c}$ if for all $f\in C(\Omega)$ and $j\in S$,

$$\varrho(f) = \int_{\Omega_{\{j\}^c}} \varrho(dx) \int_{\Omega_j} \mu_j(ds|x)\, f(s\times x)\,.$$

Some measurability condition must be imposed on $\mu_j(ds|x)$ here; for convenience, we will assume it is continuous as a function from $\Omega_{\{j\}^c}$ to $\mathcal{M}(\Omega_j)$ with the weak*-topology. In statistical mechanics the conditional probabilities are usually given, and we wish to study probability measures ("Gibbs states") with those conditional probabilities. Assuming such a measure exist (as it will in the applications to statistical mechanics; in general some consistency conditions are required), we ask first whether it is unique. One sufficient condition for this is given by Dobrushin [1]. Next, if the conditional probabilities are varied analytically, we ask whether the Gibbs state varies analytically in an appropriate sense. This leads us to study complex "conditional measures".

Note [10] that we can not expect the expectation $\varrho(f)$ of every $f\in C(\Omega)$ to vary analytically: by a well-known theorem of Dunford ([7], Theorem 76) this would imply that ϱ varies continuously in the norm topology. On the other hand, in typical cases the distance between Gibbs states for different values of a parameter is always 2. What we actually expect is that $\varrho(f)$ should be analytic for f in a suitable dense subspace of $C(\Omega)$, e.g. the "local" functions: those that depend only on the configuration at a finite number of sites.

A system of complex "conditional measures" $\mu_j(ds|x)$ will be assumed to have the following properties:

(i) For each $j\in S$, $x\mapsto\mu_j(ds|x)$ is a continuous function from $\Omega_{\{j\}^c}$ to $\mathcal{M}(\Omega_j)$ with the weak* topology.

(ii) For each $j\in S$ and $x\in\Omega_{\{j\}^c}$, $\mu_j(1|x)=1$.

An operator $\tau_j: C(\Omega)\to C(\Omega_{\{j\}^c})$ is defined by

$$\tau_j f(x) = \int_{\Omega_j} \mu_j(ds|x) f(s\times x) \quad \text{for} \quad x\in\Omega_{\{j\}^c}\,.$$

Later we will write $\mu_j^{(z)}(ds|x)$, and correspondingly $\tau_j^{(z)}$, depending on a parameter z. If the $\mu_j(ds|x)$ are probability measures, τ_j is a contraction on $C(\Omega)$, but in the general case it is more difficult to control. We now introduce a space of functions with a norm in which τ_j will be a contraction under certain conditions.

Let $r>0$. We define $\mathcal{E}^r$ as the space of functions $f\in C(\Omega)$ which can be written as $f=\sum_X f_X$ (the sum being over finite subsets of S), with $f_X\in C(\Omega_X)$ and

$$\sum_X e^{r|X|}\,\|f_X\| < \infty\,.$$

A norm on $\mathscr{E}^r$ is given by

$$|||f|||_r \equiv \inf\left\{\sum_X e^{r|X|}\,\|f_X\| : f = \sum_X f_X,\ f_X \in C(\Omega_X)\right\}.$$

Note that $\|f\| \leqq |||f|||_r$, and that $\mathscr{E}^r$ is a Banach algebra with pointwise multiplication and the norm $|||\cdot|||_r$.

Lemma II.1. *Suppose there are continuous functions $y \mapsto \mu_j^Y(ds|y)$ from Ω_Y to $\mathscr{M}(\Omega_j)$ with the weak* topology, such that*

$$\mu_j(ds|x) = \sum_{Y \subset \{j\}^c} \mu_j^Y(ds|x) \quad \textit{for} \quad x \in \Omega_{\{j\}^c} \quad \textit{and} \quad \sum_Y e^{r(|Y|-1)} \sup_{y \in \Omega_Y} \|\mu_j^Y(ds|y)\| \leqq 1\,. \tag{1}$$

Then τ_j is a contraction on $\mathscr{E}^r$.

Proof. If $f = \sum_X f_X$ as above,

$$\begin{aligned} |||\tau_j f|||_r &\leqq \sum_{X \not\ni j} |||f_X|||_r + \sum_{X \ni j} \sum_Y |||\textstyle\int \mu_j^Y(ds|\cdot) f_X(s \times \cdot)|||_r \\ &\leqq \sum_{X \not\ni j} e^{r|X|}\,\|f_X\| + \sum_{X \ni j} \sum_Y e^{r(|X|+|Y|-1)}\,\|\textstyle\int \mu_j^Y(ds|\cdot) f_X(s \times \cdot)\| \\ &\leqq \sum_{X \not\ni j} e^{r|X|}\,\|f_X\| + \left(\sum_{X \ni j} e^{r|X|}\,\|f_X\|\right)\left(\sum_Y e^{r(|Y|-1)} \sup_{y \in \Omega_Y} \|\mu_j^Y(ds|y)\|\right). \end{aligned}$$ □

For distinct sites $i, j \in S$, let

$$\varrho_{ji} = \tfrac{1}{2} \sup\{\|\mu_j(ds|x) - \mu_j(ds|x')\| : x, x' \in \Omega_{\{j\}^c},\ x_k = x'_k \text{ for } k \neq i\}.$$

In [1], Dobrushin showed that if the $\mu_j(ds|x)$ are probability measures and there is $\alpha < 1$ such that $\sum_{i \in \{j\}^c} \varrho_{ji} \leqq \alpha$ for all $j \in S$, then there is at most one measure with conditional probabilities $\mu_j(ds|x)$. This condition is related to the hypotheses of Lemma II.1 as follows:

Lemma II.2. *Suppose the hypotheses of Lemma II.1 hold with* (1) *replaced by*

$$\sum_Y e^{r(|Y|-1)} \sup_{y \in \Omega_Y} \|\mu_j^Y(ds|y)\| \leqq \alpha < 1\,. \tag{2}$$

Then $\sum_{i \in \{j\}^c} \varrho_{ji} \leqq \alpha$.

Proof. Let $a_Y = \sup_{y \in \Omega_Y} \|\mu_j^Y(ds|y)\|$. Then $\varrho_{ji} \leqq \sum_{Y \ni i} a_Y$, so that $\sum_{i \in \{j\}^c} \varrho_{ji} \leqq \sum_Y |Y| a_Y$. Let $f(t) = \sum_Y e^{t|Y|} a_Y - \alpha e^t$. By assumption $f(r) \leqq 0$, while $f(0) \geqq 0$ since $\mu_j(1|x) = 1$. Therefore for some $t > 0$ we have

$$0 \geqq f'(t) = \sum_Y |Y| a_Y e^{t|Y|} - \alpha e^t \geqq e^t\left(\sum_Y |Y| a_Y - \alpha\right).$$ □

In his proof [6] of Dobrushin's result, Lanford uses the space $\mathscr{F}$ of functions $f \in C(\Omega)$ with $\sum_{i \in S} \delta_i(f) < \infty$, where

$$\delta_i(f) = \sup\{|f(x) - f(x')| : x, x' \in \Omega,\ x_k = x'_k \text{ for } k \neq i\}.$$

It is easily seen that our space $\mathscr{E}^r$ is contained in $\mathscr{F}$.

Theorem II.3. *Let $\mu_j^{(z)}(ds|)$ depend analytically on the parameter z in some open subset W of $\mathbb{C}^n$ (so that for fixed x and j, the $\mathscr{M}(\Omega_j)$-valued function $z \to \mu_j^{(z)}(ds|x)$ is analytic). Assume that:*

(i) *For each $z \in W$ there are $\alpha < 1$ and $\mu_j^Y(ds|y)$ as in Lemma II.1 for which* (2) *is satisfied.*

(ii) *$W \cap \mathbb{R}^n$ is a set of uniqueness for W.*

(iii) *For $z \in W \cap \mathbb{R}^n$ the $\mu_j^{(z)}(ds|x)$ are probability measures, and are the conditional probabilities for some probability measure $\varrho^{(z)}$ (which is unique by Dobrushin's result).*

Then if $f \in \mathscr{E}^r$ there is an analytic function $g(z)$ on W with $|g| \leqq |||f|||_r$ and $g(z) = \varrho^{(z)}(f)$ for $z \in W \cap \mathbb{R}^n$.

Proof. As shown in ([6], p. 110), for each $z \in W \cap \mathbb{R}^n$ and $f \in \mathscr{F}$ there is a sequence (j_n) in S for which

$$\sum_{i \in S} \delta_i(\tau_{j_n}^{(z)} \dots \tau_{j_1}^{(z)} f) \to 0 \,. \tag{3}$$

Since

$$\sum_{i \in S} \delta_i(\tau_j^{(z)} f) \leqq \sum_{i \in S} \delta_i(f) \quad \text{for} \quad f \in \mathscr{F} \quad \text{and} \quad z \in W \cap \mathbb{R}^n$$

it is easily seen that given $f \in \mathscr{F}$ and a countable dense subset C of $W \cap \mathbb{R}^n$, there is a sequence (j_n) in S for which (3) holds for all $z \in C$. Now fix $x \in \Omega$, $f \in \mathscr{E}^r$, and consider the sequence of analytic functions of $z \in W$ given by

$$\tau_{j_n}^{(z)} \dots \tau_{j_1}^{(z)} f(x) \,.$$

By Lemma II.1, these are uniformly bounded by $|||f|||_r$. For $z \in C$, they converge to $\varrho^{(z)}(f)$ as in Lanford's proof. By Vitali's Theorem, they converge uniformly on compact subsets of W to an analytic function $g(z)$, which agrees with $\varrho^{(z)}(f)$ on C. Note that $g(z)$ is independent of the choice of x, since this is true on the set of uniqueness C. Therefore, for $z \in W \cap \mathbb{R}^n$ we have

$$\varrho^{(z)}(f) = \lim_{n \to \infty} \varrho^{(z)}(\tau_{j_n}^{(z)} \dots \tau_{j_1}^{(z)} f) = g(z) \,. \qquad \square$$

The conditional measures in statistical mechanics arise from "interactions". Let $\tilde{\mathscr{B}}_r$ be the space of (complex) interactions Φ defined by the following conditions:

(i) For each nonempty finite subset X of S, $\Phi(X)$ is a complex-valued function in $C(\Omega_X)$.

(ii) The norm in $\tilde{\mathscr{B}}_r$ is $\|\Phi\|_r = \sup_{j \in S} \sum_{X \ni j} e^{r(|X|-1)} \|\Phi(X)\| < \infty$. The space of real interactions in $\tilde{\mathscr{B}}_r$ is denoted $\mathscr{B}_r$. For $\Phi \in \tilde{\mathscr{B}}_r$ we define "conditional Hamiltonians"

$$H_j^{\Phi}(s|x) = \sum_{X \ni j} \Phi(X)(s \times x) \qquad s \in \Omega_j \,, \qquad x \in \Omega_{\{j\}^c} \tag{4}$$

and the corresponding conditional measures

$$\mu_j^{\Phi}(ds|x) = \frac{e^{-H_j^{\Phi}(s|x)} \nu_j(ds)}{\int_{\Omega_j} e^{-H_j^{\Phi}(s'|x)} \nu_j(ds')} \tag{5}$$

where v_j is a fixed "a priori" probability measure on Ω_j, and the denominator in (5) is assumed to be nonzero.

It will be convenient to modify the algebra $\mathscr{E}^r$ and its norm by giving special treatment to a fixed site j, leaving it out of the exponential factor in the norm. Thus we define

$$|||f|||_{r,j} = \inf\left\{\sum_X e^{r|X\setminus j|}\, \|f_X\| : f = \sum_X f_X,\ f_X \in C(\Omega_X)\right\}$$

and $\mathscr{E}^{r,j} = \{f \in C(\Omega) : |||f|||_{r,j} < \infty\}$. This is clearly still a Banach algebra.

The following theorem shows that at "high temperature" Theorem II.3 is applicable.

Theorem II.4. *Let $\Phi = \Psi + \hat{\Phi} \in \tilde{\mathscr{B}}_r$, where Ψ is real with $\Psi(X) = 0$ for $|X| > 1$, while $\|\hat{\Phi}\|_r < \ln(2/(1+e^{-r}))$. Then there are $\alpha < 1$ and $\mu_j^Y(ds|y)$ for which condition (2) is satisfied. If W is the set of Φ as above, then the Gibbs state ϱ^Φ is unique for $\Phi \in W \cap \mathscr{B}_r$, and for $f \in \mathscr{E}^r$ the function $\Phi \to \varrho^\Phi(f)$ extends to a function $g_f(\Phi)$ on W which is analytic on $W \cap \mathscr{N}$ for each finite-dimensional complex linear subspace $\mathscr{N}$ of $\tilde{\mathscr{B}}_r$, with $|g_f(\Phi)| \leqq |||f|||_r$.*

Proof. We can ignore Ψ, since its only effect is to change $v_j(ds)$ to a new probability measure

$$\frac{e^{-\Psi(\{j\})(s)} v_j(ds)}{\int\limits_{\Omega_j} e^{-\Psi(\{j\})(s')} v_j(ds')}.$$

Therefore we will take $\Psi = 0$. Let

$$h_j(s|x) = \frac{e^{-H_j^\Phi(s|x)}}{\int\limits_{\Omega_j} e^{-H_j^\Phi(s'|x)} v_j(ds')}.$$

We need to estimate $|||h_j|||_{r,j}$. It is clear that

$$|||e^{-H_j^\Phi}|||_{r,j} \leqq e^{|||H_j^\Phi|||_{r,j}} \leqq e^{\|\Phi\|_r}.$$

Moreover

$$\left|\left|\left|1 - \int\limits_{\Omega_j} e^{-H_j^\Phi(s'|\cdot)} v_j(ds')\right|\right|\right|_{r,j} = \left|\left|\left|\int\limits_{\Omega_j} (1 - e^{-H_j^\Phi(s'|\cdot)})\, v_j(ds')\right|\right|\right|_{r,j}$$
$$\leqq |||1 - e^{-H_j^\Phi}|||_{r,j} \leqq e^{\|\Phi\|_r} - 1.$$

If the latter is less than 1, $\int\limits_{\Omega_j} e^{-H_j^\Phi(s'|\cdot)} v_j(ds')$ is invertible and

$$\left|\left|\left|\left(\int\limits_{\Omega_j} e^{-H_j^\Phi(s'|\cdot)} v_j(ds')\right)^{-1}\right|\right|\right|_{r,j} \leqq \frac{1}{2 - e^{\|\Phi\|_r}}.$$

Thus we obtain $|||h_j|||_{r,j} \leqq \dfrac{e^{\|\Phi\|_r}}{2 - e^{\|\Phi\|_r}} < e^r$

if

$$\|\Phi\|_r < \ln \frac{2}{1 + e^{-r}}.$$

The rest follows directly from Theorem II.3. □

Theorem II.4 was independent of the one-body ("chemical potential" or "external magnetic field") terms. Next we will show how analyticity can be obtained by adding to a given interaction a sufficiently strong one-body term favoring one configuration. Although it is possible to obtain somewhat greater generality, it is convenient to formulate this result in a "translation-invariant" context (to ensure that certain estimates are uniform on S). Thus we assume that S is a group, and that all the Ω_j are copies of each other, with the same "a priori" measure ν_j. Thus we have a group of "translations" α_j on $C(\Omega)$, with $\alpha_j(C(\Omega_\Lambda))=C(\Omega_{j\cdot\Lambda})$ and $\nu_{j\cdot i}\circ\alpha_j=\nu_i$. The interaction Φ is said to be *translation-invariant* if

$$\Phi(j\cdot X)=\alpha_j(\Phi(X)) \quad \text{for all} \quad j\in S \quad \text{and} \quad X\subset S \quad \text{finite}.$$

Spaces of translation-invariant interactions will be denoted by $\mathscr{B}_r^I$ and $\tilde{\mathscr{B}}_r^I$.

Theorem II.5. *Let $\Phi\in\tilde{\mathscr{B}}_r^I$, and $\Psi\in\mathscr{B}_r^I$ with $\Psi(X)=0$ for $|X|>1$, such that $\Psi(\{j\})$ attains its minimum at exactly one point s_0, which is in the support of ν_j. Then for $t>0$ sufficiently large, there is a neighborhood W of $\Phi+t\Psi$ in $\tilde{\mathscr{B}}_r$ in which the conclusions of Theorem II.4 are valid.*

Proof. As before, the effect of Ψ is to change ν_j. We let

$$\nu_j^t(ds)=\frac{e^{-t\Psi(\{j\})(s)}\nu_j(ds)}{\int\limits_{\Omega_j} e^{-t\Psi(\{j\})(s')}\nu_j(ds')}$$

and

$$h_j^t(s\times x)=\frac{e^{-H_j^\Phi(s\times x)}}{\int\limits_{\Omega_j} e^{-H_j^\Phi(s'\times x)}\nu_j^t(ds')}.$$

Given $\delta>0$, there is a neighborhood U of s_0 such that

$$\sum_{X\ni j} e^{r(|X|-1)}\,|\Phi(X)\,(s\times x)-\Phi(X)\,(s_0\times x)|<\delta \quad \text{for} \quad s\in U\,, \qquad x\in\Omega_{\{j\}^c}.$$

For t sufficiently large, $\nu_j^t(U)>1-\delta$. We then have

$$\begin{aligned}&\left|\!\left|\!\left|1-e^{H_j^\Phi(s_0|\cdot)}\int\limits_{\Omega_j} e^{-H_j^\Phi(s'|\cdot)}\nu_j^t(ds')\right|\!\right|\!\right|_{r,j}\\ &\leqq \int\limits_U |\!|\!|1-e^{H_j^\Phi(s_0|\cdot)-H_j^\Phi(s'|\cdot)}|\!|\!|_{r,j}\,\nu_j^t(ds')+\delta(e^{2\|\Phi\|_r}-1)\\ &\leqq e^\delta-1+\delta(e^{2\|\Phi\|_r}-1)\equiv\varepsilon\,.\end{aligned}$$

Thus we can write $h_j^t=\sum\limits_{Y\ni j} h_Y$ with $h_Y\in C(\Omega_Y)$ and

$$\sum_Y e^{r|Y\setminus j|}\,\|h_Y\|\leqq(1-\varepsilon)^{-1}\,e^{2\|\Phi\|_r}$$

while

$$\sum_Y e^{r|Y\setminus j|}\,\|h_Y(s\times\cdot)\|\leqq(1-\varepsilon)^{-1}\,|\!|\!|e^{H_j^\Phi(s_0|\cdot)-H_j^\Phi(s|\cdot)}|\!|\!|_{r,j}<(1-\varepsilon)^{-1}e^\delta \quad \text{for} \quad s\in U\,.$$

Taking $\mu_j^Y(ds|y) = h_{Y\cup\{j\}}(s \times y)\nu_j^t(ds)$, we have

$$\sum_Y e^{r|Y|} \sup_{y\in\Omega_Y} \|\mu_j^Y(ds|y)\| \leqq (1-\varepsilon)^{-1}(e^\delta + \delta e^{2\|\Phi\|_r}) < e^r$$

if δ is chosen sufficiently small. □

If $S = \mathbb{Z}^\nu$ there is a continuous convex function P on $\mathscr{B}_r^I$ traditionally called the "pressure", and the translation-invariant Gibbs states are obtained from functionals tangent to its graph (see [6]). Whenever the invariant Gibbs state for Φ is unique,

$$\frac{d}{dt} P(\Phi + t\Phi')_{|t=0} = -\varrho^\Phi(A_{\Phi'})$$

where

$$A_{\Phi'} = \sum_{X\ni 0} \frac{\Phi'(X)}{|X|}.$$

Therefore we have.

Corollary II.6. *If $S = \mathbb{Z}^\nu$ and $W \subset \tilde{\mathscr{B}}_r^I$ is an open starshaped set in which the conclusions of Theorem II.4 are valid, then the pressure P extends from $W \cap \mathscr{B}_r^I$ to a function on W which is analytic on $W \cap \mathscr{N}$ for each finite-dimensional complex linear subspace $\mathscr{N}$ of $\tilde{\mathscr{B}}_r^I$.*

Proof. Consider $g_{A_{\Phi'}}(\Phi)$ as a holomorphic differential 1-form on W; on $W \cap \mathscr{B}_r^I$ it is the derivative of the pressure, so by analytic continuation it is a closed form, and we can integrate it. □

III. An Integral Equation

With additional structure on the configuration space, we can replace $\mathscr{E}^r$ by a Banach algebra of absolutely converging Fourier series, and obtain a linear equation in the dual of this algebra which the Gibbs state must satisfy. This will give us a "high temperature" result for a somewhat different space of interactions: the exponential factor in the norm is no longer needed, but the uniform norm of $\Phi(X)$ is replaced by the ℓ^1 norm of its Fourier series. In most cases the bounds on critical temperatures obtained by this method are much better than those given by Theorem II.4.

For simplicity, we first consider the case where the configuration spaces Ω_i, and thus also the product Ω, are compact abelian groups, with normalized Haar measure denoted ds. A character of Ω is either 1 or of the form $\gamma_X = \gamma_{i_1} \otimes \ldots \otimes \gamma_{i_n}$ where $X = \{i_1, \ldots, i_n\}$ is a finite subset of S and $\gamma_{i_j} \neq 1$ is a character of Ω_{i_j}. The dual groups of Ω and Ω_i will be denoted by Γ and Γ_i respectively. Let $A(\Omega)$ be the space of complex functions f on Ω with absolutely convergent Fourier series, i.e. $f = \sum_{\gamma\in\Gamma} a_\gamma \gamma$ with $|||f||| \equiv \sum_{\gamma\in\Gamma} |a_\gamma| < \infty$. The dual $A(\Omega)^*$ of $A(\Omega)$ is naturally identified with the space $\ell^\infty(\Gamma)$ of bounded sequences indexed by Γ, with the uniform norm. It is well known that $A(\Omega)$ is a Banach algebra under pointwise operations.

Consider a system of conditional measures $\mu_j(ds|x)$ as in Section II, and suppose each $\mu_j(\gamma_j|\cdot)\in A(\Omega)$ with $\sup_{\gamma_j\in\Gamma_j} |||\mu_j(\gamma_j|\cdot)||| < \infty$. Then τ_j is a bounded operator on $A(\Omega)$ with norm $\sup_{\gamma_j\in\Gamma_j} ||| \mu_j(\gamma_j|\cdot) |||$. Let S be ordered so that each finite subset X is given a first element. Then we can define a linear operator K on $A(\Omega)$ by

$$\begin{aligned} &K\gamma_X = \tau_j \gamma_X \quad \text{where } j \text{ is the first element of } X \\ &K1 = 0 . \end{aligned} \tag{6}$$

It is clear that K has norm at most $\sup_{j\in S} \sup_{\gamma_j\neq 1} |||\mu_j(\gamma_j|\cdot)|||$. Our "integral equation" is

$$K^*\varphi + \delta = \varphi \tag{7}$$

where $\delta(1)=1$ and $\delta(\gamma_X)=0$ for all other characters. If the $\mu_j(ds|x)$ are probability measures, any Gibbs state must satisfy (7). This is the equation studied by Gruber and Merlini [3] for the spin $-\frac{1}{2}$ case ($\Omega_j=\mathbb{Z}_2$). If $1-K^*$ is invertible, the unique solution of (7) is given by $\varphi=(1-K^*)^{-1}\delta$. If in addition the conditional measures depend analytically on a parameter $z\in W\subset\mathbb{C}^n$, then the operator $K(z)$ is also analytic in z, and in the open set where $1-K(z)$ is invertible, $\varphi(z)=(1-K(z)^*)^{-1}\delta$ is also analytic.

We define new spaces $\mathscr{B}_A$ and $\tilde{\mathscr{B}}_A$ of real and complex interactions respectively, by requiring $\Phi(X)\in A(\Omega_X)$ with the norm

$$|||\Phi||| \equiv \sup_{j\in S} \sum_{X\ni j} |||\Phi(X)||| < \infty .$$

The conditional Hamiltonians and conditional measures are as in (4) and (5), but we require $\nu_j(ds)$ to be Haar measure ds.

Theorem III.1. *Suppose $\Phi\in\tilde{\mathscr{B}}_A$ with $|||\Phi||| < \ln\frac{3}{2}$. Then the operator $K(\Phi)$ of (6) has norm less than 1. For $f\in A(\Omega)$ the function $\varphi^\Phi(f)$ is analytic on the intersection of the set W of such Φ with any finite-dimensional complex linear subspace of $\tilde{\mathscr{B}}_A$ (where φ^Φ is the unique solution of the equation $K(\Phi)^*\varphi+\delta=\varphi$).*

Proof. For $\gamma_j\neq 1$ we have

$$|||\mu_j^\Omega(\gamma_j|\cdot)||| = \left|\left|\left| \frac{\int_{\Omega_j} (1-e^{-H_j^\Phi(s|\cdot)})\gamma_j(s)ds}{\int_{\Omega_j} e^{-H_j^\Phi(s|\cdot)}ds} \right|\right|\right| \leqq \frac{|||1-e^{-H_j^\Phi}|||}{1-|||1-e^{-H_j^\Phi}|||}$$

and

$$|||1-e^{-H_j^\Phi}||| \leqq e^{|||H_j^\Phi|||}-1 \leqq e^{|||\Phi|||}-1 < \tfrac{1}{2} . \qquad \square$$

In many cases we can obtain improved estimates on $|||\mu_j^\Phi(\gamma_j|\cdot)|||$. The simplest is the spin $-\frac{1}{2}$ Ising model with interaction of the form

$$\Phi(X) = -J(X)\sigma_X (J(X)\in\mathbb{C}, |||\Phi||| = \sup_{j\in S} \sum_{X\ni j} |J(X)| < \infty) .$$

Letting $u_j = \sum_{X \ni j} J(X)\sigma_{X \setminus j}$, we obtain $\mu_j(\sigma_j|\cdot) = \tanh u_j$. Now in the Taylor series $\tanh z = \sum_{n=0}^{\infty} b_n z^{2n+1}$, the sign of b_n is $(-1)^n$. Thus

$$|||\tanh u_j||| \leqq \sum_{n=0}^{\infty} |b_n| \, |||u_j|||^{2n+1} = \tfrac{1}{i} \tanh \, (i|||u_j|||) = \tan |||u_j|||\,.$$

Thus the conclusions of Theorem III.1 hold for this model if

$$\sup_{j \in S} |||u_j||| \; (= |||\Phi|||) < \tfrac{\pi}{3}\,.$$

On the other hand, the "mean field" value is $|||\Phi||| = 1$; it follows from Dobrushin's result quoted in Section II that if Φ is a real pair interaction with $|||\Phi||| < 1$ (and in general if $\sup_{j \in S} \sum_{X \ni j} (|X| - 1) \tanh |J(X)| < 1$) then Φ has a unique Gibbs state.

The plane rotator model can be treated in a similar fashion. Here Ω_j is the unit circle with characters $\sigma_j^n = e^{in\theta_j}$. We restrict our attention to the usual pair interaction, of the form

$$\Phi(\{i, j\})\,(\theta_i, \theta_j) = -J(i, j)\sigma_1 \cdot \sigma_2 = -\tfrac{1}{2} J(i, j)\,(\sigma_i \sigma_j^{-1} + \sigma_i^{-1} \sigma_j)\,.$$

For convenience we also take the $J(i, j)$ real. Now if we write

$$\sum_{X \ni j} J(i, j)\sigma_i = u_j = re^{i\theta}$$

we obtain

$$\mu_j(\sigma_j^k|\cdot) = e^{ik\theta} \frac{I_k(r)}{I_0(r)}$$

where I_k is the *k-th* order modified Bessel function

$$I_k(r) = \sum_{n=0}^{\infty} \frac{(r/2)^{2n+k}}{n!\,(n+k)!} \; (k \geqq 0)$$

$$I_{-k}(r) = I_k(r)\,.$$

The Taylor series $I_k(z)/I_0(z) = \sum_{n=0}^{\infty} b_n(k) z^{2n+k}$ (for $k > 0$) has coefficients $b_n(k)$ of sign $(-1)^n$, so that

$$|||\mu_j(\sigma_j^k|\cdot)||| \leqq \sum_{n=0}^{\infty} |b_n(k)| \, |||u_j|||^{2n+k} = \frac{J_k(|||u_j|||)}{J_0(|||u_j|||)}$$

where $J_k(z) = i^k I_k(-iz)$ is the *k-th* order (unmodified) Bessel function. For $0 \leqq z \leqq 2$ this is a decreasing function of $|k|$. Numerically solving $J_0(z) = J_1(z)$, we find that for this model the conclusions of Theorem III.1 hold (in some neighborhood of Φ) if

$$\sup_{j \in S} \sum_{i \in S} |J(i, j)| < 1.4347\,.$$

This compares to a "mean field" value of 2 for this model.

Another interesting example is a "continuous Ising model" with the interval $[-1, 1]$ (with normalized Lebesgue measure) as configuration space at each site, and an interaction of the form

$$\Phi(\{i, j\})(s_i, s_j) = -J(i, j)s_i s_j.$$

We can not use the usual Fourier series on the interval, because the interaction terms do not have absolutely convergent Fourier series. Instead, we use an expansion in Walsh functions: these are essentially the characters of $\prod_{n=1}^{\infty} \mathbb{Z}_2$, which is mapped onto $[-1, 1]$ by the measure-preserving transformation $(\sigma_n) \mapsto \sum_{n=1}^{\infty} 2^{-n}\sigma_n$. A similar analysis to that used for the spin $-\frac{1}{2}$ model can be performed, with the result that the conclusions of Theorem III.1 hold if

$$\sup_{j \in S} \sum_{i \neq j} |J(i, j)| < \frac{\pi}{2}.$$

This compares to a mean field value of 3 for this model.

The estimates for these models can be improved still more for interactions of "nearest-neighbor" type, as was done for the spin $-\frac{1}{2}$ model in [3].

Now we go to the non-abelian case. Here Ω_j is assumed to admit a transitive action by a compact group G_j of homeomorphisms preserving the a priori measure ν_j; thus it can be identified with G_j/H_j where H_j is the isotropy subgroup of an arbitrary point, and ν_j is the probability measure induced by Haar measure on G_j (therefore we will write it as ds). Similarly, Ω is identified with G/H, where $G = \prod_{j \in S} G_j$ and $H = \prod_{j \in S} H_j$. To replace the dual group of the abelian case, we have the continuous irreducible unitary representations of G. Let Σ_j be the set of equivalence classes (for unitary equivalence) of irreducible unitary representations of G_j. For each $\sigma \in \Sigma_j$ we suppose some member $U^{(\sigma)}$ of σ has been chosen. The equivalence classes of irreducible unitary representations of G are given by sequences $\boldsymbol{\sigma} = (\sigma_j)_{j \in S}$ where all but finitely many $\sigma_j = 1$ (the trivial representation), $\boldsymbol{\sigma}$ denoting the equivalence class of the representation $\bigotimes_j U^{(\sigma_j)} \equiv U^{(\boldsymbol{\sigma})}$ ([4], Theorem 27.43). The set of such equivalence classes is denoted Σ.

The space $A(G)$ of functions on G with absolutely converging Fourier series consists of functions of the form

$$f = \sum_{\boldsymbol{\sigma}} \mathrm{Tr}(A_{\boldsymbol{\sigma}} U^{(\boldsymbol{\sigma})}) \quad \text{with} \quad |||f||| \equiv \sum_{\boldsymbol{\sigma}} \|A_{\boldsymbol{\sigma}}\|_1 < \infty$$

where $A_{\boldsymbol{\sigma}}$ is an operator in the representation space $\mathscr{H}_{\boldsymbol{\sigma}}$, and $\|A\|_1$ is the trace-class norm $\mathrm{Tr}|A|$. See [4], §34, or [5]. The space $A(\Omega)$ of absolutely converging Fourier series on Ω consists of functions f on Ω such that $f \circ q \in A(G)$, where $q: G \to \Omega$ is the quotient map. The norm on $A(\Omega)$ is $|||f||| = |||f \circ q|||$. An equivalent formulation is as follows: for each $\boldsymbol{\sigma}$, define a function $W^{(\boldsymbol{\sigma})}$ from Ω to operators on $\mathscr{H}_{\boldsymbol{\sigma}}$ by

$$W^{(\boldsymbol{\sigma})}(gH) = \int_H dh U^{(\boldsymbol{\sigma})}(gh) = U^{(\boldsymbol{\sigma})}(g) P^{(\boldsymbol{\sigma})}$$

where $P^{(\sigma)} = \int_H dh U^{(\sigma)}(h)$ is an orthogonal projection. Then $A(\Omega)$ consists of all functions of the form

$$f = \sum_\sigma \mathrm{Tr}(C_\sigma W^{(\sigma)}) \quad \text{with} \quad P^{(\sigma)} C_\sigma = C_\sigma \quad \text{and} \quad |||f||| \equiv \sum_\sigma \|C_\sigma\|_1 < \infty .$$

$A(G)$ and $A(\Omega)$ are commutative Banach algebras under pointwise operations, and are dense in $C(G)$ and $C(\Omega)$ respectively (see [5]).

Suppose our conditional measures $\mu_j(ds|x)$ are of the form $g_j(s \times x)\, ds$ with $g_j \in A(\Omega)$. Then τ_j is an operator on $A(\Omega)$ of norm at most $|||g_j|||$. A linear operator K on $A(\Omega)$ is defined by

$$\begin{aligned} &K1 = 0 \\ &K f_X = \tau_j f_X \quad \text{where} \quad j \quad \text{is the first element of} \quad X \end{aligned} \tag{8}$$

and $f_X = \mathrm{Tr}(C_\sigma W^{(\sigma)})$, $X = \{i \in S : \sigma_i \neq 1\}$. This has norm at most $\sup_{j \in S} |||g_j|||$. Again we have an "integral equation" as in (7), and Banach spaces $\mathscr{B}_A$ and $\bar{\mathscr{B}}_A$ of interactions with $\Phi(X) \in A(\Omega_X)$.

Theorem III.2. *The results of Theorem III.1 hold also for the non-abelian case as formulated above.*

Proof. Suppose $f_X = \mathrm{Tr}(C_\sigma W^{(\sigma)}) \in A(\Omega)$ with $X = \{i \in S : \sigma_i \neq 1\}$, and j is the first element of X. Then $\int_{\Omega_j} f_X(s \times \cdot)\, ds = 0$, so that

$$|||\tau_j f_X||| = \left|\left|\left| \frac{\int_{\Omega_j} (1 - e^{-H_j^\Phi(s|\cdot)}) f_X(s \times \cdot)\, ds}{\int_{\Omega_j} e^{-H_j^\Phi(s|\cdot)} ds} \right|\right|\right| \leqq \frac{|||1 - e^{-H_j^\Phi}|||}{1 - |||1 - e^{-H_j^\Phi}|||} |||f_X||| .$$

As in Theorem III.1, for $|||\Phi||| < \ln \frac{3}{2}$ we obtain

$$|||1 - e^{-H_j^\Phi}||| \leqq e^{|||\Phi|||} - 1 < \tfrac{1}{2} \quad \text{so that} \quad \|K(\Phi)\| < 1 .$$

An important example is the classical Heisenberg model, where $\Omega_j = \mathbb{S}^2 = SO(3,\mathbb{R})/SO(2,\mathbb{R})$. Here the "Fourier series" is an expansion in spherical harmonics; if $f(x) = \sum_{l,m} a_{lm} \prod_j Y_{m_j}^{l_j}(x_j)$ then

$$|||f||| = \sum_l \left(\prod_j (2l_j + 1)^{1/2} \right) \left(\sum_m |a_{lm}|^2 \right)^{1/2} .$$

Here we are taking Y^l normalized so $\int_{\mathbb{S}^2} ds |Y^l(s)|^2 = 1$ (with ds normalized to 1, *not* 4π); the factor $(2l+1)^{1/2}$ arises because the matrix elements of $U^{(l)}$ have L^2 norm $(2l+1)^{-1/2}$ by the Schur orthogonality relations. The verification that $|||fg||| \leqq |||f|||\, |||g|||$ is recommended as an exercise in Clebsch-Gordan coefficients.

IV. Cluster Properties

The methods of Section III can be used to yield cluster properties for the correlation functions $\varphi^\Phi(f)$ where $f \in A(\Omega)$. The results of this section deal with a slight generalization of finite-range interactions. We conjecture that results similar to those of [8] could be obtained for exponentially-decreasing interactions.

Definition. For fixed interaction $\Phi \in \tilde{\mathscr{B}}_A$ and subsets X, Y of S, let $N^\Phi(X, Y)$ be the least n for which there is a sequence $\{j_0, j_1, \dots, j_n\}$ in S with $j_0 \in X$, $j_n \in Y$, such that there are sets $Z_i \subset S$ with $\Phi(Z_i) \neq 0$ and $\{j_{i-1}, j_i\} \subset Z_i$ for $i=1, \dots, n$. (If $X \cap Y \neq \emptyset$ then $N^\Phi(X, Y)=0$).

If we are given a metric on S and Φ has range at most R (i.e. $\Phi(X)=0$ whenever $\operatorname{diam}(X)>R$), then $N^\Phi(X, Y) \geqq R^{-1} \operatorname{dist}(X, Y)$.

In the following lemma we use for the first time the hypothesis that the selection of first elements comes for an ordering of S, so that if j is the first element of X and $j \in Y$, then j is the first element of $X \cap Y$.

Lemma IV.1. *If $f \in A(\Omega_X)$ and $g \in A(\Omega_Y)$ with $N^\Phi(X, Y)>n$ then*

$$\delta(K(\Phi)^n(fg)) = \sum_{k=0}^{n} \delta(K(\Phi)^k f)\delta(K(\Phi)^{n-k} g).$$

Proof. We use induction on n. For $n=0$, $\delta(fg)=\delta(f)\delta(g)$ if $X \cap Y=\emptyset$. Suppose the lemma is true for $n-1$. We can assume f and g are of the form $\operatorname{Tr}(C_\sigma W^{(\sigma)})$ with $\sigma_i \neq 1$ for $i \in X$ or Y respectively. If X or Y is empty the result is clear. Now $K(\Phi)(fg)$ is either $f\tau_j(g)=fK(\Phi)g$ or $g\tau_j(f)=gK(\Phi)f$, depending on whether the first element j of $X \cup Y$ is in Y or X respectively. Suppose $j \in Y$. Then $\tau_j g \in A(\Omega_{Y'})$ with $N^\Phi(X, Y') \geqq N^\Phi(X, Y)-1$, so by the induction hypothesis

$$\delta(K(\Phi)^n(fg)) = \delta(K(\Phi)^{n-1}(fK(\Phi)g)) = \sum_{k=0}^{n} \delta(K(\Phi)^k f)\delta(K(\Phi)^{n-k} g)$$

where the $k=n$ term is zero since $\delta(g)=0$. □

Theorem IV.2. *Let $\Phi \in \mathscr{B}_A$ with $\|K(\Phi)\|<1$, $f \in A(\Omega_X)$ and $g \in A(\Omega_Y)$ with $N^\Phi(X, Y)=N$. Then*

$$|\Phi^\Phi(fg)-\varphi^\Phi(f)\varphi^\Phi(g)| \leqq \left(\frac{2-\|K(\Phi)\|+N(1-\|K(\Phi)\|)}{(1-\|K(\Phi)\|)^2}\right)\|K(\Phi)\|^N \,|||f|||\,|||g|||.$$

Proof. Using Lemma IV.1 and the series expansion

$$\varphi^\Phi = (1-K(\Phi)^*)^{-1}\delta = \sum_{n=0}^{\infty} K(\Phi)^{*n}\delta$$

we obtain

$$\varphi^\Phi(fg)-\varphi^\Phi(f)\varphi^\Phi(g) = \sum_{n=N}^{\infty}\left(\delta(K(\Phi)^n(fg)) - \sum_{k=0}^{n}\delta(K(\Phi)^k f)\delta(K(\Phi)^{n-k} g)\right)$$

so

$$\begin{aligned}|\varphi^\Phi(fg)-\varphi^\Phi(f)\varphi^\Phi(g)| &\leqq \sum_{n=N}^{\infty}(2+n)\|K(\Phi)\|^n \,|||f|||\,|||g|||\\ &= \left(\frac{2-\|K(\Phi)\|+N(1-\|K(\Phi)\|)}{(1-\|K(\Phi)\|)^2}\right)\|K(\Phi)\|^N \,|||f|||\,|||g|||.\end{aligned}$$

□

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Remark. After the completion of this manuscript the author received a copy of [9], in which some of our results are obtained for the spin $-\frac{1}{2}$ model.

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