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Phase Transitions for φ_2^4 Quantum Fields*

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Abstract. Phase transitions for the quantum field interaction $\lambda\varphi^4 + m_0^2\varphi^2$, $m_0^2/\lambda \ll 1$ are established in two dimensional space time.

1. Introduction

We present a direct proof of the existence of phase transitions in quantum field theory. We consider here the simplest interaction for which a phase transition is expected, namely the

$$\lambda\varphi^4 + \tfrac{1}{2} m_0^2\varphi^2, \qquad m_0^2/\lambda \ll 1, \tag{1.1}$$

perturbation of the free field of mass m_0. We give a complete proof in space time dimension $d=2$. Our same methods apply in principle to arbitrary even $P(\varphi)_2$ models without cutoff.

To define the interaction (1.1) for $d=2$ we require Wick ordering. We denote Wick ordering of P with respect to the covariance $(-\Delta + m_0^2)^{-1}$ by $:P:_{m_0}$. Then scaling and re Wick ordering leads to an equivalent theory with the bare mass $O(\sigma)^{-1}$ and the interaction which we study, see [13], is

$$:P(\varphi):_{\sigma^{-1}} = :(\varphi^2 - \sigma^2)^2/\sigma^2:_{\sigma^{-1}}, \qquad \sigma \gg 1. \tag{1.2}$$

It is the occurrence of two distinct minima, separated by a large barrier, which suggests the occurrence of phase transitions for the interaction (1.2). The two pure phases are ground states localized (in φ space) near the two minima $\varphi = \pm\sigma$.

In the case we consider, the polynominal $P(\varphi)$ is invariant under the symmetry transformation $\varphi \to -\varphi$, while the pure phases are interchanged by the symmetry. We note, however, that symmetry breaking is a distinct issue from the existence of phase transitions. Just as in statistical mechanics, where phase transitions may occur without symmetry breaking [11], we expect phase transitions in field theory for certain $P(\varphi)$ models which do not possess a symmetry group, such as

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for the interaction

$$\sigma^2 P(\varphi) = (\varphi^2 - \sigma^2)^4 + \varepsilon\varphi^3 - \mu\varphi\,,$$

for $\sigma \gg 1$, $\varepsilon \ll 1$, and $\mu = \mu(\varepsilon, \sigma)$.

The method we use here to study the interaction (1.2) is to generalize the φ^j bounds [3, 8, 2, 6, 5] in order to estimate perturbations of the measures for the interacting ground state. These estimates follow from the Osterwalder-Schrader positivity and the Euclidean invariance of the theory, combined with bounds we prove in Section 2 on the vacuum energy per unit volume. The existence of long range order (phase transitions and symmetry breaking) follow by a variant of a statistical mechanics argument due to Peierls. As a consequence, we find that the Dirichlet φ^4 theory (1.2) has a non-unique vacuum for $\sigma \gg 1$. The two phases are separated by an external field $-\mu\varphi$ added to $P(\varphi)$ of (1.2).

In a separate article [7], we generalize our convergent weak coupling expansion to yield a cluster expansion for the interaction (1.2). This expansion allows us to introduce boundary conditions (other than zero Dirichlet) which yield two distinct pure phases. In each pure phase we obtain from the cluster expansion a model with a unique vacuum, which satisfies the Wightman axioms with a mass gap. Of course, it would be of interest to pursue the particle structure of the model (1.2) via the cluster expansion.

In contrast to our detailed study based on the cluster expansion, our present paper gives a simple, direct proof that phase transitions occur. An alternative approach to the problem of phase transitions has been announced in [1], but the proof has not appeared.

We now introduce some notation and explain the Peierls argument. Let $\Lambda \subset R^2$ be a square, centered at the origin, and let Δ_Λ be the Laplace operator with zero Dirichlet boundary conditions on the boundary $\partial\Lambda$ of Λ. Let $d\varphi_{\varepsilon,\Lambda}$ denote the Gaussian measure over $\mathscr{S}'(R^2)$ wit mean zero and covariance $(-\Delta_\Lambda + \varepsilon^2)^{-1}$. Let

$$V(\Lambda) = \int_\Lambda :P(\varphi(x)):dx\,, \tag{1.3}$$

where $:P:$ denotes Wick ordering with respect to the covariance $(-\Delta_\Lambda + \varepsilon^2)^{-1}$. We let

$$\langle A \rangle = \lim_{\Lambda \nearrow R^2} \frac{\int A(\varphi) e^{-V(\Lambda)} d\varphi_{\varepsilon,\Lambda}}{\int e^{-V(\Lambda)} d\varphi_{\varepsilon,\Lambda}}\,, \tag{1.4}$$

and in order to ensure that P defined by (1.2) is equivalent to (1.1), we also choose $\varepsilon = \sigma^{-1}$.

For P given by (1.2), plus a linear term, the limit $\Lambda \nearrow R^2$ in (1.4) exists for

$$A = \int \varphi(x_1) \ldots \varphi(x_r) v(x_1, \ldots, x_r) dx\,, \qquad v \in \mathscr{S}\,,$$

and also for $A = \exp Q$ with Q defined by (1.9), as well as for

$$A = \prod \chi_\pm(\Delta_j)$$

defined below.

For each unit lattice square $\Delta = \Delta_j$, centered at $j \in Z^2$, we define the averaged field

$$\varphi(\Delta) = \int_\Delta \varphi(x) dx .$$

Essentially $\varphi(\Delta)$ is the low momentum field in Δ, so we expect that $\varphi(\Delta)$ characterizes phase transitions in Δ. We define a localizing function of the averaged field

$$\chi_{(a,b)}(\Delta) = \begin{cases} 1 & \text{if} \quad \varphi(\Delta) \in (a, b) \\ 0 & \text{otherwise.} \end{cases}$$

We use the particular functions

$$\chi_+ = \chi_{(0,\infty)} \quad \text{and} \quad \chi_- = \chi_{(-\infty,0)}$$

to separate phases and to establish the existence of long range order.

Lattice squares Δ_i, Δ_j are nearest neighbors if $|i-j| = 1$. Let $\mathcal{N}$ be a collection of nearest neighbor pairs and let $|\mathcal{N}|$ denote the cardinality of $\mathcal{N}$. Assume there exists a constant $K > 0$ such that

$$\langle \prod_{(\Delta,\Delta') \in \mathcal{N}} \chi_-(\Delta) \chi_+(\Delta') \rangle \leqq e^{-K|\mathcal{N}|} . \tag{1.5}$$

Theorem 1.1. *Consider a $P(\varphi)_2$ quantum field theory defined by* (1.3)–(1.4). *If* (1.5) *holds for K sufficiently large and for all $\mathcal{N}$, then for any two lattice squares Δ_i and Δ_j*

$$\langle \chi_+(\Delta_i) \chi_-(\Delta_j) \rangle \leqq 1/8 .$$

Corollary 1. *If the field φ has a $\varphi \to -\varphi$ symmetry* (*for example if P and the boundary conditions are both even*) *then $\langle \chi_\pm(\Delta) \rangle = \frac{1}{2}$. In this case,*

$$|\langle \chi_+(\Delta_i) \chi_-(\Delta_j) \rangle - \langle \chi_+(\Delta_i) \rangle \langle \chi_-(\Delta_j) \rangle| \geqq 1/8 ,$$

which establishes the existence of long range order.

In order to establish symmetry breaking, we consider the model

$$\begin{aligned} :P(\varphi, \mu): &= :(\varphi^2 - \sigma^2)^2/\sigma^2: - \mu\varphi \\ &= :P(\varphi): - \mu\varphi , \end{aligned} \tag{1.6}$$

with $:P:$ given by (1.2). The estimates in this paper are uniform for $0 \leqq \mu < \sigma^{-2}$. Then we find

Corollary 2. *For the interaction defined by* (1.3), (1.4), *and* (1.6),

$$\lim_{\mu \searrow 0} \langle \chi_+(\Delta) \rangle_\mu > 0.8 .$$

Proof. For $\mu \neq 0$, the $:P(\varphi, \mu):$ model has a unique vacuum, as a corollary of the Lee-Yang theorem [12]. Thus as $\operatorname{dist}(\Delta, \Delta') \to \infty$,

$$\begin{aligned} \langle \chi_+(\Delta) \chi_-(\Delta') \rangle_\mu &\to \langle \chi_+(\Delta) \rangle_\mu \langle \chi_-(\Delta') \rangle_\mu \\ &= \langle \chi_+(\Delta) \rangle_\mu (1 - \langle \chi_+(\Delta) \rangle_\mu) , \end{aligned}$$

where we use clustering and translation invariance. Thus by the theorem,

$$\langle\chi_+(\Delta)\rangle_\mu(1-\langle\chi_+(\Delta)\rangle_\mu)\leqq 1/8\,,$$

which has the solution $\langle\chi_+(\Delta)\rangle_\mu > 0.8$. By FKG inequalities, $\langle\chi_+(\Delta)\rangle_\mu$ is monotone in μ, as $\mu\searrow 0$. Thus $\lim\limits_{\mu\searrow 0}\langle\chi_+(\Delta)\rangle_\mu$ exists and is greater than $0.8>\frac{1}{2}$.

Theorem 1.2. *Let K be given, and let a φ^4 quantum field be defined by* (1.3), (1.4), *and* (1.6). *For σ sufficiently large,* (1.5) *holds for all collections $\mathcal{N}$ of nearest neighbor pairs.*

Proof of Theorem 1.1. Let Λ_0 be a square, centered at the origin, containing Δ_i, Δ_j and large enough (depending on i and j) so that

$$2+\operatorname{dist}(\Delta_i,\Delta_j)\leqq \frac{1}{9}|\partial\Lambda_0|\leqq \operatorname{dist}(\partial\Lambda_0,\{\Delta_i,\Delta_j\})\,. \tag{1.7}$$

A configuration is defined as a map from lattice squares in Λ_0 to ± 1. Given a configuration c, a subset $Y\subset c^{-1}(+1)$ is said to be $+$ connected if any two lattice squares of Y can be connected by a path of nearest neighbor squares belonging to Y. The definition of $-$ connected is analogous. Each configuration introduces a decomposition of Λ_0 into $+$ and $-$ connected components $\{X_k(c)\}$. Let X_i and X_j be components containing Δ_i and Δ_j respectively. Let γ_i be the outermost contour in ∂X_i. Technically, γ_i is defined to be the boundary of the component of $R^2\backslash X_i$ which contains the point at ∞. At least one of the contours γ_i or γ_j must separate Δ_i from Δ_j. Supposing this contour is γ_i, we define

$$\gamma=\gamma_i\backslash\partial\Lambda_0\,,$$

i.e. the portion of γ_i not contained in $\partial\Lambda_0$.

Letting $|\gamma|$ denote the length of γ, we assert that

$$|\gamma_i|+\operatorname{dist}(\gamma,\{\Delta_i,\Delta_j\})\leqq 11|\gamma|\,. \tag{1.8}$$

If $\gamma_i\cap\partial\Lambda_0=\emptyset$, then there is nothing to prove, since the shortest contour with a given radius ($\geqq\operatorname{dist}(\gamma,\{\Delta_i,\Delta_j\})$) is a square, and for this case, the circumference is eight times the radius. Now suppose $\gamma_i\cap\partial\Lambda_0\neq\emptyset$. Then

$$\begin{aligned}\operatorname{dist}(\gamma,\{\Delta_i,\Delta_j\})&\leqq \tfrac{1}{2}\operatorname{dist}(\Delta_i,\Delta_j)\leqq\frac{1}{18}|\partial\Lambda_0|\\&\leqq\tfrac{1}{2}\operatorname{dist}(\partial\Lambda_0,\{\Delta_i,\Delta_j\})\\&\leqq\tfrac{1}{2}|\gamma|\end{aligned}$$

by (1.7). Thus

$$\begin{aligned}|\gamma_i|+\operatorname{dist}(\gamma,\{\Delta_i,\Delta_j\})&\leqq|\gamma|+|\partial\Lambda_0|+\tfrac{1}{2}|\gamma|\\&\leqq 11|\gamma|\,.\end{aligned}$$

Since

$$\textstyle\prod_{\Delta\subset\Lambda_0}(\chi_+(\Delta)+\chi_-(\Delta))=1\,,$$

we have

$$\langle\chi_+(\Delta_i)\chi_-(\Delta_j)\rangle = \sum_c' \langle\chi_+(\Delta_i)\chi_-(\Delta_j)\prod_\Delta \chi_{c(\Delta)}(\Delta)\rangle\,,$$

where $\sum_c'$ is the sum over configurations c with $c(\Delta_i)=1$, $c(\Delta_j)=-1$ and $\prod_\Delta$ omits Δ_i and Δ_j. Let $\gamma=\gamma(c)$ be defined as above, and let $\mathcal{N}=\mathcal{N}(\gamma)$ be the set of nearest neighbor pairs bordering γ. Then

$$\begin{aligned}\langle\chi_+(\Delta_i)\chi_-(\Delta_j)\rangle &= \sum_\gamma \sum_{\{c:\gamma(c)=\gamma\}} \langle\chi_+(\Delta_i)\chi_-(\Delta_j)\prod_\Delta \chi_{c(\Delta)}(\Delta)\rangle \\ &\leqq \sum_\gamma \langle\chi_+(\Delta_i)\chi_-(\Delta_j)\prod_{\Delta,\Delta'\in\mathcal{N}(\gamma)}\chi_+(\Delta)\chi_-(\Delta')\rangle \\ &\leqq \sum_\gamma e^{-K|\gamma|}\,.\end{aligned}$$

The sum ranges over γ's resulting from contours γ_i enclosing Δ_i and separating Δ_i from Δ_j. There are at most $3^{|\gamma_i|}$ such contours of length $|\gamma_i|$, with a fixed starting point. We choose as starting point the point of γ_i lying on the line (Δ_i, Δ_j) and closest to Δ_i. With $|\gamma_i|$ fixed, this can be chosen in at most $11|\gamma|$ ways, by (1.8). Thus $\langle\chi_+(\Delta_i)\chi_-(\Delta_j)\rangle < 1/8$ for K sufficiently large.

In order to prove Theorem 1.2, we study perturbations of the interacting Euclidean measure defined by polynomials of the form

$$Q(\xi, X) \equiv \sum_{\nu=1}^{4} Q_\nu(\xi^{(\nu)}, X)\,, \tag{1.9}$$

where

$$Q_1(\xi^{(1)}, X) = \sigma^{-1}\sum_{\Delta_j\subset X}\xi_j^{(1)}\int_{\Delta_j}(:\varphi^2(x): - \sigma^2)dx\,, \tag{1.10}$$

$$Q_2(\xi^{(2)}, X) = (\ln\sigma)^{-1}\sum_{\Delta_j\subset X}\xi_j^{(2)}\int_{\Delta_j}:\varphi^2(x) - \varphi(\Delta_j)^2:dx\,, \tag{1.11}$$

$$Q_3(\xi^{(3)}, X) = \sum_{\substack{\Delta_i,\Delta_j\subset X\\ |i-j|=1}}\xi_{ij}^{(3)}(\varphi(\Delta_i)-\varphi(\Delta_j))\,, \tag{1.12}$$

$$Q_4(\xi_j^{(4)}, \kappa, \kappa', X) = \sigma^{-4}(\kappa+1)^\delta\sum_{\Delta_j\subset X}\xi_j^{(4)}\sum_{j=1}^{4}\int_{\Delta_i}:\varphi_\kappa^j(x) - \varphi_{\kappa'}^j(x):dx\,. \tag{1.13}$$

In (1.13), κ and κ' denote ultraviolet cutoffs, and

$$0\leqq\kappa\leqq\kappa'\leqq\infty\,,\qquad \kappa<\infty\,.$$

For $\kappa=0$, $:\varphi_\kappa^j:\equiv 0$. Let $|X|$ denote the area of the subset $X\subset R^2$. Our main technical estimate is the following theorem, which will be proved in Section 3.

Theorem 1.3. *There is a constant K_1 and a $\delta>0$ such that for all Q defined by* (1.9)–(1.13), *with $|\xi_\alpha^{(\beta)}|\leqq 1$ and for $\langle\cdot\rangle$ defined by* (1.2)–(1.4), *and* (1.6) *with σ sufficiently large*

$$|\langle e^{Q(\xi,X)}\rangle| \leqq e^{K_1(\log\sigma)^2|X|}\,.$$

Proof of Theorem 1.2, assuming Theorem 1.3. For each pair $(\Delta_i, \Delta_j)\in\mathcal{N}$ we write

$$\chi_+(\Delta_i)\chi_-(\Delta_j) = (\chi_{(0,\sigma/2)} + \chi_{(\sigma/2,\infty)})(\chi_{(-\infty,-\sigma/2)} + \chi_{(-\sigma/2,0)})$$

as a sum of four terms. We shall bound each term for any even M by $[\sigma^{-1}Q_\nu(\varphi)]^M$, $\nu=1,2,3$. By using the Cauchy formula for derivatives in the variable ξ as in [2] we have

$$\sigma^{-M}\langle Q_\nu^M\rangle = \sigma^{-M}\left[\frac{d}{d\xi^{(\nu)}}\right]^M\langle e^Q\rangle \leqq \frac{M!\,\sigma^{-M}}{2\pi}\left|\oint\frac{\langle e^Q\rangle}{[\xi^\nu]^{M+1}}d\xi\right|\,. \tag{1.15}$$

In each case we choose $M=2[\sigma/8]$ to optimize the above bound. Here $[x]$ denotes the integer part of x.

For instance, we bound the term $\chi_{(\sigma/2,\infty)}(\Delta_i)\chi_{(-\infty,-\sigma/2)}(\Delta_j)$ by

$$[\sigma^{-1}(\varphi(\Delta_i)-\varphi(\Delta_j))]^M=\sigma^{-M}\left[\frac{d}{d\xi_{ij}^{(3)}}\right]^M e^{Q_3}\Big|_{\xi=0}.$$

Each of the remaining three terms is bounded by $\chi_{(0,\sigma/2)}(\Delta_i)$ or $\chi_{(-\sigma/2,0)}(\Delta_j)$. Note that on the support of $\chi_{(0,\sigma/2)}(\Delta)$,

$$\begin{aligned}\chi_{(0,\sigma/2)}(\Delta)&\leqq 1\leqq 4/3\left[1-\left(\frac{\varphi(\Delta)}{\sigma}\right)^2\right]\\&\leqq \frac{4}{3}\sigma^{-2}[\sigma^2-\textstyle\int_\Delta :\varphi(x)^2:dx]+\frac{4}{3}\sigma^{-2}[\textstyle\int_\Delta :\varphi(x)^2:dx-:\varphi(\Delta)^2:+O(\ln\sigma)].\end{aligned}$$

Here $O(\ln\sigma)$ is a Wick ordering constant. Since $\sigma^{-2}O(\ln\sigma)\to 0$ one has for sufficiently large σ either

$$\chi_{(0,\sigma/2)}(\Delta_i)\leqq\sigma^{-M}\left[\frac{4}{\sigma}(\textstyle\int_{\Delta_i}:\varphi(x)^2:dx-\sigma^2)\right]^M$$

or

$$\chi_{(0,\sigma/2)}(\Delta_i)\leqq\sigma^{-2M}[4\textstyle\int_{\Delta_i}:\varphi(x)^2:dx-:\varphi(\Delta)^2:]^M.$$

The factor $\chi_{(-\sigma/2,0)}(\Delta_j)$ is bounded by the same expressions with Δ_j replacing Δ_i.

Now we can apply (1.15) to each pair of squares. Since a single square Δ_i occurs in Q_1 or Q_2 in at most four pairs $(\Delta_i,\Delta_j)\in\mathcal{N}$, the factorials are bounded by

$$(4M!)^{|\mathcal{N}|/4}\leqq([\sigma]!)^{|\mathcal{N}|/4}.$$

The theorem now follows from

$$\begin{aligned}\langle\textstyle\prod\chi_+(\Delta_i)\chi_-(\Delta_j)\rangle&\leqq e^{O(\log\sigma)^2|\mathcal{N}|}[[\sigma]!\,\sigma^{-\sigma}]^{|\mathcal{N}|/4}\\&\leqq e^{O(\log\sigma)^2|\mathcal{N}|}e^{-\sigma|\mathcal{N}|/4}.\end{aligned}$$

In the last line we have used Stirling's formula (with $n=[\sigma]$),

$$n!\leqq(2\pi n)^{\frac{1}{2}}(n/e)^n.$$

Note that the weakly divergent bound $\exp(O(\ln^2\sigma))$ per square from Theorem 1.3 is bounded by the strongly convergent factor $\exp(-O(\sigma))$.

2. Bounds on Vacuum Energy per Unit Volume

In this section we give upper and lower bounds on the vacuum energy per unit volume. The upper bound is more elementary, and we consider it first. Let $|\partial\Lambda|$ denote the length of the boundary of Λ.

Proposition 2.1. *There is a constant $O(1)$ independent of σ and Λ such that*

$$e^{-O(1)|\Lambda|}\leqq\int e^{-V(\Lambda)}d\varphi_{\sigma^{-1},\Lambda}\tag{2.1}$$

for all Λ satisfying $\sigma|\partial\Lambda|\leqq|\Lambda|$.

Proof. We translate φ by the function $\sigma f(x)$, where

$$0 \leqq f(x) = \sigma^{-2} \int_\Lambda (-\Delta_\Lambda + \sigma^{-2})^{-1}(x, y) dy \leqq 1 .$$

(Note that the Dirichlet Green's function is less than the free Green's function.) By convexity of the exponential,

$$\int e^W d\varphi \geqq \exp(\int W d\varphi) .$$

Hence for $W(\varphi)$ a Wick ordered polynomial without a constant term,

$$\int W d\varphi = 0 \quad \text{and} \quad \int e^W d\varphi \geqq 1 .$$

Thus we need only compute the constant in the exponent after translation. The purpose of the translation is to eliminate the leading term $O(\sigma^2|\Lambda|)$. There is a constant term associated with $V(\Lambda)$ and a constant associated with the change of measure $d\varphi \to d(\varphi + \sigma f)$. The constant associated with $V(\Lambda)$ of (1.3) is

$$\begin{aligned} \int_\Lambda P(\sigma f(x)) dx &= \sigma^2 \int_\Lambda (f(x)^2 - 1)^2 dx \\ &\leqq 2\sigma^2 \int_\Lambda (1 - f(x)) dx \\ &= 2 \int_{\Lambda \times \Lambda} [(-\Delta + \sigma^2)^{-1} - (-\Delta_\Lambda + \sigma^{-2})^{-1}](x, y) dx dy \\ &\quad + 2 \int_{R^2 \setminus \Lambda} \int_\Lambda (-\Delta + \sigma^{-2})^{-1}(x, y) dx dy . \end{aligned}$$

In the first term, Δ and Δ_Λ agree except on $\partial\Lambda$, so that the difference kernel contributes only along the boundary $\partial\Lambda$. Scaling the length scale σ back to one shows that this term is $O(\sigma|\partial\Lambda|)$. The second term is integrated with one variable in and one variable outside of Λ. Again scaling the length back to one we obtain a contribution $O(\sigma|\partial\Lambda|)$. Thus

$$\int_\Lambda P(\sigma f(x)) dx \leqq O(\sigma|\partial\Lambda|) \leqq O(|\Lambda|) ,$$

where the last inequality follows by our restriction on the volumes.

The constant associated with the translation of the Gaussian measure $d\varphi_{\sigma^{-1},\Lambda}$ is

$$\tfrac{1}{2}\sigma^2 \langle f, (-\Delta_\Lambda + \sigma^{-2}) f \rangle = \tfrac{1}{2} \int_\Lambda f(x) dx = O(|\Lambda|) ,$$

and the proof for $\mu = 0$ is complete. For $\mu > 0$, we use the Griffiths inequality

$$\frac{d}{d\mu} \int e^{-[V(\Lambda) - \mu\varphi(\Lambda)]} d\varphi \geqq 0$$

to complete the proof.

The lower bound on the vacuum energy concerns the perturbed Euclidean measure

$$\int e^{-[V(\Lambda) + Q(\xi, \Lambda)]} d\varphi_{\sigma^{-1},\Lambda} . \tag{2.2}$$

This bound, which is slowly divergent as $\sigma \to \infty$ (see Theorem 1.3), is sufficient for the present paper. Bounds uniform in σ may be found in [7]. By Schwarz' inequality, it is sufficient to establish separately the bounds

$$\int e^{4Q_\nu} d\varphi_{\sigma^{-1},\Lambda} \leqq e^{O(1)|\Lambda|} , \qquad \nu = 2, 3 , \tag{2.3}$$

$$\int e^{-2V(\Lambda)} e^{4Q_\nu} d\varphi_{\sigma^{-1},\Lambda} \leqq e^{O(1) ln^2 \sigma |\Lambda|} , \qquad \nu = 1, 4 . \tag{2.4}$$

Proposition 2.2. *There is a constant independent of* σ *and* Λ *such that* (2.3) *and* (2.4) *hold.*

Proof. The main point is that the low momentum modes of Q_2 and Q_3 are suppressed, thereby introducing an effective mass into the covariance. In technical terms, this idea emerges as the following fact: The Laplace operator of a bounded region, with Neumann boundary data, has $\lambda_1 = 0$ as the lowest eigenvalue, and the next eigenvalue λ_2 is strictly positive. The eigenspace corresponding to $\lambda_1 = 0$ is the space of functions which are constant on each connected component of the bounded region.

We write Q_3 (and $\xi^{(3)}$) as a sum of four terms so that in a single term, each lattice square Δ occurs only once in the sum over nearest neighbor pairs. By Hölders inequality, it is sufficient to consider a single such term. Let $\mathcal{N} = \{(\Delta, \Delta')\}$ be the corresponding set of nearest neighbor pairs of lattice squares, and let $\Delta_{\mathcal{N}}$ be the Laplace operator with Neumann data on

$$\bigcup_{(\Delta,\Delta')\in\mathcal{N}} \partial(\Delta\cup\Delta').$$

Let $d\varphi_{\mathcal{N}}$ be the Gaussian measure with covariance $(-\Delta_{\mathcal{N}} + \sigma^{-2})^{-1}$. By conditioning [10] and factorization of $d\varphi_{\mathcal{N}}$ across boundary lines, it is sufficient to consider a single pair of lattice squares. The required bound reduces to

$$\int e^{16\,\mathrm{Re}\,\xi(\varphi(\Delta)-\varphi(\Delta'))} d\varphi_{\mathcal{N}} \leqq \exp[\tfrac{1}{2}(16\,\mathrm{Re}\,\xi)^2 \langle \chi, (-\Delta_{\mathcal{N}} + \sigma^{-2})^{-1}\chi\rangle] = O(1),$$

where $\chi = \chi_\Delta - \chi_{\Delta'}$ is the difference of the characteristic functions of Δ and Δ'. To establish the bound $O(1)$, independent of σ, we use the fact that χ is perpendicular to the functions which are constant on $\Delta\cup\Delta'$, and so the lowest eigenvalue $\lambda_1 = 0$ of $\Delta_{\mathcal{N}}$ does not contribute.

For the case $\nu = 2$, it suffices to estimate

$$\int e^{4\xi :\varphi^2(\Delta) - \varphi(\Delta)^2:/\log\sigma} d\varphi_{\mathcal{N}},$$

where the Gaussian measure $d\varphi_{\mathcal{N}}$ has a covariance corresponding to Neumann boundary conditions on the boundary $\partial\Delta$ of the lattice square Δ. Let P_Δ be the orthogonal projection onto the subspace of L_2 spanned by χ_Δ and let

$$A = (\log\sigma)^{-1}(-\Delta_{\mathcal{N}} + \sigma^{-2})^{-1}(\chi_\Delta - P_\Delta).$$

Then by a standard computation (see also [7])

$$\int e^{4\xi:(\varphi^2(\Delta)-\varphi(\Delta)^2):/\log\sigma} d\varphi_{\mathcal{N}} = |e^{-\frac{1}{2}\,\mathrm{tr}\ln(I-A)} e^{-\frac{1}{2}\,\mathrm{tr}\,A}| \leqq e^{\frac{1}{2}\|A\|_2^2},$$

where $\|\cdot\|_2^2$ is the Hilbert-Schmidt norm and we use $0 \leqq A < I$, true for $\sigma \gg 1$. In fact, since $\chi_\Delta - P_\Delta$ annihilates the eigenspace of $\Delta_{\mathcal{N}}$ corresponding to the eigenvalue $\lambda_1 = 0$, we see that for large σ

$$\|A\|_2^2 \leqq \left(\textstyle\sum_{i=2}^\infty \lambda_i^{-2}\right)/\ln^2\sigma < \tfrac{1}{2}.$$

This completes the proof.

For the case $\nu = 1$, we note that when $|\xi| \leqq 1$,

$$-O(1)\sigma^2(1+\ln^2\kappa) \leqq :(\varphi_\kappa^2(x) - \sigma^2)^2: + \xi\sigma :\varphi_\kappa^2(x) - \sigma^2:, \tag{2.5}$$

with Wick ordering with respect to a unit mass. By standard methods the vacuum energy per unit volume corresponding to the polynomial on the right is $O(\sigma^2 \ln^2 \sigma)$, see [9]. The proof of (2.4) for $\mu = 0$ then reduces to scaling by σ^{-2}. We introduce the external field $\mu\varphi$ directly into (2.5). Again, for μ sufficiently small, we obtain the lower bound (2.5) by removing Wick ordering, and this completes the proof of (2.4). We omit the discussion of $\nu = 4$.

3. Local Perturbations of the Interaction

In this section we extend the φ^j bounds to localized perturbations of the type $Q(\xi, X)$, and thereby prove Theorem 1.3. We make two modifications to the basic φ^j bound argument [3, 8, 2, 5]. The first is that we consider perturbations of the measure such as

$$\varphi(\Delta)^2 - \int_\Delta :\varphi(x)^2: dx\,,$$

which, while localized in a bounded region Δ, cannot be expressed as a local perturbation of the Hamiltonian. Secondly, by using Euclidean invariance of the physical measure ($\Lambda = R^2$), we show that it suffices to consider the case where X is a large rectangle. In the case $X =$ rectangle, we work in a finite volume and obtain bounds uniform in Λ. This allows the passage to the limit $\Lambda \nearrow R^2$. The φ bounds have the form

$$\pm \varphi(f) \leqq H + O(1)|\mathit{supp}\, f|$$

rather than the cruder estimate

$$\pm \varphi(f) \leqq H + O(1)|\mathrm{diam}\; \mathit{supp}\, f|\,.$$

We do not require estimates on β_∞ [cf. 8, 2] which measures surface effects on the vacuum energy due to removing or adding parts of the interaction associated with a bounded region. (We remark that estimates on β_∞ are poor because of the small mass σ^{-1} in the covariance.) Dirichlet data provide a further complication to such estimates.

Let

$$H_l(\sigma) = H_{0,l}(\sigma) + \sigma^{-2} \int_{-l}^{l} :(\varphi(x)^2 - \sigma^2)^2: dx \tag{3.1}$$

be the Hamiltonian with Dirichlet boundary conditions at $x = \pm l$. Here $H_{0,l}(\sigma)$ is the free Hamiltonian with the same boundary conditions and mass σ^{-1}. Then $H_{0,l}(\sigma)$ and $H_l(\sigma)$ act on the time zero Fock space

$$\mathscr{F} = L^2(\mathscr{S}'(R^1), d\mu_{0,l})\,. \tag{3.2}$$

Here $d\mu_{0,l}$ is the Gaussian measure defined by the condition that the vacuum $\Omega_{0,l}$ for $H_{0,l}$ is represented by the function 1. Let Ω_l be the vacuum for $H_l(\sigma)$ and let $E_l(\sigma)$ be the vacuum energy, so that

$$(H_l(\sigma) - E_l(\sigma))\Omega_l = 0\,.$$

Recall that $H_l(\sigma)$ is a direct sum of a free Hamiltonian associated with the region $|x| \geqq l$ and an operator with compact resolvent associated with the region $|x| \leqq l$, so that the existence of Ω_l follows.

Lemma 3.1. *There is a constant independent of σ and a sequence $l_i \to \infty$ (possibility dependent on σ) such that*

$$-E_{l_i-a}(\sigma) \leqq -E_{l_i}(\sigma) + \text{const}\, a\,, \tag{3.3}$$

for any $a \geqq 0$.

Proof. By Proposition 2.1, there is a constant independent of σ and l such that

$$l \leqq e(l) \equiv -E_l + \text{const}\, l \tag{3.4}$$

for l sufficiently large. Choose l_i to be a one-sided maximum of e, so that for positive $l \leqq l_i$,

$$e(l) \leqq e(l_i)\,.$$

Since $e(l) \to \infty$ as $l \to \infty$, there is a sequence of such l_i, $l_i \to \infty$. This completes the proof.

The Euclidean Fock space is

$$\mathscr{K} = \mathscr{K}(-l, l) = L^2(\mathscr{S}'(R^2), d\varphi_{\sigma^{-1}, \Lambda_l})\,,$$

where $\Lambda_l = R \times [-l, l]$. For each time t there is a natural imbedding $J_t : \mathscr{F} \to \mathscr{K}$ from $\mathscr{F}$ to the time t subspace of $\mathscr{K}$. We also introduce spacetime rectangles

$$Y_{T,l} = [-T, 0] \times [-l, l]$$
$$X_{T,l} = [-T, T] \times [-l, l]\,.$$

For each $a \in (-l, l)$, we also introduce the Fock spaces

$$\mathscr{F}(a, l) = L_2(\mathscr{S}'(R^1), d\mu_{a,l})$$

and

$$\mathscr{K}(-l+2a, l),\ a>0$$
$$\mathscr{K}(-l, l-2a),\ a>0\,.$$

The space $\mathscr{K}(-l+2a, l)$ is determined by placing zero Dirichlet data on the line $x=l$ and also on the symmetric line $x=-l+2a$ obtained by reflecting the line $x=l$ about the line $x=a$, for $a>0$. The definition of $\mathscr{K}(-l, l-2a)$, $a>0$, is similar. The measure $d\mu_{a,l}$ is Gaussian, and its covariance is given explicitly in [5]. It is characterized by the following property: Let I_a be the natural injection from polynominals on $\mathscr{S}'(R^1)$ to generalized polynomials on $\mathscr{S}'(R^2)$ defined by the formula

$$I_a\phi(f) = \phi(f \otimes \delta_a)\,.$$

Here $f \in \mathscr{S}(R^1)$ and ϕ on the left above is a field (linear function) defined on $\mathscr{S}'(R^1)$, while ϕ on the right is a field on $\mathscr{S}'(R^2)$. We extend I_a to polynomials,

$$I_a(Q(\phi(f_1), \dots, \phi(f_n)) = Q(I_a(\phi(f_1)), \dots I_a(\phi(f_n)))\,,$$

so that I_a can be regarded as an identification map. Then $d\mu_{a,l}$ is characterized by the fact that I_a is isometric from $\mathscr{F}(a, l)$ into $\mathscr{K}(-l+2a, l)$ $(a>0)$ or $\mathscr{K}(-l, l+2a)$ $(a<0)$. We use the fact that conditional expectation onto the line $x=a$, $a>0$, of a function supported in the region $a \leqq x \leqq l$, is independent of the location of Dirichlet boundary data in the region $x<a$. In particular these conditional expectations, defined in $\mathscr{H}(-l+2a, l)$ and in $\mathscr{H}(-l, l)$ coincide, see [5]. In order to give this statement a meaning, we identify the $x=a$ functions of the fields in these two spaces. However, this identification is not a unitary map between the Hilbert spaces.

Proposition 3.2. *Let $l=l_i$ be as in Lemma 3.1. There is a constant independent of ξ, a, σ, and i such that for $a \geqq O(1)\sigma$ and for l sufficiently large,*

$$\|J_{-2}^* e^{-V(Y_{2,l})} e^{+Q(\xi, Y_{2,a})} J_0\|_{\mathscr{F}}^2 e^{E_l} \leqq e^{\text{const}\, a(\ln\sigma)^2}.$$

Proof. Let

$$A = J_{-2}^* e^{-V(Y_{2,l})} e^{-Q(\xi, Y_{2,a})} J_0,$$

and let $Q^{\sim}(\xi, X_{2,a})$ be the sum of $Q(\xi, Y_{2,a})$ and its time reflextion about the x-axis. Then

$$AA^* = J_{-2}^* e^{-V(X_{2,l})} e^{-Q^{\sim}(\xi, X_{2,a})} J_2.$$

Since AA^* is self adjoint and positivity preserving, there is a unit vector $v \in L_\infty(\mathscr{S}'(R^1), d\mu_{0,l})$ such that

$$\begin{aligned}
\|A\|^2 = \|AA^*\| &= \lim_{N\to\infty} \langle v, (AA^*)^N v\rangle_{\mathscr{F}}^{1/N} \\
&\leqq \lim_{N\to\infty} \|v\|_{L_\infty}^{2/N} \langle \Omega_{0,l}, (AA^*)^N \Omega_{0,l}\rangle_{\mathscr{F}}^{1/N} \\
&= \lim_{N\to\infty} \langle \Omega_{0,l}, (AA^*)^N \Omega_{0,l}\rangle_{\mathscr{F}}^{1/N}.
\end{aligned}$$

The inner product can be written in Euclidean form, yielding

$$\begin{aligned}
\|A\|^2 &\leqq \lim_{N\to\infty} \left(\int e^{-V(X_{2N,l})} e^{Q^{\sim}(\xi, X_{2N,a})} d\varphi_{\sigma^{-1}, A_l}\right)^{1/N} \\
&\leqq \lim_{N\to\infty} \left(\int e^{-V(X_{2N,l-a})} d\varphi_{\sigma^{-1}, A_{l-a}}\right)^{1/N} \\
&\quad \cdot \|I_{-a}^* e^{-V(X_{2N,a})} e^{Q^{\sim}(\xi, X_{2N,a})} I_a\|^{1/N} \\
&\leqq e^{-E_{l-a}} \lim_{N\to\infty} \|\ldots\|^{1/N} \\
&\leqq e^{O(a)-E_l} \lim_{N\to\infty} \|\ldots\|^{1/N},
\end{aligned} \tag{3.5}$$

where we use Lemma 3.1. Here the norm $\|\ldots\|$ is the operator norm from $\mathscr{F}(a, l)$ to $\mathscr{F}(-a, l)$. The norm is estimated by an integral. For vectors $v_\pm \in \mathscr{F}(\pm a, l)$, we have by the Schwarz inequality

$$\begin{aligned}
|\langle I_{-a} v_-, e^{-V(X_{2N,a})} e^{Q^{\sim}(\xi, X_{2N,a})} I_a v_+\rangle| &\leqq \|(I_{-a}v_-)(I_a v_+)\|_{L_2(d\varphi_{\sigma^{-1}, A_l})} \\
&\quad \cdot \|e^{-V(X_{2N,a})} e^{Q^{\sim}(\xi, X_{2N,a})}\|_{L_2(d\varphi_{\sigma^{-1}, A_l})}.
\end{aligned}$$

By Proposition 2.2, the second factor is bounded by $\exp[\text{const}(\ln\sigma)^2 Na]$. We bound the first factor as in [5]. For a $\geqq O(1)\sigma$ and l large, we have the hypercontractive bound

$$\int |(I_{-a}v_-)(I_a v_+)|^2 d\varphi_{\sigma^{-1}, A_l} \leqq \prod_{+,-} \|I_{\pm a} v_\pm\|_{L_2(d\varphi_{\sigma^{-1}, A_l})}^2$$

Thus the norm $\|\ldots\|$ in (3.5) satisfies

$$\|\ldots\|^{1/N} \leqq \exp[O(a)(\ln\sigma)^2]$$

and using (3.5), the proof is complete.

Theorem 3.3. *The limit* $\Lambda \nearrow R^2$ *in* (1.4) *exists for*

$$A = e^{zQ(\xi,X)} \quad \text{and} \quad A = \prod_{\Delta_j} \chi_\pm(\Delta_j)\,.$$

Proof. We prove convergence for $\exp(zQ)$ in two steps. First in the ultraviolet cutoff case $Q = Q_\kappa$, the existence of the limit $\Lambda \nearrow R^2$ for $A = Q_\kappa^n$ exists by Griffiths' inequality. (Because of the use of full Dirichlet data, this argument is special to φ^4.) We expand $\exp(zQ_\kappa)$ in a power series in z. Using term by term convergence, and a φ^j bound to give a uniform bound on the sum, we obtain convergence for $\exp(zQ_\kappa)$. We then remove the momentum cutoff by considering the difference $\exp(zQ_\kappa) - \exp(zQ_{\kappa'})$ as a power series. By Proposition 2.2 (and 3.2) we have a uniform bound which is $o(1)$ as $\kappa, \kappa' \to \infty$. Here we use the bound on Q_4, cf. [6, Chapters 2 and 5].

To prove the convergence for $A = \Pi\chi_\pm(\Delta)$ note that since $\langle \prod_j \exp(i\xi_j\varphi(\Delta))\rangle_\Lambda$ converges so does $\langle \psi(\varphi(\Delta))\rangle$ for $\tilde{\psi} \in L_1$. Using the fact that $\chi_\pm \leqq 1$ we need only establish

$$\langle |(\chi_+ - \psi_r)(\varphi(\Delta))|\rangle_\Lambda \leqq \langle (\chi_+ - \psi_r)^2(\varphi(\Delta))\rangle_\Lambda^{\frac{1}{2}} \to 0$$

uniformly in Λ for a suitable sequence $\tilde{\psi}_r \in L_1$. To deal with the singularities of $\tilde{\chi}_\pm$ it suffices to show

(a) $\dfrac{d^n}{d\xi^n}\langle e^{i\xi\varphi(\Delta)}\rangle_\Lambda$

(b) $\xi^n \langle e^{i\xi\varphi(\Delta)}\rangle_\Lambda$

are bounded uniformly in ξ and Λ. (a) is an immediate consequence of the φ bound. (b) follows from integration by parts, e.g.

$$\begin{aligned}\langle \xi e^{i\varphi(\Delta)}\rangle_\Lambda &= -i\int_\Delta dx \left\langle \frac{\delta}{\delta\varphi(x)}\, e^{i\xi\varphi(\Delta)}\right\rangle_\Lambda \\ &= -i\int_\Delta dx \langle e^{i\xi\varphi(\Delta)} \int dy(-\Delta + \sigma^{-2})^{-1}(x,y) : P'(\varphi(y)) :\rangle_\Lambda\end{aligned}$$

and Proposition 3.2.

For the remainder of this paper, we take $\Lambda = R^2$. The bounds of Proposition 3.2 extend to this case, and prove Theorem 1.3 for (sufficiently long) rectangles (cf. [4]). The problem is to allow X to be an arbitrary union of lattice squares. Let T_a denote time translation, $(t,x) \to (t+a,x)$ and let $\mathcal{R}$ denote time inversion, $(t,x) \to (-t,x)$, and let T_a and $\mathcal{R}$ act on functions of the Euclidean field $\varphi(t,x)$.

Proposition 3.4. *Let A, B, and C be functions of φ localized in the time intervals* $(-\infty, -1)$, $(-1, 0)$, *and* $(0, \infty)$ *respectively, and suppose* $B \geqq 0$. *Then*

$$|\langle ABC\rangle| \leqq \langle C\mathcal{R}C^-\rangle^{\frac{1}{2}} \langle (T_1A)\mathcal{R}(T_1A)^-\rangle^{\frac{1}{2}} \lim_{N\to\infty} \langle \prod_{i=1}^N T_{2i}(B\mathcal{R}B^-)\rangle^{1/2N}$$

Proof. We use the positivity condition of Osterwalder-Schrader (positivity of the inner product in the physical Hilbert space). The result then follows from the Schwarz inequality and the density of L_∞ in L_2.

Proof of Theorem 1.3. Let $i=(i_0, i_1)\in Z^2$. Given ξ, let $\xi(i_0, N)$ and $\xi(i, N)$ be multiindices defined by

$$\xi(i_0, N)_j = \begin{cases} \xi_i & \text{if } \; i_1 = j_1 \;\text{ and }\; |i_0 - j_0| \leqq N \\ 0 & \text{otherwise} \end{cases}$$

$$\xi(i, N)_j = \begin{cases} \xi_i & \text{if } \; |i_0 - j_0| \leqq N \;\text{ and }\; |i_1 - j_1| \leqq N \\ 0 & \text{otherwise}. \end{cases}$$

Consider the case $Q_3=0$. By repeated use of Proposition 3.4 in the x_0 direction and again after Euclidean rotation in the x_1-direction we conclude that when X_N is an $N \times N$ rectangle

$$\begin{aligned} \langle e^{Q(\xi, X_N)} \rangle &\leqq \prod\nolimits_{i_0} \lim_{N\to\infty} \langle e^{Q(\xi(i_0, N))} \rangle^{1/N} \\ &\leqq \prod\nolimits_{i\in X} \lim_{N\to\infty} \langle e^{Q(\xi(i, N))} \rangle^{1/N^2}. \end{aligned}$$

By Theorem 3.3 with $\Lambda = \Lambda_l = [-l, l] \times R$ we have

$$\langle e^{Q(\xi(i, N))} \rangle = \lim_{l\to\infty} \langle e^{Q(\xi(i, N))} \rangle_{\Lambda_l}.$$

To express the last term in Fock space let

$$A = J_{-2}^* e^{-V(Y_{2,l})} e^{-Q(\xi(i, N), Y_{2,N})} J_0.$$

Then

$$\begin{aligned} \langle e^{Q(\xi(i, N))} \rangle_{\Lambda_l} &= \langle \Omega_l A^{N/2} \Omega_l \rangle_{\mathscr{F}} \\ &\leqq \|A\|^{N/2} \\ &\leqq e^{\mathrm{const}\, N^2 (\log\sigma)^2}. \end{aligned}$$

Here we have applied Proposition 3.2 with $a=N$. Hence combining these estimates we have

$$\langle e^{Q(\xi, X)} \rangle \leqq \prod\nolimits_{i\in Z^2 \cap X} e^{K_1 (\log\sigma)^2} = e^{K_1 (\log\sigma)^2 |X|}.$$

In the case of Q_3 we apply the same argument with appropriate reflections, as in Proposition 3.2.

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