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Renormalization of the Abelian Higgs-Kibble Model

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Abstract. This article is devoted to the perturbative renormalization of the abelian Higgs-Kibble model, within the class of renormalizable gauges which are odd under charge conjugation. The Bogoliubov Parasiuk Hepp-Zimmermann renormalization scheme is used throughout, including the renormalized action principle proved by Lowenstein and Lam. The whole study is based on the fulfillment to all orders of perturbation theory of the Slavnov identities which express the invariance of the Lagrangian under a supergauge type family of non-linear transformations involving the Faddeev-Popov ghosts. Direct combinatorial proofs are given of the gauge independence and unitarity of the physical S operator. Their simplicity relies both on a systematic use of the Slavnov identities as well as suitable normalization conditions which allow to perform all mass renormalizations, including those pertaining to the ghosts, so that the theory can be given a setting within a fixed Fock space. Some simple gauge independent local operators are constructed.

Introduction

The latest achievements on the renormalization of Lagrangian models involving gauge fields, mostly due to t'Hooft, Lee, Veltman, Zinn-Justin [1], were primarily based on the use of a gauge invariant regularization procedure, the most popular of which being the so called dimensional regularization [2]. The gauge structure could thus conveniently be respected by fulfilling the so called Slavnov identities [3] through the renormalization procedure. There resulted finite Green's functions which could not however be directly given an interpretation relevant to an operator theory in some Fock space, were it be in a perturbative sense, because of the lack of the finite mass renormalizations which would have been necessary for this purpose. As will be seen here, an operator interpretation is quite convenient for any discussion involving asymptotic concepts concerning e.g. the unitarity of the S operator, the construction of gauge invariant local operators etc.

We shall treat here the simplest model involving gauge fields in which no infrared problem occurs, namely the abelian Higgs-Kibble model [4] within the class of gauges advocated by t'Hooft. The algebraic complications which occur in the non abelian cases are deferred to later publications.

We shall make full use of the combinatiorial knowledge or renormalized perturbation theory that has been acquired through the work of Zimmermann [5] (effective Lagrangians, normal products, Wilson expansions), Lowenstein [6] and

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Lam [7] (renormalized action principle), which has been successfully applied in other cases (massive quantum electrodynamics [8], σ models [9] abelian Higgs-Kibble model in the Stueckelberg gauge [10]).

This well developed machinery, which relies on the locality and power counting properties of perturbation theory, is most effectively put to work by intensive use of the implicit function theorem for formal power series [11] through which, as we shall see, most symmetry aspects of the perturbation series can be read off on the classical Lagrangian on which the theory is based, including the possible occurrence of anomalies. This possibly surprising statement will be widely illustrated in the present work and in reviews now in preparation [12].

The main reason why such a favourable situation prevails in the present case is that the model is almost entirely specified by an invariance property even after the introduction of the necessary Faddeev-Popov ghosts [13]. Namely, at the classical level, the Lagrangian is invariant under transformations of the super-gauge type [14], which we have called Slavnov transformations. In the abelian case treated here, one has however also to impose the full degeneracy of the ghost masses in order to implement spontaneous breaking. This is a particular feature of the abelian case which in a sense makes things more complicated.

Section I is thus devoted to a study of some crucial aspects of the tree approximation. The role of the invariance under Slavnov transformations and the particular expression of spontaneous breaking are stressed.

In Section II the model is defined to all orders of a perturbation expansion in powers of a parameter, $\hbar$, which counts the number of loops in Feynman diagrams. Namely, we show that both renormalized Slanov identities and the normalization conditions on Green's functions which hold in the tree approximation can be fulfilled to all orders. The compactness of the proofs is due to a repeated use of the implicit function theorem for formal power series [11]. The logic of the construction also makes clear how anomalies, which do not occur in the present model, can be produced.

In Section III, one proves the independence of the physical scattering operator against a change of the parameters which label the gauge function, by suitably generalizing the argument given by Lowenstein and Schroer [8] in the case of massive quantum electrodynamics.

Section IV is devoted to a direct combinatorial proof of the unitarity of the physical S operator.

Several appendices are devoted to a number of technical questions:

Appendix I deals with the structure of the Slavnov identities at the classical level in the non abelian case.

Appendix II is devoted to a brief description of the implicit function theorem for formal power series [11].

Appendices III, IV, and V give some computational details which would have obscured the line of argument in the body of the article.

Appendix VI deals with the construction of some local gauge invariant operators of dimension smaller than or equal to four.

Appendix VII extends the theory to quadratic gauges odd under charge conjugation.

I. The Tree Approximation

As is well known, a classical Lagrangian, $\mathscr{L}^{\text{cl}}(\underline{\varphi})$, which will be assumed to be of the renormalizable type, defines the tree approximation of a quantum Green's functional

$$Z(\underline{J}) = \exp \frac{i}{\hbar} Z_C(\underline{J}) \tag{1}$$

where $\underline{J}$ denotes collectively a set of sources linearly coupled to the field variables $\underline{\varphi}$ from which $\mathscr{L}^{\text{cl}}$ is constructed. The Legendre transform [15] $\Gamma(\underline{\varphi})$ of the connected Green's functional $Z_C(\underline{J})$ defined through

$$Z_C(\underline{J}) = \Gamma(\underline{\varphi}) + \int (\underline{J} \cdot \underline{\varphi})(x)\, dx \Big|_{\left(\frac{\delta \Gamma}{\delta \underline{\varphi}} + \underline{J}\right)(x) = 0} \tag{2}$$

coïncides with $\int dx\, \mathscr{L}^{\text{cl}}(\underline{\varphi})(x)$ in the lowest approximation of a perturbative expansion in powers of $\hbar$, and, in higher orders, generates "proper" Feynman graphs.

Let us now consider a classical Lagrangian

$$\begin{aligned} \mathscr{L}^{\text{cl}}(\underline{\varphi}_i \underline{J}) &\equiv \mathscr{L}^{\text{cl}}(\underline{\varphi}) + \underline{J} \cdot \underline{\varphi} \\ &= \mathscr{L}_{\text{inv}}(\underline{\varphi}) - \frac{\mathscr{G}^2}{2\alpha} + \underline{J} \cdot \underline{\varphi} \end{aligned} \tag{3}$$

where $\mathscr{L}_{\text{inv}}(\underline{\varphi})$ is invariant under local abelian gauge transformations of the second kind;

$$\delta \underline{\varphi}(x) = \int \frac{\delta \underline{\varphi}(x)}{\delta \Lambda(y)} \delta \Lambda(y)\, dy\,. \tag{4}$$

$\mathscr{G}$ is a gauge function which breaks gauge invariance, and α is a numerical parameter, as they occur for instance in quantum electrodynamics. Noether's theorem yields the following Ward identity:

$$\int dx \left[\underline{J}(x) \cdot \frac{\delta \underline{\varphi}(x)}{\delta \Lambda(y)} - \mathscr{M}(x,y) \frac{\mathscr{G}(x)}{\alpha} \right] = 0 \tag{5}$$

where the substitution

$$\underline{\varphi} = \frac{\delta Z_C(\underline{J})}{\delta \underline{J}} \tag{6}$$

has to be made, and where

$$\mathscr{M}(x,y) = \frac{\delta \mathscr{G}(x)}{\delta \Lambda(y)} \tag{7}$$

is the kernel of a field dependent differential operator of hyperbolic character whenever $\mathscr{G}$ is a perturbed version of the divergence of the gauge vector field associated with the gauge transformations under consideration. We shall from now on limit ourselves to this situation. The Ward identity (5) can conveniently be solved for $\mathscr{G}$ upon introducing scalar charged Faddeev-Popov ($\Phi\Pi$) ghost fields [13] and the corresponding sources into the initial Lagrangian

$$\begin{aligned}\mathscr{L}^{\text{cl}}(x) &= \mathscr{L}_{\text{inv}}(x) - \frac{1}{\alpha}\left(\frac{\mathscr{G}^2(x)}{2} + \int dy\, \bar{c}(y)\, \mathscr{M}(x,y)\, c(x)\right) + [\underline{J}\cdot\underline{\varphi} + \bar{\xi} c + \xi \bar{c}](x) \\ &= \mathscr{L}(\underline{\varphi}, c, \bar{c})(x) + [\underline{J}\cdot\underline{\varphi} + \bar{\xi} c + \xi\bar{c}](x)\,.\end{aligned} \tag{8}$$

The Fermi statistics conventionally assigned to these fields while preserving locality introduces new sources of indefinite metric into the quantum interpretation of such a system and, at the same time exhibits crucial properties connected with the structure of the gauge transformations, which are best observed in the non abelian case described in Appendix I. The new Ward identity reads:

$$W(x)(Z_C) = \int dy \left[\underline{J}(y)\cdot \frac{\delta\underline{\varphi}(y)}{\delta\Lambda(x)} - \frac{1}{\alpha}\mathscr{M}(y,x)\,\mathscr{G}(y) - \frac{1}{\alpha}\int dz\, \bar{c}(z)\, \frac{\delta\mathscr{M}(y,z)}{\delta\Lambda(x)}\, c(y)\right] = 0\,. \tag{9}$$

Integrating through $\bar{c}$ yields the so-called Slavnov identity (3), which, in the present, abelian, case, reads:

$$S(Z_C) = \int dx \left[\bar{\xi}(x)\,\mathscr{G}(x) + \int dy\, \underline{J}(x)\cdot\frac{\delta\underline{\varphi}(x)}{\delta\Lambda(y)}\,\bar{c}(y)\right] = 0 \tag{10}$$

where use has been made of the equations of motion for the $\Phi\Pi$ fields, and of their anticommutativity, whereby the last term in the Ward identity (9) drops out in view of the abelianness of the gauge transformations. In the non abelian case treated in Appendix I, this last term contributes however in a way which is characterized in terms of the structure constants of the Lie algebra involved.

The Slavnov identity can be interpreted as expressing the invariance of $\mathscr{L}$ under the following transformations of the supergauge type [14], which we shall call Slavnov transformations:

$$\begin{aligned}\delta_\lambda \underline{\varphi}(x) &= \lambda \int \frac{\delta\underline{\varphi}(x)}{\delta\Lambda(y)}\,\bar{c}(y)\,dy \\ \delta_\lambda c(x) &= \lambda\mathscr{G}(x) \\ \delta_\lambda \bar{c}(x) &= 0\end{aligned} \tag{11}$$

where λ is an infinitesimal, space time independent, gauge parameter of the Fermi type. The vanishing of the variation of $\bar{c}$ is due to the abelian character of the gauge transformations and is suitably altered in the non abelian case as shown in Appendix I. The Fermi character of the $\Phi\Pi$ field linearizes the gauge "group" since

$$\begin{aligned}\delta_{\lambda_1}\delta_{\lambda_2}\underline{\varphi}(x) &= 0 \\ \delta_{\lambda_1}\delta_{\lambda_2}\bar{c}(x) &= 0 \\ \delta_{\lambda_1}\delta_{\lambda_2} c(x) &= \lambda_1\lambda_2 \int \mathscr{M}(y,x)\,\bar{c}(y)\,dy\end{aligned} \tag{12}$$

so that

$$S^2(Z_C) \equiv \int dx\,dy\,\bar{\xi}(y)\,\mathscr{M}(y,x)\,\bar{c}(x) = 0 \tag{13}$$

and

$$(S\cdot S^2 - S^2\cdot S)(Z_C) = 0\,. \tag{14}$$

One should realize the lack of equivalence, in general, between the Ward identity (5) and the Slavnov identity (10): if one adds to $\mathscr{L}$ a breaking term of the form

$$-\frac{B}{\alpha} \tag{15}$$

such that

$$\frac{\delta B(x)}{\delta A(y)} = \mathscr{N}(x,y)\,\mathscr{G}(x) \tag{16}$$

where $\mathscr{M}$ is a possibly field dependent differential operator which does not upset the hyperbolic character of $\mathscr{M}$, the Lagrangian

$$\begin{aligned}\mathscr{L}_B^{\mathrm{cl}} = \mathscr{L}_{\mathrm{inv}}(x) - \frac{1}{\alpha}\left(\frac{\mathscr{G}^2(x)}{2} + B(x) + \int dy\,\bar{c}(y)\,(\mathscr{M}(x,y)+\mathscr{N}(x,y))\,c(x)\right) \\ + \underline{J}(x)\cdot\underline{\varphi}(x) + \bar{\xi}(x)\,c(x) + \xi(x)\,\bar{c}(x)\end{aligned} \tag{17}$$

will lead to the same Slavnov identity whereas the Ward identity is modified according to

$$\begin{aligned}W^{(B)}(x)\,[Z_C] = \int dy\Bigg[\underline{J}(y)\cdot\frac{\delta\underline{\varphi}(y)}{\delta A(x)} - \frac{1}{\alpha}\,(\mathscr{M}(y,x)+\mathscr{N}(y,x))\,\mathscr{G}(y) \\ - \frac{1}{\alpha}\int dz\,\bar{c}(z)\,\frac{\delta(\mathscr{M}(y,z)+\mathscr{N}(y,z))}{\delta A(x)}\,c(y)\Bigg] = 0\,.\end{aligned} \tag{18}$$

This pathological situation is due to the abelianness of the gauge transformations which insures the absence from the Slavnov identity of a contribution involving the last term of the Ward identity.

A concrete example of this phenomenon will be given in the context of the abelian Higgs-Kibble model treated within a family of linear, charge conjugation odd gauges.

The basic fields and sources are given in Table 1.

Table 1. Fields and sources

Field	Behaviour under charge conjugation	Source
φ_1	even	J_1
φ_2	odd	J_2
A_μ	odd	J_μ
c } $\varphi\Pi$ ghosts	even	$\bar{\xi}$
$\bar{c}$ } $\varphi\Pi$ ghosts	even	ξ

One may choose for the Slavnov transformation:

$$\begin{aligned}\delta\varphi_1 &= -\lambda e_1^0\varphi_2\bar{c} \\ \delta\varphi_2 &= +\lambda e_2^0(\varphi_1+v^0)\,\bar{c} \\ \delta A_\mu &= \lambda\partial_\mu\bar{c} \\ \delta c &= \lambda(a^0\,\partial_\mu A_\mu + \varrho^0\varphi_2) \\ \delta\bar{c} &= 0\end{aligned} \tag{19}$$

where v^0 is a field translation parameter e_1^0, e_2^0 are charge parameters, a^0 and ϱ^0 characterize the gauge function. The corresponding Slavnov identity reads:

$$\begin{aligned} S(Z_C) \equiv \int dx \{ & J_\mu \partial_\mu \delta_\xi Z_C - e_1^0 J_1 \delta_{J_2} Z_C \delta_\xi Z_C + e_2^0 J_2 (\delta_{J_1} Z_C \delta_\xi Z_C + v^0 \delta_\xi Z_C) \\ & - \bar{\xi} [a^0 \partial_\mu \delta_{J_\mu} Z_C + \varrho^0 \delta_{J_2} Z_C]\} (x) = 0 \,. \end{aligned} \tag{20}$$

Equation (20) can be linearized by introducing into the Lagrangian the source terms:

$$\eta_1 (z_1 \varphi_1 \bar{c} + z' \bar{c}) + \eta_2 z_2 \varphi_2 \bar{c} \tag{21}$$

where η_1, η_2 are Fermi type sources:

$$S(Z_C) \equiv \mathscr{S} Z_C = 0 \,. \tag{22}$$

Now Z_C also depends on η_1, η_2 whereas the Lagrangian is the partial Legendre transform of Z_C with respect to $J_1, J_2, J_\mu, \xi, \bar{\xi}$.

$\mathscr{S}$ is now a linear functional partial differential operator of the form:

$$\mathscr{S} = \int dx [J_\mu \partial_\mu \delta_\xi - e_1 J_1 \delta_{\eta_2} + e_2 J_2 \delta_{\eta_1} + m J_2 \delta_\xi - \bar{\xi}(a \partial_\mu \delta_{J\mu} + \varrho \delta_{J_2})] (x) \,. \tag{23}$$

The transformation law (19) is easily converted by translation and renormalization of the field variables into the more conventional one

$$\begin{aligned} \delta \varphi_1 &= -\lambda e \varphi_2 \bar{c} \\ \delta \varphi_2 &= \lambda e (\varphi_1 + v) \bar{c} \\ \delta A_\mu &= \lambda \partial_\mu \bar{c} \\ \delta c &= \lambda (\partial_\mu A_\mu + \varrho \varphi_2) \end{aligned} \tag{24}$$

($e_1^0 = e_2^0 = e$, $v = v^0$, $\varrho = \varrho^0$, $a^0 = 1$), where we keep however a field translation parameter explicit.

One may ask oneself what is the most general Lagrangian of the renormalizable type which is invariant under such a transformation, even under charge conjugation and carrying zero $\Phi\Pi$ charge.

This problem is a purely algebraic one. The most general Lagrangian of the renormalizable type which carries the vacuum quantum numbers is, up to a divergence, a linear combination of the following twenty six monomials:

0)	φ_1	10)	$\varphi_2 \partial_\mu A_\mu$	20)	$\bar{c} c$
1)	φ_1^2	11)	$A_\mu \varphi_1 \partial_\mu \varphi_2$	21)	$\bar{c} \varphi_1 c$
2)	φ_2^2	12)	$A_\mu \varphi_2 \partial_\mu \varphi_1$	22)	$\bar{c} \varphi_1^2 c$
3)	φ_1^3	13)	$A_\mu A_\mu$	23)	$\bar{c} \varphi_2^2 c$
4)	$\varphi_2^2 \varphi_1$	14)	$A_\mu A_\mu \varphi_1$	24)	$\bar{c} A_\mu A_\mu c$
5)	φ_1^4	15)	$A_\mu A_\mu \varphi_1^2$	25)	$\bar{c} \Box c$
6)	φ_2^4	16)	$A_\mu A_\mu \varphi_2^2$		
7)	$\varphi_1^2 \varphi_2^2$	17)	$(\partial_\mu A_\mu)^2$		
8)	$\partial_\mu \varphi_1 \partial_\mu \varphi_1$	18)	$(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial_\mu A_\nu - \partial_\nu A_\mu)$		
9)	$\partial_\mu \varphi_2 \partial_\mu \varphi_2$	19)	$(A_\mu A_\mu)^2$		

(25)

Its variation $s\mathscr{L}$ under the Slavnov transformation (24) is of maximal dimension five, carrying the $\Phi\Pi$ charge of a $\bar{c}$ ghost, odd under charge conjugation. It is therefore a combination of the 23 monomials:

$$
\begin{array}{llllll}
1) & \bar{c}\varphi_2 & 11) & \bar{c}\partial_\mu A_\mu & 21) & \bar{c}A_\mu A_\mu \varphi_2 \\
2) & \bar{c}\varphi_1\varphi_2 & 12) & \bar{c}A_\mu \partial_\mu \varphi_1 & 22) & \bar{c}A_\mu A_\mu \varphi_1\varphi_2 \\
3) & \bar{c}\varphi_2^3 & 13) & \bar{c}\partial_\mu A_\mu \varphi_1 & 23) & \bar{c}A_\mu A_\mu \partial_\nu A_\nu \\
4) & \bar{c}\varphi_1^2\varphi_2 & 14) & \bar{c}\partial_\mu A_\mu \varphi_1 & & \\
5) & \bar{c}\Box\varphi_2 & 15) & \bar{c}A_\mu \varphi_1 \partial_\mu \varphi_1 & & \\
6) & \bar{c}\varphi_1^3\varphi_2 & 16) & \bar{c}\partial_\mu A_\mu \varphi_2 & & \\
7) & \bar{c}\varphi_1\varphi_2^3 & 17) & \bar{c}A_\mu \varphi_2 \partial_\mu \varphi_2 & & \\
8) & \bar{c}\varphi_2\Box\varphi_1 & 18) & \bar{c}\Box\partial_\mu A_\mu & & \\
9) & \bar{c}\varphi_1\Box\varphi_2 & 19) & \bar{c}A_\mu \partial_\mu \bar{c}c & & \\
10) & \bar{c}\partial_\mu\varphi_1\partial_\mu\varphi_2 & 20) & \bar{c}A_\mu A_\nu \partial_\mu A_\nu & &
\end{array}
\tag{26}
$$

One can however verify that the last three monomials can never occur as variations of some monomials in Eq. (25) whereas the first twenty are such variations. It follows that the requirement that $\mathscr{L}$ be invariant under Slavnov transformations is expressed via a homogeneous linear system of twenty equations whose unknowns are the coefficients of the twenty six monomials listed in Eq. (25). As a result, the most general invariant $\mathscr{L}$ can be written as a linear combination of the following six terms:

$$
\begin{array}{ll}
1) & G_{\mu\nu}G_{\mu\nu} = (\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial_\mu A_\nu - \partial_\nu A_\mu) \\
2) & (D_\mu\varphi)^* D_\mu\varphi \\
3) & \varphi^*\varphi \\
4) & (\varphi^*\varphi)^2 \\
5) & \dfrac{\mathscr{G}^2}{2} + \bar{c}\,\dfrac{\delta\mathscr{G}}{\delta\Lambda}\,c \\
6) & \dfrac{A_\mu A_\mu}{2} - \bar{c}c + \dfrac{\varrho}{c}\,\varphi_1
\end{array}
\tag{27}
$$

where

$$
\begin{array}{ll}
\varphi = \dfrac{\varphi_1 + v + i\varphi_2}{\sqrt{2}} & \mathscr{G} = \partial_\mu A_\mu + \varrho\varphi_2 \\
D_\mu = \partial_\mu - ieA_\mu & \dfrac{\delta\mathscr{G}}{\delta\Lambda} = \Box + \varrho e v + \varrho e \varphi_1 \,.
\end{array}
\tag{28}
$$

In other words, $\mathscr{L}$ is of the form

$$
\begin{aligned}
\mathscr{L} = & -\frac{Z_A}{4}\,G_{\mu\nu}G_{\mu\nu} + Z_1(D_\mu\varphi)^* D_\mu\varphi + \mu^2\varphi^*\varphi \\
& - g(\varphi^*\varphi)^2 - \frac{1}{\alpha}\left(\frac{\mathscr{G}^2}{2} + \bar{c}\,\frac{\delta\mathscr{G}}{\delta\Lambda}\,c\right) \\
& + \beta\left(\tfrac{1}{2}A_\mu A_\mu - \bar{c}c + \frac{\varrho}{c}\,\varphi_1\right).
\end{aligned}
\tag{29}
$$

The last term which is conspicuously absent from the classical Higgs-Kibble Lagrangian has precisely to do with the phenomenon previously alluded to. Its presence violates spontaneous breakdown without spoiling the Slavnov identity. As we shall see later, its absence can be imposed by requiring suitable normalization conditions on the Green functions which allow to convert the unphysical parameters $Z_A, Z_1, \mu^2, g, \alpha, \varrho, \beta$ into parameters that are needed to interpret the theory in terms of particles. In terms of the variables appropriate to the case of broken symmetry, Eq. (29) can rewritten as:

$$\begin{aligned}\mathcal{L} = &-\frac{Z_A}{4} G_{\mu\nu} G_{\mu\nu} + Z_1 [\partial_\mu \varphi_1 \partial_\mu \varphi_1 + \partial_\mu \varphi_2 \partial_\mu \varphi_2 \\ &+ 2eA_\mu(\varphi_2 \partial_\mu \varphi_1 - (\varphi_1 + v)\partial_\mu \varphi_2) + e^2 A_\mu A_\mu((\varphi_1 + v)^2 + \varphi_2^2)] \\ &+ \mu^2[(\varphi_1 + v)^2 + \varphi_2^2] - g[(\varphi_1 + v)^2 + \varphi_2^2]^2 \\ &- Z_G\left[\frac{(\partial_\mu A_\mu + \varrho\varphi_2)^2}{2} + \bar{c}(\Box + \varrho e(\varphi_1 + v))\, c\right] \\ &+ \beta\left[\frac{A_\mu A_\mu}{2} - \bar{c}c + \frac{\varrho}{e}\varphi_1\right].\end{aligned} \tag{30}$$

We shall now impose the following normalization conditions, which for reasons to be explained, we split into two groups:

$$\begin{array}{lll} \text{unphysical:} & \langle \varphi_1 \rangle = 0 & (0) \\ & \Gamma_{c\bar{c}}(p^2 = m_G^2) = 0 & (1) \\ & \Gamma'_{c\bar{c}}(p^2 = m_G^2) = \dfrac{1}{\alpha} & (2) \\ \text{physical:} & \Gamma_{\varphi_1\varphi_1}(p^2 = M^2) = 0 & (3) \\ & \Gamma'_{\varphi_1\varphi_2}(p^2 = M^2) = z_{\varphi_1} = 1 & (4) \\ & \Gamma_{A^T A^T}(p^2 = m^2) = 0 & (5) \\ & \Gamma'_{A^T A^T}(p^2 = m^2) = z_A = 1 & (6) \\ & \Gamma_{A^T A^T \varphi_1}(m^2, m^2, M^2) = 2\varepsilon m & (7) \\ & \Gamma_{c\bar{c}}^{-1} \det \begin{pmatrix} \Gamma_{A^L A^L} & \Gamma_{A^L \varphi_2} \\ \Gamma_{A^L \varphi_2} & \Gamma_{\varphi_2 \varphi_2} \end{pmatrix}\Bigg|_{p^2 = m_G^2} = 0\,. & (8) \end{array} \tag{31}$$

Here A^T (resp. A^L) denotes the transverse (resp. longitudinal part) of A; expressed in terms of the parameters specifying $\mathscr{L}$, these conditions read:

$$
\left.
\begin{aligned}
&\langle\varphi_1\rangle = 0 = 2v\mu^2 - 4v^3 g + \frac{\varrho\beta}{e} && (0)\\
&m_G^2 = \varrho ev + \beta\alpha && (1)\\
&Z_G = \frac{1}{\alpha} && (2)\\
&M^2 = \frac{2}{Z_1}(6gv^2 - \mu^2) && (3)\\
&Z_1 = z_{\varphi_1} = 1 && (4)\\
&m^2 = [Z_1 e^2 v^2 + \beta]\, Z_A^{-1} && (5)\\
&Z_A = z_A = 1 && (6)\\
&e = \varepsilon && (7)
\end{aligned}
\right.
\qquad (32)
$$

$$
\Gamma_{c\bar{c}}^{-1} \det \begin{pmatrix} \Gamma_{A^L A^L} & \Gamma_{A^L\varphi_2} \\ \Gamma_{A^L\varphi_2} & \Gamma_{\varphi_2\varphi_2} \end{pmatrix}\Bigg|_{p^2+m_G^2}
$$

$$
= -\frac{1}{\alpha}\,\frac{1}{p^2-\varrho ev-\beta\alpha}\left((p^2-\varrho ev-\beta\alpha)\left(p^2-\varrho ev-\frac{\beta\varrho}{ev}\right)\right)\Bigg|_{p^2=m_G^2}
$$

$$
= \beta\left(\frac{\varrho}{ev}-\alpha\right) = 0\,. \qquad (8)
$$

This last normalization condition is well defined because the $\Phi\Pi$ ghost mass turns out to be degenerate with at least one of the coupled (A^L, φ_2) ghost system. This is a consequence of the Slavnov identity, as shown in Appendix III. On the other hand complete degeneracy of the ghost masses is precisely the condition for spontaneous breakdown, $(\beta = 0)$, except if $\varrho = \alpha ev$, which characterizes the restricted t'Hooft gauge, excluded here and eventually recovered by a limiting procedure.

The system is an algebraic system which is invertible and allows to solve for the coefficients in the Lagrangian in terms of the parameters occurring in the normalization conditions. This leads to a particle interpretation of the theory in a Fock space carrying an indefinite metric due to the Fermi character of the $\Phi\Pi$ ghosts and the non positive definiteness of the (A^L, φ_2) coupled propagator matrix.

One can easily generalize this analysis to the case where $e_1^0 \neq e_2^0$, $a^0 \neq 1$ where the theory is again determined by the Slavnov identity and normalization conditions, e_2^0 and a^0 being left free. Although the corresponding algebra is not illuminating and will not be reported here, the possibility of such a generalization should be kept in mind for further reference.

We are now able to describe the scattering theory: the Fock space is determined by the quadratic part of $\mathscr{L}$, the corresponding in fields being solutions of the derived Euler Lagrange equations. Within this Fock space we may select a physical subspace generated from vacuum by application of $(\varphi_{1,\mathrm{in}} A^T_{\mathrm{in}})$. Physical states should actually be equivalence classes of such states modulo some zero norm states whose structure will be mentioned later in connection with the questions of the unitarity of the physical S operator and of the existence of physical local observables.

The restriction to the above defined physical subspace of the connected scattering operator is given by the LSZ formula:

$$\begin{aligned}
S^C_{\mathrm{phys}} = {} & \colon \exp\left[i \int dx\, dy \left(\varphi_{1,\mathrm{in}}(x)\, K_1(x,y) \frac{\delta}{\delta J_1(y)} \right.\right. \\
& \left.\left. + A^T_{\mu,\mathrm{in}}(x)\, K_{\mu\nu}(x,y) \frac{\delta}{\delta J_\nu(y)} \right) \right] \colon Z_C[\underline{J}]|_{J_\mu = J_1 = J_2 = \bar{\xi} = \xi = 0} \\
\equiv {} & \Sigma_{\mathrm{phys}} Z_C(\underline{J})|_{\underline{J}=0}
\end{aligned} \tag{33}$$

where, in view of Eq. (31, 3–6)

$$\begin{aligned}
K_1(x,y) &= (\Box + M^2)\, \delta(x-y) \\
K_{\mu\nu}(x,y) &= (\Box g_{\mu\nu} - \partial_\mu \partial_\nu + m^2 g_{\mu\nu})\, \delta(x-y)\,.
\end{aligned} \tag{34}$$

It is typical of the spontaneously broken theory that the physical scattering operator does not depend on the parameters which specify the gauge. In other words,

$$\frac{\partial S_{\mathrm{phys}}}{\partial \varrho} = 0\,, \qquad \frac{\partial S_{\mathrm{phys}}}{\partial \alpha} = 0\,. \tag{35}$$

The first relation can be proved as follows:

$$\begin{aligned}
\frac{\partial S^C_{\mathrm{phys}}}{\partial \varrho} &= \Sigma_{\mathrm{phys}} \left. \frac{\partial Z_C[\underline{J}]}{\partial \varrho} \right|_{J=0} \\
&= \Sigma_{\mathrm{phys}} \left(-\frac{1}{\alpha} \right) \int dx [\mathscr{G} \varphi_2 + \bar{e} e^0_2 (\varphi_1 + v^0)\, c]\, (x)|_{\underline{J}=0} = 0
\end{aligned} \tag{36}$$

since the expectation value of $\mathscr{G}$ between physical states vanishes because of the Slavnov identity and those of c and $\bar{c}$ because of $\Phi\Pi$ charge conservation. Similarly,

$$\begin{aligned}
\frac{\partial S^C_{\mathrm{phys}}}{\partial \alpha} &= \Sigma_{\mathrm{phys}} \left. \frac{\partial Z_C[\underline{J}]}{\partial \alpha} \right|_{\underline{J}=0} \\
&= \Sigma_{\mathrm{phys}} \frac{1}{\alpha^2} \int dx \left(\frac{\mathscr{G}^2}{2} + \bar{c} \mathscr{M} c \right) (x)|_{\underline{J}=0} \\
&= \Sigma_{\mathrm{phys}} \frac{1}{\alpha^2} \int dx \left(\frac{\mathscr{G}^2 + \bar{c} \mathscr{M} c}{2} + \bar{\xi} c \right) (x)|_{\underline{J}=0} = 0\,.
\end{aligned} \tag{37}$$

This concludes our review of the tree approximation.

II. Perturbation Theory to All Orders: The Slavnov Identities

The extension of the model beyond the tree approximation, proceeds in the spirit of the $BPHZ$ [5] renormalization scheme, via an effective Lagrangian of the form

$$\begin{aligned}\mathscr{L}^{\text{eff}}(\underline{\varphi}, \underline{J}, \underline{\eta}) = \mathscr{L}_4^{\text{eff}}(\underline{\varphi}) &+ \eta_1(z_1 N_2[\bar{c}\varphi_1] + z_1' \bar{c}) \\ &+ \eta_2 z_2 N_2[\bar{c}\varphi_2] + J_i\varphi_1 + J_2\varphi_2 + J_\mu A_\mu \\ &+ \bar{\xi} c + \xi \bar{c} \\ &= \mathscr{L}^{\text{eff}}(\underline{\varphi}, \underline{\eta}) + \underline{J}\underline{\varphi}\,.\end{aligned} \tag{38}$$

The corresponding Green functionals

$$Z(\underline{J}, \underline{\eta}) = \left\langle T \exp\left[\frac{i}{\hbar} \int \mathscr{L}_{\text{int}}^{\text{eff}}(\underline{\varphi}, \underline{J}, \underline{\eta})(x)\, d^4x\right]\right\rangle \tag{39}$$

and

$$Z_C(\underline{J}, \underline{\eta}) = \frac{\hbar}{i} \ln Z(\underline{J}, \underline{\eta}) = \frac{\hbar}{i}\left\langle T \exp\left[\frac{i}{\hbar} \int d^4x\, \mathscr{L}_{\text{int}}^{\text{eff}}(\underline{\varphi}, \underline{J}, \underline{\eta})(x)\right]\right\rangle^C \tag{40}$$

are expressed in terms of Feynman graphs in which the propagators are defined by the quadratic part $\mathscr{L}_0$ of $\mathscr{L}$ [Eqs. (30, 31)], and the vertices are given by

$$\mathscr{L}_{\text{int}}^{\text{eff}} = \mathscr{L}^{\text{eff}}(\underline{\varphi}, \underline{J}, \underline{\eta}) - N_4 \mathscr{L}_0\,. \tag{41}$$

The substraction procedure which defines the time ordering symbol T in Eqs. (39, 40) being specified by the N prescriptions indicated in Eq. (38). The coefficients of the Wick monomials in $\mathscr{L}^{\text{eff}}$ are to be considered as formal power series in $\hbar$, and, of course, $\mathscr{L}^{\text{eff}}$ should coïncide in zeroth order with $\mathscr{L}$ [Eqs. (31)].

We shall also clearly restrict ourselves to effective Lagrangians even under charge conjugation and carrying no $\Phi\Pi$ charge.

One can furthermore immediately specialize Eq. (38) by making the choice

$$z_1 = z_2 = 1\,, \qquad z_1' = 0 \tag{42}$$

which corresponds to fixing normalization conditions on the fields coupled to $\underline{\eta}$. We can also define $\mathscr{L}^{\text{eff}}(\underline{\varphi})$ so that no linear term is present, thus automatically fulfilling the normalization condition (31,0)

$$\langle \varphi_1 \rangle = 0\,. \tag{31,0}$$

We shall have however to keep in mind in the following that the allowed class of Lagrangians is that written down in Eq. (38) and $\mathscr{L}^{\text{eff}}(\underline{\varphi})$ is a linear combination of 25 terms which are listed in Eq. (25) [excluding Eq. (25,0) in view of Eq. (31,0)].

The question is now whether one can determine $\mathscr{L}^{\text{eff}}$ so that $Z_C(\underline{J}, \underline{\eta})$ fulfills a renormalized Slavnov identity:

$$\begin{aligned}\mathscr{S} Z_C(\underline{J}, \underline{\eta}) = \int dx\,[J_\mu \partial_\mu \delta_\xi - \bar{e}_1 J_1 \delta_{\eta_2} + \bar{e}_2 J_2 \delta_{\eta_1} + \bar{m} J_2 \delta_\xi \\ - \bar{\xi}(\bar{a}\partial_\mu \delta_{J_\mu} + \bar{\varrho}\delta_{J_2})](x)\, Z_C(\underline{J}, \underline{\eta}) = 0\end{aligned} \tag{43}$$

where $\bar{a}$, $\bar{\varrho}$, $\bar{e}_1$, $\bar{e}_2$, $\bar{m} = \bar{e}_2 \bar{v}$ are formal power series in $\hbar$. We shall eventually require that the normalization conditions (31) be fulfilled.

Now, according to Lam's [7] renormalized action principle, the Slavnov identity (43) expresses the invariance of the effective Lagrangian under the renormalized Slavnov transformation

$$\begin{aligned} \delta\varphi_1 &= \lambda\bar{e}_2 N_2[\varphi_2\bar{e}] \\ \delta\varphi_2 &= -\lambda(\bar{e}_1 N_2[\varphi_1\bar{c}] - \bar{m}\bar{c}) \\ \delta A_\mu &= \lambda\partial_\mu\bar{c} \\ \delta c &= \lambda(\bar{a}\partial_\mu A_\mu + \bar{\varrho}\varphi_2) \\ \delta\bar{c} &= 0\,. \end{aligned} \tag{44}$$

Indeed performing on an arbitrary effective Lagrangian the quantum variation (44) according to the quantum action principle yields:

$$\mathscr{S} Z_C(\underline{J},\underline{\eta}) = \Delta Z_C(\underline{J},\underline{\eta}) \tag{45}$$

where the left hand side comes from the variation of the source terms, and where $-\Delta$ is precisely the insertion of the quantum variation of the effective Lagrangian $\mathscr{L}^{\mathrm{eff}}(\underline{\varphi},\underline{\eta})$. It is a consequence of Lam's analysis that:

$$\Delta = -\int dx\, N_5[s\mathscr{L}^{\mathrm{eff}}(\underline{\varphi},\underline{\eta}) + \hbar Q](x) \tag{46}$$

where $s\mathscr{L}^{\mathrm{eff}}$ is the naïve variation of the Lagrangian, and $\hbar Q$ sums up the quantum corrections. Because of power counting and selection rules. Δ is a linear combination of twenty three monomials listed in Eq. (26), the coefficients being formal power series in $\hbar$ and in the coefficients of $\mathscr{L}^{\mathrm{eff}}$ as well as in those appearing in Eq. (44). The symmetry condition we are looking for is

$$\Delta = 0\,. \tag{47}$$

It can be partially satisfied by requiring that the coefficients of the 20 first monomials vanish to all orders in $\hbar$ the parameters of the Slavnov identity being left arbitrary to all orders. The argument is that if

$$\mathscr{L}^{\mathrm{eff}} = \underline{c}_\natural \underline{\mathscr{L}}^\natural + \underline{c}_\flat \underline{\mathscr{L}}^\flat \tag{48}$$

where:

$$s\underline{\mathscr{L}}^\natural = 0\,, \qquad s\underline{\mathscr{L}}^\flat \neq 0 \tag{49}$$

we can write:

$$\Delta = \underline{d}_\flat s\underline{\mathscr{L}}^\flat + \hbar\mathscr{R} \tag{50}$$

where $\underline{d}_\flat = \underline{c}_\flat + \hbar\underline{\Phi}_\flat$, $\underline{\Phi}_\flat$ being a formal power series in $\hbar$, $\underline{c}_\natural$, $\underline{c}_\flat$ and the coefficient $\underline{s}$ of the Slavnov identity, and the quantum correction $\hbar\mathscr{R}$ is not of the form $s\underline{\mathscr{L}}^\flat$, namely it involves the last three monomials in Eq. (26). By the implicit function theorem for formal power series (cf. Appendix II), the system:

$$\underline{c}_\flat + \hbar\underline{\Phi}_\flat(\hbar, \underline{c}_\natural, \underline{c}_\flat, \underline{s}) = 0 \tag{51}$$

is soluble for $\underline{c}_\flat$.

$\mathscr{L}^{\mathrm{eff}}$ is thus now determined in terms of five parameters [because of (31,0)] and of the five coefficients involved in the Slavnov identity which now reads

$$\mathscr{S} Z_C(\underline{J},\underline{\eta}) = (c_1\Delta_1 + c_2\Delta_2 + c_3\Delta_3)\, Z_C(\underline{J},\underline{\eta}) \tag{52}$$

where $\Delta_1, \Delta_2, \Delta_3$ (previously numbered 21, 22, 23) are the last three terms in (26), affected with the N_5 prescription.

Now, obviously, the right hand side (52) has to fulfill the compatibility condition implied by the structure of the left hand side [cf. Eqs. (12, 13)] namely:

$$\begin{aligned}\mathcal{S}^2 Z_C &\equiv -\int dx[\bar{\xi}((\bar{a}\Box + \bar{\varrho}\bar{m})\,\delta_\xi + \bar{\varrho}\bar{e}_2\,\delta_{\eta_1})](x)\,Z_C \\ &\equiv -\int dx[\bar{\xi}(\mathcal{M}\delta_\xi)](x)\,Z_C \\ &= \mathcal{S}(c_1\Delta_1 + c_2\Delta_2 + c_3\Delta_3)\,Z_C \\ &= [\mathcal{S}, c_1\Delta_1 + c_2\Delta_2 + c_3\Delta_3]\,Z_C\,.\end{aligned} \tag{53}$$

Now:

$$[\mathcal{S}, \Delta_i]\,Z_C = [s\Delta_i + \hbar P_i]\,Z_C \tag{54}$$

where $s\Delta_i$ is the naïve variation of the monomial Δ_i under a Slavnov transformation, to which dimension six is assigned, whereas P_i is a dimension six insertion, carrying two $\bar{c}$ charges, even under charge conjugation and whose coefficients are formal power series in $\hbar$ and in the so far undetermined power series coefficients occuring in $\mathcal{L}^{\text{eff}}$ and $\mathcal{S}$, as a consequence of Zimmermann's reduction formulae.

On the other hand, the $\Phi\Pi$ ghost equation of motion is of the form:

$$(\mathcal{N}\delta_\xi)(x)\,Z_C \equiv [(\bar{a}\Box + \bar{\mu}^2)\,\delta_\xi + d\delta_{\eta_1} + \vec{f}\cdot\delta_{\vec{\beta}}](x)\,Z_C|_{\vec{\beta}=0} = \bar{\alpha}\bar{\xi}(x) \tag{55}$$

where $\bar{a}, \bar{\mu}, d, \vec{f}, \bar{\alpha}$ are formal power series, and $\vec{\beta}$ is a source coupled to

$$\vec{\mathscr{C}} = \{N_3[\varphi_1^2\bar{c}],\, N_3[\varphi_2^2\bar{c}],\, N_3[A_\mu A_\mu\bar{c}]\}\,.$$

Thus, integrating Eq. (52) through $\bar{\xi}$, one gets

$$\int dx[\bar{\xi}(\mathcal{N}\delta_\xi)](x)\,Z_C = 0\,. \tag{56}$$

Noticing that

$$\int dx(\bar{\xi}\delta_\xi)(x)\,Z_C = 0 \tag{57}$$

as a consequence of the invariance of $\mathcal{L}^{\text{eff}}(\underline{J}, \underline{\eta})$ under the variation

$$\delta c = \lambda\bar{c}$$

and substracting Eq. (56) from Eq. (53) yields:

$$\begin{aligned}&\int dx[\bar{\xi}((\bar{\varrho}\bar{e}_2 - d)\,\delta_{\eta_1} - \vec{f}\cdot\delta_{\vec{\beta}})](x)\,Z_C(\underline{J}, \underline{\eta}, \vec{\beta})|_{\vec{\beta}=0} \\ &= \sum_1^3{}_i\, c_i(s\Delta_i + \hbar P_i)\,Z_C(\underline{J}, \underline{\eta})\,.\end{aligned} \tag{58}$$

We now express Z_C in terms of Γ by Legendre transform, thus obtaining:

$$\begin{aligned}&\int dx[(d - \bar{\varrho}\bar{e}_2)\,\delta_c\Gamma\delta_{\eta_1}\Gamma + \vec{f}\delta_c\Gamma\delta_{\vec{\beta}}\Gamma](x)|_{\vec{\beta}=0} \\ &= \sum_1^3{}_i\, c_i(s\Delta_i + \hbar P_i)\,\Gamma = 0|_{\vec{\beta}=0}\,.\end{aligned} \tag{59}$$

Let us now write

$$\Gamma = \mathcal{L}^{\text{eff}}(\underline{J}, \underline{\eta}) + \vec{\beta}\cdot\vec{\mathscr{C}} - \hbar\Gamma' \tag{60}$$

where it has been explicitly noted that the corrections to $\mathscr{L}^{\text{eff}}$ occuring in Γ are necessarily radiative corrections. Equation (59) can be cast into the form:

$$\begin{aligned}\int dx\{\Box\bar{c}[(d-\bar{\varrho}\bar{e}_2)\varphi_1\bar{c}+f_1\varphi_1^2\bar{c}+f_2\varphi_2^2\bar{c}+f_3A_\mu A_\mu\bar{c}+2c_1\bar{e}A_\mu\partial_\mu\bar{c}\varphi_2\\ +2c_2A_\mu\partial_\mu\bar{c}\varphi_1\varphi_2+C_3(2\bar{c}\partial_\mu\bar{c}A_\mu\partial_\nu A_\nu+\bar{c}A_\mu A_\mu\Box\bar{c})]\}(x)\\ =\hbar\Phi(d-\bar{\varrho}\bar{e}_2,f_1,f_2,f_3,c_1,c_2,c_3)\end{aligned} \tag{61}$$

where Φ is a functional of the fields which is linear in the indicated arguments and lumps together contributions from Γ' and from the P_i's. Differenciating in turn Eq. (61) with respects to the fields occuring in each indicated monomial, and setting all fields equal to zero, yields, in view of the independance of these monomials:

$$\begin{aligned}d-\bar{\varrho}\bar{e}_2&=\hbar\Phi_0(d-\bar{\varrho}\bar{e}_2,\underline{f},\underline{c})\\ f_1&=\hbar\Phi_1(d-\bar{\varrho}\bar{e}_2,\underline{f},\underline{c})\\ f_2&=\hbar\Phi_2(d-\bar{\varrho}\bar{e}_2,\underline{f},\underline{c})\\ f_3&=\hbar\Phi_3(d-\bar{\varrho}\bar{e}_2,\underline{f},\underline{c})\\ c_1&=\hbar\Psi_1(d-\bar{\varrho}\bar{e}_2,\underline{f},\underline{c})\\ c_2&=\hbar\Psi_2(d-\bar{\varrho}\bar{e}_2,\underline{f},\underline{c})\\ c_3&=\hbar\Psi_3(d-\bar{\varrho}\bar{e}_2,\underline{f},\underline{c})\end{aligned} \tag{62}$$

where Φ_i $(i=1,2,3)$, Ψ_i $(i=1,2,3)$ are linear in the indicated arguments, formal power series in $\hbar$ and in the remaining parameters. The situation occuring in the tree approximation and application of the theorem in Appendix II yield:

$$d-\bar{\varrho}\bar{e}_2=c_1=c_2=c_3=f_1=f_2=f_3=0\,. \tag{63}$$

Hence, the Slavnov identity holds, and, up to the mass term the $\Phi\Pi$ equation of motion involves the same coefficients and monomials as those occuring in $\mathscr{S}^2$. The equality of the two mass terms will be proved in Appendix III in connection with the normalization conditions we shall now consider.

Namely, we shall show that the normalization conditions (31) can be fulfilled, whereby all parameters are determined except $\bar{a}$ and $\bar{e}_2$. Eq. (31,0) is already fulfilled. Next we try to impose Eq. (31,1–7). Looking at the algebraic system which is soluble in the tree approximation, we can apply once more the theorem of Appendix II, because this system is perturbed as allowed by this theorem by higher order terms occuring in $\Gamma_{\varphi_1\varphi_1},\Gamma_{A^TA^T},\Gamma_{c\bar{c}},\underline{\Gamma}_{A^TA^T\varphi_1}$.

The last normalization condition (31,8) is more delicate: one has first to show, to all orders in $\hbar$ that $\underline{\Gamma}_{c\bar{c}}^{-1}(\underline{\Gamma}_{A^LA^L}\underline{\Gamma}_{\varphi_2\varphi_2}-\underline{\Gamma}_{A^L\varphi_2}^2)$ is finite at $p^2=m_G^2$. The proof, based on the Slavnov identity and Eq. (31,2) is given in Appendix III. As a by product, as previously announced, one obtains the last equation connecting $\mathscr{S}^2$ and the $\Phi\Pi$ equation of motion [Eq. (52)] namely the $\Phi\Pi$ equation reads:

$$(\mathscr{N}\delta_\xi)(x)\,Z_C\equiv(\mathscr{M}\delta_\xi)(x)\,Z_C=\bar{\alpha}\bar{\xi}(x)\,. \tag{64}$$

In conclusion, once the Slavnov identities and the normalization conditions have been fulfilled, there remain two free parameters, $\bar{a}$ and $\bar{e}_2$, which will not be specialized any further.

III. Perturbation Theory to All Orders: Gauge Invariance of the Physical Scattering Operator

The normalization conditions Eq. (31) allow to interpret the theory, in the sense of formal power series, within the Fock space defined in the tree approximation, and the formula giving S^C_{phys} in terms of $Z_C(\underline{J})$ [Eq. (31)], or similarly S_{phys} in terms of $Z(\underline{J})$ remains unchanged. For a technical reason which will appear later we shall from now on work with the non connected Green functional.

We now wish to evaluate

$$\frac{\partial S_{\text{phys}}}{\partial \varrho}, \quad \frac{\partial S_{\text{phys}}}{\partial \alpha}.$$

Using Lowenstein's [6] renormalized action principle we see that

$$\frac{\hbar}{i}\frac{\partial}{\partial\lambda} Z(\underline{J},\underline{\eta}) = \Delta_\lambda Z(\underline{J},\underline{\eta}) \tag{65}$$

where λ is one of the parameters α, m_G^2 and Δ_λ is a dimension four insertion obtained by differenciating $\mathscr{L}^{\text{eff}}(\underline{\varphi},\underline{\eta})$, with respect to λ, namely an operation which alters infinitesimally $\mathscr{L}^{\text{eff}}(\underline{\varphi},\underline{\eta})$ within the class (38). Using the Slavnov identity, we are going to show that Δ_λ can be written as

$$\Delta_\lambda = \sum_1^8{}_i c_\lambda^{0,i}\Delta_i^0 + \sum_1^6{}_i c_\lambda^{S,i}\Delta_i^S \tag{66}$$

where the Δ_i^0's $(i=1,\ldots,8)$ are eight insertions such that

$$\Sigma_{\text{phys}}\Delta_i^0 Z(\underline{J},\underline{\eta})|_{\underline{J}=\underline{\eta}=0} = 0 \tag{67}$$

and leaving unchanged the physical normalization condition (31,8).

The other physical normalization conditions (31,3–7) are left unaltered as a consequence of Eq. (67). In the following, we shall call these insertions non physical. The Δ_i^S, $(i=1,\ldots,6)$, are six symmetric insertions, namely such that

$$\mathscr{S}\Delta_i^S Z(\underline{J},\underline{\eta}) = 0 \qquad (i=1,\ldots,6). \tag{68}$$

Thus applying Eq. (65) to the physical normalization conditions (31,3–8) yields a linear homogeneous system of equations of the form

$$\sum_1^6{}_i c_\lambda^{S,i}\Delta_i^{S,j} = 0 \qquad (j=3,4,5,6,7,8). \tag{69}$$

The forthcoming analysis shows that

$$\det\|\Delta_i^{S,j}\| \neq 0 \tag{70}$$

since this happens to be true in the tree approximation. Hence it follows that

$$c_\lambda^{S,i} = 0 \tag{71}$$

and the gauge invariance of the physical S-operator follows from Eq. (67). We now construct the decomposition of Δ_λ given in Eq. (67).

From the definition of Δ_λ we have

$$\partial_\lambda(\mathscr{S}Z) = \partial_\lambda \mathscr{S} Z + \frac{i}{\hbar} \mathscr{S} \Delta_\lambda Z = 0 \tag{72}$$

so that

$$\frac{i}{\hbar}[\Delta_\lambda, \mathscr{S}] = \partial_\lambda \mathscr{S}. \tag{73}$$

Thus we can write

$$\Delta_\lambda = \hat{\Delta}_\lambda + \Delta_\lambda^S \tag{74}$$

where $\hat{\Delta}_\lambda$ is a particular solution of Eq. (73) and Δ_λ^S is a symmetrical insertion.

We shall first construct a non physical $\hat{\Delta}_\lambda$, and we shall show that any Δ_λ^S is a linear combination of nine symmetrical insertions three of which are non physical the remaining six satisfying Eq. (68).

Let us denote

$$\mathscr{S} = \sum_0^5 {}_i c_i s_i \tag{75}$$

where

$$\begin{aligned} c_0 &= 1 & s_0 &= \int dx (J_\mu \partial_\mu \delta_\xi)(x) \\ c_1 &= -\bar{e}_1 & s_1 &= \int dx (J_1 \delta_{\eta_2})(x) \\ c_2 &= \bar{e}_2 & s_2 &= \int fx (J_2 \delta_{\eta_1})(x) \\ c_3 &= \bar{m} & s_3 &= \int dx (J_2 \delta_\xi)(x) \\ c_4 &= -\bar{a} & s_4 &= \int dx (\bar{\xi} \partial_\mu \delta_{J_\mu})(x) \\ c_5 &= -\bar{\varrho} & s_5 &= \int dx (\bar{\xi} \delta_{J_2})(x). \end{aligned} \tag{76}$$

So that Eq. (73) now reads:

$$\frac{i}{\hbar}[\hat{\Delta}_\lambda, \mathscr{S}] = \sum_1^5 {}_i \partial_\lambda c_i s_i. \tag{77}$$

Now, there exists a basis of covariant non physical insertions Δ_i: i ($i = 1, \ldots, 5$) satisfying:

$$\frac{i}{\hbar}[\Delta_i, \mathscr{S}] = s_i.$$

Indeed let us consider:

$$Q_{i,\varepsilon} = \frac{\hbar^2 c_i}{\bar{\alpha}} \int dx (\delta_{g_i(\varepsilon)} \delta_{\bar{\xi}(-\varepsilon)})(x) \qquad (i = 4, 5) \tag{78}$$

where the $c_i s$ are defined in Eq. (76) and $\bar{\alpha}$ in Eq. (55).

The symbols δ_{g_i} are defined by:

$$\begin{aligned} \delta_{g_4} &= \partial_\mu \delta_{J_\mu} \\ \delta_{g_5} &= \delta_{J_2}. \end{aligned} \tag{79}$$

The indices ($\pm\varepsilon$) indicate translations by the e.g. space like small vectors $\pm\varepsilon$.

We introduce the insertions:

$$\Delta_{i,\varepsilon} Z = \mathscr{S} Q_{i,\varepsilon} Z = [\mathscr{S}, Q_{i,\varepsilon}] Z \tag{80}$$

and we have:

$$\frac{i}{\hbar}[\Delta_{i,\varepsilon}, \mathcal{S}]Z = -\frac{i}{\hbar}[\mathcal{S}^2, Q_{i,\varepsilon}]Z$$

$$= -\frac{i}{\hbar}[Q_{i,\varepsilon}, \int dx[\bar{\xi}(\bar{\mathcal{M}}\delta_{\xi})](x)Z = \frac{\hbar}{i}\frac{c_i}{\bar{\alpha}}\int dx[\delta_{g_i(\varepsilon)}(\bar{\mathcal{M}}\delta_{\xi})(-\varepsilon)](x)Z$$

$$= c_i \int dx[\bar{\xi}(-\varepsilon)\,\delta_{g_i(\varepsilon)}](x)Z \tag{81}$$

where the connection between $\mathcal{S}$ and the $\Phi\Pi$ equation of motion [Eq. (64)] has been used.

It is shown in Appendix IV that, in the limit $\varepsilon \to 0$ the finite part Δ_i of $\Delta_{i,\varepsilon}$ has the same covariance as $\Delta_{i,\varepsilon}$, namely

$$c_1 s_i Z(\underline{J}, \underline{\eta}) = \lim_{\varepsilon\to 0} c_i \int dx[\bar{\xi}(-\varepsilon)\,\delta_{g_i(\varepsilon)}](x)\, Z(\underline{J}, \underline{\eta})\,. \tag{82}$$

It is furthermore shown, in Appendix IV, that by substracting a symmetric insertion, which therefore does not alter the covariance $c_i s_i$, one obtains non physical insertions which we denote Δ_i^0.

We now look for other non physical insertions which are easily obtained by applying the renormalized action principle [6].

The following variations whose covariances are indicated provide us with the desired insertions:

1) $\delta c \propto c$ yields the insertion

$$\Delta_c = \frac{\hbar}{i}\int dx(\bar{\xi}\delta_{\bar{\xi}})(x) \tag{83}$$

with

$$\frac{i}{\hbar}[\Delta_c, \mathcal{S}] = c_4 s_4 + c_5 s_5\,, \tag{84}$$

2) the operation

$$\Delta_{\eta_1} = \frac{\hbar}{i}\int dx(\eta_1\delta_{\eta_1})(x) \tag{85}$$

corresponds to a variation of z_1 in the neighbourhood of $z_1 = z_2 = 1$, $z_1' = 0$ and its covariance is given by

$$\frac{i}{\hbar}[\Delta_{\eta_1}, \mathcal{S}] = -c_2 s_2\,, \tag{86}$$

3) the operation

$$\Delta_{\eta_2} = \frac{\hbar}{i}\int dx(\eta_2\delta_{\eta_2})(x) \tag{87}$$

corresponds to a variation of z_2. Its covariance is given by

$$\frac{i}{\hbar}[\Delta_{\eta_2}, \mathcal{S}] = -c_1 s_1\,, \tag{88}$$

4) the operation

$$\Delta'_{\eta_1} = \frac{\hbar}{i} \int dx (\eta_1 \delta_\xi)(x) \tag{89}$$

corresponds to a variation of z'_1. Its covariance is given by

$$\frac{i}{\hbar} [\Delta'_{\eta_1}, \mathscr{S}] = -c_2 s_3 . \tag{90}$$

It is obvious that all of these four insertions leave all physical normalization conditions (31,3–8) unchanged.

$\hat{\Delta}_\lambda$ is thus a linear combination of $\Delta_4^0, \Delta_5^0, \Delta_{\eta_1}, \Delta_{\eta_2}, \Delta'_{\eta_1}$ which solves part of Eq. (74).

We are thus left with finding a basis of symmetrical insertions. We know that, given the Slavnov identity $\mathscr{L}^{\text{eff}}(\underline{\varphi}, \underline{\eta})$ depends on nine parameters, namely six to specify $\mathscr{L}^{\text{eff}}(\underline{\varphi})$, three to specify the external field dependence (i.e., z_1, z_2, z'_1).

This is indeed true in the tree approximation and therefore, by the theorem of Appendix II, to all orders. [Of course, this counting does not take into account any of the normalization conditions (31), including (31,0).] As a consequence, there are nine independent symmetric insertions.

We first construct those which respect the physical normalization conditions: The first one is:

$$\Delta_0^{0,S} = \Delta_4^0 + \Delta_5^0 - \Delta_c . \tag{91}$$

The second one is generated by the variation $\delta\varphi_1 = \text{const.}$:

$$\Delta_1^{0,S} = \int J_1(x)\, dx . \tag{92}$$

The third one is obtained by considering

$$\Delta_{\varphi_2} = \frac{\hbar}{i} \int dx (J_2 \delta_{J_2})(x) \tag{93}$$

whose covariance is given by

$$\frac{i}{\hbar} [\Delta_{\varphi_2}, \mathscr{S}] = c_2 s_2 + c_3 s_3 - c_5 s_5 . \tag{94}$$

From the foregoing analysis:

$$\Delta_2^{0,S} = \Delta_{\varphi_2} + \Delta_{\eta_1} + \frac{c_3}{c_2} \Delta_{\eta_1}^1 + \Delta_5^0 \tag{95}$$

is symmetric, leaves the physical normalization conditions unchanged, and is non zero as can be seen by a direct calculation at the tree level.

We are thus left with finding six independent symmetric insertions. By the general theorem of Appendix II, five of them are determined by the terms of $\mathscr{L}$ (excluding the one which leads to $\Delta_0^{0,S}$). The sixth one involves

$$\Delta_A = \frac{\hbar}{i} \int dx (J_\mu \delta_{J_\mu})(x) \tag{96}$$

since

$$\frac{i}{\hbar}[\Delta_A, \mathscr{S}] = c_0 s_0 - c_4 s_4 = -c_1 s_1 - c_2 s_2 - c_3 s_3 - 2c_4 s_4 - c_5 s_5 \tag{97}$$

it follows that

$$\Delta_6^S = \Delta_A - \Delta_{\eta_2} - \Delta_{\eta_1} - \frac{c_3}{c_2}\Delta_{\eta_1}^1 + 2\Delta_c - \Delta_5^0 \tag{98}$$

is symmetric.

It is straightforward but tedious to verify in the tree approximation that these six insertions alter independently the six physical normalization conditions (31,3–8).

The gauge invariance proof is thus completed. It is extended in Appendix VII to gauges which contain a quadratic term odd under charge conjugation.

IV. Unitary of the S Operator

Let us first define

$$S_{\text{phys}}(\underline{J}) = \Sigma_{\text{phys}} Z(\underline{J}) \tag{99}$$

where the notations are the same as in Eq. (33). According to the reduction formula, the physical S operator is given by

$$S_{\text{phys}} = S_{\text{phys}}(\underline{J})|_{\underline{J}=0}\,. \tag{100}$$

The contribution of non physical particle states to physical unitarity is explicitly given in the expression:

$$\begin{aligned} &S^\dagger_{\text{phys}}(\underline{J}) \exp\{i\hbar \textstyle\int dx [\overleftarrow{\delta}_{\underline{J}^*} \overleftarrow{L} S_+ * \overrightarrow{L}\overrightarrow{\delta}_{\underline{J}}](x)\}_{\text{Ghost}} S_{\text{phys}}(\underline{J})|_{\underline{J}=0} \\ &\quad \equiv S_{\text{phys}}(\underline{J}) \exp \mathscr{A} S_{\text{phys}}(\underline{J})|_{\underline{J}=0}\,. \end{aligned} \tag{101}$$

Here L and $i\hbar S_+$ are respectively the differential operator occuring in the asymptotic field equations and the positive frequency part of the asymptotic field commutator.

The proof consists in considering

$$S^\dagger_{\text{phys}}(\underline{J}) \mathscr{U}(\lambda) S_{\text{phys}}(\underline{J})|_{J=0} = S^\dagger_{\text{phys}}(J) \exp(\lambda\mathscr{A}) S_{\text{phys}}(\underline{J})|_{\underline{J}=0} \tag{102}$$

and evaluating

$$\partial_\lambda S^\dagger_{\text{phys}}(\underline{J}) \mathscr{U}(\lambda) S_{\text{phys}}(\underline{J})|_{\underline{J}=0}\,. \tag{103}$$

It is shown in Appendix V, by extensive use of the Slavnov identity that

$$\begin{aligned} &\partial_\lambda S^\dagger_{\text{phys}}(\underline{J}) \mathscr{U}(\lambda) S_{\text{phys}}(\underline{J})|_{\underline{J}=0} \\ &= \lim_{\varepsilon\to 0} S^\dagger_{\text{phys}}(\underline{J}) \{\textstyle\int i\hbar [\overleftarrow{\delta}_{J(\mathscr{G})} (\overleftarrow{L} S^\varepsilon_+ * \overrightarrow{L})_{\mathscr{G}\mathscr{G}} \overrightarrow{\delta}_{J(\mathscr{G})}](x)\, dx\} \mathscr{U}(\lambda) S_{\text{phys}}(\underline{J})|_{J=0} \end{aligned} \tag{104}$$

where

$$\delta_{J(\mathscr{G})} = \overline{a}\partial_\mu \delta_{J_\mu} + \overline{\varrho}\delta_{J_2}\,. \tag{105}$$

S^{ε}_{+} denotes a regularized version of S_{+} around the ghost mass shell and the index $\mathscr{G}\mathscr{G}$ labels the "gauge-gauge" matrix element of $\overleftarrow{L}S^{\varepsilon}_{+}\overrightarrow{L}$ as indicated in Appendix V.

Integrating Eq. (104) with respect to λ yields

$$\begin{aligned} &S^{\dagger}_{\text{phys}}(\underline{J})\,\mathscr{U}(1)\,S_{\text{phys}}(\underline{J})|_{\underline{J}=0} \\ &\quad = \lim_{\varepsilon\to 0} S^{\dagger}_{\text{phys}}(\underline{J})\exp\{i\hbar\int dx[\overleftarrow{\delta}_{J(\mathscr{G})}(\overleftarrow{L}S^{\varepsilon}_{+} * \overrightarrow{L})_{\mathscr{G}\mathscr{G}}\,\delta_{J(\mathscr{G})}](x)\}\,S_{\text{phys}}(\underline{J})|_{J=0}\,. \end{aligned} \tag{106}$$

Further use of the Slavnov identity according to which, the gauge operator decouples from physical states finishes the proof:

$$S^{\dagger}_{\text{phys}}(\underline{J})\,\mathscr{U}(1)\,S_{\text{phys}}(\underline{J})|_{\underline{J}=0} = S^{\dagger}_{\text{phys}}(\underline{J})\,S_{\text{phys}}(\underline{J})|_{\underline{J}=0} = S^{\dagger}_{\text{phys}}S_{\text{phys}}\,. \tag{107}$$

Unitary follows from the hermiticity of the Lagrangian.

Conclusion – Outlook

The gauge invariance problem has been solved for the abelian Higgs-Kibble model treated in a family of gauges odd under charge conjugation. Emphasis was put on the fulfillment of normalization conditions which allow the interpretation of the theory within a Fock space with indefinite metric. This has in particular allowed us to prove the unitarity of the physical scattering operator and to construct some physical local observables. We feel however that one should make a more complete study of the zero norm states that are allowed in the definition of physical states as equivalence classes. From the technical point of view, it was encouraging to see that the theory was widely controlled by the algebraic structure of its tree approximation thanks to the repeated application of the implicit function theorem for formal power series. This situation looks quite favorable to a future treatement of the non abelian situations, at least when no fermion anomalies are potentially present. This last case will doubtlessly call for more refined techniques, involving the Callan-Symanzik equations which have not been included here.

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Appendix I. Non Abelian Gauge Transformations: Classical Theory

Here are a few details concerning the classical theory of non abelian gauge transformations: the gauge parameters Λ as well as the $\Phi\Pi$ ghost field $\bar{c}$ are labelled by the indices of the dual of a Lie algebra $\mathscr{F}$ with structure constants f. The $\Phi\Pi$ ghost field c and the gauge function $\mathscr{G}$ are labelled by the Lie algebra itself. $\mathscr{M} = \dfrac{\delta\mathscr{G}}{\delta\Lambda}$ is labelled as a linear operator from $\mathscr{F}$ into $\mathscr{F}$. The square of $\mathscr{G}$ is the Killing form of $\mathscr{F}$, at least for the non degenerate part.

Going from the Ward identity to the Slavnov identity now involves an extra term:

$$- \int dx\, dy\, dz\, c_\alpha(x) \left(\frac{\delta^2 \mathcal{G}_\alpha(x)}{\delta A^\gamma(z)\, \delta A^\beta(y)} \right) \bar{c}^\gamma(z)\, \bar{c}^\beta(y) \tag{AI.1}$$

which, using the group law together with the anticommutativity of $\bar{c}$ boils down to

$$- \tfrac{1}{2} \int dx\, dy\, c_\alpha(x) \frac{\delta \mathcal{G}_\alpha(x)}{\delta A^\delta(y)} f^\delta_{\beta\gamma}\, \bar{c}^\gamma(y)\, \bar{c}^\beta(y) \tag{AI.2}$$

or, using the equation of motion:

$$\tfrac{1}{2} \int dx\, \bar{c}^\alpha(x)\, f^\gamma_{\alpha\beta}\, \bar{c}^\beta(x)\, \xi_\gamma(x)\,. \tag{AI.3}$$

The corresponding Slavnov identity can then be interpreted as expressing the invariance of the lagrangian under the transformation law:

$$\begin{aligned} \delta\varphi_i(x) &= \lambda \int dy \frac{\delta\varphi_i(x)}{\delta A^\alpha(y)}\, \bar{c}^\alpha(y) \\ \delta c_\alpha(x) = \lambda \mathcal{G}_\alpha(x), &\qquad \delta \bar{c}^\alpha(x) = \frac{\lambda}{2} f^\alpha_{\beta\gamma}\, \bar{c}^\beta(x)\, \bar{c}^\gamma(x) \end{aligned} \tag{AI.4}$$

where λ is a space time independent anticommuting parameter carrying no index.

Appendix II. The Implicit Function Theorem for Formal Power Series

This appendix is devoted to the statement and proof of an easy theorem (11) which has been repeatedly used to reduce the proof of a property to all orders of perturbation theory down to the verification of a simple algebraic property of the tree approximation:

Theorem. *Let $F_i(x_1, \ldots, x_n; y_1, \ldots, y_p) = 0$ $(i = 1, \ldots, n)$ be a set of algebraic (analytic) equations which has a unique solution $x_i = \varphi_i(y_1, \ldots, y_p)$ analytic in $(y_1, \ldots, y_p)$ in some neighbourhood of $(y_1^0, \ldots, y_p^0)$.*

Then the perturbed system

$$F_i(\underline{x}_1, \ldots, \underline{x}_n; \underline{y}_1, \ldots, \underline{y}_p) = \hbar\, f_i(\underline{x}_1, \ldots, \underline{x}_n; \underline{y}_1, \ldots, \underline{y}_p; \hbar) \qquad (i = 1, \ldots, n)$$

where $\underline{y}_1, \ldots, \underline{y}_p$ are formal power series in $\hbar$ whose lower order terms are $y_1^0, \ldots, y_p^0$ and the f_i's are formal power series in $\underline{x}_1, \ldots, \underline{x}_n$, $y_1, \ldots, \underline{y}_p$, $\hbar$, possesses a unique solution

$$\underline{x}_i = \underline{\varphi}_i(\hbar, \underline{y}_1, \ldots, \underline{y}_p)$$

where the $\underline{\varphi}_i$'s are formal power series in $\hbar$, $\underline{y}_1, \ldots, \underline{y}_p$.

Proof. Let

$$\underline{\xi}_i = \underline{x}_i - \varphi_i(\underline{y}_1, \ldots, \underline{y}_p)$$

F_i can be expanded into a formal power series in $\underline{\xi}_i$, $\underline{y}_i$, whose term linear in $\underline{\xi}$ is

$$\left.\frac{\partial F_i}{\partial x_j}\right|_{x_k=\varphi_k(\underline{y}_1,\ldots,\underline{y}_p)} \underline{\xi}_j$$

where, by the hypothesis

$$\det\left\|\frac{\partial F_i}{\partial x_j}\right\|$$

is invertible in the sense of formal power series. Hence the initial system can be cast into the form

$$\left.\frac{\partial F_i}{\partial x_j}\right|_{x_k=\varphi_k(\underline{y}_1,\ldots,\underline{y}_p)} \underline{\xi}_j=\Phi_i(\underline{\xi}_1,\ldots,\underline{\xi}_n;\underline{y}_1,\ldots,\underline{y}_p;\hbar)$$

where the formal power series Φ_i are such that $\Phi_i(0,\ldots,0;\underline{y}_1,\ldots,\underline{y}_p;0)=0$

$$\text{i.e.}\quad \underline{\xi}_j=\psi_j(\underline{\xi}_1,\ldots,\underline{\xi}_n;\underline{y}_1,\ldots,\underline{y}_p;\hbar)$$

with the same conditions on ψ_j. This system is easily solved by iteration.

Appendix III

We show here that, as a consequence of the Slavnov identity, $\Gamma_{c\bar{c}}^{-1}(\Gamma_{A^LA^L}\Gamma_{\varphi_2\psi_2}-\Gamma^2_{A^L\varphi_2})$ is finite at $p^2=m_G^2$ and thus can be required to vanish. In other words, the $\Phi\Pi$ ghost mass is always degenerate with one of the A^L, φ_2 ghost masses, complete degeneracy then characterizing spontaneous breakdown. We first write the Slavnov identity in terms of the vertex functional:

$$\begin{aligned}\mathscr{S}(\Gamma)\equiv \int dx(&-\bar{c}\,\partial_\mu\delta_{A_\mu}\Gamma-\bar{e}_1\delta_{\varphi_1}\Gamma\delta_{\eta_2}\Gamma+\bar{e}_2\delta_{\varphi_2}\Gamma\delta_{\eta_1}\Gamma\\&+\bar{m}\bar{e}\delta_{\varphi_2}\Gamma-\bar{a}A_\mu\partial_\mu\delta_c\Gamma+\bar{\varrho}\varphi_2\delta_c\Gamma)(x)=0\,.\end{aligned}\tag{AIII.1}$$

Within the A_μ, φ_2 channel, we get:

$$\begin{aligned}ip_\mu\Gamma_{A_\mu\varphi_2}(p)-\bar{\varrho}\Gamma_{\bar{c}c}(p^2)-\gamma(p^2)\Gamma_{\varphi_2\varphi_2}(p^2)=0\\ p_\mu p_\nu\Gamma_{A_\mu A_\nu}(p)-\bar{a}p^2\Gamma_{\bar{c}c}(p^2)+i\gamma(p^2)p_\mu\Gamma_{A_\mu\varphi_2}(p)=0\end{aligned}\tag{AIII.2}$$

where

$$\gamma(p^2)=\bar{e}_2\Gamma_{\bar{c}\eta_1}(p^2)+\bar{m}\,.\tag{AIII.3}$$

Thus

$$\begin{aligned}\Gamma_{\varphi_2\varphi_2}(p^2)&=\frac{1}{\gamma(p^2)}[ip_\mu\Gamma_{A_\mu\varphi_2}(p)-\bar{\varrho}\Gamma_{\bar{c}c}(p^2)]\\ p_\mu p_\nu\Gamma_{A_\mu A_\nu}(p)&=i\gamma(p^2)p_\mu\Gamma_{A_\mu\varphi_2}(p)+\bar{a}p^2\Gamma_{\bar{c}c}(p^2)\,.\end{aligned}\tag{AIII.4}$$

Hence:

$$\begin{aligned}D=\det\begin{pmatrix}\Gamma_{A^LA^L} & \Gamma_{A^L\varphi_2}\\ \Gamma_{A^L\varphi_2} & \Gamma_{\varphi_2\varphi_2}\end{pmatrix}&=\frac{1}{p^2\gamma(p^2)}[-\bar{\varrho}\bar{a}p^2(\Gamma_{\bar{c}c}(p^2))^2]\\ &+ip_\mu\Gamma_{A_\mu\varphi_2}(p)\Gamma_{\bar{c}c}(p^2)\,(\bar{a}^2p^2-\bar{\varrho}\gamma(p^2))\,.\end{aligned}\tag{AIII.5}$$

Thus, first

$$\Gamma_{\bar{c}c}^{-1} D|_{p^2=m_G^2}$$

is finite and furthermore its vanishing implies

$$ap^2 - \bar{\varrho}(\bar{m} + \bar{e}_2 \Gamma_{\bar{c}\eta_1}(p^2))|_{p^2=m_G^2} \equiv ap^2 - \bar{\varrho}\gamma(p^2)|_{p^2=m_G^2} = 0, \tag{AIII.6}$$

since $\Gamma_{A_\mu\varphi_2} \neq 0$, provided one stays away from the restricted t'Hooft gauge ($\varrho = \alpha e v$).

Looking now at the $\Phi\Pi$ propagator equation

$$[\bar{a}p^2 - (\varrho\bar{m} + \bar{\mu}^2)]\, G_{\bar{c}c}(p^2) - \varrho\bar{e}_2\, G_{\bar{c}\eta_1}(p^2) = \bar{\alpha} \tag{AIII.7}$$

the absence of a pole in the left hand side at $p^2 = m_G^2$ yields:

$$\bar{a}p^2 - (\varrho\bar{m} + \bar{\mu}^2) - \varrho\bar{e}_2\, \Gamma_{\bar{c}\eta_1}(p^2)|_{p^2=m_G^2} = 0 \tag{AIII.8}$$

after multiplication of (AIII.7) through $\Gamma_{\bar{c}c}$ and use of

$$G_{\bar{c}\eta_1} = \Gamma_{\bar{c}\eta_1} G_{\bar{c}c}\,. \tag{AIII.9}$$

Hence comparing with Eq. (AIII.6), we get:

$$\bar{\mu} = 0\,. \tag{AIII.10}$$

Appendix IV

We have shown in Chapter III [Eq. (81)] that, in the limit $\varepsilon \to 0$, the commutator of $\Delta_{i,\varepsilon}$ with $\mathscr{S}$ is equal to $c_i s_i$. We thus infer that the infinite part of $\Delta_{i,\varepsilon}$ as given by the Zimmermann-Wilson expansion is a symmetric insertion with coefficients going to infinity as $\varepsilon \to 0$. The finite part Δ_i will then be given by $[\mathscr{S}, Q_i]$ where Q_i is the finite part of $Q_{i,\varepsilon}$.

It is possible but lengthy to verify these statements by looking at the Zimmermann-Wilson expansion of $Q_{i,\varepsilon}$. In the case of $Q_{5,\varepsilon}$ the calculation is however reasonably simple:

$$\begin{aligned}
&\int dx [T\varphi_2(\varepsilon)\, c(-\varepsilon)]\,(x) \\
&\quad = \frac{i}{\hbar} \langle \int dx\; T[\varphi_2(\varepsilon)\, c(-\varepsilon)]\,(x)\, N_2[\bar{c}\varphi_2]\,(0)\rangle \int dx \eta_2(x) \\
&\quad\quad + \langle \int dx\; T[\varphi_2(\varepsilon)\, c(-\varepsilon)]\,(x)\, \underset{\sim}{\tilde{\varphi}}_2(0)\, \underset{\sim}{\bar{c}}(0)\rangle \cdot \int dx \{TN_2[\varphi_2(\varepsilon)\, c(-\varepsilon)]\,(x)\}
\end{aligned} \tag{AIV.1}$$

where the second coefficient is amputated on its $\tilde{\varphi}_2\bar{c}$ arguments. The only singular coefficient in this expansion is

$$\langle \int dx\; T[\varphi_2(\varepsilon)\, c(-\varepsilon)]\,(x)\, N_2[\bar{c}\varphi_2]\,(0)\rangle \tag{AIV.2}$$

which diverges logarithmically. The singular part of $\Delta_{5,\varepsilon}$ is thus proportional to $[\mathscr{S}, \int dx \eta_2(x)] \sim \int dx J_1(x)$ which is symmetrical [cf. Eq. (92)] as expected.

By a similar but more involved analysis one can evaluate the singular part of $\Delta_{4,\varepsilon}$ which assumes the form

$$\omega(\varepsilon)\int dx J_1(x) \\ +\omega'(\varepsilon)\left\{\int dx\left(J_1\delta_{J_1}+\eta_2\delta_{\eta_2}+J_2\delta_{J_2}+\eta_1\delta_{\eta_1}+\frac{\overline{m}}{\overline{e}_2}\eta_1\delta_\xi\right)(x)+\Delta_5\right\} \tag{AIV.3}$$

where ω and ω' are, in the limit $\varepsilon\to 0$ logarithmically divergent.

The resulting finite parts are however not suitable for our purpose because, due to the occurence of graphs which are φ_1 one particle reducible they do not vanish upon application of the operator Σ_{phys} [cf. (Eq. (33)]. Since the $Q_{i,\varepsilon}$'s carry the quantum numbers of $c\varphi_2$ we have:

$$\begin{aligned}\Sigma_{\text{phys}}\mathscr{S}Q_i Z &= -\bar{e}_1\Sigma_{\text{phys}}\int dx(J_1\delta_{\eta_2})(x)Q_i Z\\ &= -\bar{e}_1\Sigma_{\text{phys}}\int dp\tilde{J}_1(p)\Gamma^{(Q_i)}(p)\delta_{\tilde{J}_1(p)}Z\\ &= -\bar{e}_1\Gamma^{(Q_i)}(M^2)\Sigma_{\text{phys}}\int dp\tilde{J}_1(p)\delta_{\tilde{J}_1(p)}Z\\ &= -\bar{e}_1\Gamma^{(Q_i)}(M^2)\Sigma_{\text{phys}}\int dx(J_1\delta_{J_1}+\eta_2\delta_{\eta_2})(x)Z\end{aligned} \tag{AIV.4}$$

where

$$\Gamma^{(Q_i)}(p)=\frac{i}{\hbar}\langle TN_2[\bar{c}\varphi_2](0)Q_i\tilde{\varphi}_1(p)\rangle \tag{AIV.5}$$

is involved in the expansion:

$$\begin{aligned}\delta_{\eta_2(p)}Q_i Z &= \frac{i}{\hbar}\langle TN_2[\bar{c}\varphi_2](0)Q_i\tilde{\varphi}_1(p)\rangle\delta_{\tilde{J}_1(p)}Z\\ &\quad+\frac{i}{\hbar}\langle TN_2[\widetilde{\bar{c}\varphi_2}](p)Q_i\rangle^{I_1}Z\end{aligned} \tag{AIV.6}$$

where the upperscript I_1 denotes the set of graphs which are one particle irreducible with respect to the pair $\bar{c}\varphi_2, Q_i$.

Since $\int dx[J_1\delta_{J_1}+\eta_2\delta_{\eta_2}](x)$ is obviously a symmetric insertion, adding

$$\bar{e}_1\Gamma^{(Q_i)}(M^2)\int dx[J_1\delta_{J_1}+\eta_2\delta_{\eta_2}](x)$$

does not change the covariance of $[\mathscr{S}, Q_i]$ and produces insertions which leave the physical normalization conditions [Eq. 31 (3–7)] invariant. We now want to show that the insertions Δ_i^0 leave the normalization condition Eq. (31.8) unchanged. These insertions can be replaced by $\Delta_i=[\mathscr{S}, Q_i]$ modulo terms which trivially do not contribute to the calculation.

We shall show that:

$$\Gamma_{\bar{c}c}^{-2}\det\begin{pmatrix}\Gamma_{A^LA^L} & \Gamma_{A^L\varphi_2}\\ \Gamma_{A^L\varphi_2} & \Gamma_{\varphi_2\varphi_2}\end{pmatrix}\equiv G_{c\bar{c}}^2 D \tag{AIV.7}$$

remains regular at the $\Phi\Pi$ mass, upon insertion of Δ_i.

$$\Delta_i(G^2_{c\bar{c}}D)$$
$$= G_{c\bar{c}}D\left(2[\Delta_i G_{c\bar{c}}] + G_{c\bar{c}}\,\mathrm{Tr}\left\{\begin{pmatrix}\Gamma_{A^LA^L} & \Gamma_{A^L\varphi_2}\\ \Gamma_{A^L\varphi_2} & \Gamma_{\varphi_2\varphi_2}\end{pmatrix}\left[\Delta_i\begin{pmatrix}G_{A^LA^L} & G_{A^L\varphi_2}\\ G_{\varphi_2A^L} & G_{\varphi_2\varphi_2}\end{pmatrix}\right]\right\}\right) \tag{A IV.8}$$

where the matrix $\|G_{ij}\|$ is the inverse of the matrix $\|\Gamma_{ij}\|$. By commuting $\delta_\xi\delta_{\bar{\xi}}$ through $\mathscr{S}$ we get:

$$\begin{aligned}[\Delta_i G_{c\bar{c}}](p) &= \langle T\tilde{\bar{c}}(p)\,\mathscr{G}(0)\,Q_i\rangle\\ &= \mathrm{G}_{c\bar{c}}(p^2)\,\langle \mathrm{T}\;\tilde{\underline{\bar{c}}}(p)\,\mathscr{G}(0)\,\mathrm{Q}_i\rangle\end{aligned} \tag{A IV.9}$$

where underlining means amputation.

Similarly

$$\begin{aligned}[\Delta_i G^2_{c\bar{c}}D](p) = 2G^2_{c\bar{c}}(p^2)D(p^2)\,[&\langle T\tilde{\underline{\bar{c}}}(p)\,\mathscr{G}(0)\,Q_i\rangle\\ &+\langle T\tilde{\underline{\varphi}}_2(p)\,[\bar{e}_2N_2[\bar{c}\varphi_1]+\bar{m}\bar{c}]\,(0)\,Q_i\rangle\\ &-p^2\langle T\,\widetilde{\partial_\mu\underline{A}_\mu}(p)\,\bar{c}(0)\,Q_i\rangle]\,.\end{aligned} \tag{A IV.10}$$

Thus we only have to make sure that the bracket is regular at the $\Phi\Pi$ squared mass. Now the first term is singular due to the occurence of the $\mathscr{G}$ propagator, and the last two terms are singular due the occurence of the $\Phi\Pi$ propagators. The $\mathscr{G}$ propagator can be factorized using the Slavnov identities:

$$\begin{aligned}\langle T\tilde{\mathscr{G}}(p)\,\varphi_2(0)\rangle &= -\langle T(\bar{e}_2N_2[\bar{c}\varphi_1]+\bar{m}\bar{c})\,(0)\,\tilde{c}(p)\rangle\\ \langle T\tilde{\mathscr{G}}(p)\,\partial_\mu A_\mu(0)\rangle &= p^2\langle T\,\bar{c}(0)\,\tilde{c}(p)\rangle\,,\end{aligned} \tag{A IV.11}$$

so that:

$$\begin{aligned}\langle T\tilde{\underline{\bar{c}}}(p)\,\mathscr{G}(0)\,Q_i\rangle = &-\langle T\tilde{\underline{\bar{c}}}(p)\,\varphi_2(0)\,Q_i\rangle\,\langle T(\bar{e}_2N_2[\bar{c}\varphi_1]+\bar{m}\bar{c})\,(0)\,\tilde{c}(p)\rangle\\ &+p^2\langle T\tilde{\underline{\bar{c}}}(p)\,\partial_\mu A_\mu(0)\,Q_i\rangle\,\langle T\,\bar{c}(0)\,\tilde{c}(p)\rangle\,.\end{aligned} \tag{A IV.12}$$

Now the $\Phi\Pi$ equation of motion allows to replace $[\bar{e}_2N_2[\bar{c}\varphi_1]+\bar{m}\bar{c}]\,(0)$ by a term proportional to $\Box\,\bar{c}(0)$ up to a regular term, so that the factors $\langle T\,\bar{c}(0)\,\tilde{c}(p)\rangle$ undo the $\bar{c}$ amputation involved in their factors and produce an exact cancellation with the last two terms in Eq. (A IV.10).

Appendix V

This appendix deals with a number of details in the unitarity proof of Chapter IV. We first discuss the properties of the asymptotic ghost field wave operators L and of the corresponding asymptotic field two point function S_+. Within the coupled $(\partial_\mu A_\mu,\,\varphi_2)$ channels L can be taken as a polynominial approximation to the matrix

$$\begin{pmatrix}\Gamma_{\mathscr{G}\mathscr{G}} & \Gamma_{\bar{\mathscr{G}}\mathscr{G}}\\ \Gamma_{\bar{\mathscr{G}}\mathscr{G}} & \Gamma_{\bar{\mathscr{G}}\bar{\mathscr{G}}}\end{pmatrix}$$

where

$$\begin{aligned} \mathscr{G} &= \quad \bar{a}\,\partial_\mu A_\mu + \bar{\varrho}\,\varphi_2 \\ \bar{\mathscr{G}} &= -\bar{a}\,\partial_\mu A_\mu + \bar{\varrho}\,\varphi_2\,. \end{aligned} \tag{AV.1}$$

Denoting $x = p^2 - m_G^2$, and taking into account: (i) the normalization condition (31.8), which implies the occurence of a double zero in $\det L$ at $x=0$ (ii) the lack of singularity in the $\mathscr{G}\mathscr{G}$ propagator, which follows from the Slavnov identity and implies the occurence of a double zero in $\Gamma_{\bar{\mathscr{G}}\bar{\mathscr{G}}}$ at $x=0$. We can parametrize L in the following form:

$$L = \begin{pmatrix} A+\beta x & \gamma x \\ \gamma x & 0 \end{pmatrix} + O(x^2), \tag{AV.2}$$

the last term giving corrections of order x^2.

The corresponding matrix propagator is:

$$\begin{pmatrix} G_{\mathscr{G}\mathscr{G}} & G_{\mathscr{G}\bar{\mathscr{G}}} \\ G_{\mathscr{G}\bar{\mathscr{G}}} & G_{\bar{\mathscr{G}}\bar{\mathscr{G}}} \end{pmatrix} = -\frac{1}{\gamma^2 x^2}\begin{pmatrix} 0 & -\gamma x \\ -\gamma x & A+\beta x \end{pmatrix} + \text{Regular terms} \tag{AV.3}$$

and the S_+ operator is given by:

$$\widetilde{i\hbar S_+} = \theta(p_0)\begin{pmatrix} 0 & \dfrac{1}{\gamma}\,\delta(x) \\ \dfrac{1}{\gamma}\,\delta(x) & -\dfrac{\beta}{\gamma^2}\,\delta(x) + \dfrac{A}{\gamma^2}\,\delta'(x) \end{pmatrix} \tag{AV.4}$$

so that

$$i\hbar(\overleftarrow{L}S_+\overrightarrow{L})_{\mathscr{G}} = \theta(p_0)\begin{pmatrix} A[\overleftarrow{x}\,\delta(x) + \delta(x)\overrightarrow{x} + \overleftarrow{x}\,\delta'(\overleftarrow{x})\overrightarrow{x}] + \beta\overleftarrow{x}\,\delta(x)\overrightarrow{x} & \gamma\overleftarrow{x}\,\delta(x)\overrightarrow{x} \\ \gamma\overleftarrow{x}\,\delta(x)\overrightarrow{x} & 0 \end{pmatrix} \tag{AV.5}$$

where the symbols $\overrightarrow{x}$, $\overleftarrow{x}$ are to remind that the usual identities: $x\,\delta(x) = x^2\,\delta'(x) = 0$ cannot be used here because this kernel is to be tested with functions which have poles at $x=0$.

Concerning the Faddeev-Popov fields, let us define

$$\begin{aligned} \mathscr{C} &= c \\ \bar{\mathscr{C}} &= 2\bar{\varrho}[\bar{m} + \bar{e}_2\Gamma_{\bar{c}\eta_1}(m_G^2)]\,\bar{c}\,. \end{aligned} \tag{AV.6}$$

Using the results of Appendix III we get:

$$G_{\mathscr{C}\bar{\mathscr{C}}} = \frac{1}{\gamma x} + \text{Regular terms} \tag{AV.7}$$

and

$$i\hbar(\overleftarrow{L}S_+\overrightarrow{L})_{\Phi\Pi} = \gamma\overleftarrow{x}\,\delta(x)\overrightarrow{x}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{AV.8}$$

The $\mathscr{A}$ operator Eq. (101) can now be written

$$\begin{aligned} \mathscr{A} = \int dp\,\theta(p_0)\,\{&\overleftarrow{\delta}_{J_{\mathscr{G}}(-p)}[A(\overleftarrow{x}\,\delta(x) + \delta(x)\overrightarrow{x} + \overleftarrow{x}\,\delta'(x)\overrightarrow{x}) + \beta\overleftarrow{x}\,\delta(x)\overrightarrow{x}]\,\overrightarrow{\delta}_{J_{\mathscr{G}}(p)} \\ &+ \gamma\overleftarrow{x}\,\delta(x)\overrightarrow{x}[\overleftarrow{\delta}_{J_{\bar{\mathscr{G}}}(-p)}\overrightarrow{\delta}_{J_{\mathscr{G}}(p)} + \overleftarrow{\delta}_{J_{\mathscr{G}}(p)}\overrightarrow{\delta}_{J_{\bar{\mathscr{G}}}(p)} + \overleftarrow{\delta}_{J_{\bar{\mathscr{C}}}(-p)}\overrightarrow{\delta}_{J_{\bar{\mathscr{C}}}(p)} \\ &- \overleftarrow{\delta}_{J^*_{\mathscr{C}}(-p)}\overrightarrow{\delta}_{J_{\mathscr{C}}(p)}]\}_{x=p^2-m_{\mathscr{G}}^2} \end{aligned} \tag{AV.9}$$

where J^*_φ is the source of the field $\varphi^\dagger$ that is used in the definition of the antitime ordered functional.

Before pursuing, let us regularize the δ functions according to

$$\delta(x) \to \delta_\varepsilon(x) = \frac{e^{-\frac{x^2}{\varepsilon}}}{\sqrt{\pi\varepsilon}} \tag{AV.10}$$

so that we may forget about the arrows on the $\vec{x}$ variables. Owing to the invariance of the lagrangian under the transformation:

$$\begin{aligned} c &\to -\bar{c} \\ \bar{c} &\to c \end{aligned} \tag{AV.11}$$

we have the identity

$$\overleftarrow{\delta}_{\tilde{J}^*_{\bar{\mathscr{C}}}(-p)} \overrightarrow{\delta}_{\tilde{J}_{\bar{\mathscr{C}}}(p)} - \overleftarrow{\delta}_{\tilde{J}^*_{\mathscr{C}}(p)} \overrightarrow{\delta}_{\tilde{J}_{\mathscr{C}}(p)} = \overleftarrow{\delta}_{\tilde{J}^*_{\mathscr{C}}(-p)} \overrightarrow{\delta}_{\tilde{J}_{\bar{\mathscr{C}}}(p)} + \overleftarrow{\delta}_{\tilde{J}^*_{\bar{\mathscr{C}}}(-p)} \overrightarrow{\delta}_{\tilde{J}_{\mathscr{C}}(p)} \,. \tag{AV.12}$$

Taking into account [Eq. (43)] we get:

$$\begin{aligned} &[\overrightarrow{\delta}_{\tilde{J}_{\bar{\mathscr{G}}}(p)}, \overrightarrow{\mathscr{S}}] = \delta_{\tilde{J}_{\bar{\mathscr{C}}}(p)} + O(x) \\ &[\overrightarrow{\delta}_{\tilde{J}_{\mathscr{G}}(p)}, \overrightarrow{\mathscr{S}}] = -\frac{\bar{\alpha}}{\gamma} x \delta_{\tilde{J}_{\bar{\mathscr{C}}}(p)} + O(x^2) \\ &\{\overrightarrow{\delta}_{\tilde{J}_{\mathscr{C}}(p)}, \overrightarrow{\mathscr{S}}\} = \delta_{\tilde{J}_{\mathscr{G}}(p)} \\ &\{\overrightarrow{\delta}_{\tilde{J}_{\bar{\mathscr{C}}}(p)}, \overrightarrow{\mathscr{S}}\} = 0 \end{aligned} \tag{AV.13}$$

which yields

$$\begin{aligned} [\mathscr{A}, \overrightarrow{\mathscr{S}}] = \int dp\, \theta(p_0) \Bigg\{ & \overleftarrow{\delta}_{\tilde{J}_{\mathscr{G}}(-p)} [2Ax\,\delta(x) + Ax^2\,\delta'(x) + \beta x^2\,\delta(x)] \left(-\frac{\bar{\alpha}}{\gamma} x\right) \overrightarrow{\delta}_{\tilde{J}_{\mathscr{G}}(p)} \\ & + \gamma x^2\,\delta(x) \left[\overleftarrow{\delta}_{\tilde{J}_{\bar{\mathscr{G}}}(-p)} \left(-\frac{\bar{\alpha}}{\gamma} x\right) \overrightarrow{\delta}_{\tilde{J}_{\bar{\mathscr{C}}}(p)} + \overleftarrow{\delta}_{\tilde{J}_{\mathscr{G}}(-p)} \overrightarrow{\delta}_{\tilde{J}_{\bar{\mathscr{C}}}(p)} + \overleftarrow{\delta}_{\tilde{J}^*_{\bar{\mathscr{C}}}(-p)} \overrightarrow{\delta}_{\tilde{J}_{\mathscr{G}}(p)} \right] \Bigg\} \\ & + O(\varepsilon) \,. \end{aligned} \tag{AV.14}$$

Since the propagator attached to the $\mathscr{G}$ and $\bar{\mathscr{C}}$ legs have only simple poles the β dependent term in the righthand side of Eq. (AV.14) is of order ε. This does not happen for the term involving $\overleftarrow{\delta}_{\tilde{J}_{\bar{\mathscr{G}}}}$ because the $(\bar{\mathscr{G}}, \bar{\mathscr{G}})$ propagator has a double pole.

However we have

$$x\,G_{\bar{\mathscr{G}}\bar{\mathscr{G}}}(p^2) = -\frac{A}{\gamma} G_{\mathscr{G}\bar{\mathscr{G}}}(p^2) + \text{Regular terms} \,. \tag{AV.15}$$

Since the $(\mathscr{G}, \mathscr{G})$ propagator has no pole we have:

$$x\,\overrightarrow{\delta}_{\tilde{J}_{\bar{\mathscr{G}}}(p)} Z = -\frac{A}{\gamma} \overrightarrow{\delta}_{J_{\mathscr{G}}(p)} Z + \text{Regular terms} \,. \tag{AV.16}$$

Thus

$$[\mathscr{A},\mathscr{S}] = \int dp\,\theta(p_0)\left\{\overleftarrow{\delta}_{\tilde{J}_{\mathscr{G}}(-p)}[2Ax\,\delta(x) + Ax^2\,\delta'(x) - Ax\,\delta(x)]\left(-\frac{\overline{\alpha}}{\gamma}x\right)\overrightarrow{\delta}_{\tilde{J}_{\bar{\mathscr{C}}}(p)}\right.$$
$$\left. + \gamma x^2\,\delta(x)\,[\overleftarrow{\delta}_{\tilde{J}_{\mathscr{G}}(-p)}\overrightarrow{\delta}_{\tilde{J}_{\bar{\mathscr{G}}}(p)} + \overleftarrow{\delta}_{\tilde{J}^*_{\bar{\mathscr{C}}}(-p)}\overrightarrow{\delta}_{\tilde{J}_{\mathscr{G}}(p)}]\right\} + O(\varepsilon) \tag{AV.17}$$
$$= \int dp\,\theta(p_0)\,\{\gamma x^2\,\delta(x)\,[\overleftarrow{\delta}_{\tilde{J}_{\mathscr{G}}(-p)}\overrightarrow{\delta}_{\tilde{J}_{\bar{\mathscr{G}}}(p)} + \overleftarrow{\delta}_{\tilde{J}^*_{\bar{\mathscr{C}}}(-p)}\overrightarrow{\delta}_{\tilde{J}_{\mathscr{G}}(p)}]\} + O(\varepsilon)\,.$$

The vanishing of the A dependent terms is due to the absence of double poles in the $\mathscr{G}$ and $\mathscr{C}$ propagators.

As a consequence we get:

$$[\mathscr{A},\overrightarrow{\mathscr{S}}] = [\overleftarrow{\mathscr{S}}^*,\mathscr{A}] + O(\varepsilon) \tag{AV.18}$$

where $\overleftarrow{\mathscr{S}}^*$ is the Slavnov operator which characterizes the anti-time ordered functional.

Indeed Eq. (AV.18) is a consequence of the symmetry of both $\mathscr{A}$ and $[\mathscr{A},\overrightarrow{\mathscr{S}}]$ with respect to the transposition and the complex conjugation of the sources. We are now in condition to state the following identity:

$$\begin{aligned}
&S^\dagger_{\text{phys}}(\underline{J})\,\{\textstyle\int dp\,\theta(p_0)\gamma x^2\,\delta(x)\,[\overleftarrow{\delta}_{\tilde{J}_{\mathscr{G}}(-p)}\overrightarrow{\delta}_{\tilde{J}_{\bar{\mathscr{G}}}(p)} + \overleftarrow{\delta}_{\tilde{J}_{\bar{\mathscr{G}}}(-p)}\overrightarrow{\delta}_{\tilde{J}_{\mathscr{G}}(p)} \\
&\quad + \overleftarrow{\delta}_{\tilde{J}^*_{\bar{\mathscr{C}}}(-p)}\overrightarrow{\delta}_{\tilde{J}_{\mathscr{C}}(p)} + \overleftarrow{\delta}_{\tilde{J}^*_{\mathscr{C}}(-p)}\overrightarrow{\delta}_{\tilde{J}_{\bar{\mathscr{C}}}(p)}]\}\;e^{\lambda\mathscr{A}}\,S_{\text{phys}}(\underline{J})|_{\underline{J}=0} \\
&= S^\dagger_{\text{phys}}(\underline{J})\;\textstyle\int dp\,\theta(p_0)\gamma x^2\,\delta(x)\,[\{\overleftarrow{\delta}_{\tilde{J}^*_{\mathscr{C}}(-p)},\overleftarrow{\mathscr{S}}^*\}\,\overrightarrow{\delta}_{\tilde{J}_{\bar{\mathscr{G}}}(p)} \\
&\quad + \overleftarrow{\delta}_{\tilde{J}_{\bar{\mathscr{G}}}(-p)}\{\overrightarrow{\delta}_{\tilde{J}_{\mathscr{C}}(p)},\overrightarrow{\mathscr{S}}\} + [\overleftarrow{\mathscr{S}}^*,\overleftarrow{\delta}_{\tilde{J}_{\bar{\mathscr{G}}}(-p)}]\,\overrightarrow{\delta}_{\tilde{J}_{\mathscr{C}}(p)} \\
&\quad + \overleftarrow{\delta}_{\tilde{J}^*_{\mathscr{C}}(-p)}[\overrightarrow{\delta}_{\tilde{J}_{\bar{\mathscr{G}}}(p)},\overrightarrow{\mathscr{S}}]] + O(\varepsilon)\quad e^{\lambda\mathscr{A}}\,S_{\text{phys}}(\underline{J})|_{\underline{J}=0} \\
&= S^\dagger_{\text{phys}}(\underline{J})\,\{\textstyle\int dp\,\theta(p_0)\gamma x^2\,\delta(x)\,[\overleftarrow{\delta}_{\tilde{J}^*_{\mathscr{C}}(-p)}[\overleftarrow{\mathscr{S}}^*,e^{\lambda\mathscr{A}}]\,\overrightarrow{\delta}_{\tilde{J}_{\bar{\mathscr{G}}}(p)} \\
&\quad - \overleftarrow{\delta}_{\tilde{J}_{\bar{\mathscr{G}}}(-p)}[\overleftarrow{\mathscr{S}}^*,e^{\lambda\mathscr{A}}]\,\overrightarrow{\delta}_{\tilde{J}_{\mathscr{C}}(p)} + \overleftarrow{\delta}_{\tilde{J}_{\bar{\mathscr{G}}}(-p)}[e^{\lambda\mathscr{A}},\overrightarrow{\mathscr{S}}]\,\overrightarrow{\delta}_{\tilde{J}_{\mathscr{C}}(p)} \\
&\quad - \overleftarrow{\delta}_{\tilde{J}^*_{\mathscr{C}}(-p)}[e^{\lambda\mathscr{A}},\overrightarrow{\mathscr{S}}]\,\overrightarrow{\delta}_{\tilde{J}_{\bar{\mathscr{G}}}(p)}] + O(\varepsilon)\}\;S_{\text{phys}}(\underline{J})|_{\underline{J}=0} = O(\varepsilon)\,.
\end{aligned} \tag{AV.19}$$

In the first step of this reduction, the $O(\varepsilon)$ term takes into account contributions of the kind:

$$S^\dagger_{\text{phys}}(\underline{J})\int dp\,\theta(p_0)\gamma x^3\,\delta(x)\,\overleftarrow{\delta}_{\tilde{J}^*_{\bar{\mathscr{C}}}(-p)}\overrightarrow{\delta}_{\tilde{J}_{\mathscr{C}}(p)}e^{\lambda\mathscr{A}}\,S_{\text{phys}}(\underline{J})|_{\underline{J}=0}\,.$$

The second step makes use of the Slavnov identities

$$\begin{aligned}
\overrightarrow{\mathscr{S}}\,S_{\text{phys}}(\underline{J}) &= 0 \\
S^\dagger_{\text{phys}}(\underline{J}^*)\,\overleftarrow{\mathscr{S}}^* &= 0
\end{aligned} \tag{AV.20}$$

and takes advantage of the zero source condition by commuting $\overrightarrow{\mathscr{S}}$ to the left and $\overleftarrow{\mathscr{S}}^*$ to the right.

The last step is a trivial consequence of Eq. (AV.18). Going back to the expression for $\mathscr{A}$, and taking into account the symmetry property, one gets Eq. (104) of Section IV.

Appendix VI. A Class of Local Gauge Invariant Operators

In order to define a local operator $\Theta(x)$ of dimension d, we shall first consider an effective lagrangian.

$$\mathscr{L}_{\text{eff}}^{(\omega)}(x) = \mathscr{L}_{\text{eff}}(\underline{\varphi}, \underline{J}, \underline{\eta})(x) + \omega(x) N_d[\Theta(x)] \tag{AVI.1}$$

where ω is a classical field of Dimension 4-d.

The first criterion for gauge invariance is

$$\mathscr{S} Z_c(\underline{J}, \underline{\eta}, \omega) = O(\omega^2) \tag{AVI.2}$$

where $\mathscr{S}$ is the operator defined in Eq. (43). Assuming that Eq. (AVI.2) has solutions, they are in one to one correspondence with those found at the tree level. They will be further specified by as many physical normalization conditions as are necessary to specify physical operators of this type at the tree level (namely modulo the ideal generated by operators which vanish on the physical subspace). It follows that Θ is ambiguous up to a linear combination of operators whose tree approximations vanish on the physical subspace.

The proof of gauge invariance then proceeds as usual (Chapter III). Keeping terms of the first order in ω, one looks for the most general solution of

$$\partial_\lambda Z(\underline{J}, \underline{\eta}, \omega) = \frac{i}{\hbar} \Delta_\lambda^{(\omega)} Z(\underline{J}, \underline{\eta}, \omega) \tag{AVI.3}$$

which is of the form

$$\Delta_\lambda^{(\omega)} = \Delta_\lambda + \int dx\, \omega(x) [\Sigma_i f_i \Theta_i^S(x)] \tag{AVI.4}$$

where the second term in the right hand side of Eq. (AVI.4) is a perturbation of the ω dependent part of the solutions of Eq. (AVI.2) in the tree approximation. Testing now Eq. (AVI.4) with the physical normalization conditions which specify $\Theta(x)$ shows that the problem reduces to check that the perturbations of operators which have null physical restriction in the tree approximation retain this property to all orders.

Finally the stability of the physical subspace under application of $\Theta(x)$, up to the zero norm states is a consequence of Eq. (AVI.2) as follows from a slight generalization of the argument in Chapter IV: defining $S(\omega)$ by replacing $Z(\underline{J}, \underline{\eta})$ by $Z(\underline{J}, \underline{\eta}, \omega)$ in the LSZ definition of S in the overall Fock space, $\Theta(x)$ is defined according to

$$\begin{aligned} \Theta(x) &= \frac{\hbar}{i} S^{-1} \left. \frac{\delta S}{\delta \omega(x)} \right|_{\omega=0} \\ \Theta^\dagger(x) &= -\frac{\hbar}{i} \left. \frac{\delta S^\dagger}{\delta \omega(x)} S \right|_{\omega=0} \end{aligned} \tag{AVI.5}$$

Let E_0 be the projector on the physical subspace generated by φ_1 and A_μ^T quanta. One wishes to show that

$$E_0\,\Theta^\dagger(x)\,E_0\,\Theta(y)\,E_0 = E_0\,\Theta^\dagger(x)\Theta(y)E_0 \tag{A VI.6}$$

i.e.

$$E_0\frac{\delta S^\dagger}{\delta\omega(x)}\frac{\delta S}{\delta\omega(y)}E_0\bigg|_{\omega=0} = E_0\frac{\delta S^\dagger}{\delta\omega(x)}S\,E_0\,S^\dagger\frac{\delta S}{\delta\omega(y)}E_0\bigg|_{\omega=0} \tag{A VI.7}$$

where the unitarity of S has been used.

Equation (A VI.7) follows simply from

$$E_0\,S\,E_0\,S^\dagger\,E_0 = E_0 \tag{A VI.8}$$

which is the result of Chapter IV and from the identities

$$\begin{aligned} E_0\frac{\delta S^\dagger}{\delta\omega(x)}\frac{\delta S}{\delta\omega(y)}E_0\bigg|_{\omega=0} &= E_0\frac{\delta S^\dagger}{\delta\omega(x)}E_0\frac{\delta S}{\delta\omega(y)}E_0\bigg|_{\omega=0} \\ E_0\frac{\delta S^\dagger}{\delta\omega(x)}S\,E_0\bigg|_{\omega=0} &= E_0\frac{\delta S^\dagger}{\delta\omega(x)}E_0\,S\,E_0\bigg|_{\omega=0} \end{aligned} \tag{A VI.9}$$

which are consequences of the first criterion for gauge invariance:

$$\left[\frac{\delta}{\delta\omega},\mathscr{S}\right]=0 \tag{A VI.10}$$

and of the argument in Chapter IV.

Example. a) $d=2$ $C=+1$.

Θ is a linear combination of $\{\varphi_1, \varphi_1^2, \varphi_2^2, A_\mu A_\mu, \bar{c}c\}$,

$\mathscr{S}\Theta$ is a linear combination of $\{\varphi_2\bar{c}, \varphi_1\varphi_2\bar{c}, \bar{c}\partial_\mu A_\mu, \partial_\mu\bar{c}A_\mu\}$; so is the term $O(\omega)$ in $\mathscr{S}Z(\underline{J},\underline{\eta},\omega)$.

Thus there is no anomaly i.e. there exists one invariant local operator which is a perturbation of

$$\varphi_1^2+\varphi_2^2+2v\varphi_1\,,$$

which is non zero in the physical subspace. This operator is completely determined by e.g.

$$\langle\Omega,\Theta(x)\varphi_{1,\mathrm{in}}(y)\Omega\rangle = i\Delta_M^+(x-y)$$

and can serve as a gauge invariant interpolating field operator for $\varphi_{1,\mathrm{in}}$.

b) $d=3$ $C=-1$ vector operator.

It is trivial that $\partial_\mu G_{\mu\nu}$ solves the problem:

$$[(\Box g_{\mu\nu}-\partial_\mu\partial_\nu)\,\delta_{J_\nu(x)},\mathscr{S}]=0$$

$$\partial_\lambda\prod_1^n{}_i(\Box g_{\mu_i\nu_i}-\partial_{\mu_i}\partial_{\nu_i})\,\delta_{J_{\nu_i}(x_i)}\Sigma_{\mathrm{phys}}Z|_{J=\eta=0}=0 \qquad (\lambda=\alpha, m_G^2)\,.$$

Appendix VII

This appendix is devoted to the main steps involved in the treatement of an extended class of gauges involving quadratic terms odd under charge conjugation. In the tree approximation, the Slavnov transformation [cf. Eq. (24)] is now taken to be:

$$\begin{aligned}\delta A_\mu &= \lambda\, \partial_\mu \bar{c}\\ \delta\varphi_1 &= -\lambda e \varphi_2 \bar{c}\\ \delta\varphi_2 &= \lambda e(\varphi_1 + v)\bar{c}\\ \delta c &= \lambda(\partial_\mu A_\mu + \varrho\varphi_2 + \sigma\varphi_1\varphi_2) = \lambda\mathscr{G}\\ \delta\bar{c} &= 0\,.\end{aligned} \tag{A VII.1}$$

The most general lagrangian fulfilling the corresponding Slavnov identity is now:

$$\begin{aligned}\mathscr{L} = &-\frac{Z_A}{4} G_{\mu\nu}G_{\mu\nu} + Z_1(D_\mu\varphi)^* D_\mu\varphi + \mu^2\varphi^*\varphi - g(\varphi^*\varphi)^2\\ &-\frac{1}{\alpha}\left[\frac{\mathscr{G}^2}{2} + \bar{c}\frac{\delta\mathscr{G}}{\delta A}c\right] + \beta\left[\frac{A_\mu A_\mu}{2} - \bar{c}c + \frac{\sigma}{e}\frac{\varphi_1^2}{2} + \frac{\varrho}{e}\varphi_1\right]\end{aligned} \tag{A VII.2}$$

where now

$$\frac{\delta\mathscr{G}(x)}{\delta A(y)} = \mathscr{M}(x,y) = [\Box + \varrho e v + e(\varrho + \sigma v) + e\sigma(\varphi_1^2 - \varphi_2^2)]\,(x)\,\delta(x-y)\,. \tag{A VII.3}$$

Keeping the normalization conditions Eq. (31) unchanged Eq. (32) is unchanged except for Eq. (32.3) which now reads

$$M^2 = \frac{2}{Z_1}\left(6gv^2 - \mu^2 - \frac{\beta\sigma}{2c}\right). \tag{A VII.4}$$

But due to Eq. (32.8) ($\beta = 0$), the overall algebraic system Eq. (32) is unchanged. We now turn to the details of the Slavnov identity which we shall express in linear form as in Eq. (43). Before doing so we need to introduce at least one external field γ coupled to $\varphi_1\varphi_2$, to which we assign dimension two and odd charge conjugation quantum number. The corresponding term however undergoes a variation under the Slavnov transformation (A VII.1), which forces us also to introduce at least one field coupled to $(\varphi_1^2 - \varphi_2^2)\bar{c}$. However, for later use, we shall right away introduce three fields of dimension one, $\vec{\beta} \equiv (\beta_1, \beta_2, \beta_3)$ coupled to three independent linear combinations of: $\vec{\bar{\mathscr{C}}} \equiv (\bar{\mathscr{C}}_1, \bar{\mathscr{C}}_2, \bar{\mathscr{C}}_3) \equiv (\varphi_1^2\bar{c}, \varphi_2^2\bar{c}, A_\mu A_\mu \bar{c})$ and also a field of dimension dimension zero τ coupled to $A_\mu \bar{c}\,\partial_\mu \bar{c}$. Thus we have introduced external fields coupled to a system of operators which is closed under Slavnov transformations. The most general lagrangian invariant under charge conjugation,

$\Phi\Pi$ neutral and consistent with power counting is now

$$\begin{aligned}\mathscr{L}(\underline{\varphi}, \underline{J}, \underline{\eta}, \gamma, \vec{\beta}, \tau) = {} & \mathscr{L}(\underline{\varphi}) + \eta_1(z_1 \bar{c}\varphi_1 + z_1' \bar{c}) + \eta_2 z_2 \bar{c}\varphi_2 \\ & + \gamma(a_1\varphi_1\varphi_2 + a_2\varphi_2 + a_3\,\partial_\mu A_\mu) \\ & + \beta_i(B_{ij}\overline{\mathscr{C}}_j + M_i\varphi_1\bar{c} + v_i\bar{c} + w_i\,\Box\,\bar{c}) \\ & + \tau a_4 A_\mu \bar{c}\,\partial_\mu \bar{c} + a_5\gamma^2 + J_1\varphi_1 + J_2\varphi_2 + J_\mu A_\mu + \bar{\xi}c + \xi\bar{c}\end{aligned} \tag{AVII.5}$$

and the Slavnov identity assumes the general form which will be needed later:

$$\begin{aligned}\mathscr{S}_\sigma Z_C = \mathscr{S} Z_C + \textstyle\int dx [& -\bar{\sigma}\bar{\xi}\delta_\gamma \\ & + \gamma\{\lambda_1\,\Box\,\delta_\xi + \lambda_2\delta_\xi + \lambda_3\delta_{\eta_1} + \vec{b}\,\delta_{\vec{\beta}}\} + \vec{d}\cdot\vec{\beta}\delta_\tau](x) Z_C = 0\end{aligned} \tag{AVII.6}$$

where however in spite of the huge number of parameters involved in (AVII.5) the coefficients in $\mathscr{S}_\sigma$ are always constrained by

$$\vec{d}\cdot\vec{b} = 0 \tag{AVII.7}$$

because $b_i \propto (B^{-1})_{1i} - (B^{-1})_{2i}$ and $d_i \propto B_{i3}$ $(i=1,2,3)$. One can in fact verify directly that this is the only constraint on the coefficients of the Slavnov identity.

We are now ready to prove that one can fulfill a Slavnov identity of the type (AVII.6) to all orders in $\hbar$, with coefficients constrained by Eq. (AVII.7). Performing a Slavnov variation of the type (AVII.1) on an effective lagrangian of the form[1] $N_4\mathscr{L}(\underline{\varphi}, \underline{J}, \underline{\eta}, \gamma, \vec{\beta}, \tau)$ [cf. Eq. (AVII.5)] yields:

$$\mathscr{S}_\sigma Z = \sum_{i=1}^{32} c_i \Delta_i Z \tag{AVII.8}$$

where the first 23 Δ_i's are listed in Eq. (26), and the last nine Δ_i's are:

$$\begin{aligned} &(24) && \textstyle\int dx[\gamma\bar{c}](x) \\ &(25) && \textstyle\int dx[\gamma\,\Box\,\bar{c}](x) \\ &(26) && \textstyle\int dx[\gamma N_3[\bar{c}\varphi_1]](x) \\ &(27) && \textstyle\int dx[\gamma N_3[\bar{c}\varphi_1^2]](x) \\ &(28) && \textstyle\int dx[\gamma N_3[\bar{c}\varphi_2^2]](x) \\ &(29) && \textstyle\int dx[\gamma N_3[\bar{c}A_\mu A_\mu]](x) \\ &(30,31,32) && \textstyle\int dx[\beta_i N_4[A_\mu \bar{c}\,\partial_\mu \bar{c}]](x)\,. \end{aligned} \tag{AVII.9}$$

After elimination of the Δ_i's which are naïve variations, one is left with an $\mathscr{L}^{\mathrm{eff}}$, including source terms such that only Δ_{21}, Δ_{22}, Δ_{23} and a linear combination of Δ_{30}, Δ_{31}, Δ_{32}, namely: $\Delta_{33} = \vec{b}\int dx[\vec{\beta} N_4[A_\mu\bar{c}\,\partial_\mu\bar{c}]](x)$ remain on the right hand side of the Slavnov identity which reads:

$$\mathscr{S} Z_C = (c_{21}\Delta_{21} + c_{22}\Delta_{22} + c_{23}\Delta_{23} + c_{33}\Delta_{33}) Z_C\,. \tag{AVII.10}$$

Now recall that $\mathscr{S}$ is the naïve Slavnov identity associated with the Slavnov transformation we started with, hence $\vec{d}\cdot\vec{b} = 0$. Now compute $\mathscr{S}^2 Z_C$, which

[1] If a monomial is of the form $\varepsilon M(\varphi)$ where ε is an external field to which dimension d was assigned, $N_4[\varepsilon M(\varphi)]$ means $\varepsilon N_{4-d}[M(\varphi)]$.

because of this condition has the same form as the $\Phi\Pi$ equation of motion integrated through $\bar{\xi}$.

We have:

$$\mathscr{S}^2 Z_C = \mathscr{S}[c_{21}\Delta_{21} + c_{22}\Delta_{22} + c_{23}\Delta_{23} + c_{33}\Delta_{33}] Z_C . \qquad \text{(AVII.11)}$$

The same argument as before shows that $\mathscr{S}^2 Z_C$ has coefficients identical with those occuring in the $\Phi\Pi$ equation of motion, except for the mass term, and that

$$c_{21} = c_{22} = c_{23} = c_{33} = 0 . \qquad \text{(AVII.12)}$$

At this point the Lagrangian depends on 24 parameters since 28 relations were imposed on the initial 52 parameters. Together with the 14 independent parameters of the Slavnov identity we have 38 parameters which can be fixed by the 9 normalization conditions in Eq. (31)[2] together with 26 others fixing the couplings with the external fields[3]. It is a matter of routine to verify that the corresponding system is soluble the condition $\vec{d} \cdot \vec{b} = 0$ being preserved. Three parameters are then left free: e_2, a, σ. The gauge parameter σ could be fixed by imposing an extra normalization condition on $\Gamma_{\varphi_1^2 \varphi_2^2}$. We now extend the proof of the gauge invariance of the scattering operator. In order to do so, we shall decompose again the insertion Δ_λ generating an infinitesimal variation of the gauge parameter λ according to

$$\Delta_\lambda = \hat{\Delta}_\lambda + \Delta_\lambda^S \qquad \text{(AVII.13)}$$

and we shall show that it is possible to choose the two insertions $\hat{\Delta}_\lambda$ and Δ_λ^S satisfying the same requirements as in Chapter III, Eq. (68) and Eq. (73). First of all, let us write Eq. (AVII.6) in the form.

$$\mathscr{S}_\sigma Z_C \equiv \left\{ \sum_{0}^{9}{}_i\, c_i s_i + \int dx [\gamma \vec{b} \cdot \delta_{\vec{\beta}} + \vec{d} \cdot \vec{\beta} \delta_\tau](x) \right\} Z_C \qquad \text{(AVII.14)}$$

the first six s_i's are listed in Eq. (76) the remaining four are:

$$\begin{aligned} s_6 &= \int dx [\bar{\xi} \delta_\gamma](x) \\ s_7 &= \int dx [\gamma \Box \delta_{\xi}](x) \\ s_8 &= \int dx [\gamma \delta_{\xi}](x) \\ s_9 &= \int dx [\gamma \delta_{\eta_1}](x) . \end{aligned} \qquad \text{(AVII.15)}$$

The derivative of $\mathscr{S}_\sigma$ with respect to the parameter λ is obtained by differenciating the c_i's and the vectors $\vec{b}$ and $\vec{d}$. Since we know that $\vec{b} \cdot \vec{d} = 0$ [Eq. (AVII.7)] independently on λ, we have the equation:

$$\vec{d} \cdot \partial_\lambda \vec{b} + \vec{b} \partial_\lambda \vec{d} = 0 . \qquad \text{(AVII.16)}$$

---

[2] Using the same kind of arguments as in Appendix III it can be shown that the condition given in Eq. (31.8) is a suitable normalization condition and that the mass term in the $\Phi\Pi$ equation of motion has the same coefficient as the corresponding term in $\mathscr{S}^2 Z_C$.

[3] The simplest additional normalization conditions are:

$$z_1 = z_2 = a = B_{11} = B_{22} = B_{33} = a_4 = 1$$

$$z_1' = a_2 = a_3 = a_5 = \vec{u} = \vec{v} = \vec{w} = B_{i \neq j} = 0 .$$

We can parametrize $\partial_\lambda \vec{b}$ and $\partial_\lambda \vec{d}$ by introducing the two cartesian triplets:

$$\begin{array}{c} \vec{b}, \vec{b}_1, \vec{b}_2 \\ \vec{d}, \vec{d}_1, \vec{d}_2 \end{array} \tag{A VII.17}$$

in the form:

$$\begin{aligned} \partial_\lambda \vec{b} &\equiv \sum_{i=1,2} x_i^{(\lambda)} \vec{d}_i + z^{(\lambda)} b^2 \vec{d} \\ \partial_\lambda \vec{d} &= \sum_{i=1,2} y_i^{(\lambda)} \vec{b}_i - z^{(\lambda)} d^2 \vec{b} . \end{aligned} \tag{A VII.18}$$

Thus we have to find a non physical $\hat{\Delta}_\lambda$ satisfying the equation:

$$\begin{aligned} i/\hbar[\hat{\Delta}_\lambda \mathscr{S}_\sigma] &= \sum_1^9{}_i\, \partial_\lambda c_i^{(\lambda)} s_i \\ + \int dx \Big\{ \sum_{i=1,2} x_i^{(\lambda)} \gamma \vec{d}_i \cdot \delta_{\vec{\beta}} &+ \sum_{i=1,2} y_i^{(\lambda)} \vec{b}_i \cdot \vec{\beta} \delta_\tau + z^{(\lambda)} (b^2 \gamma \vec{d} \cdot \delta_{\vec{\beta}} - d^2 \vec{b} \cdot \vec{\beta} \delta_\tau) \Big\} (x) . \end{aligned} \tag{A VII.19}$$

The insertion $\hat{\Delta}_\lambda$ is a linear combination of a basis of covariant non physical insertions which can be found as follows.

First we introduce, in analogy with Eq. (78) three operators $Q_{i,\varepsilon}$ with $\delta_{g_6} = \delta_\gamma$. By the same construction as in Chapter III we get three insertions Δ_i^0 of covariances $c_i s_i$ $(i = 4, 5, 6)$.

Then using the generalized action principle [6] we can complete the basis of covariant non physical insertions as indicated in Table (A VII.1).

Table A VII.1

Insertion	Covariance
$\Delta_{\eta_2} = \frac{\hbar}{i} \int dx [\eta_2 \delta_{\eta_2}] (x)$	$-c_1 s_1$
$\Delta_{\eta_1} = \frac{\hbar}{i} \int dx [\eta_1 \delta_{\eta_1}] (x)$	$-c_2 s_2 - c_9 s_9$
$\Delta'_{\eta_1} = \frac{\hbar}{i} \int dx [\eta_1 \delta_\xi] (x)$	$-c_1 s_3 - c_9 s_8$
$\Delta_\beta = \frac{\hbar}{ib^2} \int dx [\vec{b} \cdot \vec{\beta} \square \delta_\xi] (x)$	$-s_7$
$\Delta'_\beta = \frac{\hbar}{ib^2} \int dx [\vec{b} \cdot \vec{\beta} \delta_\xi] (x)$	$-s_8$
$\Delta''_\beta = \frac{\hbar}{ib^2} \int dx [\vec{b} \cdot \vec{\beta} \delta_{\eta_1}] (x)$	$-s_9$
$\Delta^i_{\beta^2} = \frac{\hbar}{ib^2} \int dx [\vec{b} \cdot \vec{\beta}\, \vec{d}_i \cdot \delta_{\vec{\beta}}] (x)$	$-\int dx [\gamma \vec{d}_i \delta_{\vec{\beta}}] (x)$
$\Delta'^i_{\beta^2} = \frac{\hbar}{id^2} \int dx [\vec{b}_i \cdot \vec{\beta}\, \vec{d} \cdot \delta_{\vec{\beta}}] (x)$	$\int dx [\vec{b}_i \cdot \vec{\beta} \delta_\tau] (x)$
$\Delta''^i_{\beta^2} = \frac{\hbar}{i} \int dx [\vec{b} \cdot \vec{\beta}\, \vec{d} \cdot \delta_{\vec{\beta}}] (x)$	$\int dx [d^2 \vec{b} \cdot \vec{\beta} \delta_\tau - b^2 \gamma \vec{d} \cdot \delta_{\vec{\beta}}] (x)$

It is evident that one can find a particular solution of Eq. (A VII.19) as a linear combination of these insertions and of the three Δ_i^0's. In the same way one can get other non physical insertions which are listed in Table (A VII.2).

Table A VII.2

Insertions	Covariance
$\Delta_1^0 = \int dx\, J_1(x)$	0
$\Delta_c = \frac{\hbar}{i} \int dx [\xi \delta_{\bar{\xi}}](x)$	$c_4 s_4 + c_5 s_5 + c_6 s_6$
$\Delta_{\varphi_2} = \frac{\hbar}{i} \int dx [J_2 \delta_{J_2}](x)$	$c_2 s_2 + c_3 s_3 - c_5 s_5$
$\Delta_\gamma = \frac{\hbar}{i} \int dx [\gamma \delta_\gamma](x)$	$-c_6 s_6 + \sum_{\iota=7,8,9} c_i s_\iota + \gamma \vec{b} \cdot \delta_\beta$
$\Delta_\gamma'' = \frac{\hbar}{i} \int dx [\gamma \delta_{J_2}](x)$	$c_3 s_8 + c_2 s_9 - c_6 s_5$
$\Delta_\gamma'' = \frac{\hbar}{i} \int dx [\gamma \partial_\mu \delta_{J_\mu}](x)$	$s_7 - c_6 s_4$
$\Delta_\beta^i = \frac{\hbar}{i} \int dx [\vec{b}_i \cdot \vec{\beta} \Box \delta_\xi](x)$	0
$\Delta_\beta'^i = \frac{\hbar}{i} \int dx [\vec{b}_i \cdot \vec{\beta} \delta_\xi](x)$	0
$\Delta_\beta''^i = \frac{\hbar}{i} \int dx [\vec{b}_i \cdot \vec{\beta} \delta_{\eta_1}](x)$	0
$\Delta_{\beta^2}^{ij} = \frac{\hbar}{i} \int dx [\vec{b}_i \cdot \vec{\beta}\, \vec{d}_j \cdot \delta_\beta](x)$	0
$\Delta_\tau = \frac{\hbar}{i} \int dx [\tau \delta_\tau](x)$	$-\int dx [\vec{d} \cdot \vec{\beta} \delta_\tau](x)$
$\Delta_{\gamma^2} = \int dx\, \gamma^2(x)$	$-\bar{\sigma} \int dx [\xi \gamma](x) = -\frac{\hbar \bar{\sigma}}{i \bar{\alpha}} \int dx [\gamma (\cdot \bar{\mathscr{M}} \delta_\xi)](x)$

It is clear that combining linearly the insertions listed in Table (A VII.2) with those previously considered we obtain 10 symmetrical non physical insertions; in fact, because of the orthogonality condition Eq. (A VII.7) $\vec{b}$ is a linear combination of the $\vec{d}_i$'s and $\vec{d}$ of the $\vec{b}_i$'s ($i = 1, 2$).

Now, following the same procedure as in Chapter III, we complete the construction of Δ_λ by studying a basis of symmetrical insertions Δ^S. Since we know that, given the Slavnov identity, the complete lagrangian [Eq. (A VII.5)] depends on 24 parameters (6 of them fixing the propagators and the couplings of the quantized fields, and 18 specifying the external field dependence), it follows that there are 24 independent symmetrical insertions. We have already constructed 18 independent Δ^S's which are non physical. Thus to complete the proof of gauge invariance we have to find six symmetrical insertions satisfying Eq. (70). Five of them are determined by the independent terms of the tree approximation lagrangian [Eq. (AVII.2)] excluding $\frac{\mathscr{G}^2}{2} + \bar{c}\mathscr{M}c$. The sixth one is the analog of Δ_6^S [Eq. (98)]. They verify Eq. (70) as can be seen in the tree approximation.

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