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A Remark to Harris's Theorem on Percolation

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Abstract. Harris's theorem on percolation is generalized to a dependent case by his own method.

Let T be the set of bonds in the plane square lattice Z^2. We adopt the following

Notations.

$\Omega = \{0, 1\}^T$; the set of configurations of 0 and 1 in T.
$X_t(\omega) = \omega(t)$ for $t \in T$ and $\omega \in \Omega$.
$\boldsymbol{X} = \{X_t; t \in T\}$.
$\boldsymbol{X}^{-1}(i) \equiv \boldsymbol{X}^{-1}(i, \omega) = \{t \in T; X_t(\omega) = i\}$ for $i = 0, 1$.
P is a probability measure on Ω.

Harris [3] proved following

Theorem. *If a random field $(\Omega, P; \boldsymbol{X})$ is independent at each $t \in T$, and if $P(X_t = 0) = P(X_t = 1) = 1/2$ for every t, then neither $\boldsymbol{X}^{-1}(0)$ nor $\boldsymbol{X}^{-1}(1)$ has infinite connected components a.s.*

His method can be applied to generalize the above to a dependent case. For $V \subset T$, let $\mathscr{B}_V$ be the σ-algebra generated by $\{X_t; t \in V\}$, and let $\mathscr{B}_\infty = \bigcap_V \mathscr{B}_{V^c}$, where V runs over the set of all finite subsets of T. Let ∂V be the set of bonds which meet bonds in V at right angles.

Theorem. *We assume that a random field $(\Omega, P; \boldsymbol{X})$ satisfies the following conditions;*

(1) *(Spatial symmetry) P is invariant under shift, rotation by right angles and reflection in the axis in T.*

(2) *(Symmetry of configurations) P is invariant under interchange of* 0 *and* 1.

(3) *P is everywhere dense.*

(4) *$\mathscr{B}_\infty$ is trivial, if it is measured by P.*

(5) *(The* FKG *inequality) If $f(\omega)$ and $g(\omega)$ are non-decreasing functions of $\omega \in \Omega$, then*

$$\int_\Omega f(\omega) g(\omega) P(d\omega) \geqq \int_\Omega f(\omega) P(d\omega) \cdot \int_\Omega g(\omega) P(d\omega).$$

(6) *(Markovian property) For each $A \in \mathscr{B}_V$,*

$$P(A | \mathscr{B}_{V^c}) = P(A | \mathscr{B}_{\partial V}).$$

Then, neither $\boldsymbol{X}^{-1}(0)$ nor $\boldsymbol{X}^{-1}(1)$ has infinite connected components a.s.

These conditions are satisfied by the Ising model with the nearest neighbour attraction at high temperature for a suitable value of the chemical potential. (For details, see [1, 2] and [4].)

Lemma 1. (Cf. Lemmata 5.1, 2, and 3 in [3].) *Let* R_1 *be the probability that the origin* $(0,0)$ *belongs to an infinite connected component of* $\boldsymbol{X}^{-1}(1)\cap\{x\geqq 0, y\geqq 0\}$. *Then,* $R_1=0$.

Proof. Suppose $R_1>0$. For a positive integer j, let E_j be the event that a point $(0,j)$ belongs to an infinite connected component of $\boldsymbol{X}^{-1}(1)\cap\{x\geqq 0, y\leqq j\}$. By spatial symmetry (1), we have $P(E_j)=R_1>0$. The point-wise ergodic theorem of Birkhoff combined with (4) assures that frequency of occurence of events $E_1, E_2, \dots$ is equal to R_1 a.s. Therefore, infinitely many E_j's occur a.s.

Assume that the event E_j occurs. For a positive integer k, let B_k be the event that $X_t=0$ for each bond t in the set $\{k\leqq x\leqq k+1, 0\leqq y\leqq j\}$. Since $P(B_k)>0$ by (3), Birkhoff's ergodic theorem again implies that infinitely many B_k's occur a.s. A connected component of $\boldsymbol{X}^{-1}(1)\cap\{x\geqq 0, 0\leqq y\leqq j\}$ containing $(0,j)$ is blocked by barricades B_k, so that the infinite connected component appearing in the event E_j crosses the x-axis. a.s. Thus we a.s. have infinitely many chains of 1-bonds which connect points on the x-axis to points on the y-axis.

Denote by T^* the lattice dual to T. Let $X^*_{t^*}(\omega)=X_t(\omega)$ for a bond t^* in T^* which crosses a bond t in T. A random field $\boldsymbol{X}^*=\{X^*_{t^*}; t^*\in T^*\}$ is homomorphic to $\boldsymbol{X}$.

Therefore, we a.s. have infinitely many chains of bonds in $\boldsymbol{X}^{*-1}(1)\cap\{x\geqq -\frac{1}{2}, y\geqq -\frac{1}{2}\}$ which connect points on an axis $\{y=-\frac{1}{2}\}$ to points on an axis $\{x=-\frac{1}{2}\}$. By symmetry of configurations (2), we a.s. have infinitely many chains of bonds in $\boldsymbol{X}^{*-1}(0)\cap\{x\geqq -\frac{1}{2}, y\geqq -\frac{1}{2}\}$ with the same property as above. Since chains in $\boldsymbol{X}^{-1}(1)$ and those in $\boldsymbol{X}^{*-1}(0)$ can not intersect with each other, there exist chains in $\boldsymbol{X}^{*-1}(0)\cap\{x\geqq -\frac{1}{2}, y\geqq -\frac{1}{2}\}$ which connect points on $\{y=-\frac{1}{2}\}$ to points on $\{x=-\frac{1}{2}\}$ and do not pass a point $(-\frac{1}{2}, -\frac{1}{2})$.

The infinite connected component of $\boldsymbol{X}^{-1}(1)\cap\{x\geqq 0, y\geqq 0\}$ containing the origin necessarily crosses the above mentioned chains of bonds in $\boldsymbol{X}^{*-1}(0)\cap\{x\geqq -\frac{1}{2}, y\geqq -\frac{1}{2}\}$, which is absurd. Thus, we have $R_1=0$.

Lemma 2. (Cf. Lemma 6.1 in [3].) *Let* R_2 *be the probability that the origin belongs to an infinite connected component of* $\boldsymbol{X}^{-1}(1)\cap\{x\geqq 0\}$. *Then,* $R_2=0$.

Proof. Suppose $R_2>0$. For a positive integer j, let W_j be the event that a point $(j,0)$ belongs to an infinite connected component of $\boldsymbol{X}^{-1}(1)\cap\{y\geqq 0\}$, a point $(j,-1)$ belongs to an infinite connected component of $\boldsymbol{X}^{-1}(1)\cap\{y\leqq -1\}$ and $X_{t_j}=1$ where t_j is a bond connecting $(j,0)$ to $(j,-1)$. By the FKG inequality and symmetry (1) and (2), we have

$$P(W_j)\geqq R_2^2\cdot P(X_{t_j}=1)=R_2^2/2>0\,.$$

Therefore, Birkhoff's ergodic theorem implies that infinitely many W_j's occur a.s.

Assume that the event W_j occurs. The infinite connected components of 1-bonds which appears in W_j necessarily cross the y-axis both above and strictly below the origin a.s. by Lemma 1. We a.s. have chains of bonds in $\boldsymbol{X}^{-1}(1)\cap\{x\geqq 0\}$ connecting a point on the y-axis above the origin to a point on the y-axis strictly below the origin. Therefore by homomorphism between X and X^* and by symmetry (2), we have chains of bonds in $X^{*-1}(0)\cap\{x\geqq-\frac{1}{2}\}$ connecting a point on $\{x=-\frac{1}{2}\}$ above the x-axis to a point on $\{x=-\frac{1}{2}\}$ strictly below the x-axis.

The infinite connected component of $X^{-1}(1)\cap\{x\geqq 0\}$ containing the origin necessarily crosses the above chains of bonds in $X^{*-1}(0)\cap\{x\geqq-\frac{1}{2}\}$, which is absurd. Thus, we have $R_2=0$.

A set $V=\{|x|\leqq n, |y|\leqq n\}$ is called a *box*. The integer n in the definition of V is denoted by $|V|$. A chain of bonds which starts at a point on the y-axis, ends at a point on the y-axis strictly below the starting point and no other bond of which crosses the y-axis is called a *half-circuit*.

Lemma 3. (Cf. Lemma 7.1 in [3].) *For any box V, there exists a box V' such that with probability $>1/2$ there exists a half-circuit lying in $X^{-1}(1)\cap\{x\geqq 0\}\cap(V'\setminus V)$, starting above the box V and ending below it.*

Proof. A connected component of $X^{*-1}(0)\cap\{x\geqq\frac{1}{2}\}$ containing $(1/2, j+1/2)$ is finite a.s. by Lemma 2, so that the point $(1/2, j+1/2)$ is surrounded a.s. by half-circuits lying in $X^{-1}(1)\cap\{x\geqq 0\}$. The union of all the half-circuits in $X^{-1}(1)\cap\{x\geqq 0\}$ surrounding $(1/2, j+1/2)$ is divided into connected components $C_1^j, C_2^j, \ldots$, each of which is finite a.s. by Lemma 2. The event that there exist infinitely many components $C_1^j, C_2^j, \ldots$ belongs to $\mathscr{B}_\infty$, whose probability is 0 or 1 by (4).

Suppose it is 0, i.e., there exist only finitely many components a.s. Then there exists a maximal component C_∞^j. For any j and j', we have $C_\infty^j=C_\infty^{j'}$ or $C_\infty^j\cap C_\infty^{j'}=\phi$. Along the boundary of $\bigcup_{j=-\infty}^{\infty} C_\infty^j$, there extends an infinite chain of bonds in $X^{*-1}(0)\cap\{x\geqq-\frac{1}{2}\}$, which is impossible by Lemma 2. Therefore, there a.s. exist infinitely many components $C_1^0, C_2^0, \ldots$ surrounding the origin, all except finitely many of which are outside of the box V. Taking a box V' large enough, we get the result.

Let R be the probability that the origin belongs to an infinite connected component of $X^{-1}(1)$.

Lemma 4. (Cf. Lemma 7.2 in [3].) *Suppose $R>0$. For any box V and sufficiently large i, there exists a box V'' with $|V''|>i>|V|$ such that with probability $>R^2/2^5$ there exists a chain in $X^{-1}(1)\cap(V''\setminus V)$ connecting $(0,i)$ to $(0,-i)$.*

Proof. Let A be the event that $X_t=0$ for all $t\in V$, let H_i be the event that $(0,i)$ belongs to an infinite connected component of $X^{-1}(1)$ and let C_i be the event that $(0,i)$ belongs to an infinite connected component of $X^{-1}(1)\cap V^c$. Noting that A is decreasing, i.e., $\omega\leqq\omega'\in A$ implies $\omega\in A$, we have by (5)

$$P(H_i\cap A)=P(C_i\cap A)\leqq P(C_i)P(A),$$

i.e. $P(H_i|A)\leqq P(C_i)$. The left-hand side of the inequality converges to $P(H_0)=R$, as $i\to\infty$. For sufficiently large i, we have $R/2\leqq P(C_i)$.

Take any box V' such that $|V'|>i$. By Lemma 3, there exists a box V'' such that with probability $>1/2$ there exists a half-circuit lying in $X^{-1}(1)\cap\{x\geqq 0\}\cap(V''\setminus V')$, starting above V' and ending below V'. Let $C=C(\omega)$ be the maximal one of those half-circuits.

Let S be a half-circuit in $\{x\geqq 0\}\cap(V''\setminus V')$ from above to below V', let S_1 be the union of S and its reflection in the y-axis and let S_1^{int} be the interior of the circuit S_1. Let $D(i, S)$ be the event that there exists a chain of bonds in $X^{-1}(1)\cap(S_1^{\text{int}}\setminus V)$ connecting $(0, i)$ to a bond in S. From symmetry (1), we have, $P(D(i, S))\geqq P(C_i)/2\geqq R/2^2$ and $P(D(-i, S))\geqq R/2^2$.

Let E_S be the event that S is the maximal half-circuit $C(\omega)$, and let E'_S be the event that $X_t=1$ for all t on the circuit S_1 and outside of S_1. Then, we have

$$\begin{aligned}&P(D(i, S)\cap D(-i, S)|E_S)=P(D(i, S)\cap D(-i, S)|E'_S)\\&\geqq P(D(i, S)\cap D(-i, S))\geqq P(D(i, S))\cdot P(D(-i, S))\geqq R^2/2^4\,.\end{aligned}$$

The equality above is valid by the Markovian property (6) and the first two inequalities are those of FKG. Thus, we have

$$P(D(i, S)\cap D(-i, S))\cap E_S)\geqq P(E_S)\cdot R^2/2^4\,,$$

from which follows the desired result, since $\sum_S P(E_S)>1/2$.

Corollary 1. *In Lemma 4, with probability* $>R^2/2^6$ *we can take the chain clockwise.*

Proof is obvious from spatial symmetry (1).

Corollary 2. (Cf. Lemma 8.1 in [3].) *Let* $F_{V'',V}$ *be the event that there exists a circuit in* $X^{-1}(1)\cap(V''\setminus V)$ *surrounding the origin. For any box* V, *there exists a box* V'' *such that* $P(F_{V'',V})\geqq R^4/2^{12}$.

Proof. By Corollary 1, spatial symmetry (1) and the FKG inequality (5), for a box V and sufficiently large i, there exists a box V'' with $|V''|>i>|V|$ such that with probability $>(R^2/2^6)^2$ there exist both clockwise chains and counterclockwise ones in $X^{-1}(1)\cap(V''\setminus V)$ connecting $(0, i)$ to $(0, -i)$.

Lemma 5. *Suppose* $R>0$. *Then, there exists a circuit in* $X^{-1}(1)$ *around the origin a.s.*

Proof. Let G be the event above. By Corollary 2 to Lemma 4, there exists a sequence $V_1\subset V_2\subset\ldots$ of boxes such that $P(F_{V_{k+1},V_k})\geqq R^4/2^{12}$. Let $F_k=F_{V_{k+1},V_k}$. Take arbitrary $\varepsilon>0$. By (4), there exists a sub-sequence $k_1<k_2<\ldots$ such that

$$|P(F^c_{k_n}\cap F^c_{k_{n-1}}\cap\ldots\cap F^c_{k_1})-P(F^c_{k_n})P(F^c_{k_{n-1}}\cap\ldots\cap F^c_{k_1})|<\varepsilon\,.$$

Noting that $G^c\subset\bigcap_{n=1}^N F^c_{k_n}$ for arbitrary N, we have

$$\begin{aligned}P(G^c)&\leqq P(\textstyle\bigcap_{n=1}^N F^c_{k_n})\\&\leqq P(F^c_{k_N})P(\textstyle\bigcap_{n=1}^{N-1}F^c_{k_n})+|P(F^c_{k_N})P(\bigcap_{n=1}^{N-1}F^c_{k_n})-P(\bigcap_{n=1}^N F^c_{R_n})|\\&\leqq(1-R^4/2^{12})P(\textstyle\bigcap_{n=1}^{N-1}F^c_{k_n})+\varepsilon\,.\end{aligned}$$

Repeating this procedure, we have

$$P(G^c)\leqq(1-R^4/2^{12})^{N-1}+\varepsilon/(R^4/2^{12})\,.$$

Letting $N\to\infty$ and $\varepsilon\to 0$, we have $P(G^c)=0$.

Proof of Theorem. Suppose $R>0$. Then, by homomorphism between $\boldsymbol{X}$ and $\boldsymbol{X}^*$ and symmetry of configurations (2), there exists a circuit of 0-bonds in $\boldsymbol{X}^*$ around $(-\frac{1}{2}, -\frac{1}{2})$ a.s.

An infinite connected component of $X^{-1}(1)$ containing the origin crosses the above circuit in $\boldsymbol{X}^*$, which is impossible.

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Note Added in Proof. Prof. G. Gallavotti kindly informed me that six conditions in our Theorem *characterize* the Ising model which has the nearest neighbour attraction, the zero external field and $T \geqq T_c$.