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Sommerfeld-Watson Representation for Double-Spectral Functions

I. Potential Scattering without Regge Poles

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Abstract. We demonstrate the existence of elastically unitary scattering amplitudes, corresponding to a given Born term, when the latter satisfies certain conditions of Hölder-continuity. The results are directly applicable to nonrelativistic potential scattering; and they will also be useful later in connection with the construction of relativistic, crossing-symmetric amplitudes. In this preliminary study, no singularities are allowed in the right half of the complex angular-momentum plane; but it is intended eventually to introduce such singularities (Regge poles and branch-points).

I. Introduction

During the years 1968–1970, it was proved [1–3] that there exists an infinity of crossing-symmetric amplitudes that satisfy the Mandelstam representation for pion-pion scattering. These amplitudes satisfied elastic unitary exactly between the elastic and inelastic thresholds, while the inelastic unitary inequality was observed above the latter. Although the full extent of the infinite family of such amplitudes was not investigated, it was established that there is a completely arbitrary contribution to the inelastic double spectral function [1], that there is in general an undetermined subtraction constant [2], and that the Castillejo-Dalitz-Dyson (CDD) ambiguity occurs [3].

Despite the plethora of possible solutions, it is hoped that the Mandelstam representation may be used to build a phenomenological pion amplitude. By injecting sufficient experimental information into the system of equations, it should be possible to restrict the set of solutions to a narrow class, from which other experimental paramaters may be predicted. The work of Basdevant, Froggatt, and Petersen, with the simpler equations of Roy [4], already indicates that such a scheme may well have predictive value.

A serious shortcoming of the work so far is that it has proved impossible to construct amplitudes with full crossing, elastic unitary up to the inelastic threshold, and positivity in the inelastic region, if the elastic part of the amplitude needs more than one subtraction in its double dispersion relation. Indeed, it is not known in such a case how one can keep the partial-wave amplitudes bounded as the energy tends to infinity. Since it is known that Regge trajectories do rise above one, it is of the utmost importance to solve the problem of the boundedness of the partial-wave amplitudes.

The most reasonable way solving the problem is to work directly with $A(s, l)$, the continuation of the partial-wave amplitude into the complex angular-momentum plane. It is possible to write equations for $A(s, l)$ that guarantee crossing symmetry for the full amplitude, $F(s, t)$, and the idea is to control the l-plane singularities in such a way that the boundedness of $|A(s, l)|$ as $s \to \infty$ is assured. The problem of positivity is probably too delicate to be handled analytically; but it is not unreasonable to try to guarantee boundedness, and to relegate positivity to a numerical investigation.

This article is planned as the first in a series, in which the above programme will hopefully be implemented. In this paper we consider the very simplest case which exhibits the structure of the Sommerfeld-Watson representation: that of potential scattering without Regge poles. In Section 2, we set up the equations, which involve a Froissart-Gribov representation for $A(s, l)$, and the double spectral part of the Sommerfeld-Watson integral. We define a Banach space of doubly Hölder-continuous functions, and explain how, with the help of the contraction mapping principle, one may show that the equations have a solution, if the partial-wave Born term is small enough in norm. Sections 3 and 4 are devoted to the technical aspects of the demonstration and some bounds on Legendre functions and upon integrals are to be found in the two appendices. In Section 5, we collect the various constraints that have to be satisfied, if our proof is to work: these results duplicate some of the findings of Blankenbecler, Goldberger, Khuri, and Treiman [5], but our methods are capable of considerable generalization. We discuss the extension to relativistic pion-pion scattering in Section 5.

2. Sommerfeld-Watson Transform

The usual Sommerfeld-Watson transform is

$$F(s, t) = \frac{i}{2} \int_{-\frac{1}{2}+\varepsilon-i\infty}^{-\frac{1}{2}+\varepsilon+i\infty} \frac{dl(2l+1\, A(s, l)}{\sin \pi l} P_l(-z)\,. \tag{2.1}$$

Here $F(s, t)$ is the scattering amplitude written as a function of $s = 4(q^2+1)$ and $t = -2q^2(1-z)$, where $q(s) = (s-4)^{\frac{1}{2}}/2$ is the momentum and z is the cosine of the scattering angle. (We use the relativistic definition of s, rather than the more usual nonrelativistic $s=q^2$, simply for convenience later, when we turn to pion scattering with full crossing symmetry.) The contour of integration is a straight line parallel to the imaginary axis in the complex l-plane, at the position $\mathrm{Re}\, l = -\frac{1}{2}+\varepsilon$, with $0<\varepsilon<\frac{1}{2}$, and $A(s,l)$ is the partial-wave amplitude, which is supposed to exist as a function holomorphic in l, for $\mathrm{Re}\, l \geqq -\frac{1}{2}+\varepsilon$. We shall in fact construct functions $A(s,l)$ with this property, such that $F(s, l\to t)$ is well-defined by (2.1).

It is convenient to transform to the variable y, defined by

$$l = -\tfrac{1}{2}+\varepsilon+iy\,, \tag{2.2}$$

and to use the reduced amplitude

$$B(s,y)=(s-4)^{-l}A(s,l)\,. \tag{2.3}$$

The t-discontinuity of the s-discontinuity of $F(s,t)$, for $t\geqq 4, s\geqq\sigma(t) = 4\dfrac{t-3}{t-4}$, may be written [5]

$$\varrho(s,t)=\theta(s-\sigma(t))\,iq(s)\int_{-\infty}^{\infty}dy(y-i\varepsilon)P_{-\frac{1}{2}+\varepsilon+iy}(z)\,(s-4)^{-1+2\varepsilon+2iy} \cdot B(s_+,y)B(s_-,y) \tag{2.4}$$

where the suffices $\pm$ mean that the boundary values of the function must be taken respectively above and below the cut on the real s-axis. In Eq. (2.4), use of the unitary condition has been made. Define

$$D(s,t)=\frac{1}{\pi}\int_{\sigma(t)}^{\infty}\frac{ds'}{s'-s}\varrho(s',t)\,, \tag{2.5}$$

and

$$\bar{B}(s,y)=\frac{2}{\pi}(s-4)^{-\frac{1}{2}-\varepsilon-iy}\int_{4}^{\infty}dt\,Q_{-\frac{1}{2}+\varepsilon+iy}(z)\,D(s,t)+V(s,y), \tag{2.6}$$

where V is the partial-wave Born term, which may be calculated explicitly from the Schrödinger potential.

In this work, we shall be interested in studying the mathematical structure of the Eqs. (2.4)–(2.6). We shall show that, if the partial-wave Born term, $V(s,y)$, satisfies certain conditions, then there is a fixed-point, $\bar{B}(s,y)=B(s,y)$, from which the following unitary amplitude may be constructed:

$$F(s,t)-F_B(s,t)=\frac{1}{\pi^2}\int_{4}^{\infty}\frac{ds'}{s'-s}\int_{\sigma(s')}^{\infty}\frac{dt'}{t'-t}\varrho(s',t')\,. \tag{2.7}$$

Here we may allow the Born term, F_B, to depend on s as well as t, with the restriction that it be real for $s \geqq 4$. It may be written

$$F_B(s,t) = i \int_{-\infty}^{\infty} dy \frac{(y-i\varepsilon)(s-4)^{-\frac{1}{2}+\varepsilon+iy} V(s,y)}{\cosh \pi(y-i\varepsilon)} P_{-\frac{1}{2}+\varepsilon+iy}(-z). \quad (2.8)$$

We shall quote sufficient conditions on $V(s, y)$, such that the integrals in (2.7) are well-defined. It is to be noted that (2.7) will agree with (2.1).

After some experimentation, we have found it satisfactory to look for solutions in a Banach space of doubly Hölder-continuous functions, $f(s, y)$, which is specified by means of the following norm:

$$\|f\| = \sup \{s_2^\lambda |y_2+i|^{\frac{1}{2}+\nu} |f(s_2, y_2)|\}$$

$$+ \sup \left\{ s_2^\lambda |y_2+i|^{\frac{1}{2}+\nu} \frac{|f(s_1,y_1)-f(s_1,y_2)-f(s_2,y_1)+f(s_2,y_2)|}{\left|\frac{s_1-s_2}{s_1}\right|^\mu \left|\frac{y_1-y_2}{y_1+i}\right|^\varrho} \right\}, \quad (2.9)$$

where the suprema are to be taken over $s_1 > s_2 \geqq 4$, $-\infty < y_2 < \infty$, $|y_1| > |y_2|$. The indices are subject to a number of restrictions, which we shall give in detail in Section 5. It should be noted that, if $\|f\| < \infty$, then automatically

$$|f(s,y)| \leqq \|f\| s^{-\lambda} |y+i|^{-\frac{1}{2}-\nu}, \quad (2.10a)$$

$$|f(s_1,y)-f(s_2,y)| \leqq \|f\| s_2^{-\lambda} \left|\frac{s_1-s_2}{s_1}\right|^\mu |y+i|^{-\frac{1}{2}-\nu}, \quad (2.10b)$$

$$|f(s,y_1)-f(s,y_2)| \leqq \|f\| s^{-\lambda} |y_2+i|^{-\frac{1}{2}-\nu} \left|\frac{y_1-y_2}{y_1+i}\right|^\varrho, \quad (2.10c)$$

$$|f(s_1,y_1)-f(s_1,y_2)-f(s_2,y_1)+f(s_2,y_2)|$$

$$\leqq \|f\| s_2^{-\lambda} \left|\frac{s_1-s_2}{s_1}\right|^\mu |y_2+i|^{-\frac{1}{2}-\nu} \left|\frac{y_1-y_2}{y_1+i}\right|^\varrho. \quad (2.10d)$$

We shall devote most of the rest of the paper to a proof that, if $\|B\| < \infty$ and $\|V\| < \infty$, then $\bar{B}(s, y)$ is well defined by (2.4)–(2.6), and moreover that a constant, κ, exists such that

$$\|\bar{B}\| \leqq \kappa \|B\|^2 + \|V\|. \quad (2.11)$$

By an immediate extension of the method of proof, it follows that, for any two functions $B_a(s, y)$ and $B_b(s, y)$ that belong to the space, the corresponding image functions $\bar{B}_a(s, y)$ and $\bar{B}_b(s, y)$ satisfy

$$\|\bar{B}_a - \bar{B}_b\| \leqq \kappa \{\|B_a\| + \|B_b\|\} \|B_a - B_b\|. \quad (2.12)$$

It follows easily from the contraction mapping theorem that if $\|V\| < (4\kappa)^{-1}$, then there is a fixed point $B = \bar{B}$ in the ball

$$\|B\| \leqq \frac{1 - [1 - 4\kappa \|V\|]^{\frac{1}{2}}}{2\kappa}, \tag{2.13}$$

and that there are no other fixed points in the larger open ball $\|B\| < (2\kappa)^{-1}$ [1].

To begin the proof of (2.11), we shall combine (2.4)–(2.6) together and change orders of integration. This change may be justified, under the restriction that $\|B\| < \infty$, most easily by considering a continuation into the lower half y-plane. The result is

$$\begin{aligned} \bar{B}(s,y) - V(s,y) = {} & \frac{2i}{\pi^2} \int_4^\infty ds' \frac{q(s')}{s'-s} \cdot \\ & \cdot \int_{-\infty}^{\infty} dy' \, (y' - i\varepsilon) B(s'_+, y') B(s'_-, y') \, (s'-4)^{-\frac{1}{2}+\varepsilon+2iy'} \cdot \\ & \cdot \int_{\sigma(s')}^{\infty} \frac{dt}{(s-4)^{\frac{1}{2}+\varepsilon+iy}(s'-4)^{\frac{1}{2}-\varepsilon}} Q_{-\frac{1}{2}+\varepsilon+iy}\left(1 + \frac{2t}{s-4}\right) \\ & \cdot P_{-\frac{1}{2}+\varepsilon+iy'}\left(1 + \frac{2t}{s'-4}\right). \end{aligned} \tag{2.14}$$

The t-integral here needs very careful handling when $y' \simeq y$ and $s' \simeq s$. We have the formula

$$\begin{aligned} & \int_{z_0}^{\infty} dz \, Q_{-\frac{1}{2}+\varepsilon+iy}(z) P_{-\frac{1}{2}+\varepsilon+iy'}(z) \\ & = \frac{1}{(y'-y)(y'+y-2i\varepsilon)} \, \Xi\left(z_0; -\tfrac{1}{2}+\varepsilon+iy, -\tfrac{1}{2}+\varepsilon+iy'\right) \end{aligned} \tag{2.15}$$

where

$$\Xi(z_0; l, l') = (z_0^2 - 1) \left[Q_l(z_0) P'_{l'}(z_0) - Q'_l(z_0) P_{l'}(z_0) \right]. \tag{2.16}$$

The formula (2.15) is in the first place derived for y' real and y in the lower half-plane (the z-integral is then absolutely convergent), and then y is allowed to tend to the real axis (the point $y = y'$ being excluded). Unfortunately, the arguments of the two Legendre functions in (2.14) are different. Accordingly we write the t-integral in (2.14) in the form

$$\Lambda_1(s'; y, y') + \Lambda_2(s, s'; y, y'), \tag{2.17}$$

where

$$\Lambda_1(s'; y, y')$$
$$= (s'-4)^{-1-iy} \int_{\sigma(s')}^{\infty} dt\, Q_{-\frac{1}{2}+\varepsilon+iy}\left(1+\frac{2t}{s'-4}\right) P_{-\frac{1}{2}+\varepsilon+iy'}\left(1+\frac{2t}{s'-4}\right)$$
$$= \tfrac{1}{2}(s'-4)^{-iy} \frac{\Xi[1+8(s'-3)/(s'-4)^2; -\frac{1}{2}+\varepsilon+iy, -\frac{1}{2}+\varepsilon+iy']}{(y'-y)(y'+y-2i\varepsilon)} \tag{2.18}$$

and where Λ_2 is the remainder, namely

$$\Lambda_2(s, s'; y, y')$$
$$= \int_{\sigma(s')}^{\infty} dt \left[(s-4)^{-\frac{1}{2}-\varepsilon-iy} Q_{-\frac{1}{2}+\varepsilon+iy}\left(1+\frac{2t}{s-4}\right) - (s'-4)^{-\frac{1}{2}-\varepsilon-iy}\right.$$
$$\left.\cdot Q_{-\frac{1}{2}+\varepsilon+iy}\left(1+\frac{2t}{s'-4}\right)\right](s'-4)^{-\frac{1}{2}+\varepsilon} P_{-\frac{1}{2}+\varepsilon+iy'}\left(1+\frac{2t}{s'-4}\right). \tag{2.19}$$

The purpose of the factors $(s-4)^{-\frac{1}{2}-\varepsilon-iy}$ and $(s'-4)^{-\frac{1}{2}-\varepsilon-iy}$ is to ensure that the asymptotic behaviour of the integrand in the Λ_2-integral is reduced by t^{-1}, as compared with that in the Λ_1-integral. The Λ_2-integral converges absolutely for both y and y' real, and we can be quite brutal in our majorizations here, in a way that would not have been possible with the original t-integral in Eq. (2.14).

We write (2.14) in the form

$$\bar{B}(s, y) - V(s, y) = B_1(s, y) + B_2(s, y), \tag{2.20}$$

where B_1 corresponds to the right side of (2.14), with Λ_1 in place of the t-integral, and where B_2 is the remainder term, i.e. with Λ_2 in place of the t-integral. In the following sections, we shall study the boundedness and Hölder-continuity of B_1 and B_2.

3. Study of $B_1(s, y)$

We may write

$$B_1(s, y) = -\frac{16i}{\pi^2} \int_4^{\infty} \frac{ds'}{s'-s} D(s', y) \tag{3.1}$$

where

$$D(s', y) = (s'-4)^{\varepsilon-iy} s'^{-\frac{11}{4}} (s'-3)(s'-2)^2 (y+i)^{-1}$$
$$\cdot [I_1(s', y) Q'_{-\frac{1}{2}+\varepsilon+iy}(z'_0)(s'-4)^{-3-2\varepsilon} s'^{2+2\varepsilon} \tag{3.2}$$
$$- I'_1(s', y) Q_{-\frac{1}{2}+\varepsilon+iy}(z'_0)(s'-4)^{-1-2\varepsilon} s'^{\frac{3}{4}+2\varepsilon}]$$

with

$$z_0' = 1 + 8(s'-3)(s'-4)^{-2}, \tag{3.3}$$

and with

$$I_n(s', y) = (y+i)\int_{-\infty}^{\infty} dy' \left[\frac{1}{y'-y} + \frac{1}{y'+y-2i\varepsilon}\right] G_n(s', y') \tag{3.4a}$$

$$= \int_{-\infty}^{\infty} dy' \left[\frac{y'+i}{y'-y} - \frac{y'-i-2i\varepsilon}{y'+y-2i\varepsilon}\right] G_n(s', y') \tag{3.4b}$$

for $n = 1,2$. In Eq. (3.4),

$$G_1(s', y') = B(s'_+, y')B(s'_-, y')P_{-\frac{1}{2}+\varepsilon+iy'}(z_0')(s'-4)^{-1+2\varepsilon+2iy'}s'^{\frac{3}{4}-2\varepsilon}, \tag{3.5}$$

$$G_2(s', y') = B(s'_+, y')B(s'_-, y')P'_{-\frac{1}{2}+\varepsilon+iy'}(z_0')(s'-4)^{-3+2\varepsilon+2iy'}s'^{2-2\varepsilon}. \tag{3.6}$$

In Eqs. (3.2), (3.5), and (3.6) the powers of s' and $(s'-4)$ are such that the Legendre functions multiplied by these powers can be estimated by s'-independent bounds; this can be seen from Eqs. (3.3) and (A.19)–(A.24).

We shall now outline the essential steps in showing that $B_1(s, y)$ is bounded and doubly Hölder-continuous and, in fact, that

$$\|B_1\| \leqq \kappa \|B\|^2. \tag{3.7}$$

Here, and in what follows, κ is a generic constant, which may change from one line to the next: the important point is that there exists such a number, and that it depends only on the various indices. The details of the proof were given in a recent preprint [7] with the same title as this paper.

It is straightforward, from Eqs. (2.10) and (A.19)–(A.24), to derive an optimum bound on $G_1(s', y')$. To show that $G_1(s', y')$ is y'-Hölder continuous, one uses the mean value theorem, and one needs a bound on the derivative of a Legendre function with respect to y [see the discussion in Appendix A, following Eq. (A.25)]. Equation (3.4b) and Theorem 1 of Appendix B can then be used to show that $I_1(s', y)$ is bounded and y-Hölder continuous.

Because of the presence of the s'-singular integral in Eq. (3.1), we see, by analogy with Theorem 2 of Appendix B, that we must also show that $G_1(s', y')$ is s'-Hölder continuous and, in fact, doubly Hölder continuous in s' and y'. The s'-Hölder continuity is established by using the mean value theorem. To show that $G_1(s', y')$ is doubly Hölder continuous, one uses the double mean value theorem to derive the bound

$$\begin{aligned}&|R(s_1', y_1') - R(s_1', y_2') - R(s_2', y_1') + R(s_2', y_2')|\\ &\leqq 2^{\varrho-\mu}[|R(s_1', y_1')| + |R(s_1', y_2')| + |R(s_2', y_1')| + |R(s_2', y_2')|]^{1-\varrho}\\ &\cdot \sup\left|\frac{d}{dy'}R(s', y')\right|^{\varrho-\mu} \cdot \sup\left|\frac{d^2}{dy'ds'}R(s', y')\right|^{\mu} |s_1'-s_2'|^{\mu}|y_1'-y_2'|^{\varrho}\end{aligned} \tag{3.8}$$

for $\varrho \geqq \mu$, where the suprema are taken independently over the ranges $[y'_2, y'_1]$ for y' and $[s'_2, s'_1]$ for s'.

Here

$$R(s', y') = P_{-\frac{1}{2}+\varepsilon+iy'}(z'_0)\,(s'-4)^{-1+2\varepsilon+2iy'}\,s'^{\frac{3}{4}-2\varepsilon}, \tag{3.9}$$

and one needs as well the properties of $B(s, y)$ given in Eq. (2.10). Theorem 2 can then be used to show that $I_1(s', y)$ is doubly Hölder continuous in s' and y. A similar argument can be used to show that $I_2(s', y)$ is bounded and in fact doubly Hölder continuous in s' and y; in the treatment of $I_2(s', y)$ it turns out to be more convenient to use the definition in Eq. (3.4a).

From the properties of the Legendre functions in Appendix A it is then straightforward to show that $D(s', y)$ is bounded and doubly Hölder continuous in s' and y. It is important to note that $|D(s, y)|$ behaves like $(s'-4)^{\varepsilon}$ as $s' \downarrow 4$ and thus the threshold requirement for Theorem 2 of Appendix A is satisfied [Eq. (B.11)]. The s'-Hölder difference, $|D(s'_1, y) - D(s'_2, y)|$, and the double Hölder difference however behave like $(s_2-4)^{\varepsilon-\mu}$ as $s_2 \downarrow 4$ where $s_1 \geqq s_2 \geqq 4$. Therefore, to apply Theorem 2 we must make the requirement

$$0 < \mu < \varepsilon < \tfrac{1}{2}. \tag{3.10}$$

Then we find that

$$|B_1(s, y)| \leqq \kappa \, \|B\|^2 s^{-2\lambda+\varepsilon+\frac{1}{4}+\delta} |y+i|^{-\alpha}, \tag{3.11a}$$

$$\begin{aligned} &|B_1(s_1, y_1) - B_1(s_2, y_1) - B_1(s_1, y_2) + B_1(s_2, y_2)| \\ &\leqq \kappa \, \|B\|^2 s^{-2\lambda+\varepsilon+\frac{1}{4}+\delta} \left|\frac{s_1-s_2}{s_1}\right|^{\mu} |y_2+i|^{-\alpha+\varrho} \left|\frac{y_1-y_2}{y_1+i}\right|^{\varrho}. \end{aligned} \tag{3.11b}$$

In the following we shall choose μ and ϱ to lie in the intervals

$$0 < \mu < \min\{\tfrac{1}{8}, \varepsilon\}, \tag{3.12}$$

$$0 < \varrho < \tfrac{1}{16}. \tag{3.13}$$

Finally we see that Eq. (3.11) implies that Eq. (3.7) is valid on condition that the powers of s and $|y+i|$ in the above inequalities are not greater than the corresponding powers in Eq. (2.10). Moreover, we require that the power of s be greater than -1. These restrictions may be expressed as follows:

$$\tfrac{1}{4} + \varepsilon < \lambda \leqq \tfrac{5}{8} + \varepsilon/2 \tag{3.14}$$

and either

(a) $$0 \leqq \nu < \tfrac{1}{4} + \tfrac{1}{2}\varrho \text{ and } \alpha = 1 + 2\nu - \varrho - \mu \tag{3.15a}$$

or

(b) $$\tfrac{1}{4} + \tfrac{1}{2}\varrho < \nu < 1 - \mu - \varrho \text{ and } \alpha = \tfrac{3}{2} - \mu - \delta, \tag{3.15b}$$

where $\delta > 0$ may be as small as one pleases.

4. Treatment of $B_2(s, y)$

It can be proved from Eq. (2.14), with Λ_2 replacing the t-integral, that $B_2(s, y)$ is bounded and s-Hölder continuous. To do this, we show first that

$$E(s, s' y) =$$

$$\tfrac{1}{2}(s'-4)^{\varepsilon} \int_{-\infty}^{\infty} dy'(y'-i\varepsilon)B(s'_+, y')B(s'_-, y')\,(s-4)^{2iy'}\Lambda_2(s, s'; y, y') \tag{4.1}$$

satisfies the conditions on the spectral function in Theorem 3 of Appendix B (generalized to the case when the spectral function also depends on a parameter y). Thus we must demonstrate that $E(s, s', y)$ is bounded s'-Hölder continuous and s-Hölder continuous. It is somewhat simpler to establish the y- and double Hölder continuity of $B_2(s, y)$ than was the case for $B_1(s, y)$, since now the y' integral is not singular. Thus one does not need to establish Hölder continuity with respect to y', but only with respect to y, which is simply a parameter, insofar as the s'-integral is concerned.

We begin by obtaining bounds on $\Lambda_2(s, s'; y, y')$, which was defined in Eq. (2.20). As we have already remarked, the t-integral here is absolutely convergent for y and y' real; and so we may hope to be able to treat this term in a more cavalier manner than was possible for Λ_1. However, although we shall be able to obtain a bound on $\|B_2\|$, this will be at the expense of constraints on the indices that are more stringent than those that we imposed in Section 3. A straightforward bound on the integrand in (2.20) would yield the estimate

$$|\Lambda_2(s, s'; y, y')| \leqq \kappa |y'+i|^{-\frac{1}{2}}\, |y+i|^{-\frac{1}{2}+\delta}(s+s')^{\delta} s'^{r} \tag{4.2}$$

where

$$r = \begin{cases} 0 & 0 < \varepsilon \leqq \tfrac{1}{4} \\ \varepsilon - \tfrac{1}{4} & \tfrac{1}{4} < \varepsilon < \tfrac{1}{2}\,. \end{cases} \tag{4.3}$$

Here and in what follows, δ will be taken to be a generic small positive number which may be as small as one pleases, and which may change from one equation in which it occurs to the next. This estimate does not have the required large-y behaviour, namely$|y+i|^{-\frac{1}{2}-\nu}$ [see Eq. (2.10)]. In order to improve the bound (4.2), we integrate (2.20) by parts, obtaining

$$\begin{aligned} \Lambda_2(s, s'; y, y') = &-\tfrac{1}{2}\, P_{-\frac{1}{2}+\varepsilon+iy'}(z'_s(\sigma(s')))\,(s'-4)^{-\frac{1}{2}+\varepsilon}\Phi(s, s', y, \sigma(s')) \\ &+ \int_{\sigma(s')}^{\infty} dt\, P'_{-\frac{1}{2}+\varepsilon+iy'}(z'_s(t))\,(s'-4)^{-\frac{3}{2}+\varepsilon}\,\Phi(s, s', y, t)\,. \end{aligned} \tag{4.4}$$

It is convenient to express $\Phi(s, s', y, t)$ directly in terms of the hypergeometric functions:

$$\begin{aligned}\Phi&(s, s', y, t)\\ &= -(\tfrac{1}{2}\pi)^{\frac{1}{2}}\Gamma(3/2+\varepsilon+iy)\,[\Gamma(1+\varepsilon+iy)(-\tfrac{1}{2}+\varepsilon+iy)(\tfrac{1}{2}+\varepsilon+iy)]^{-1}\,. \\ &\quad\cdot(2t)^{\frac{1}{2}-\varepsilon+iy}[G(s', y, t)-G(s, y, t)]\end{aligned} \tag{4.5}$$

where

$$\begin{aligned}G(s, y, t) &= \left[1+\frac{s-4}{t}\right]^{\frac{1}{4}}\left[1+\frac{s-4}{2t}+\left\{1+\frac{s-4}{t}\right\}^{\frac{1}{2}}\right]^{-\varepsilon-iy}\\ &\quad\cdot F(\tfrac{3}{2}, -\tfrac{1}{2}, 1+\varepsilon+iy, \zeta_s(t)),\end{aligned} \tag{4.6}$$

F is the standard standard hypergeometric function and

$$\zeta_s(t) = -\tfrac{1}{2}\,[((z_s(t))^2-1)^{\frac{1}{2}}\,(z_s(t)+((z_s(t))^2-1)^{\frac{1}{2}})]^{-1}\,, \tag{4.7a}$$

$$z_s(t) = 1+\frac{2t}{s-4}\,;\; z_s'(t) = 1+\frac{2t}{s'-4}\,. \tag{4.7b}$$

In both the terms in Eq. (4.4) we have improved the y-behaviour considerably, since $\Phi(s, s', y, t)$ behaves like $y^{-\frac{3}{2}}$ for large y. However, in the unintegrated term, the derivative of the P-function behaves like $y'^{\frac{1}{2}}$ for large y', which leads to stringent constraints on the index v. By using the bounds on the Legendre and hypergeometric functions given in Appendix A it is then possible to derive an alternative bound on Λ_2 from Eq. (4.4). However, it proves advantageous to use a bound which is a compromise between Eq. (4.2) and the above estimate on Λ_2, namely,

$$\begin{aligned}|\Lambda_2(s, s'; y, y')| &= |\Lambda_2|^{\eta}\,|\Lambda_2|^{1-\eta}\\ &\leqq \kappa|y'+i|^{\frac{1}{2}-\eta}\,|y+i|^{-\frac{3}{2}+\eta+\delta}(s+s')^{p(1-\eta)+\delta}s'^{r\eta}\end{aligned} \tag{4.8}$$

with $0\leqq\eta\leqq 1$. In Eq. (4.8),

$$p = \begin{cases}\tfrac{1}{2}-\varepsilon & \text{if}\quad 0<\varepsilon\leqq\tfrac{1}{4}\\ \tfrac{1}{4} & \text{if}\quad \tfrac{1}{4}<\varepsilon<\tfrac{1}{2}\,.\end{cases} \tag{4.9}$$

By using the mean value theorems it is possible to show in a similar way that $\Lambda_2(s, s'; y, y')$ is doubly Hölder continuous in the variables sy, $s'y$. From Eq. (2.10) it then follows that $E(s, s', y)$ is doubly Hölder continuous in sy, $s'y$. Finally, we find, on applying Theorem 3 of Appendix A, that

$$|B_2(s, y)| \leqq \kappa \|B\|^2 s^{-a} |y+i|^{-\frac{3}{2}+\eta+\mu}, \tag{4.10a}$$

$$\begin{aligned} &|B_2(s_1, y_1) - B_2(s_2, y_1) - B_2(s_1, y_2) + B_2(s_2, y_2)| \\ &\leqq \kappa \|B\|^2 s_2^{-a+\mu} \left|\frac{s_1 - s_2}{s_1}\right|^{\mu} |y_2 + i|^{-\frac{3}{2}+\eta+\mu+\varrho+\delta} \left|\frac{y_1 - y_2}{y_1 + i}\right|^{\varrho}, \end{aligned} \tag{4.10b}$$

where

$$a = 2\lambda - (r\eta + p(1-\eta) + \varepsilon + \delta) \tag{4.11}$$

and

$$0 < \mu < a < 1. \tag{4.12}$$

The above set of inequalities implies that

$$\|B_2\| \leqq \kappa \|B\|^2, \tag{4.13}$$

provided that the y' integral in Eq. (4.1) converges and that the powers of s and $|y+i|$ in Eq. (4.10) are not greater than the corresponding powers that occur in Eq. (2.10). If we combine these constraints with Eq. (4.12) and note that $\delta > 0$ can be chosen as small as we like, we find that we must impose the requirements

$$r\eta + p(1-\eta) + \varepsilon + \mu < \lambda \leqq \tfrac{1}{2}(1 + r\eta + p(1-\eta) + \varepsilon), \tag{4.14}$$

$$\tfrac{3}{4} - \eta/2 + \mu/2 < v < 1 - \eta - \mu - \varrho, \tag{4.15}$$

and

$$0 < \eta < \tfrac{1}{2} - 3\mu - 2\varrho \tag{4.16}$$

where μ and ϱ are restricted as in Eqs. (3.12) and (3.13).

5. Conclusion and Generalization

In Section 3 we proved that $\|B_1\|$ exists [Eq. (3.7)] and in Section 4 that $\|B_2\|$ exists [Eq. (4.13)]. If we now assume that

$$\|V\| < \infty \tag{5.1}$$

it follows immediately from (2.20) that

$$\|\bar{B}\| \leqq \kappa \|B\|^2 + \|V\|, \tag{5.2}$$

provided that there are ranges of the indices for which both Eqs. (3.7) and (4.13) are valid. First we see from Eqs. (3.14) and (4.14) that we require

$$\max\{\tfrac{1}{4} + \varepsilon, r\eta + p(1-\eta) + \varepsilon + \mu\} < \lambda \leqq \tfrac{5}{8} + \varepsilon/2 \tag{5.3}$$

where r is defined in Eq. (4.3), p in Eq. (4.9). The inequalities which ε, μ, and η must satisfy are given in Eqs. (3.10), (3.12), and (4.16). We also see that, since ϱ satisfies (3.13), Eqs. (3.15) and (4.15) imply that v must simply satisfy inequality (4.15). Thus, as we mentioned at the beginning of Section 4, the strongest constraints on v come from the analysis carried out in that section. In the proof, it is in fact sufficient to take μ and ϱ very small and positive, which means that v can range from slightly larger than $\frac{1}{2}$ to just less than 1. Similarly, for sufficiently small μ, and for suitable values of η and ε, λ can range from slightly larger than $\frac{1}{4}$ to just less than $\frac{7}{8}$.

Thus, when the above restrictions on the indices hold, there is a fixed point

$$\bar{B}(s, y) = B(s, y) \tag{5.4}$$

if $\|V\|$ is small enough [see the discussion after Eq. (2.12)]. The restriction $\|V\| < (4\kappa)^{-1}$ [where κ now refers specifically to the constant appearing in Eq. (5.2)] defines the class of Born terms for which our proof is valid: it would be possible in principle to calculate κ explicitly, but we have been content simply to show that such a κ exists. Further, since we are mainly interested in the structure of the Sommerfeld-Watson equation, with a view to eventual generalization (which we discuss below), we shall not re-express the restrictions on $V(s, y)$ in terms of restrictions on the potential.

A simple generalization, which can be discussed in the framework of potential scattering, is the situation that obtains when there is an exchange potential. In the case that there is symmetry between the s- and u-channels, the only change is that Eq. (2.5) is replaced by

$$D(s, t) = \frac{1}{\pi} \int_4^\infty ds' \left[\frac{1}{s'-s} + \frac{1}{s'-u}\right] \varrho(s', t), \tag{5.5}$$

where $u = 4 - s - t$ (we use relativistic notation for convenience). The whole proof goes through with very little change, since the denominator $(s'-u)$ is quite harmless. We have the following term, in addition to that on the right-hand side of (2.14):

$$\frac{2i}{\pi^2}(s-4)^{-\frac{1}{2}-\varepsilon-iy} \int_4^\infty ds' q(s') \int_{-\infty}^\infty dy'(y'-i\varepsilon)$$

$$\cdot B(s'_+, y') B(s'_-, y') (s'-4)^{-1+2\varepsilon+2iy'}$$

$$\cdot \int_{\sigma(s')}^\infty \frac{dt}{s'+t+s-4} Q_{-\frac{1}{2}+\varepsilon+iy}\left(1+\frac{2t}{s-4}\right) P_{-\frac{1}{2}+\varepsilon+iy'}\left(1+\frac{2t}{s'-4}\right). \tag{5.6}$$

Because of the occurence of t in the denominator $(s'+t+s-4)$, the t-integral here converges absolutely, for y and y' real, and so we may treat the whole term much as we treated the B_2-term in Section 4. Note that there is no singular integral here, either in y' or in s'.

It will be shown in a subsequent paper [8] how our proof may be generalized to the case of relativistic pion-pion scattering. The first change is the replacement of the nonrelativistic momentum in (2.4) and (2.14) by the relativistic phase-space factor

$$\tilde{q}(s)=\left(\frac{s-4}{s}\right)^{\frac{1}{2}}. \tag{5.7}$$

Secondly, the Legendre function in (2.1) becomes

$$\tfrac{1}{2}\left[P_l(-z)+P_l(z)\right], \tag{5.8}$$

in order to ensure $t\,u$ crossing. The Legendre function in (2.4) remains as it is, but a factor of $\frac{1}{2}$ must be introduced, because of (5.8). Now ϱ is only the elastic contribution to the double spectral function, and (5.5) must be further generalized by writing

$$D(s,t)=\frac{1}{\pi}\int ds'\left[\frac{1}{s'-s}+\frac{1}{s'-u}\right]\left[\varrho(s',t)+\varrho(t,s')\right]. \tag{5.9}$$

An extra factor 2 comes into (2.6), because of the presence of the u-channel.

It is shown in Ref. [8] how the norm may be generalized to handle the extra term $\varrho(t,s)$. In this connection, one simplification is that a Hölder-continuous cut-off function, $h(s)$, with the property that it is equal to unity in the elastic region, may be inserted before the integral in (2.4). This is possible in the relativistic case, since elastic unitarity does not hold for $s>16$; and the effect is simply to redefine $V(s,y)$, which is now no longer simply a Born term, but contains a crossing-symmetric contribution from the deep inelastic double-spectral function.

Next we plan to generalize the system still further by introducing Regge poles, so that an unsubtracted Mandelstam representation will no longer be valid. By handling the pole terms explicitly, we hope to treat the Sommerfeld-Watson background integral by means of the same norm as that used in Ref. [8]. There is a good possibility that, by explicitly imposing analyticity in angular-momentum, we can solve the problem of the violation of the inelastic unitarity bounds that seemed unavoidable in earlier work [1, 2] when singularities move to the right of $\mathrm{Re}\,l=1$.

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Appendix A. Bounds for Legendre and Hypergeometric Functions

In this appendix we establish bounds on the Legendre and hypergeometric functions, some of which we have not been able to find in the mathematical literature. The associated Legendre functions satisfy the relationship

$$P_\nu^m(z)\cos\pi\nu = \frac{1}{\pi}\sin\pi\nu\,[Q_\nu^m(z) - Q_{-\nu-1}^m(z)] \tag{A.1}$$

where, for the purposes of this paper, $z \geqq 1$, $m = 0$, 1 or 2 and $\nu = -\frac{1}{2} \pm (\varepsilon + iy)$ where y is real and $0 < \varepsilon < \frac{1}{2}$. The functions $Q_\nu^m(z)$ are defined in terms of $Q_\nu(z)$, the Legendre function of the second kind, by

$$Q_\nu^m(z) = (z^2-1)^{\frac{1}{2}m}\frac{d^m Q_\nu(z)}{dz^m} \qquad m = 1, 2 \tag{A.2}$$

and in terms of the hypergeometric functions, by

$$\begin{aligned} Q_\nu^m(z) = {} & (-1)^m(\tfrac{1}{2}\pi)^{\frac{1}{2}}\frac{\Gamma(\nu+m+1)}{\Gamma(\nu+\frac{3}{2})}(z^2-1)^{-\frac{1}{4}}[z+(z^2-1)^{\frac{1}{2}}]^{-(\nu+\frac{1}{2})} \\ & \cdot F(\tfrac{1}{2}+m, \tfrac{1}{2}-m, \nu+\tfrac{3}{2}, \zeta) \qquad m = 0, 1, 2 \end{aligned} \tag{A.3}$$

where

$$\zeta = -\tfrac{1}{2}\,[(z^2-1)^{\frac{1}{2}}(z+(z^2-1)^{\frac{1}{2}})]^{-1} \qquad (\leqq 0)\,. \tag{A.4}$$

We begin by establishing a bound on the ratio of the gamma functions in Eq. (A.3). From the expression for the logarithm of the gamma function given in Eqs. (8.8.24) and (8.8.37) of Hille [9] vol I, it can be shown, by using Maclaurin's Theorem, that

$$\frac{\Gamma(\frac{1}{2}\pm(\varepsilon+iy)+m)}{\Gamma(1\pm(\varepsilon+iy))} \leqq \kappa|y+i|^{-\frac{1}{2}+m}, \tag{A.5a}$$

$$\frac{\Gamma(1\pm(\varepsilon+iy))}{\Gamma(\frac{1}{2}\pm(\varepsilon+iy)+m)} \leqq \kappa|y+i|^{\frac{1}{2}-m} \qquad m = 0, 1, 2\,, \tag{A.5b}$$

where κ is some generic constant which may change from line to line.

Next, we consider the hypergeometric function which, when $\operatorname{Re} c > \operatorname{Re} b > 0$, has the integral representation

$$F(a,b,c,\zeta) = \Gamma(c)\,[\Gamma(b)\Gamma(c-b)]^{-1}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-t\zeta)^{-a}\,dt\,. \tag{A.6}$$

Also, from the series representation of F, it can be shown that[1]

$$F(a,b,c,\zeta)-\sum_{r=0}^{n}\frac{\Gamma(a+r)\Gamma(b+r)\Gamma(c)}{\Gamma(c+r)\Gamma(a)\Gamma(b)\Gamma(r+1)}\zeta^{r} \tag{A.7}$$

$$=\frac{\Gamma(c)\Gamma(a+n+1)\zeta^{n+1}}{\Gamma(b)\Gamma(c-b)\Gamma(a)\Gamma(n+1)}\int_0^1\int_0^1 ds\,dt\, t^{b+n}(1-t)^{c-b-1}(1-s)^{n}(1-st\zeta)^{-a-n-1}$$

provided that $\mathrm{Re}(c+n+1)>\mathrm{Re}(b+n+1)>0$. Both Eqs. (A.6) and (A.7) are valid, in particular, for $\zeta\leqq 0$.

Of the 3 hypergeometric functions needed in Eq. (A.3), $F(\frac{1}{2},\frac{1}{2},1\pm(\varepsilon+iy),\zeta)$ is the most difficult to handle. It can however be shown from Eq. (A.6) that

$$|F(\tfrac{1}{2},\tfrac{1}{2},1\pm(\varepsilon+iy),\zeta)|\leqq\kappa\,. \tag{A.8}$$

The method consists in first making the change of variable

$$w=-\log(1-t)$$

in Eq. (A.6) and then using the following theorem, which was proved in Ref. [7]. If

$$I(y)=\int_0^1 dw\, e^{iwy}f(w)$$

where

$$|f(w)|\leqq\kappa w^{-\alpha}\,,$$

$$|f(w_1)-f(w_2)|\leqq\kappa\,|w_1-w_2|^{\mu}\,w_2^{-\alpha-\beta}\,,$$

$0<\mu\leqq 1$, $\alpha+\mu\leqq 1$, $0\leqq\beta\leqq\mu$, $w_1\geqq w_2>0$ and y is real, then

$$|I(y)|\leqq\kappa|y|^{-\mu}\,.$$

To obtain a bound on $F(\frac{3}{2},-\frac{1}{2},1\pm(\varepsilon+iy),\zeta)$, we cannot use Eq. (A.6); but Eq. (A.7), with $n=0$, is valid. Thus, with the aid of Eq. (A.5a) we have

$$|F(\tfrac{3}{2},-\tfrac{1}{2},1\pm(\varepsilon+iy),\zeta)-1|\leqq\kappa|y+i|^{-\frac{1}{2}}(-\zeta)^{\frac{1}{2}}\,. \tag{A.9}$$

Similarly, we find from Eq. (A.7) with $n=1$ that

$$\begin{aligned}&|F(\tfrac{5}{2},-\tfrac{3}{2},1\pm(\varepsilon+iy),\zeta)-1+\frac{15}{4}[1\pm(\varepsilon+iy)]^{-1}\zeta|\\&\leqq\kappa|y+i|^{-\frac{3}{2}}(-\zeta)^{\frac{3}{2}}\,.\end{aligned} \tag{A.10}$$

Further, using the fact that

$$\left|\frac{d}{dy}\log\left\{\frac{\Gamma(1\pm(\varepsilon+iy))}{\Gamma(\frac{1}{2}\pm(\varepsilon+iy)+m)}\right\}\right|\leqq\kappa\,,\qquad m=0,1,2 \tag{A.11}$$

[1] Note that the corresponding expression given in Ref. [10] contains a misprint.

we have

$$\left|\frac{d}{dy}F\left(\tfrac{1}{2},\tfrac{1}{2},1\pm(\varepsilon+iy),\zeta\right)\right|\leqq\kappa\,, \tag{A.12}$$

$$\left|\frac{d}{dy}F\left(\tfrac{3}{2},-\tfrac{1}{2},1\pm(\varepsilon+iy),\zeta\right)\right|\leqq\kappa\,|y+i|^{-\frac{1}{2}}(-\zeta)^{\frac{1}{2}}, \tag{A.13}$$

and

$$\left|\frac{d}{dy}F\left(\tfrac{5}{2},-\tfrac{3}{2},1\pm(\varepsilon+iy),\zeta\right)\right|\leqq\kappa\,|y+i|^{-\frac{3}{2}}(-\zeta)^{\frac{3}{2}}. \tag{A.14}$$

Finally we shall need the estimates

$$\left|\frac{d}{d\zeta}F\left(\tfrac{1}{2},\tfrac{1}{2},1\pm(\varepsilon+iy),\zeta\right)\right|\leqq\kappa|y+i|^{-\frac{1}{2}}, \tag{A.15}$$

$$\left|\frac{d}{d\zeta}F\left(\tfrac{3}{2},-\tfrac{1}{2},1\pm(\varepsilon+iy),\zeta\right)\right|\leqq\kappa|y+i|^{-\frac{1}{2}} \tag{A.16}$$

$$\left|\frac{d^2}{d\zeta\,dy}F\left(\tfrac{1}{2},\tfrac{1}{2},1\pm(\varepsilon+iy),\zeta\right)\right|\leqq\kappa\,|y+i|^{-\frac{1}{2}}, \tag{A.17}$$

$$\left|\frac{d^2}{d\zeta\,dy}F\left(\tfrac{3}{2},-\tfrac{1}{2},1\pm(\varepsilon+iy),\zeta\right)\right|\leqq\kappa\,|y+i|^{-\frac{1}{2}}. \tag{A.18}$$

We are now in a position to obtain bounds on the Legendre functions. From Eqs. (A.8)-(A.10) and (A.4), together with the definitions of the Legendre functions in Eqs. (A.1)–(A.3), we find that

$$|Q_{-\frac{1}{2}\pm(\varepsilon+iy)}(z)|\leqq\kappa\,|y+i|^{-\frac{1}{2}}\left(\frac{z}{z-1}\right)^{\frac{1}{4}}z^{-\frac{1}{2}\mp\varepsilon}, \tag{A.19}$$

$$|Q'_{-\frac{1}{2}\pm(\varepsilon+iy)}(z)|\leqq\kappa\,|y+i|^{\frac{1}{2}}\left(\frac{z}{z-1}\right)z^{-\frac{3}{2}\mp\varepsilon} \tag{A.20}$$

$$|Q''_{-\frac{1}{2}\pm(\varepsilon+iy)}(z)|\leqq\kappa\,|y+i|^{\frac{3}{2}}\left(\frac{z}{z-1}\right)^{2}z^{-\frac{5}{2}\mp\varepsilon} \tag{A.21}$$

$$|P_{-\frac{1}{2}+\varepsilon+iy}(z)|\leqq\kappa\,|y+i|^{-\frac{1}{2}}\left(\frac{z}{z-1}\right)^{\frac{1}{4}}z^{-\frac{1}{2}+\varepsilon}, \tag{A.22}$$

$$|P'_{-\frac{1}{2}+\varepsilon+iy}(z)|\leqq\kappa\,|y+i|^{\frac{1}{2}}\left(\frac{z}{z-1}\right)z^{-\frac{3}{2}+\varepsilon} \tag{A.23}$$

$$|P''_{-\frac{1}{2}+\varepsilon+iy}(z)|\leqq\kappa\,|y+i|^{\frac{3}{2}}\left(\frac{z}{z-1}\right)^{2}z^{-\frac{5}{2}+\varepsilon}. \tag{A.24}$$

To obtain estimates for the corresponding derivatives with respect to y, we note that

$$\begin{aligned}
&\frac{d}{dy} Q^m_{-\frac{1}{2}\pm(\varepsilon+iy)}(z) \\
&\quad = \mp i \log[z+(z^2-1)^{\frac{1}{2}}]\, Q^m_{-\frac{1}{2}\pm(\varepsilon+iy)}(z) \\
&\quad + \frac{d}{dy}\left[\log\frac{\Gamma(\frac{1}{2}\pm(\varepsilon+iy)+m)}{\Gamma(1\pm(\varepsilon+iy))}\right] Q^m_{-\frac{1}{2}\pm(\varepsilon+iy)}(z) \\
&\quad + (-1)^m (\tfrac{1}{2}\pi)^{\frac{1}{2}} \frac{\Gamma(\frac{1}{2}\pm(\varepsilon+iy)+m)}{\Gamma(1\pm(\varepsilon+iy))} (z^2-1)^{-\frac{1}{4}} [z+(z^2-1)^{\frac{1}{2}}]^{\mp(\varepsilon+iy)} . \\
&\quad \cdot \frac{d}{dy} F(\tfrac{1}{2}+m, \tfrac{1}{2}-m, 1\pm(\varepsilon+iy), \zeta) \qquad m=0,1,2 .
\end{aligned} \tag{A.25}$$

Thus, from Eqs. (A.11)–(A.14) we see that the derivatives with respect to y of the Legendre functions on the left-hand sides of Eqs. (A.19)–(A.24) can be bounded by the corresponding right-hand sides multiplied by $(1+\log z)$.

From the Laplace representation for $P_\nu(z)$, one can obtain the alternative bound

$$|P_{-\frac{1}{2}+\varepsilon+iy}(z)| \leqq \kappa z^{-\frac{1}{2}+\varepsilon} \quad \text{for} \quad z \geqq 1 . \tag{A.26}$$

Appendix B. Hölder-Continuity of Integrals

In this appendix, we shall state a number of theorems concerning the Hölder-continuity of integrals. The proofs are straightforward generalizations of a demonstration contained in Muskhelishvili [11] and we shall not give the details.

Theorem 1. *Let us first consider a function $f(y)$, defined on the real line $(-\infty, \infty)$, and let us introduce the norm*

$$\|f\|_1 = \sup\{|y_2+i|^\beta\, |f(y_2)|\} + \sup\left\{ |y_2+i|^\beta \frac{|f(y_1)-f(y_2)|}{\left|\frac{y_1-y_2}{y_1+i}\right|^\varrho} \right\} \tag{B.1}$$

where $1>\beta\geqq\varrho>0$ and the suprema are taken over $-\infty<y_2<\infty$, $|y_1|>|y_2|$. If $\|f\|_1<\infty$, and

$$I(y) = P\int_{-\infty}^{\infty} \frac{dy'}{y'-y} f(y') , \tag{B.2}$$

then it may be shown that a constant, κ, exists such that

$$\|I\|_1 \leqq \kappa \|f\|_1 . \tag{B.3}$$

The same result holds if one replaces y, on the right-hand side of (B.2), by $y+i\eta, \eta$ real (when of course there is no principal value). A special case in obtained if one, or both of the end-points of integration is finite. It suffices then to require also that $f(y')$ vanish at such finite end-points, for then $f(y')$ may simply be defined to be zero beyond the end-points, without destroying the Hölder-continuity. One is then formally back to the case (B.2). A very simple special case is the following:

$$I(y) = \int_b^\infty \frac{dy'}{y'+y} f(y') , \tag{B.4}$$

$y \geqq b > 0$. Here one can show that $I(y)$ is Hölder-continuous, given only that $f(y')$ is suitably bounded; but if $\|f\|_1 < \infty$, then again (B.3) holds.

Theorem 2. *Consider now a function of two variables, $f(s, y)$. For generality we shall allow s to range over $(-\infty, \infty)$, and we introduce the two-dimensional norm:*

$$\begin{aligned} \|f\|_2 = \sup \{|s_2+i|^\lambda |y_2+i|^\beta |f(s_2, y_2)|\} \\ + \sup \left\{ |s_2+i|^\lambda |y_2+i|^\beta \frac{|f(s_1,y_1)-f(s_1,y_2)-f(s_2,y_1)+f(s_2,y_2)|}{\left|\dfrac{s_1-s_2}{s_1+i}\right|^\mu \left|\dfrac{y_1-y_2}{y_1+i}\right|^\varrho} \right\}, \end{aligned} \tag{B.5}$$

$>\beta \geqq \varrho > 0$, $>\gamma \geqq \mu > 0$, *with suprema over* $-\infty < y_2 < \infty$, $|y_1| > |y_2|$. $-\infty < s_2 < \infty$, $|s_1| > |s_2|$. *Define*

$$I(s, y) = P \int_{-\infty}^\infty \frac{dy'}{y'-y} f(s, y') , \tag{B.6}$$

and consider the functions

$$g(y) = f(s_2, y) , \tag{B.7a}$$

$$h(y) = f(s_1, y) - f(s_2, y) , \tag{B.7b}$$

where s_1 and s_2 are to be regarded as parameters. Then, by considering $I(s_2, y)$ as an integral over $g(y')$ and $I(s_1, y) - I(s_2, y)$ as an integral over $h(y')$, it may be shown from the earlier results that

$$\|I\|_2 \leqq \kappa \|f\|_2 . \tag{B.8}$$

Of course if

$$J(s, y) = P \int_{-\infty}^\infty \frac{ds'}{s'-s} I(s', y) , \tag{B.9}$$

then

$$\|J\|_2 \leqq \kappa \|f\|_2 . \tag{B.10}$$

In the body of the paper, the s-integrals have the semi-infinite range $[4, \infty)$, so we need the additional condition

$$f(4, y) = 0\,, \tag{B.11}$$

since then we can freely define $f(s, y) = 0, s < 4$, without destroying Hölder-continuity. Again, all the above results hold if y is replaced by $y + i\eta$, and the P in (B.6) is dropped.

Theorem 3. *Finally we consider the case when both the Cauchy kernel and the weight function depend on s. Consider a function $g(s', s)$ defined on $4 \leqq s' < \infty$, $4 \leqq s < \infty$ and introduce the norm*

$$\begin{aligned}\|g\|_3 &= \sup\{(s_2 + s_2')^{-a} s_2'^{\lambda} |g(s_2', s_2)|\} \\ &+ \sup\left\{(s_2 + s_2')^{-a} s_2'^{\lambda} \frac{|g(s_1', s_2) - g(s_2', s_2)|}{\left|\frac{s_1' - s_2'}{s_1'}\right|^{\mu}}\right\} \\ &+ \sup\left\{s_1^{-b} s_2'^{\lambda} \frac{|g(s_2', s_1) - g(s_2', s_2)|}{|s_1 - s_2|^{\mu+\delta}}\right\}\end{aligned} \tag{B.12}$$

where $\max\{a + \mu, b + \mu\} < \lambda < 1$, $\mu > 0$, $a \geqq 0$, $b \geqq 0$ *and δ is a generic small positive number which can be chosen as small as one pleases. The suprema are taken over $s_1' \geqq s_2' \geqq 4$, $s_1 \geqq s_2 \geqq 4$. If $\|g\|_3 < \infty$, $g(4, s) = 0$ and*

$$G(s) = P \int_4^{\infty} \frac{ds'}{s' - s}\, g(s', s) \tag{B.13}$$

then it can be shown that $\|G\|_4 < \infty$ where

$$\begin{aligned}\|G\|_4 &= \sup\{s_2^{\lambda - a} |G(s_2)|\} \\ &+ \sup\left\{s_2^{\lambda - c} \frac{|G(s_1) - G(s_2)|}{\left|\frac{s_1 - s_2}{s_1}\right|^{\mu}}\right\}\end{aligned} \tag{B.14}$$

and $c = \max\{a, b + \mu + \delta\}$. *The suprema in* (B.14) *are taken over* $s_1 \geqq s_2 \geqq 4$.

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