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Unbounded Derivations of C^*-Algebras II

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Abstract. It is demonstrated that a closed symmetric derivation δ of a C^*-algebra $\mathfrak{A}$ generates a strongly continuous one-parameter group of automorphisms of a C^*-algebra $\mathfrak{A}$ if and only if, it satisfies one of the following three conditions

1. $(\alpha\delta+1)(D(\delta))=\mathfrak{A}$, $\alpha\in\mathbb{R}\backslash\{0\}$.
2. δ possesses a dense set of analytic elements.
3. δ possesses a dense set of geometric elements.

Together with one of the following two conditions

1. $\|(\alpha\delta+1)(A)\|\geqq\|A\|$, $\alpha\in\mathbb{R}$, $A\in D(\delta)$.
2. If $\alpha\in\mathbb{R}$ and $A\in D(\delta)$ then $(\alpha\delta+1)(A)\geqq 0$ implies $A\geqq 0$.

Other characterizations are given in terms of invariant states and the invariance of $D(\delta)$ under the square root operation of positive elements.

1. Introduction

A derivation δ of a C^*-algebra $\mathfrak{A}$ is defined to be a linear mapping from a dense *-subalgebra $D(\delta)\subseteqq\mathfrak{A}$, the domain of δ, to a subspace $R(\delta)\subseteqq\mathfrak{A}$, the range of δ, satisfying the property

$$\delta(AB)=\delta(A)B+A\delta(B),\qquad A,B\in D(\delta)\,.$$

A derivation of this type is called symmetric if

$$\delta(A)^*=\delta(A^*),\qquad A\in D(\delta)\,.$$

A general derivation δ always has a decomposition

$$\delta=\delta_1+i\delta_2$$

in terms of symmetric derivations.

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The principal interest of symmetric derivations of the above type is that they arise as infinitesimal generators of strongly continuous one-parameter groups of $*$-automorphisms of $\mathfrak{A}$. Let

$$A \in \mathfrak{A} \mapsto \tau_t(A) \in \mathfrak{A}$$

be a one-parameter group of automorphisms of the foregoing type and define δ by

$$\delta(A) = \operatorname{unif.}\lim_{t\to 0} (\tau_t(A) - A)/t\,.$$

where $D(\delta)$ is the set of $A \in \mathfrak{A}$ such that the limit exists. It can be checked that δ is a derivation of $\mathfrak{A}$ satisfying the symmetry condition. We will refer to derivations which arise in the above manner as the infinitesimal generator of a group τ, or, more briefly, as an infinitesimal generator; the automorphic and continuity properties of the group will be left implicit.

One basic property of infinitesimal generators is that they are closed in the Banach space sense, i. e. if $A_\alpha \in D(\delta)$ is such that

$$\lim_\alpha \|A_\alpha - A\| = 0 \quad \text{and} \quad \lim_\alpha \|\delta(A_\alpha) - B\| = 0$$

then $A \in D(\delta)$ and $\delta(A) = B$.

In Section 2 we discuss some algebraic properties of the domains of closed derivations, and in Sections 3 and 4 we consider characterizations of infinitesimal generators. This material extends previous work of the authors [1]. Related material is contained in [2–5].

2. Closed Derivations

If this section we consider a closed derivation δ. If the C^*-algebra $\mathfrak{A}$ does not contain an identity element $\mathbb{1}$ then it can be embedded in a canonical fashion in a larger C^*-algebra, $\tilde{\mathfrak{A}} = \mathfrak{A} + \mathbb{C}\mathbb{1}$, with identity. One may then extend δ to a closed derivation $\tilde{\delta}$ of $\tilde{\mathfrak{A}}$ by setting $D(\tilde{\delta}) = D(\delta) + \mathbb{C}\mathbb{1}$ and

$$\tilde{\delta}(A + C\mathbb{1}) = \delta(A), \qquad A \in D(\delta), \qquad C \in \mathbb{C}\,.$$

If, however, $\mathfrak{A}$ contains an identity $\mathbb{1}$ it is not "a priori" evident whether $\mathbb{1}$ is automatically included in the domain of δ. The main result of this section is to prove that this is indeed the case. The method we use is an adaptation of the resolvent functional analysis for the domains of closed derivations developed in [1].

Theorem 1. *Let $\mathfrak{A}$ be a C^*-algebra containing an identity element $\mathbb{1}$ and let δ be a closed derivation of $\mathfrak{A}$.*

It follows that $\mathbb{1} \in D(\delta)$, and

$$\delta(\mathbb{1}) = 0\,.$$

Proof. Let δ be a strictly positive number such that $\varepsilon < 1/4$. Clearly one may find a self-adjoint $A \in \mathfrak{A}$ such that

$$(1 - 2\varepsilon)\mathbb{1} \geqq A \geqq 2\varepsilon\mathbb{1}\,.$$

As $D(\delta)$ is dense there then exists a self-adjoint $B \in D(\delta)$ such that

$$\|A - B\| < \varepsilon$$

and hence

$$(1-\varepsilon)\mathbb{1} \geqq B \geqq \varepsilon \mathbb{1} .$$

We now have the following result

Lemma 1. *Let δ be a closed derivation of a C^*-algebra $\mathfrak{A}$ and for $1/2 > \varepsilon > 0$ choose $B = B^* \in D(\delta)$ such that*

$$(1-\varepsilon)\mathbb{1} \geqq B \geqq \varepsilon \mathbb{1} .$$

Further define B_n by

$$B_n = B(\mathbb{1} + B)^{-n}, \qquad n = 1, 2, 3, \ldots .$$

It follows that $B_n \in D(\delta)$ and

$$\|\delta(B_n)\| \leqq n(1+\varepsilon)^{-n-1} \|\delta(B)\| .$$

Proof. Introduce the sequence of approximants

$$C_N = \sum_{m=0}^{N} (-1)^{mn+m}\, {}^{n}C_m B^{m+1} ,$$

where ${}^{n}C_m$ denotes the usual binomial coefficient.

This sequence converges uniformly to B_n because $\|B\| \leqq 1 - \varepsilon$.

Further one has for $N_2 > N_1$

$$\|\delta(C_{N_1}) - \delta(C_{N_2})\| \leqq \sum_{m=N_1+1}^{N_2} {}^{n+m}C_m (m+1)(1-\varepsilon)^m \|\delta(B)\| .$$

Hence $\delta(C_N)$ converges uniformly. As δ is closed one deduces that $B_n \in D(\delta)$.

Next note that

$$B_1(B + B^2) = B^2 .$$

Hence

$$\delta(B_1)B(\mathbb{1} + B) + B_1 \delta(B) = (\mathbb{1} - B_1)\delta(B^2) = (\mathbb{1} + B)^{-1}\delta(B^2)$$

which immediately yields

$$\delta(B_1)B(\mathbb{1} + B) = (\mathbb{1} + B)^{-1}\delta(B)B .$$

As B is invertible one then concludes that

$$\delta(B_1) = (\mathbb{1} + B)^{-1}\delta(B)(\mathbb{1} + B)^{-1}$$

which implies the bound

$$\|\delta(B_1)\| \leqq \|\delta(B)\| (1+\varepsilon)^{-2} .$$

Next note that

$$B_n B = B_{n-1} B_1$$

and hence, using the expression for $\delta(\mathrm{B}_1)$

$$\delta(B_n)B = -B_n\delta(B) + B_n\delta(B)(\mathbb{1}+B)^{-1} + \delta(B_{n-1})B_1$$
$$= -B_n\delta(B)B_1 + \delta(B_{n-1})B_1 .$$

Using the invertibility of B once again one finds

$$\delta(B_n) = -B_{n-1}\delta(B_1) + \delta(B_{n-1})(1+B)^{-1} .$$

Therefore

$$\|\delta(B_n)\| \leqq (1+\varepsilon)^{-n-1}\|\delta(B)\| + (1+\varepsilon)^{-1}\|\delta(B_{n-1})\|$$
$$\leqq n(1+\varepsilon)^{-n-1}\|\delta(B)\| ,$$

where the second inequality follows by iteration.

Let us now return to the proof of the theorem.

Consider the sequence of elements

$$F_N = \sum_{n=0}^{N} B_n \in D(\delta) .$$

This sequence converges uniformly to the identity.

But

$$\delta(F_N) = \sum_{n=0}^{N} \delta(B_n)$$

and the estimate of Lemma 1 establish that $\delta(F_N)$ also converges uniformly. Therefore $\mathbb{1} \in D(\delta)$.

The last statement of the theorem follows from the relation $\mathbb{1} = \mathbb{1}^2$ which yields

$$\delta(\mathbb{1}) = \delta(\mathbb{1})\mathbb{1} + \mathbb{1}\delta(\mathbb{1})$$

and the relation

$$A = A\mathbb{1} = \mathbb{1}A, \qquad A \in \mathfrak{A} .$$

Combining these relations one has

$$\delta(\mathbb{1}) = 2\delta(\mathbb{1})$$

i. e. $\delta(\mathbb{1}) = 0$.

The foregoing result allows us to assume there exists an identity $\mathbb{1} \in D(\delta)$ without loss of generality. Throughout the sequel we will make this assumption.

Analysis of the foregoing type was used in [1] to obtain functional properties of the domains of closed derivations δ. In particular if $A = A^* \in D(\delta)$ then $(\lambda\mathbb{1} - A)^{-1} \in D(\delta)$ and

$$\delta((\lambda\mathbb{1} - A)^{-1}) = (\lambda\mathbb{1} - A)^{-1}\delta(A)(\lambda\mathbb{1} - A)^{-1} .$$

Consequently functions $f(A)$ representable by Cauchy integrals

$$f(A) = 1/2\pi i \int_\Gamma d\lambda f(\lambda)(\lambda\mathbb{1} - A)^{-1}$$

also satisfy $f(A) \in D(\delta)$ and

$$\delta(f(A)) = 1/2\pi i \int_\Gamma d\lambda f(\lambda)(\lambda\mathbb{1} - A)^{-1}\delta(A)(\lambda\mathbb{1} - A)^{-1} .$$

Manipulation of such integrals allows one to reproduce a result of Powers [3] that for $A=A^*\in D(\delta)$ one has $\exp\{itA\}\in D(\delta)$ for $t\in\mathbb{R}$ and

$$\delta(e^{itA})=it\int_0^1 dr e^{itrA}\delta(A)e^{it(1-r)A}.$$

Consequently if f is a function of one real variable such that

$$|||\tilde{f}|||=\int dp|p\tilde{f}(p)|<\infty,$$

where $\tilde{f}$ is the Fourier transorm of f, then $f(A)\in D(\delta)$ and

$$\delta(f(A))=i\int dp\tilde{f}(p)p\int_0^1 dr e^{iprA}\delta(A)e^{ip(1-r)A}.$$

Hence

$$\|\delta(f(A))\|\leqq|||\tilde{f}|||\,\|\delta(A)\|. \tag{$*$}$$

[This result is most easily derived for smooth f and then exploiting ($*$) to extend the result to general f. In [3] a stronger result is stated but the proof is erroneous (the inequality stated on Line 1 of p. 91 is false for the algebra of 4×4 matrices).]

3. Infinitesimal Generators I

In the introduction we mentioned that symmetric derivations arise as infinitesimal generators of groups of automorphisms. In this section we attempt to characterize such derivations and establish some properties of their domains.

There are two aspects to this analysis, the linear or vector space aspect, and the algebraic aspect. The characterization of infinitesimal generators of groups of transformations of a Banach space is well understood (see, for example [6–8]) although several aspects such as the properties or analytic and geometric elements have not been fully esploited. We will analyze these properties in the first subsection of this section. The C^*-algebraic structure of the Banach space adds a new element to the analysis and can be utilized to strengthen the results obtained purely from the Banach space structure; this will be discussed in the second subsection.

a) Banach Space Theory

Let $\mathfrak{B}$ denote a Banach space, possibly complex, and τ a one-parameter family of mappings of $\mathfrak{B}$ into itself $B\in\mathfrak{B}\mapsto\tau_t(B)\in\mathfrak{B}$, $t\in\mathbb{R}$.

The family τ is defined to be a (quasi-bounded) group of type $G(M,\beta)$ if

$$\tau_{t+s}(B)=\tau_t\tau_s(B),\qquad t,s\in\mathbb{R},\ B\in\mathfrak{B},$$

$$\tau_0(B)=B,\qquad B\in\mathfrak{B},$$

$$\lim_{t\to 0}\|\tau_t(B)-B\|=0,\qquad B\in\mathfrak{B},$$

and, finally,

$$\|\tau_t(B)\|\leqq Me^{\beta|t|}\|B\|,\qquad B\in\mathfrak{B},$$

where $M\geqq 1$ and $\beta\geqq 0$.

If τ is a group of the foregoing type one may define an infinitesimal generator δ of τ by

$$\delta(B) = \lim_{t\to 0} (\tau_t(B) - B)/t$$

with the domain $D(\delta)$ of the unbounded operator δ defined to be the set of $B \in \mathfrak{B}$ such that the foregoing limit exists. It is well-known, and easily established, that δ is a closed, densely defined operator on $\mathfrak{B}$. We now aim to characterize operators on $\mathfrak{B}$ which generate groups of type $G(M, \beta)$ in the above manner. The characterization will be partly in terms of analytic and geometric elements.

Definition 1. Let δ be an operator on the Banach space $\mathfrak{B}$. An element $B \in \mathfrak{B}$ is defined to be analytic for δ if $B \in D(\delta^n)$, $n = 1, 2, 3, \ldots$, and the function

$$z \in \mathbb{C} \mapsto e_z(B) = \sum_{n \geq 0} z^n/n! \, \|\delta^n(B)\|$$

has a non-zero radius of convergence. An analytic element is defined to be geometric for δ if $B \in D(\delta^n)$, $n = 1, 2, 3, \ldots$, and

$$\|\delta^n(B)\| \leq r_\delta(B)^n \|B\|, \qquad n = 1, 2, 3, \ldots$$

for some $r_\delta(B) \in \mathbb{R}_+$.

If δ is the generator of a group τ of isometries of $\mathfrak{B}$, then the geometric elements of δ are just those elements in $\mathfrak{B}$ that has bounded Arveson spectrum with respect to τ.

An essential condition that enters all our characterizations is a bound on powers of the resolvent of the operator in question. The operator δ is said to satisfy condition $R(M, \beta)$ if $\|A\| \leq M \|(\alpha\delta + 1)^n(A)\| (1 - |\alpha|\beta)^{-n}$, for all $\alpha \in \mathbb{R}$ such that $|\alpha|\beta < 1$ and all $A \in D(\delta^n)$, $n = 1, 2, \ldots$, where $M \geq 1$ and $\beta \geq 0$. Note that condition $R(1, 0)$ simplifies to the sole condition

$$\|A\| \leq \|(\alpha\delta + 1)(A)\|, \qquad \alpha \in \mathbb{R},$$

where $A \in D(\delta)$.

Theorem 2. *Let δ be an operator on the Banach space $\mathfrak{B}$. The following condition are equivalent*

1. δ is the infinitesimal generator of a group of type $G(M, \beta)$.

2. δ is closed, $\overline{(\alpha\delta + 1)(D(\delta))} = \mathfrak{B}$ for all $\alpha \in \mathbb{R}$ such that $|\alpha|\beta < 1$ and δ satisfies condition $R(M, \beta)$.[1]

3. δ is closed, δ possesses a dense set of analytic elements, and δ satisfies condition $R(M, \beta)$.

In the particular case $M = 1$, $\beta = 0$, the above conditions are equivalent to.

4. δ is closed, δ possesses a dense set of geometric elements, and δ satisfies

$$\|\alpha\delta(A) + A\| \geq \|A\|, \qquad \alpha \in \mathbb{R}, \ A \in D(\delta).$$

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1 Here, and in the sequel, the assumption $(\alpha\delta + 1)(D(\delta)) = \mathfrak{B}$ can be replaced by $(\pm\alpha_0\delta + 1)(D(\delta) = \mathfrak{B}$ for some α_0 such that $|\alpha_0|\beta < 1$. This follows by use of the Neumann expansion

$$(\alpha\delta + 1)^{-1} = \alpha/\alpha_0 \sum_{n \geq 1} ((\alpha - \alpha_0)/\alpha)^{n-1} (\alpha_0\delta + 1)^{-n}$$

together with condition $R(M, \beta)$.

Proof. It is a well known fact of semi-group theory that Condition 2 implies Condition 1 and proofs may be found, for example, in [5–8]. We next prove that 1 implies 3.

Let τ denote the group generated by δ and for $B\in D(\delta)$ form the family of regularized elements B_λ by

$$B_\lambda = 1/\sqrt{\pi}\int dt\,\tau_{\lambda t}(B)e^{-t^2}.$$

First note that

$$\begin{aligned}B_\lambda - B &= 1/\sqrt{\pi}\int dt\,e^{-t^2}(\tau_{\lambda t}(B)-B)\\ &= 1/\sqrt{\pi}\int dt\,e^{-t^2}\int_0^{\lambda t} ds\,\tau_s(\delta(B))\end{aligned}$$

and hence for $\lambda\geqq 0$

$$\|B_\lambda - B\| \leqq 1/\sqrt{\pi}\int dt\,e^{-t^2}(e^{\beta\lambda|t|}-1)/\beta M\|\delta(B)\|.$$

But as $\lambda\to 0$ the right hand side tends to zero, i. e. B_λ converges to B.

Next one has

$$\begin{aligned}\delta^n(B_\lambda) &= (-\lambda)^{-n}\,1/\sqrt{\pi}\int dt + \tau_{\lambda t}(B)d^n/dt^n\,e^{-t2}\\ &= 1/\lambda^n\,1/\sqrt{\pi}\int dt\,\tau_{\lambda t}(B)H_n(t)e^{-t^2},\end{aligned}$$

where $H_n(t)$ is the usual Hermite function. Thus

$$\begin{aligned}\|\delta^n(B_\lambda)\|^2 &\leqq \lambda^{-2n}_1/\pi\,[\int dt\,e^{\beta|t|}|H_n(t)|e^{-t^2}]^2 M^2\|B\|^2\\ &\leqq M^2\|B\|^2\lambda^{-2n}_1/\pi\int dt\,e^{-t^2}e^{2\beta|t|}\int dt|H_n(t)|^2 e^{-t^2},\end{aligned}$$

where we have used the Schwarz inequality.

Using the normalization properties of the Hermite functions one finds that

$$\|\delta^n(B_\lambda)\|^2 \leqq 2M^2\|B\|^2 e^{\beta^2}(2/\lambda^2)^n n!.$$

Hence $e_z(B_\lambda)$ is an entire function. The desired result now follows from the density of $D(\delta)$.

Next we show that 3 implies 2. If B is analytic we may define τ by

$$\tau_t(B) = \sum_{n\geqq 0} t^n/n!\,\delta^n(B)$$

whenever $|t|<t_B$, the radius of analyticity of B. Now as B is analytic one finds that $B\in D((1+t\delta/n)^n)$ and

$$\begin{aligned}\|\tau_t(B)-(1+t\delta/n)^n(B)\| \leqq &\sum_{m=0}^{n}|t|^n/m!\,\|\delta^n(B)\|\,|1-c_{n,m}|\\ &+\sum_{m\geqq n+1}|t|^m/m!\,\|\delta^m(B)\|\end{aligned}$$

where $c_{n,m}=(1-1/n)(1-2/n)\ldots(1-(m-1)/n)$. But

$$0\leqq 1-c_{n,m}\leqq 1-(1-(m-1)/n)^{m-1}\leqq m(m-1)/n.$$

Therefore

$$\tau_t(B) = \lim_{n\to\infty}(1+t\delta/n)^n(B) \qquad |t|<t_B.$$

But one also calculates that

$$\tau_{t+s}(B)=\tau_t\tau_s(B), \qquad |t|,\,|s|<t_B/2.$$

Next from the condition $R(M, \beta)$ one has

$$\|(1+t\delta/n)^n(B)\| \geqq \|B\|(1-|t|\beta/n)^n/M$$

and hence

$$\|\tau_t(B)\| \geqq e^{-\beta|t|}\|B\|/M, \qquad |t| < t_B .$$

Using the group property one then finds that

$$\|B\| = \|\tau_{-t}\tau_t(B)\| \geqq e^{-\beta|t|}\|\tau_t(B)\|/M$$

whenever $|t| < t_B/2$ and hence

$$\|\tau_t(B)\| \leqq Me^{\beta|t|}\|B\|, \qquad |t| < t_B/2 .$$

From the definition of τ_t one concludes however that

$$\tau_t(\delta(B)) = \delta(\tau_t(B))$$

and hence it is possible to iterate the definition of τ_t

$$\tau_{t+s}(B) = \tau_t(\tau_s(B)) = \sum_{n \geqq 0} t^n/n!\,\delta^n(\tau_s(B))$$

for $|t| < t_B$ and $|s| < t_B$.

Repeating this argument one may define

$$\tau_t(B) = (\tau_{t/n})^n(B), \qquad |t| < nt_B/2 .$$

(The fraction $t_B/2$ is used in order to easily prove that this definition is independent of n). One then has

$$\|\tau_t(B)\| \leqq M^n e^{\beta|t|}\|B\|$$

or, alternatively

$$\|\tau_t(B)\| \leqq Me^{\beta' t}\|B\|$$

where

$$\beta' = 2/t_B \log M + \beta .$$

Next we proceed as in the proof that $1 \Rightarrow 3$ and form

$$B_\lambda = 1/\sqrt{\pi} \int dt \tau_{\lambda t}(B) e^{-t^2} .$$

One has once again that

$$\lim_{\lambda \to 0} \|B_\lambda - B\| = 0$$

and, further, that B_λ is an entire element for δ, i.e. $z \mapsto e_z(B_\lambda)$ is an entire function. Thus we may define

$$\tau_t(B_\lambda) = \sum_{n \geqq 0} t^n/n!\,\delta^n(B_\lambda)$$

and this definition is valid for all $t \in \mathbb{R}$.

Now, however, we can conclude that

$$\tau_t(B_\lambda) = \lim_{n \to \infty} (1 + t\delta/n)^n(B_\lambda)$$

and argue, as for B, that

$$\|\tau_t(B_\lambda)\| \leqq M e^{\beta|t|} \|B_\lambda\|$$

for all $t \in \mathbb{R}$.

If, next, $\alpha > 0$ we may define

$$g(t) = -e^{t/\alpha}/\alpha \int_t^\infty ds \mu^{-1} e^{-s^2/\mu^2} / \sqrt{\pi}\, e^{-s/\alpha}$$

and note that

$$(-\alpha d/dt + 1) g(t) = e^{-t^2/\mu^2}/\mu\sqrt{\pi}\,.$$

One estimates straightforwardly that if $0 < \alpha\beta < 1$ then

$$\int_{-\infty}^{\infty} dt e^{\beta|t|} |g(t)| \leqq e^{\mu^2/4\alpha^2}/\mu^{-1}(1-\alpha\beta)^{-1} + \mu\sqrt{\pi}/\alpha e^{\beta^2\mu^2/4}$$

and one concludes that

$$B_{g,\lambda} = \int dt g(t) \tau_t(B_\lambda)$$

is an element of the Banach space $\mathfrak{B}$ satisfying

$$\begin{aligned}(\alpha\delta + 1)(B_{g,\lambda}) &= 1/\sqrt{\pi} \int dt/\mu^{-1} e^{-t^2/\mu^2} \tau_t(B_\lambda) \\ &= 1/\sqrt{\pi} \int dt e^{-t^2} \tau_{\mu t}(B_\lambda) \\ &\xrightarrow[\mu \to 0]{} B_\lambda\,.\end{aligned}$$

But we have already proved that

$$\lim_{\lambda \to 0} \|B_\lambda - B\| = 0\,.$$

Hence $(\alpha\delta + 1)(D(\delta))$ is dense in $\mathfrak{B}$, by the assumption that the analytic vectors are dense whenever $0 < \alpha\beta < 1$. But $(\alpha\delta - 1)(D(\delta))$ is closed [by use of condition $R(M, \beta)$ with $n = 1$] and the desired conclusion is reached. A similar argument holds for $-1 < \alpha\beta < 0$.

Finally consider the isometric case $M = 1$, $\beta = 0$. Clearly Condition 4 implies Condition 3 and we next show that Condition 1 implies Condition 4. Again let τ be the group generated by δ and, for $B \in D(\delta)$ form the regularized elements

$$B_\lambda = \int dt f(t) \tau_{\lambda t}(B)\,,$$

where f is now a function, with integral one, whose Fourier transform is continuously differentiable with support in the interval $[-R, R]$. Once again

$$\begin{aligned}\|B_\lambda - B\| &\leqq \left\|\int dt f(t)(\tau_{\lambda t}(B) - B)\right\| \\ &\leqq \int dt |f(t)|\, \|\tau_{\lambda t}(B) - B\|\end{aligned}$$

and hence B_λ converges to B. But

$$\delta^n(B_\lambda) = (-1/\lambda)^n \int dt f^{(n)}(t) \tau_{\lambda t}(B)$$

and $\|\delta^n(B_\lambda)\| \leqq |\lambda|^{-n} \|B\| \int dt |f^{(n)}(t)|$.

Next one has

$$\|f^{(n)}\|_1 = \int dt|f^{(n)}(t)| = \int dt1/|1+it|\,|(1+it)f^{(n)}(t)|$$
$$\leqq \pi[\int dt|(1+it)f^{(n)}(t)|^2]^{1/2}$$

by the Schwarz inequality. Therefore by Fourier transformation

$$\|f^{(n)}\|_1 \leqq \pi[\int dp|(1-d/dp)p^n\tilde{f}(p)|^2]^{1/2}$$
$$\leqq \pi(R+1)^n \max\{\|\tilde{f}\|_\infty, \|\tilde{f}^{(1)}\|_\infty\}.$$

Thus B_λ is geometric and δ has a dense set of geometric elements by the density of $D(\delta)$.

It is often useful to have criteria for cores of generators.

Theorem 3. *Let δ be an infinitesimal generator of a group τ of type $G(M,\beta)$. Let $D \subseteqq D(\delta)$ be a dense set with the property that*

$$\tau_t(D) \subseteqq D$$

It follows that D is a core for δ, i. e. the closure, $\overline{\delta|D}$, of the restriction of δ to D satisfies

$$\overline{\delta|D} = \delta\,.$$

Proof. It suffices to demonstrate that $(\alpha\delta+1)(D)$ is dense for some α such that $0<\alpha\beta<1$. If $A \in D$ consider

$$r_\alpha(A) = \int_0^\infty dte^{-t}\tau_{-\alpha t}(A)$$

and $(\alpha\delta+1)(r_\alpha(A)) = \int_0^\infty dte^{-t}\tau_{-\alpha t}((\alpha\delta+1)(A))$. One may approximate both the integrals uniformly by Riemann sums

$$\Sigma_N(A) = \sum_{i=1}^N e^{-t_i}\tau_{-\alpha t_i}(A)(t_{i+1}-t_i) \in D$$
$$\Sigma_N((\alpha\delta+1)(A)) = \sum_{i=1}^N e^{-t_i}\tau_{-\alpha t_i}((\alpha\delta+1)(A))(t_{i+1}-t_i) \in D$$

whenever $0<\alpha\beta<1$. Hence by closure $r_\alpha(A) \in D(\overline{\delta|D})$ and

$$(\alpha\overline{\delta|D}+1)(r_\alpha(A)) = r_\alpha((\alpha\delta+1)(A))$$
$$= A$$

i. e. $D \subseteqq \overline{(\alpha\delta+1)(D)}$ and the proof is complete.

Corollary 1. *Let δ be an infinitesimal generator of a group τ of type $G(M,\beta)$. If $D \subseteqq D(\delta)$ is a dense set of analytic elements for δ such that*

$$\delta(D) \subseteqq D$$

then D is core for δ.

Proof. As $A \in D$ is analytic and $\delta(A) \in D$ [hence $\delta^2(A) \in D$, etc.] one may conclude that

$$\tau_t(A) = \sum_{n \geqq 0} t^n/n!\,\delta^n(A) \in D(\overline{\delta|D})$$

for $|t|<t_A$, the radius of convergence of $e_z(A)$. But $\|\tau_t(A)\| \leqq Me^{\beta|t|}\|A\|$ and $\delta^n\tau_t(A) = \tau_t(\delta^n(A))$ thus we may conclude that $\tau_t(A)$ is again analytic with radius of convergence t_A. Iterating this argument we deduce that $\tau_t(A) \in D(\overline{\delta|D})$ for all $t \in \mathbb{R}$ and the desired result follows from Theorem 3.

b) Algebraic Theory

Let us next consider the situation in Theorem 2 but with the extra assumption that $\mathfrak{B}$ is a Banach-algebra and δ is a symmetric derivation. If δ is an infinitesimal generator of a group τ of type $G(\mu, \beta)$ one has for $A \in D(\delta)$ that $\tau_t(A) \in D(\delta)$ and

$$d/dt\, \tau_t(A) = \delta(\tau_t(A)) = \tau_t(\delta(A)) .$$

Hence for $A, B \in D(\delta)$ one finds that

$$\begin{aligned} &d/dt\, \tau_{-t}(\tau_t(A)\tau_t(B)) \\ &= \tau_{-t}\{-\delta(\tau_t(A)\tau_t(B)) + \delta(\tau_t(A))\tau_t(B) + \tau_t(A)\delta(\tau_t(B))\} \\ &= 0 . \end{aligned}$$

It follows immediately that

$$\tau_t(AB) = \tau_t(A)\tau_t(B) .$$

It also easily follows from the symmetry of δ that

$$\tau_t(A^*) = \tau_t(A)^* .$$

Thus the first algebraic aspect is to ensure that the group τ is in fact a group of $*$-automorphisms of $\mathfrak{B}$.

If $\mathfrak{B}$ is not only a Banach*-algebra but more specifically a C^*-algebra then a second algebraic aspect enters. This is the property that $*$-automorphisms of a C^*-algebra are automatically norm-preserving. Explicitly if $A \in \mathfrak{A} \mapsto \alpha(A) \in \mathfrak{A}$ is an everywhere defined $*$-morphism of a C^*-algebra $\mathfrak{A}$.

$$\alpha(A^*A) = \alpha(A)^*\alpha(A) \geqq 0$$

i. e. α is positivity preserving. But for $B \in \mathfrak{A}$ one has

$$\mathbb{1} - B^*B/\|B\|^2 \geqq 0$$

and hence using $\alpha(\mathbb{1}) = \mathbb{1}$ one finds

$$\alpha(B)^*\alpha(B) \leqq \|B\|^2 .$$

Whence

$$\|\alpha(B)\| \leqq \|B\| .$$

If $\mathbb{1} \notin \mathfrak{A}$ then a similar result is obtained by extending α to $\tilde{\mathfrak{A}} = \mathfrak{A} + \mathbb{C}\mathbb{1}$.

These properties allows one to strengthen the results of the previous subsection in the case of C^*-algebras.

Theorem 4. *Let δ be a symmetric derivation of a C^*-algebra $\mathfrak{A}$. The following conditions are equivalent.*

1. δ is the infinitesimal generator of a strongly continuous one-parameter group of $*$-automorphisms of $\mathfrak{A}$.

2. δ is closed, $\overline{(\alpha\delta + 1)(D(\delta))} = \mathfrak{A}$ for all $\alpha \in \mathbb{R}$, and δ satisfies condition $R(M, \beta)$ for some $M \geqq 1$, $\beta \geqq 0$.

3. δ is closed, δ possesses a dense set of analytic elements, and δ satisfies condition $R(M, \beta)$ for some $M \geqq 1$ and $\beta \geqq 0$.

4. δ is closed, δ possesses a dense set of geometric elements, and δ satisfies condition $R(M, \beta)$ for some $M \geqq 1$ and $\beta \geqq 0$.

Further if δ generates a group τ and $D \subseteqq D(\delta)$ is a dense $*$-subspace of $\mathfrak{A}$ such that

$$\tau_t(D) \subseteqq D, \qquad t \in \mathbb{R},$$

or if D is a dense set of analytic elements for δ such that

$$\delta(D) \subseteqq D$$

then D is a core for δ.

This result is a corollary of Theorems 2 and 3 combined with the remarks at the beginning of this subsection. For example Condition 1 implies Condition 4 because the generator of the group of automorphisms of $\mathfrak{A}$ satisfies condition $R(1, 0)$, and hence $R(M, \beta)$. But 4 implies 3 and 3 implies 2 as in the proof of Theorem 2. Using Condition 2 one then combines the result of Theorem 2, and the first remark at the beginning of this subsection, to show that δ generates a group of $*$-automorphisms satisfying

$$\|\tau_t(A)\| \leqq M e^{\beta|t|} \|A\|, \qquad A \in \mathfrak{A}, t \in \mathbb{R}.$$

But then the positivity argument shows that

$$\|\tau_t(A)\| \leqq \|A\|$$

and the group property gives

$$\|A\| = \|\tau_{-t}(\tau_t(A))\| \leqq \|\tau_t(A)\| \leqq \|A\|, \qquad A \in \mathfrak{A}, t \in \mathbb{R}.$$

The core statements are a particular case of Theorem 3.

In the foregoing discussion the group of automorphisms τ of $\mathfrak{A}$ is defined by a limiting process. For example

$$\tau_t(A) = \lim_{n \to \infty} (1 - t\delta/n)^{-n}(A).$$

We have exploited the fact that τ is positivity preserving but we next remark that the approximants for τ are also positivity preserving and hence derive alternative characterizations of infinitesimal generators.

Theorem 5. *Let δ be a symmetric derivation of a C^*-algebra $\mathfrak{A}$. The following conditions are equivalent.*

1. δ is the infinitesimal generator of a strongly continuous one-parameter group of $*$-automorphisms of $\mathfrak{A}$.

2. δ is closed, $\overline{(\alpha\delta + 1)(D(\delta))} = \mathfrak{A}$ and

$$(\alpha\delta + 1)(A) \geqq 0$$

implies that $A \geqq 0$, where $A \in D(\delta)$, for all $\alpha \in \mathbb{R}$.

3. δ is closed, δ possesses a dense set of analytic elements and

$$(\alpha\delta + 1)(A) \geqq 0$$

implies that $A \geqq 0$, where $A \in D(\delta)$, for all $\alpha \in \mathbb{R}$.

Proof. To prove that 1 implies 2 and 3 it remains to demonstrate the positivity condition. But this follows from the representation

$$(\alpha\delta+1)^{-1}(A)=\int_0^\infty dt e^{-t}\tau_{-\alpha t}(A).$$

We next prove that 2 implies 1. This follows from Theorem 4 and the following result.

Lemma 2. *Let δ be a closed symmetric derivation of a C^*-algebra $\mathfrak{A}$ such that $(\alpha\delta+1)(D(\delta))=\mathfrak{A}$ for all $\alpha\in\mathbb{R}$. The following conditions are equivalent*

1. If $\alpha\in\mathbb{R}$ and $A\in D(\delta)$ then

$$(\alpha\delta+1)(A)\geqq 0$$

implies $A\geqq 0$.

2. If $\alpha\in\mathbb{R}$ and $(\alpha\delta+1)(A)\geqq 0$ then

$$\|(\alpha\delta+1)(A)\|\geqq A\geqq 0,$$

where $A\in D(\delta)$.

3. If $\alpha\in\mathbb{R}$ and $A\in D(\delta)$ then

$$\|(\alpha\delta+1)(A)\|\geqq\|A\|.$$

Proof. 3⇒1. For $0\leqq A=(\alpha\delta+1)(C)$ apply the inequality of Condition 3 to $\mathbb{1}+\lambda C$. One finds

$$\|\mathbb{1}+\lambda A\|\geqq\|\mathbb{1}+\lambda C\|.$$

If, however, $0\geqq\lambda\geqq-1/\|A\|$ then one has

$$1\geqq\|\mathbb{1}+\lambda C\|.$$

This gives a contradiction unless $C\geqq 0$.

1⇒2. If $C=(\alpha\delta+1)(A)$ then

$$(\alpha\delta+1)(\mathbb{1}-A/\|C\|)=\mathbb{1}-C/\|C\|\geqq 0$$

and hence

$$\|(\alpha\delta+1)(A)\|\geqq A.$$

To demonstrate that 2 implies 3, we first note that if $A\in D(\delta)$ then

$$\begin{aligned}(\alpha\delta+1)(A)^*(\alpha\delta+1)(A)&=(\alpha\delta+1)(A^*A)+\alpha^2\delta(A)^*\delta(A)\\&\geqq(\alpha\delta+1)(A^*A).\end{aligned}$$

As $(\alpha\delta+1)(D(\delta))$ is dense one may find a $B\in(\alpha\delta+1)(D(\delta))$ such that

$$(\alpha\delta+1)(A)^*(\alpha\delta+1)(A)+\varepsilon\mathbb{1}\geqq B\geqq(\alpha\delta+1)(A)^*(\alpha\delta+1)(A).$$

Let $C\in D(\delta)$ be such that

$$B=(\alpha\delta+1)(C)$$

then from combination of the above inequalities one has

$$(\alpha\delta+1)(C-A^*A)\geqq 0.$$

Hence from Condition 2

$$C \geqq A^* A .$$

Again applying Condition 2 we find

$$\|B\| \geqq C \geqq A^* A$$

or

$$\|(\alpha\delta + 1)(A^*)(\alpha\delta + 1)(A) + \varepsilon \mathbb{1}\| \geqq \|A^* A\| = \|A\|^2 .$$

Taking the limit $\varepsilon \to 0$ yields the desired result.

We remark in passing that in the above Lemma $3 \Rightarrow 1$, $1 \Rightarrow 2$ even without the assumption that $(\alpha\delta + 1)(D(\delta))$ is dense.

Returning to the proof of the theorem we see that Condition 2 implies Condition 2 of Theorem 4 with $M = 1$ and $\beta = 0$. Hence δ is a generator.

It remains to prove that Condition 3 implies Condition 1.

Let A be analytic. Define

$$\tau_t(A) = \sum_{n \geqq 0} t^n/n!\, \delta^n(A)$$

for $|t| < t_A$. One estimates, as in the proof of Theorem 2, that

$$\tau_t(A) = \lim_{n \to \infty} (1 + t\delta/n)^n(A), \qquad |t| < t_A .$$

Hence

$$\tau_t(A) \geqq 0$$

and $\varepsilon > 0$ implies that

$$(1 + t\delta/n)^n(A + \varepsilon \mathbb{1}) \geqq 0$$

for $|t| < t_A$ and all n larger than some N_ε. The positivity assumption of Condition 3 then implies that $A + \varepsilon \mathbb{1} \geqq 0$. As this argument is valid for all $\varepsilon > 0$ one has $A \geqq 0$, i. e. we have deduced that $\tau_t(A) \geqq 0$ for some $|t| < t_A$ implies that $A \geqq 0$. Next note that

$$A^* A = \tau_t(\tau_{-t}(A^* A)) \geqq 0$$

for $|t| < t_{A^*A}/2 = t_A/2$. Therefore one concludes that $\tau_{-t}(A^* A) \geqq 0$ for all $|t| < t_A/2$. Applying this argument to the positive analytic element $\mathbb{1} - A^* A/\|A\|^2$ one finds

$$0 \leqq \tau_{-t}(A^* A) \leqq \|A\|^2, \qquad |t| < t_A/2 .$$

Therefore

$$\|\tau_{-t}(A)\|^2 \leqq \|A\|^2, \qquad |t| < t_A/2 ,$$

but this then implies

$$\|\tau_t(A)\| = \|A\|, \qquad |t| < t_A/2 ,$$

by the group property. The rest of the proof then parallels that of Theorem 2 with the simplification that $M = 1$ and $\beta = 0$, i. e. one extends the definition of $\tau_t(A)$ to all $t \in \mathbb{R}$ by iteration and then one regularizes $\tau_t(A)$ with a suitable smooth function and eventually concludes that $A \in (\alpha\delta + 1)(D(\delta))$, and hence this latter set is dense.

This completes the proof of the theorem.

The foregoing analysis of infinitesimal generators shows that there are three conditions which are necessary and sufficient; δ must be closed, the resolvent $(\alpha\delta + 1)^{-1}$ should be densely defined for all $\alpha \in \mathbb{R}$ and the third condition demands that the resolvent have norm less than, or equal to, one. The analysis bears some similarity to the analysis of when a symmetric operator H on a Hilbert space $\mathscr{H}$ is self-adjoint, i. e. when does iH generate a strongly continuous one-parameter group of unitary operators on $\mathscr{H}$. This latter analysis is simplified by the fact that each symmetric operator is automatically closeable and

$$\begin{aligned}\|(i\alpha H + 1)\psi\|^2 &= \alpha^2 \|H\psi\|^2 + \|\psi\|^2 \\ &\geqq \|\psi\|^2, \qquad \alpha \in \mathbb{R},\ \psi \in D(H).\end{aligned}$$

Thus the analogues of the first and third condition mentioned above are automatically guaranteed. It is natural to ask whether the algebraic structure might be exploited to ensure a similar simplification for derivations of C^*-algrbras. The answer for general algebras is, however, no, although certain special classes of algebras might provide simplifications. In [1] we exhibited non-closeable derivations of UHF algebras and abelian algebras and the following example demonstrates that the resolvent bound is not automatic.

Example 1. Let $\mathfrak{A} = C[0, 1]$ be the C^*-algebra of continuous functions on the interval $[0, 1]$ and define δ by

$$\delta(f)(x) = df(x)/dx,$$

where $D(\delta)$ is the set of continuously differentiable functions over the interval. It follows that

a) δ is closed,

b) $(\alpha\delta + 1)(D(\delta)) = \mathfrak{A}$, $\alpha \neq 0$.

c) δ has a dense set of geometric elements, the polynomials.

d) $(\alpha\delta + 1)(e^{-x/\alpha}) = 0$ and hence $\|(\alpha\delta + 1)(A)\| \not\geqq \|A\|$ for all $A \in D(\delta)$.

e) There is no state ω over $\mathfrak{A}$ satisfying the invariance condition $\omega(\delta(A)) = 0$ for all $A \in D(\delta)$.

Points a) and b) are established by differential equation techniques, c) and d) are obvious and e) follows by noting that for the linear function $x \in D(\delta)$, $\delta(x) = \mathbb{1}$, and hence $\omega(\delta(x)) = 1$ for all states ω. This last point will be of relevance in the following section.

4. Infinitesimal Generators II

In this section we continue to discuss characterizations of infinitesimal generators but the subsequent characterizations differ from the foregoing because they only provide non-empty results for certain types of C^*-algebras; the previous results were generally applicable. We consider conditions which ensure closeability and the resolvent bound. There are two such conditions known at present;

existence of invariant states, and invariance of the domain of the derivation under the square root operation. We consider these two conditions in separate subsections.

a) Invariant States

If δ is the infinitesimal generator of a group τ of automorphisms of the C^*-algebra $\mathfrak{A}$, and if ω is a state over $\mathfrak{A}$, the invariance condition

$$\omega(\tau_t(A)) = \omega(A), \qquad A \in \mathfrak{A},\ t \in \mathbb{R},$$

is equivalent to the "infinitesimal" invariance condition

$$\omega(\delta(A)) = 0, \qquad A \in D(\delta).$$

In the same way as the invariance condition gives the unitary implementability of τ in the cyclic representation defined by ω, the "infinitesimal" condition gives an implementability of δ, in the cyclic representation, by means of a symmetric operator. The following result improves a statement occurring in [1].

Theorem 6. *Let δ be a symmetric derivation of a C^*-algebra $\mathfrak{A}$. Assume that there exists a state ω over $\mathfrak{A}$ which generates a faithful cyclic representation $(\mathcal{H}, \pi, \Omega)$ and also satisfies the invariance condition*

$$\omega(\delta(A)) = 0, \qquad A \in D(\delta).$$

It follows that
1. δ is closeable.
2. There exists a symmetric operator on $\mathcal{H}$ such that

$$D(H) = \pi(D(\delta))\Omega,$$

and

$$\pi(\delta(A))\psi = [iH, \pi(A)]\psi,$$

for all $A \in D(\delta)$ and $\psi \in D(H)$.

If, further, $\overline{(\alpha\delta + 1)(D(\delta))} = \mathfrak{A}$ for all $\alpha \in \mathbb{R}$ or, alternatively, if δ possesses a dense set $\mathfrak{A}^\alpha$ of analytic elements then $\bar{\delta}$, the closure of δ, is an infinitesimal generator.

Proof. The first two statements are proved in [1]. Either of the supplementary conditions of the theorem ensure that H is essentially self-adjoint. Let $\bar{H}$ denote the self-adjoint closure of H.

We need the following result.

Lemma 3. *Let δ be a symmetric derivation of a C^*-algebra $\mathfrak{A}$ of operators on a Hilbert space $\mathcal{H}$. Supnose there exists a strongly continuous one-parameter group of unitary operators $U(t)$ on $\mathcal{H}$, with infinitesimal generator H, such that $t \in \mathbb{R} \mapsto U(t)AU(-t)\psi$ is differentiable in the strong topology whenever $A \in D(\delta)$, $\psi \in D(H)$, and*

$$d/dt\, U(t)AU(-t)\psi = U(t)\delta(A)U(-t)\psi.$$

It follows that

$$\|\alpha\delta(A) + A\| \geqq \|A\|, \qquad \alpha \in \mathbb{R},\ A \in D(\delta).$$

Proof. Under the assumptions of the lemma

$$d/dt e^t U(\alpha t) A U(-\alpha t)\psi = e^t U(\alpha t)(\alpha\delta(A)+A)U(-\alpha t)\psi$$

and hence for $t \geqq 0$

$$\|e^t U(\alpha t) A U(-\alpha t)\psi - A\psi\| \leqq t e^t \|\alpha\delta(A)+A\| \, \|\psi\| .$$

Therefore, using the triangle inequality,

$$\|U(\alpha t) A U(-\alpha t)\psi\| e^t \leqq \{t e^t \|\alpha\delta(A)+A\| + \|A\|\} \|\psi\|$$

which immediately yields

$$\|A\|(e^t-1)/t \leqq e^t \|\alpha\delta(A)+A\| .$$

Taking the limit $t \to 0$ completes the proof.

To apply this result to the proof of the theorem note that by closure $\pi(A)D(\bar{H}) \subseteq D(\bar{H})$ for $A \in D(\delta)$. Hence with $U(t) = \exp\{i\bar{H}t\}$ one finds that $U(t)\pi(A)U(-t)\psi$ satisfies the conditions of the lemma whenever $A \in D(\delta)$, $\psi \in D(\bar{H})$. One has namely

$$\begin{aligned}&[U(t)AU(-t)-A]\psi/t\\ &=(U(t)-1)A(U(-t)-1)\psi/t\\ &\quad+1/t(U(t)-1)A\psi\\ &\quad+A1/t(U(-t)-1)\psi .\end{aligned}$$

Thus the resolvent bound is valid in the representation π. But this representation is faithful and hence the bound is valid algebraically. The last statement of the theorem now follows from Theorem 4, Parts 2 and 3.

We have previously remarked in Example 1 that for some C^*-algebras there exist closed derivations without invariant states, hence Theorem 6 is not universally applicable. It does, however, give simple criteria in a case of particular interest in mathematical physics.

Corollary 2. *Let δ be a closed symmetric derivation of a simple C^*-algebra with identity. The following conditions are equivalent*

1. *δ is an infinitesimal generator.*
2. *There exists a state ω over $\mathfrak{A}$ such that $\omega \circ \delta = 0$ and $\overline{(\alpha\delta+1)(D(\delta))} = \mathfrak{A}$ for all $\alpha \in \mathbb{R}$.*
3. *There exists a state ω over $\mathfrak{A}$ such that $\omega \circ \delta = 0$ and δ possesses a dense set of analytic elements.*

The only thing that remains to be shown is that if τ is a strongly continuous one-parameter group of $*$-automorphisms of $\mathfrak{A}$ then there exists a state ω over $\mathfrak{A}$ such that $\omega \circ \tau = \omega$. The existence of such an ω follows by applying an invariant mean to $\varphi \circ \tau$, where φ is an arbitrary state.

b) The Square-Root Operation

Theorem 7. *Let δ be a symmetric derivation of a C^*-algebra $\mathfrak{A}$ with identity. Assume that the domain $D(\delta)$ of δ is invariant under the square root operation, i. e. $A \in D(\delta)$ and $A \geqq 0$ imply that $A^{1/2} \in D(\delta)$.*

It follows that δ is closeable and satisfies the resolvent bound $\|\alpha\delta(A)+A\| \geqq \|A\|$, $\alpha \in \mathbb{R}$, $A \in D(\delta)$.

Thus, if further $\overline{(\alpha\delta+1)(D(\delta))} = \mathfrak{A}$, or, alternatively, if δ possesses a dense set of analytic elements then $\bar{\delta}$, the closure of δ, is an infinitesimal generator.

Proof. The proof of closeability is due to Powers and Sakai [4]; we elaborate on their method to derive the resolvent bound

$$\|\alpha\delta(A)+A\| \geqq \|A\|, \qquad \alpha \in \mathbb{R},\ A \in D(\delta).$$

For $A \in D(\delta)$ choose a state ω over $\mathfrak{A}$ such that

$$\omega(A^*A) = \|A\|^2 .$$

We may write

$$A^*A = \mathbb{1}\|A\|^2 - U^2$$

and

$$U = \sqrt{\mathbb{1}\|A\|^2 - A^*A} \in D(\delta)$$

by assumption. Note that we then have

$$\omega(U^2) = 0$$

But it immediately follows that

$$\begin{aligned} \omega(\delta(A^*A)) &= -\omega(\delta(\mathbb{1}\|A\|^2 - A^*A)) \\ &= -\omega(U\delta(U)) - \omega(\delta(U)U) \\ &= 0 , \end{aligned}$$

where the last step uses the Schwarz inequality.

Therefore one finds

$$\begin{aligned} \|\alpha\delta(A)+A\|^2 &\geqq \omega((\alpha\delta(A^*)+A^*)(\alpha\delta(A)+A)) \\ &= \alpha^2\omega(\delta(A^*)\delta(A)) + \alpha\omega(\delta(A^*A)) + \omega(A^*A) \\ &= \alpha^2\omega(\delta(A^*)\delta(A)) + \omega(A^*A) \\ &\geqq \omega(A^*A) \\ &= \|A\|^2 . \end{aligned}$$

The statement of the theorem now follows from Theorem 2.

Unfortunately the invariance of $D(\delta)$ under the square root operation is not a generally useful criterion. In [1] it was demonstrated that if δ is closed and $A \in D(\delta)$ is not only positive but invertible then $A^{1/2} \in D(\delta)$. Here the assumption of invertibility is essential as the following example shows.

Example 2. Let $\mathfrak{A} = C_0(\mathbb{R})$ and δ the infinitesimal generator of the group τ of translations $f(x) \mapsto f(x+a)$, i.e. the action of δ is given by

$$(\delta f)(x) = df(x)/dx$$

with $D(\delta)$ the continuously differentiable functions whose derivative vanishes at infinity. It follows that $|x|^{3/2}e^{-x^2} \in D(\delta)$ but the square root of this element is not contained in $D(\delta)$.

Thus the square root criterion has limited applicability. It does, however, apply to derivations δ of UHF algebras whose domain consists of an increasing sequence of matrix algebras.

5. Conclusion

We conclude this article with some remarks concerning the stability of infinitesimal generators. If δ_1 and δ_2 are both generators then their sum $\delta_1+\delta_2$ satisfies the resolvent bound

$$\|\alpha(\delta_1+\delta_2)(A)+A\| \geqq \|A\|, \qquad A \in D(\delta_1) \cap D(\delta_2)$$

by an argument due to Kato and Trotter (see, for example, [6], p. 499). In particular if δ_2 is bounded the resolvent bound holds for the closed derivation $\overline{\delta_1+\delta_2}$, defined in a natural manner on $D(\delta_1)$. But in this latter case perturbation theory allows one to conclude that

$$R(\alpha(\delta_1+\delta_2)+1)=R(\alpha\delta_1+1)=\mathfrak{A}.$$

Thus $\delta_1+\delta_2$ is an infinitesimal generator. One can in fact show that the groups τ^{12}, and τ^1, generated by $\delta_1+\delta_2$, and δ_1, respectively, are connected by

$$\tau_t^{12}(A)=\tau_t^1(A)+\sum_{n\geqq 1}\int\ldots\int_{0\leqq s_1\leqq s_2\leqq\cdot\leqq t} ds_1\ldots ds_n \tau_{s_1}^1\delta_2\tau_{s_2-s_1}^1\delta_2\ldots\delta_2\tau_{t-s_n}^1(A)$$

for all $A\in\mathfrak{A}$ and $t\geqq 0$, and by a similar series if $t\leqq 0$.

Thus the notion of infinitesimal generator is stable under bounded perturbations. This situation is satisfactory except in the case of abelian $\mathfrak{A}$ because no bounded derivations of an abelian C^*-algebra exist. Thus we are led to consider a more general notion of perturbation.

A natural concept is to define the derivation δ_2 to be δ_1-bounded, with δ_1-bound b, if $D(\delta_2)\supseteq D(\delta_1)$ and

$$\|\delta_2(A)\| \leqq a\|A\| + b\|\delta_1(A)\|, \qquad A\in D(\delta_1).$$

Examples of relatively bounded derivations can be given; if $\mathfrak{A}$ is the continuous functions on the circle, δ is usual differentiation and $\delta_g=g(\Theta)d/d\Theta$ with g continuous then

$$\|\delta_g(A)\| \leqq \|g\|_\infty \|\delta(A)\|, \qquad A\in D(\delta).$$

It is natural to ask whether $\delta_1+\lambda\delta_2$ is an infinitesimal generator for small λ whenever δ_1 is a generator and δ_2 is relatively bounded by δ_1. One can use the Neumann series to deduce that $R(\alpha(\delta_1+\lambda\delta_2)+1)=\mathfrak{A}$ for small enough α, whenever $|\lambda|<1/2b$ but the difficulty is to prove that $\delta_1+\lambda\delta_2$ satisfies the resolvent bound. If this can be established by independent means, for all λ, for example by showing that δ_2 is an infinitesimal generator, then $\delta_1+\lambda\delta_2$ will be a generator for $|\lambda|<1/2b$. But the argument can then be iterated by noting that

$$\|\delta_2(A)\| \leqq \alpha(1-|\lambda|\upsilon)^{-1}\|A\| + \upsilon(1-|\lambda|\upsilon)^{-1}\|\delta_1(A)+\lambda\delta_2(A)\|$$

for all $A\in D(\delta_1)$ and for $|\lambda|<1/b$. One then finds that $\delta_1+\lambda\delta_2$ is a generator for all $|\lambda|<1/b$. The crux of the matter is the resolvent bound and it does not appear that this follows without extrastructure.

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