Commun. math. Phys. 38, 299—316 (1974)

Euclidean Green's Functions for Jaffe Fields

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Received June 11, 1973; in revised form April 1, 1974

Abstract. We extend the axioms for Euclidean Green's functions recently proposed by Osterwalder and Schrader to Jaffe fields.

§ 1. Introduction

We extend in this paper results of Osterwalder and Schrader [11, 12] on Euclidean Green's (Schwinger) functions to Jaffe fields.

In the first part of their work Osterwalder and Schrader give a precise distribution-theoretic definition of the Euclidean Green's functions and investigate properties of these Schwinger functions as a consequence of the Wightman axioms (in the case of tempered fields). In the second part of their work they give a system of axioms for the Schwinger functions which allow the reconstruction of a unique Wightman theory (for tempered fields).

On the other hand, Jaffe [8] extended the Wightman theory by considering fields (in x-space) as certain classes of operator-valued ultradistributions. Correspondingly, in p-space, Jaffe fields need not be tempered.

We study in this paper Euclidean Green's functions for Jaffe fields and extend the results of Osterwalder and Schrader to these classes of fields. Several alterations have to be made in the original proofs of Osterwalder and Schrader. So, for example, some new techniques are necessary in order to handle the stronger singularities of the Fourier-Laplace transform of ultradistributions on the real axis. Also the contractivity of the time-translation operator requires a different proof, using the general form of linear continuous functionals over $\mathfrak{M}$ (see for instance [5]).

We shall use freely the notations of Osterwalder and Schrader [11, 12].

§ 2. Mathematical Background

In this chapter we present a short summary of the necessary mathematical background for Jaffe fields.

Definition. Let ω be a real-valued function on $[0, \infty)$ which satisfies the following conditions:

a) $\exp(\omega(x^2))$ is a real entire function on $\mathbb{R}^n$, i.e.

$$e^{\omega(x^2)} = \sum_{k=0}^{\infty} a_{2k}|x|^{2k}, \qquad |x| = \sqrt{x_1^2 + \cdots + x_n^2}$$

and $a_0 = 1$, $a_{2k} \geqq 0$ $(k = 1, 2, 3, \ldots)$ (regularity)

b) $$\omega(x+y) \leqq \omega(x) + \omega(y)$$

for all $x, y \in [0, \infty)$ (subadditivity)

c) $$\int_0^{\infty} \frac{\omega(t^2)}{1+t^2}\, dt < \infty \qquad \text{(Carleman's criterion)}$$

d) $$2\omega(x) \leqq \omega(Ax) + C$$

for some constants A, C, (nuclearity).

Every function satisfying a)–d) will be called a *Jaffe indicatrix.*

Remark. Condition d) is equivalent to saying (see [10]):

d') $$a_k^{-1} \leqq C L^k \Bigl(\sup_{0 \leqq l \leqq k} \{a_l a_{k-l}\} \Bigr)^{-1}$$

for some constants C, L (stability under ultradifferential operators). For a proof see the Appendix.

To avoid trivial cases, we shall assume furthermore, without mentioning it, that all Jaffe indicatrices occuring in this paper satisfy:

$$\omega(x^2) \geqq \log(1 + |x|^2)$$

Example. Let $a_{2r} = \left(\frac{1}{r!}\right)^{\alpha}$, $\alpha > 2$. The corresponding entire function

$$\sum_{r=0}^{\infty} \frac{t^{2r}}{(r!)^{\alpha}} = \exp(\omega(t^2))$$

can easily be shown to be built up from a Jaffe indicatrix [i.e. satisfies conditions a)–d)].

Now we are able to define our spaces of test functions:

Definition. Let ω be a Jaffe indicatrix. We denote by $\mathfrak{M}_{\omega}(\mathbb{R}^l)$ the space of all functions $\varphi \in \mathscr{C}^{\infty}(\mathbb{R}^l)$ for which

$$\|\varphi\|_{\alpha,\lambda}^{(\mathfrak{M})} = \sup_{p \in \mathbb{R}^l} \{e^{\lambda\omega(|p|^2)} |D^{\alpha}\varphi(p)|\}$$

is finite for each multiindex α and each constant $\lambda > 0$. A natural topology on $\mathfrak{M}$ can be defined by means of the norms $\|\cdot\|_{\alpha,\lambda}^{(\mathfrak{M})}$.

Definition. Let ω be a Jaffe indicatrix. We denote by $\mathscr{C}_\omega$ the space of Fourier transforms of functions in $\mathfrak{M}_\omega$, $\mathscr{C}_\omega = \mathscr{F}(\mathfrak{M}_\omega)$. A natural topology on $\mathscr{C}_\omega$ can be defined by means of the following norms:

$$\|\varphi\|_{\alpha,\lambda}^{(\mathscr{C})} = \sup_{p \in \mathbb{R}^l} \{e^{\lambda\omega(|p|^2)}|D^\alpha \tilde{\varphi}(p)|\}.$$

The Fourier transform is a continuous isomorphism between $\mathfrak{M}_\omega$ and $\mathscr{C}_\omega$.

Definition. Let ω be a Jaffe indicatrix. We denote by $\mathscr{D}_\omega(K)$, $K \subset \mathbb{R}^l$ compact, the space of all functions $\varphi \in \mathscr{C}^\infty$ with support in K for which

$$\|\varphi\|_\lambda^{(\mathscr{D}_\omega)} = \int e^{\lambda\omega(|x|^2)}|\tilde{\varphi}(x)|\, dx$$

is finite for each $\lambda > 0$. $\mathscr{D}_\omega(K)$ is easily seen to be a Fréchet-space under the topology given by the seminorms $\|\cdot\|_\lambda^{(\mathscr{D}_\omega)}$. Now define $\mathscr{D}_\omega(\mathbb{R}^l)$ to be the inductive limit of spaces $\mathscr{D}_\omega(K_j)$, $\bigcup_{j=1}^{\infty} K_j = \mathbb{R}^l$ and $K_j \subset K_{j+1}$. Then $\mathscr{D}_\omega(\mathbb{R}^l)$ generalizes the Schwartz space $\mathscr{D}$ of infinitely differentiable functions with compact support.

Remark. Throughout this paper the Jaffe indicatrix leading to the spaces $\mathfrak{M}_\omega$ and $\mathscr{C}_\omega$ will be held fixed, so we shall denote them simply by $\mathfrak{M}$ and $\mathscr{C}$.

An equivalent set of seminorms on $\mathscr{D}_\omega(K)$ can be given by

$$|||\varphi|||_\lambda = \sup_{x \in \mathbb{R}^l} \{e^{\lambda\omega(|x|^2)}|\tilde{\varphi}(x)|\}.$$

For a proof see [1], Theorem 1.4.1. If $\exp(\omega(t^2))$ is the entire function of the preceding example, the corresponding test function spaces $\mathfrak{M}$ and $\mathscr{C}$ coincide with the projective limits of the spaces $S_{\alpha/2,A}$, $S^{\alpha/2,A}$ respectively, introduced by Gelfand and Shilov [5].

Lemma 1. *Let ω be a Jaffe indicatrix. Then the topologies on $\mathfrak{M}$, defined by the seminorms:*

$$\|\varphi\|_{\alpha,\lambda} = \sup_{p \in \mathbb{R}^l} \{e^{\lambda\omega(|p|^2)}|D^\alpha \varphi(p)|\}$$

$$|||\varphi|||_{\alpha,\lambda} = \sup \{e^{\omega(\lambda|p|^2)}|D^\alpha \varphi(p)|\}$$

$$\langle\!\langle \varphi \rangle\!\rangle_{\alpha,\lambda,n} = \sup \{e^{\lambda\omega(|p|^2)}(1+|p|^2)^n\, |D^\alpha \varphi(p)|\}$$

$$\langle\!\langle\!\langle \varphi \rangle\!\rangle\!\rangle_{\alpha,\lambda,n} = \sup \{e^{\omega(\lambda|p|^2)}(1+|p|^2)^n\, |D^\alpha \varphi(p)|\}$$

are equal.

Proof. The proof of Lemma 1 is an easy consequence of subadditivity, the nuclearity condition and $\omega(|x|^2) \geqq \log(1+|x|^2)$.

Remark. Lemma 1 obviously holds also for the seminorms' counterparts in $\mathscr{C}$. The topology defined by $\langle\!\langle\!\langle \cdot \rangle\!\rangle\!\rangle_{\alpha,\lambda,n}$ is the topology originally introduced on $\mathfrak{M}$ by Jaffe [8]. So, infact, his test function spaces and ours are the same, if ω is a Jaffe indicatrix.

Lemma 2. *Let ω be a Jaffe indicatrix. A function $\varphi \in \mathscr{C}^\infty(\mathbb{R}^l)$ belongs to $\mathscr{C}$, if and only if for each multiindex m and each constant $h > 0$ there is a constant $C_{m,h}$ such that*

$$\sup_{x \in \mathbb{R}^l} \{|x^m \Delta^k \varphi(x)|\} \leqq C_{m,h} h^k / a_{2k}. \tag{1}$$

Proof. (See [9], Theorem 2). Let $\varphi \in \mathscr{C}^\infty$ and suppose that (1) holds. Then

$$|||\varphi|||_{\alpha,\lambda}^{(\mathscr{C})} = \sup_{p \in \mathbb{R}^l} \{e^{\omega(\lambda|p|^2)} |D^\alpha \tilde{\varphi}(p)|\} \leqq \sup_{p \in \mathbb{R}^l} \{|D^\alpha [e^{\omega(\lambda|p|^2)} \tilde{\varphi}(p)]|\}$$

$$= \sum_{k=0}^\infty a_{2k} \lambda^k \sup_{p \in \mathbb{R}^l} \{|D^\alpha [|p|^{2k} \tilde{\varphi}(p)]|\} \leqq \sum_{k=0}^\infty a_{2k} \lambda^k \int |x^\alpha \Delta^k \varphi(x)|.$$

Choose $C = \int (1 + |x|^{2l})^{-1} dx$.
Then

$$|||\varphi|||_{\alpha,\lambda} \leqq C \sum_{k=0}^\infty a_{2k} \lambda^k \sup_{x \in \mathbb{R}^l} \{|(1 + |x|^{2l}) x^\alpha \Delta^k \varphi(x)|\}.$$

Using (1) each term

$$|(1 + |x|^{2l}) x^\alpha \Delta^k \varphi(x)|$$

can be majorized by Ah^k / a_{2k} where A is a constant depending only on h. It follows:

$$|||\varphi|||_{\alpha,\lambda} \leqq C_h \sum_{k=0}^\infty \lambda^k h^k$$

which is finite, if we choose $h < \lambda^{-1}$. Hence $\varphi \in \mathscr{C}$.

Conversely, let $\varphi \in \mathscr{C}$. Then

$$\sup_{x \in \mathbb{R}^l} \{|x^m \Delta^k \varphi(x)|\} \leqq \int |D^m [|p|^{2k} \tilde{\varphi}(p)]| \, dp$$
$$\leqq (4k)^{|m|} \int (1 + |p|^{2k}) \sum_{\alpha=0}^m |D^\alpha \tilde{\varphi}(p)| \, dp.$$

Choose B, C in such a way, that $(4k)^{|m|} \leqq CB^k$. Then

$$\sum_{k=0}^\infty a_{2k} h^{-k} \sup_{x \in \mathbb{R}^l} \{|x^m \Delta^k \varphi(x)|\}$$
$$\leqq C \sum_{k=0}^\infty a_{2k} h^{-k} B^k \int (1 + |p|^{2k}) \sum_{\alpha=0}^m |D^\alpha \tilde{\varphi}(p)| \, dp$$
$$= C \int \left(e^{\omega(\frac{B}{h}|p|^2)} + e^{\omega(\frac{B}{h})}\right) \sum_{\alpha=0}^m |D^\alpha \tilde{\varphi}(p)| \, dp. \tag{2}$$

Since, by hypothesis, $\varphi \in \mathscr{C}$, i.e. $\tilde{\varphi} \in \mathfrak{M}$, the last integral is finite (see [5] for integral norms on $\mathfrak{M}$). Thus the terms of the infinite sum (2) are uniformly bounded by a constant depending on h and m only. This completes the proof.

We conclude this chapter with a fairly easy consequence of the proof of Lemma 2 (see [9], Theorem 5):

Lemma 3. *Let ω be a Jaffe indicatrix. Then the seminorms on $\mathscr{C}$:*

$$//\varphi//_{h,m} = \sum_{k=0}^{\infty} a_{2k} h^{-k} \sup_{\substack{|j|=m \\ |\alpha|=2k}} \sup_{x \in \mathbb{R}^l} \{|x^j D^\alpha \varphi(x)|\}$$

are equivalent to the set of seminorms $\|\cdot\|_{\alpha,\lambda}$.

§ 3. The Euclidean Green's Functions for Jaffe Fields

We shall first state the Wightman-Jaffe axioms for the scalar neutral field $\varphi(x)$. The expectation values

$$\mathfrak{W}_n(\underline{x}) = \mathfrak{W}_n(x_1, \ldots, x_n) = (\Omega, \varphi(x_1) \ldots \varphi(x_n)\, \Omega),$$
$$\underline{x} = (x_1, \ldots, x_n),\ x_i = (x_i^0, \vec{x}_i)$$

are supposed to obey the following axioms.

Ultradistribution Property. For each n:

$$(W0) \qquad \mathfrak{W}_n(\underline{x}) \in \mathscr{C}'(\mathbb{R}^{4n}); \qquad \mathfrak{W}_0 = 1\,.$$

Relativistic Covariance. For each n, $\mathfrak{W}_n$ is Poincaré invariant:

$$(W1) \qquad \mathfrak{W}_n(\underline{x}) = \mathfrak{W}(\underline{\Lambda x} + a)$$

for all $(a, \Lambda) \in \mathfrak{P}_+^\uparrow$ where $\underline{\Lambda x} + a = (\Lambda x_1 + a, \ldots, \Lambda x_n + a)$.

Positivity. For all finite sequences $f_0, f_1, \ldots f_N$ of test functions $f_0 \in \mathbb{C}$, $f_n \in \mathscr{C}(\mathbb{R}^{4n})$; $n = 1, \ldots, N$:

$$(W2) \qquad \sum_{n,m} \mathfrak{W}_{n+m}(f_n^* \times f_m) \geqq 0$$

where $f_n^* \times f_m$ is defined by

$$(f_n^* \times f_m)(\underline{x}, \underline{y}) = f_n^*(\underline{x}) f_m(\underline{y})$$

and

$$f_n^*(\underline{x}) = f_n^*(x_1, \ldots, x_n) = \bar{f}(x_n, \ldots, x_1) \equiv \bar{f}_n(\underline{x})\,.$$

Locality. For each n and $k = 1, \ldots, n-1$:

$$(W3) \qquad \mathfrak{W}_n(x_1, \ldots, x_k, x_{k+1}, \ldots, x_n) = \mathfrak{W}_n(x_1, \ldots, x_{k+1}, x_k, \ldots, x_n)$$

if $(x_k - x_{k+1})^2 < 0$.

Cluster Property. For any space like a and $k=1,\ldots,n-1$, $\underline{x}=(x_1,\ldots,x_k)$, $\underline{y}=(y_1,\ldots,y_{n-k})$:

$$(W4)\quad \lim_{\lambda\to\infty}\mathfrak{W}_n(\underline{x},\underline{y}+\lambda a)=\mathfrak{W}_n(\underline{x})\,\mathfrak{W}_{n-k}(\underline{y})\,.$$

Spectral Condition. Using the translation invariance of the $\mathfrak{W}_n$'s, we conclude that there exist ultradistributions $W_{n-1}\in\mathscr{C}'(\mathbb{R}^{4(n-1)})$ such that

$$\mathfrak{W}_n(\underline{x})=W_{n-1}(\underline{\xi})\,,\qquad \underline{\xi}=(\xi_1,\ldots,\xi_{n-1})$$

and $\xi_k=x_{k+1}-x_k$. Then:

$$(W5)\quad \operatorname{supp}\tilde{W}_{n-1}(\underline{q})\subset\bar{V}_+^{n-1}\equiv\{\underline{q}\,|\,q_i\in\bar{V}_+,i=1,\ldots,n-1\}$$

where $\tilde{W}_{n-1}(\underline{q})$ is the Fourier transform of $W_{n-1}(\underline{\xi})$, $\bar{V}_+$ is the closed forward light cone, and $q_k\xi_k=q_k^0\xi_k^0-\vec{q}_k\vec{\xi}_k$. From a given set of Wightman-Jaffe ultradistributions satisfying $(W0)-(W5)$ we can reconstruct the Jaffe field $\varphi(x)$, the physical Hilbert space $\mathscr{H}$, the vacuum vector Ω and a unitary representation $U(a,\Lambda)$ of $\mathfrak{P}_+^\uparrow$ in $\mathscr{H}$ by applying the reconstruction theorem. Now standard methods in Axiomatic Quantum Field theory show that $W_n(\underline{\xi})$ are boundary values of functions $W_n(\underline{\zeta})$ analytic in $\mathfrak{T}_+^n=\{\underline{\zeta}\,|\,\underline{\zeta}=\underline{\xi}+i\underline{\eta}\in\mathbb{C}^{4n},\ \underline{\eta}\in V_+^n\}$ (see [9] and [3]). $W_n(\underline{\zeta})$ can be analytically continued to the extended tube $\mathfrak{T}_{+,\mathrm{ext}}^n=\{\underline{\zeta}\,|\,\Lambda\underline{\zeta}\in\mathfrak{T}_+^n,\ \Lambda\in L_+(\mathbb{C})\}$, where $L_+(\mathbb{C})$ denotes the set of complex Lorentz transformations with det 1. The functions $\mathfrak{W}_n(\underline{z})$ defined by $\mathfrak{W}_n(\underline{z})=W_{n-1}(\underline{\zeta})$, where $\zeta_k=z_{k+1}-z_k$ are analytic in $\sigma_{\mathrm{ext}}^n=\{\underline{z}\,|\,\underline{z}\in\mathfrak{T}_{+,\mathrm{ext}}^n\}$ and have as boundary values the distributions $\mathfrak{W}_n(\underline{x})$. Finally, $\mathfrak{W}_n(\underline{z})$ can be analytically extended into the set $\sigma_{\mathrm{ext,perm}}^n\equiv\{\underline{z}\,|\,(z_{\pi(1)},\ldots,z_{\pi(n)})\in\sigma_{\mathrm{ext}}^n$ for some permutation $\pi\}$.

The set $\sigma_{\mathrm{ext,perm}}^n$ contains $\mathscr{E}^n=\{\underline{z}\,|\,\underline{z}\in\mathbb{C}^n,\mathrm{Re}\,z_k^0=0,\ \mathrm{Im}\,\vec{z}_k=0,\ z_k\neq z_1$ for all $1\leqq k<1\leqq n\}$.

Definition. Points in $\mathscr{E}^n$ are called Euclidean points.

Definition. The restriction of the Wightman functions $\mathfrak{W}_n(\underline{z})$ to $\mathscr{E}^n$ are called the n-point Euclidean Green's functions or Schwinger functions for a Jaffe field.

We set, following Osterwalder and Schrader

$$\sigma_0=\mathfrak{W}_0=1$$
$$\sigma_n(\underline{x})=\sigma_n(x_1,\ldots,x_n)=\mathfrak{W}_n((ix_1^0,\vec{x}_1),\ldots,(ix_n^0,\vec{x}_n))$$

for $\underline{x}\in\Omega^n=\{\underline{x}\,|\,x_i\neq x_j$ for all $1\leqq i<j\leqq n\}$.

From the invariance properties of the Wightman functions, we derive invariance properties of the Schwinger functions under the inhomogeneous Euclidean group iSO_4 (and under permutations), we write

$$S_{n-1}(\underline{\xi}) = W_{n-1}((i\xi_1^0, \vec{\xi}), \ldots, (i\xi_{n-1}^0, \vec{\xi}_{n-1}))$$
$$= \sigma(\underline{x}), \qquad \xi_k = x_{k+1} - x_k, \underline{x} \in \Omega^n .$$

The Schwinger functions $S_{n-1}(\underline{\xi})$ are real analytic. But they also define ultradistributions in a certain sense which we make clear in Proposition I.

We now prove some results on Fourier-Laplace transform of ultradistributions which we need in this and in the next part of this work.

Theorem 1. *Let ω be a Jaffe indicatrix, let Γ be an open convex cone in $\mathbb{R}^l$ and let F be an analytic function in $\mathfrak{T}_n = \mathbb{R}^{4l} + i\Gamma$. Suppose that for each compact subset $K \subset \Gamma$ there exists a polynomial P_K such that*

$$|F(\xi + i\eta)| \leqq P_K(\xi) .$$

Then F has a boundary value in $\mathscr{C}'$ (in the weak topology of $\mathscr{C}'$) as $\eta \to 0$, η in a subcone of Γ, if and only if for each subcone $\Gamma' \subset \Gamma$, whose intersection with the unit sphere is relatively compact in Γ, and for each $R > 0$ there exists a polynomial P and a constant $a > 0$ such that for all $\eta \in \Gamma'$, $|\eta| < R$,

$$|F(\xi + i\eta)| \leqq P(\xi)\, \Lambda_a(|\eta|)$$

where

$$\Lambda_a(t) = t \int_0^\infty e^{a\omega(x^2)} e^{-xt} dx , \qquad t > 0$$

is the one-sided Laplace transform of $e^{a\omega(x^2)}$ times t.

Proof. See [3], Theorem 5.3, and [9], Theorem 19.

Remark. One can (see [9]) equivalently define:

$$\Lambda'_a(t) = t \sum_{k=0}^{\infty} a_{2k}(2k)!\, a^k t^{-2k} = t \int_0^\infty e^{\omega(ax^2)} e^{-xt} dx .$$

Anyway, by Lemma 1 it is clear that this alternate definition does not change the result of Theorem 1.

Theorem 2. *Let ω be a Jaffe indicatrix and let F be holomorphic in the tube $\mathbb{R}^l + i\Gamma$, Γ an open convex cone in $\mathbb{R}^l$. Assume that for each subcone $\Gamma' \subset \Gamma$, whose intersection with the unit sphere is relatively compact in Γ, there is a constant $C(\Gamma')$ such that*

$$|F(\xi + i\eta)| \leqq C(\Gamma')\,(1 + \Lambda_a(|\eta|))\,(1 + |\xi + i\eta|)^n$$

for all $\eta \in \Gamma'$ and for some constants a, n (which do not depend on Γ'). Then F is the Fourier-Laplace transform of an ultradistribution $f \in \mathfrak{M}'$

with support in the dual cone Γ^*

$$\Gamma^* = \{y \in \mathbb{R}^l \,|\, xy \geqq 0, \forall x \in \Gamma\}\,.$$

Proof. From Theorem 1 it follows that the limit $\lim\limits_{\eta \to 0, \eta \in \Gamma'} F(\xi + i\eta)$ exists in $\mathscr{C}'$. We denote this limits again by F. By Theorems 5.1 and 5.2 of [3] we know that F is the Fourier-Laplace transform of a generalized function $f \in \mathfrak{M}'$,

$$F(\xi + i\eta) = F(\zeta) = \mathscr{F}[e^{-x\eta} f(x)]\,(\xi)\,.$$

We still have to prove the support property.

Let φ be a function in $\mathscr{D}_\omega$. Then we get

$$\begin{aligned}(f, \varphi) &= (e^{p\eta} \mathscr{F}^{-1}[F(\xi + i\eta)]\,(p), \varphi(p)) \\ &= (F(\xi + i\eta), \mathscr{F}^{-1}[e^{p\eta}\varphi(p)]\,(\xi)) \\ &= \frac{1}{(2\pi)^{1/2}} \int F(\xi + i\eta) \int e^{-ip\xi} e^{p\eta} \varphi(p)\, dp d\xi\,. \end{aligned} \tag{3}$$

Let p_0 be a point in the complement of Γ^*. Then there exists a unit vector $\eta_0 \in \Gamma$ with $p_0 \eta_0 < 0$. Suppose that $\operatorname{supp} \varphi$ is contained in a ball B around p_0 such that $B\eta_0 < -\delta < 0$. Then by the Paley-Wiener theorem for test functions and (3) we get an estimate of the form:

$$\begin{aligned}|(f, \varphi)| &\leqq C \int |F(\xi + i\eta)|\, e^{-\mu\omega(|\xi + i\eta|^2)} \sup_{p \in \operatorname{supp}\varphi} \{e^{p\eta}\}\, d\xi \\ &\leqq C A_a(|\eta|) \sup \{e^{p\eta}\} \int e^{-\mu\omega(|\xi + i\eta|^2)} (1 + |\xi + i\eta|)^n\, d\xi\,. \end{aligned} \tag{4}$$

Let now Γ' be the subcone $\{\eta = \lambda\eta_0 \,|\, \lambda > 0\}$ of Γ and choose μ so big that the integral in (4) exists. Then, finally,

$$|(f, \varphi)| \leqq C(\Gamma')\, A_a(\lambda)\, e^{-\lambda\delta}\,.$$

Let now $\lambda \to \infty$. Then $A_a(\lambda)$ remains bounded and we see that $(f, \varphi) = 0$ for all $\varphi \in \mathscr{D}_\omega$ with $\operatorname{supp}\varphi$ in the ball B around $p_0 \notin \Gamma^*$. Thus we have shown that f vanishes in at least one neighborhood of each point in the complement of Γ^*, therefore it vanishes globally outside Γ^*. Since $\mathscr{D}_\omega$ is dense in $\mathfrak{M}$ by Theorem 1.8.7 of [1], the Theorem is proved.

Now we introduce some notations. Let $\mathbb{R}_\pm$ denote the open half intervals $(0, \pm\infty)$ and $\overline{\mathbb{R}_\pm}$ their closure. Let $\mathfrak{M}(\mathbb{R}_\pm)$ denote the space of all functions $f \in \mathfrak{M}(\mathbb{R})$ with *supp* f in $\mathbb{R}_\pm$ equipped with the induced topology. $\mathfrak{M}(\overline{\mathbb{R}_\pm})$ will be the set of all functions defined on $\overline{\mathbb{R}_+}$, $\mathscr{C}^\infty$ on $\mathbb{R}_+$ whose derivatives all have a continuous extension to $\overline{\mathbb{R}_+}$ and are of $\exp\omega$ decrease at infinity. We introduce a topology on $\mathfrak{M}(\overline{\mathbb{R}_+})$ by the following system of seminorms

$$\|\varphi\|_{\alpha,\lambda,+}=\sup_{p>0}\{e^{\lambda\omega(p^2)}|D^\alpha\varphi(p)|\}\,.$$

Then we have

Lemma 4. *The space* $\mathfrak{M}(\overline{\mathbb{R}_+})$ *is isomorphic to the topological quotient space* $\mathfrak{M}(\mathbb{R})/\mathfrak{M}(\mathbb{R}_-)$.

Proof. The proof of this lemma is based on the Whitney extension theorem in the form given by Hörmander [7] and follows from [12] by simply replacing the polynomial growth by a growth like $\exp(\omega(p^2))$.

By Lemma 1 any element in $\mathfrak{M}(\overline{\mathbb{R}_+})$ is the restriction to $\overline{\mathbb{R}_+}$ of some element in $\mathfrak{M}(\mathbb{R})$ and a generalized function in $\mathfrak{M}'(\overline{\mathbb{R}_+})$ can be identified with a generalized function in $\mathfrak{M}'(\mathbb{R})$ with support in $\mathbb{R}_+$ (see also [12]).

For $\varphi\in\mathscr{C}(\mathbb{R}_+)$ we define

$$\check{\varphi}(p)=\int_{-\infty}^{\infty}e^{-px}\varphi(x)\,dx\restriction\overline{\mathbb{R}_+}\,.$$

Then

Lemma 5. *The mapping* $\varphi\to\check{\varphi}$ *is continuous from* $\mathscr{C}(\mathbb{R}_+)$ *into* $\mathfrak{M}(\overline{\mathbb{R}_+})$*; its range is dense in* $\mathfrak{M}(\overline{\mathbb{R}_+})$ *and its kernel is zero.*

For the proof of Lemma 5 we need some more lemmas.

Lemma 6. *Suppose* $T\in\mathfrak{M}'(\overline{\mathbb{R}_+})$. *Then*

$$T=\sum_{\alpha\leqq m}D^\alpha\mu_\alpha$$

where μ_α *are measures with support in* $[0,\infty)$ *such that some non-negative constant* λ *exists with*

$$\int e^{-\lambda\omega(t^2)}|d\mu_\alpha(t)|<\infty\,.$$

Proof. The proof of this lemma can be done along the lines of [14] and [13] by using the general form of generalized functions in $\mathfrak{M}$ [5].

Lemma 7. *Let* $\varphi\in\mathscr{C}(\mathbb{R})$ *and let* $D^\alpha\varphi(0)=0$, $\alpha=0,1,2,\ldots$. *Then for all* $a>0$

$$|A_a(|x|)\,\varphi(x)|<\infty\,,\qquad x\in\mathbb{R}\,.$$

Proof. By Taylor's formula

$$\varphi(x)=\frac{x^n}{n!}\,\varphi^{(n)}(\xi)\,,\qquad 0\leqq\xi\leqq x\,.$$

Lemma 2 then yields:

$$|\varphi^{(2n)}(\xi)|\leqq Ch^n/a_{2n}$$

and therefore

$$\frac{a_{2n}(2n)!\,|\varphi(x)|}{(2h)^n\,|x^2|^n} \leqq C\frac{1}{2^n}.$$

Letting $n \to \infty$:

$$|\varphi(x)| \sum_{n=0}^{\infty} \frac{a_{2n}(2n)!}{(2h)^n\,|x^2|^n} = |\varphi(x)|\,\Lambda'_{2h}(|x|) < \infty\,.$$

Proof of Lemma 5. The following inequality is valid

$$\|\varphi\|_{\alpha,\lambda,+} = \sup_{q>0}\{e^{\lambda\omega(q^2)}|D^\alpha\check{\varphi}(q)|\} \leqq C/\!/\varphi/\!/_{h_1,m_1}$$

by Lemma 1, Lemma 3 and the formula

$$e^{\omega(\lambda q^2)} D^\alpha \check{\varphi}(q) = \int e^{-qx} e^{\omega(\lambda\Delta)}((-x)^\alpha\,\varphi(x))\,dx \restriction \{q \geqq 0\} \tag{5}$$

where $\Delta = \dfrac{d^2}{dx^2}$ and $e^{\omega(\Delta)}$ is defined as a series in the Laplacian:

$$e^{\omega(\Delta)} = \sum_{k=0}^{\infty} a_{2k}\Delta^k\,.$$

Up to now we have shown that $\check{\varphi} \in \mathfrak{M}(\overline{\mathbb{R}_+})$ and that the mapping $\varphi \to \check{\varphi}$ is continuous, but the rest of Lemma 5 can be proved like in [12], taking into account Lemma 6 and Lemma 7.

Let now $T \in \mathfrak{M}'(\mathbb{R})$ with $supp\,T \subset \overline{\mathbb{R}_+}$. It defines also a generalized function in $\mathfrak{M}'(\overline{\mathbb{R}_+})$, again denoted by T and for some constants C, α, λ:

$$|T(\check{f})| \leqq C\|f\|_{\alpha,\lambda,+} \equiv c\|f\|'_{\alpha,\lambda}\,.$$

Now for $x > 0$ we define a real analytic function

$$S(x) = (T(p), e^{-xp}) \restriction \{p \geqq 0\})\,.$$

Then

Lemma 8. *S is an ultradistribution in $\mathscr{C}'(\mathbb{R}_+)$.*

Proof. The proof is an immediate consequence of Lemma 6, Lemma 7, and formula (5) in the proof of Lemma 5.

Now we can state a basic theorem:

Theorem 3. *Let T be a generalized function in $\mathfrak{M}'(\mathbb{R})$ with $\operatorname{supp} T \subset \overline{\mathbb{R}_+}$. Then for all $\varphi \in \mathscr{C}(\mathbb{R}_+)$:*

$$|S(\varphi)| = T(\check{\varphi}) \tag{6}$$

$$|S(\varphi)| \leqq C\|\varphi\|'_{\alpha,\lambda} \equiv C\|\check{\varphi}\|_{\alpha,\lambda,+} \leqq C'\|\varphi\|_{\alpha',\lambda'} \tag{7}$$

for some constants C, α, λ depending on T only. Conversely, if S is an ultradistribution in $\mathscr{C}'(\mathbb{R}_+)$, satisfying (7) *for some C, α, λ, then there* *exists a unique generalized function $T \in \mathfrak{M}'(\mathbb{R})$ with support in $\overline{\mathbb{R}_+}$, such that* (6) *holds.*

Proof. Follows from Lemmas 4–8. In what follows we shall need a multivariable version of Theorem 3, which, due to the nuclear theorem (see Appendix), is easy to prove. First we introduce some additional notations:

We denote by $\mathbb{R}_+^{4n}$ the set $\{\underline{\xi} \mid \xi_k^0 > 0, k = 1, \ldots, n\}$, by $\overline{\mathbb{R}}_+^{4n}$ its closure and for $\varphi \in \mathscr{C}(\mathbb{R}_+^{4n})$ we define $\check{\varphi}$ by

$$\check{\varphi}(\underline{p}) = \int \exp\left(-\sum_{k=1}^{n} (p_k^0 \xi_k^0 + i\vec{p}_k \vec{\xi}_k)\right) \varphi(\underline{\xi})\, d\underline{\xi} \restriction \overline{\mathbb{R}}_+^{4n}.$$

On $\mathfrak{M}(\overline{\mathbb{R}}_+^{4n})$ we introduce a set of norms

$$\|\varphi\|_{\alpha,\lambda,+} = \sup_{p \in \mathbb{R}_+^{4n}} \{e^{\lambda\omega(p^2)} |D^\alpha \varphi(\underline{p})|\}$$

which induce a set of seminorms on $\mathscr{C}(\mathbb{R}_+^{4n})$:

$$\|\varphi\|'_{\alpha,\lambda} = \|\check{\varphi}\|_{\alpha,\lambda,+}$$

and we immediately see that Lemma 4 holds with obvious modifications. Analoguously, for $T \in \mathfrak{M}'(\mathbb{R}^{4n})$, $\operatorname{supp} T \subset \overline{\mathbb{R}}_+^{4n}$, we define S by

$$S(\underline{\xi}) = \left(T(\underline{p}), \exp\left(-\sum_{k=1}^{n} (\xi_k^0 p_k^0 + i\vec{\xi}_k \vec{p}_k)\right) \restriction \{\underline{p} \in \mathbb{R}_+^{4n}\}\right).$$

Again, S defines an ultradistribution in $\mathscr{C}'$ $(\mathbb{R}_+^{4n})$. As stated above, the multivariable version of Theorem 3, with obvious formulation, follows by the nuclear theorem.

We introduce two more spaces of ultradistributions

$$\mathscr{C}_0(\mathbb{R}^{4n}) = \{\varphi \in \mathscr{C}(\mathbb{R}^{4n}) \mid D^\alpha(x) = 0 \text{ for all } \alpha, \text{ if } x_i = x_k \text{ for some } i \neq k\}$$

and

$$\mathscr{C}(\mathbb{R}_<^{4n}) = \{\varphi \in \mathscr{C}(\mathbb{R}^{4n}) \mid \operatorname{supp}\varphi \subset \mathbb{R}_<^{4n}\}$$

where

$$\mathbb{R}_<^{4n} = \{\underline{x} \mid 0 < x_1^0 < \cdots < x_n^0\}.$$

We can state now the following proposition which is an extension of a result by Osterwalder and Schrader to Jaffe fields:

Proposition I. *The Schwinger functions associated to a Wightman-Jaffe theory have the following properties*

Distribution Property. For each $n \geq 1$

(E0) $$\sigma_n(\underline{x}) \in \mathscr{C}'_0(\mathbb{R}^{4n}), \quad \sigma_0 = 1.$$

S_n defines an element in $\mathscr{C}'(\mathbb{R}^{4n})$ and is continuous with respect to some norm $\|\cdot\|'_{\alpha,\lambda}$ on $\mathscr{C}(\mathbb{R}^{4n}_+)$.

Euclidean Covariance. For each $n \geqq 1$ and all $(\underline{a}, R) \in iSO_4$:

(E1) $$\sigma_n(\underline{x}) = \sigma_n(\underline{Rx} + a).$$

Positivity. For all finite sequences $f_0, f_1, \ldots, f_N$ of test functions $f_n \in \mathscr{C}(\mathbb{R}^{4n}_<)$,

(E2) $$\sum_{n,m} \sigma_{n+m}(\Theta f_n^* \times f_m) \geqq 0$$

where $\Theta f_n(\underline{x}) = f_n(\underline{\vartheta x})$.

Symmetry. For all permutations π

(E3) $$\sigma_n(x_1, \ldots, x_n) = \sigma_n(x_{\pi(1)}, \ldots, x_{\pi(n)}).$$

Cluster Property. For all $n, m, f \in \mathscr{C}(\mathbb{R}^{4n}_<)$, $g \in \mathscr{C}(\mathbb{R}^{4m}_<)$, $\underline{a} = (0, \vec{a}) \in \mathbb{R}^4$

(E4) $$\lim_{\lambda \to \infty} \sigma_{n+m}(\Theta f^* \times g_{\lambda a}) = \sigma_n(\Theta f^*)\, \sigma_m(g)$$

where $g_{\lambda\underline{a}}$ is defined by $g_{\lambda\underline{a}}(\underline{x}) = g(\underline{x} + \lambda\underline{a})$.

Conversely, Schwinger "functions" obeying (E0)–(E4) *are the Schwinger functions associated with a unique Wightman theory.*

Proof. The proof follows from our discussion of the Fourier-Laplace transform for ultradistributions and from arguments used by Osterwalder and Schrader [11].

For practical use in Constructive Field Theory Osterwalder and Schrader replaced condition (E0) by another condition (E0′), which reads in the framework of Jaffe fields:

(E0′) There is a norm $\|\cdot\|'_{\alpha,\lambda}$ on $\mathscr{C}(\mathbb{R}^4_+)$ and some $L > 0$, such that for all n and for all $\varphi_k \in \mathscr{C}(\mathbb{R}^4_+)$, $k = 1, \ldots, n$,

$$|S_n(\varphi_1 \times \varphi_2 \times \cdots \times \varphi_n)| \leqq (n!)^L \prod_{k=1}^{n} \|\varphi_k\|'_{\alpha,\lambda}.$$

In the next section we are going to show that condition (E0′) together with (E1)–(E4) determine a unique Jaffe-Wightman theory.

§ 4. The Reconstruction of the Wightman Theory

Proposition II. *Schwinger functions satisfying* (E0′), (E1)–(E4) *determine a unique Wightman theory (whose Schwinger functions they are).*

Let $\underline{\mathscr{C}}_<$ be the vector space of sequences $\underline{f} = (f_0, f_1, \ldots)$, where $f_0 \in \mathbb{C}$, $f_n \in \mathscr{C}(\mathbb{R}^{4n}_<)$, for $1 \leqq n \leqq N$, and $f_n = 0$ for $n > N$ for some finite N.

Let

$$\langle f, g\rangle = \sum_{n,m} \sigma_{n+m}(\Theta f_n^* \times g_m)\,; \qquad \underline{f}, \underline{g} \in \underline{\mathscr{C}}_< .$$

This defines a semi-definite inner product and the completion of $\underline{\mathscr{C}}_</\mathscr{N}$, $\mathscr{N} = \{\underline{f} \mid \underline{f} \in \underline{\mathscr{C}}_<, \|\underline{f}\|^2 = \langle \underline{f}, \underline{f}\rangle = 0\}$ defines a Hilbert space $\mathscr{H}$ (the physical Hilbert space). Let Φ be the natural injection of $\underline{\mathscr{C}}_<$ into $\mathscr{H}$. We obtain

$$(\Phi(\underline{f}), \Phi(\underline{g})) = \langle \underline{f}, \underline{g}\rangle\,, \qquad \underline{f}, \underline{g} \in \underline{\mathscr{C}}_< .$$

We set $\Omega = \Phi((1, 0, \ldots))$. If $\underline{f}$ has only one non-vanishing component $\underline{f} \equiv f_n \in \mathscr{C}(\mathbb{R}^{4n}_<)$, we write formally

$$\Phi(\underline{f}) = \Phi_n(f) = \int \Phi_n(\underline{x})\, f(\underline{x})\, d^{4n}x$$

where $\Phi_n(x)$ is a vector-valued ultradistribution in $\mathscr{C}(\mathbb{R}^{4n}_<)$. We also define

$$\Psi_n^E(x_1, \underline{\xi}) = \Phi_n(\underline{x}) \quad \text{where} \quad \xi_k = x_{k+1} - x_k .$$

Thus Ψ_n^E is a vector valued ultradistribution in $\mathscr{C}'(\mathbb{R}^{4n}_+)$ and by positivity

$$(\Psi_n^E(x, \underline{\xi}), \Psi_m^E(x', \underline{\xi}')) = S_{n+m-1}(-\overleftarrow{\underline{\vartheta\xi}}, -\vartheta x + x', \underline{\xi}')$$

we define

$$\Psi_n^E(x^0, \underline{\xi}^0 \mid g) = \int \Psi_n^E(x, \underline{\xi})\, g(\vec{x}, \underline{\vec{\xi}})\, \overrightarrow{dx}\, d\overrightarrow{\underline{\xi}}$$

$$S_{n+m-1}(\overleftarrow{\underline{\xi}}^0, x^0 + x'^0, \underline{\xi}'^0(gh)) = (\Psi_n^E(x^0, \underline{\xi}^0 \mid g), \Psi_m^E(x'^0, \underline{\xi}'^0 \mid h))$$

where $g \in \mathscr{C}(\mathbb{R}^{3n})$, $h \in \mathscr{C}(\mathbb{R}^{3m})$. Let $\mathbb{C}_+ = \{z \mid \operatorname{Re} z > 0\}$ and $\mathbb{C}_+^k = (\mathbb{C}_+)^k$.

Now following Osterwalder and Schrader the proof of Proposition II follows immediately from Theorem 2 and

Theorem 4. *For fixed $g \in \mathscr{C}(\mathbb{R}^{3n})$, $h \in \mathscr{C}(\mathbb{R}^{3m})$, the distributions $S_{n+m-1}(\xi^0, gh)$ are restrictions to the product of positive real half axis of functions $S_{n+m-1}(\zeta^0 \mid gh)$, analytic in $\mathbb{C}_+^{n+m-1}$. There exist vector-valued functions $\Psi_n^E(z^0, \underline{\zeta}^0 \mid g)$ analytic in $\mathbb{C}_+^n$, such that*

$$S_{n+m-1}(\underline{\zeta}^0, z^0 + z'^0, \underline{\zeta}'^0 \mid gh) = ((\Psi_n^E(z^0, \underline{\zeta}^0 \mid g), \Psi_m^E(z'0, \underline{\zeta}'^0 \mid h)) .$$

Furthermore $S_{n+m-1}(\zeta^0 \mid gh)$ satisfies for $\zeta^0 \in \mathbb{C}_+^{n+m-1}$:

$$|S_{n+m-1}(\underline{\zeta}^0 \mid gh)| \leq c\, \|g\|^{(\mathscr{C})}\, \|h\|^{(\mathscr{C})}\, (1 + |\underline{\zeta}^0|)^a \left[1 + \Lambda_b\!\left(\min_k \operatorname{Re} \zeta_k^0\right)\right] \tag{8}$$

for some norms $\|\cdot\|^{(\mathscr{C})}$ in $\mathscr{C}$ and some constants a, b, c depending on $n+m-1$.

Proof. The proof of the corresponding theorem for tempered fields [11] has three parts: A – constructing the Hamiltonian, B – the analytic continuations and C – estimating $S_k(\underline{\zeta})$.

The proof of A in our case is based on the following

Lemma 9. *Let $f \in \mathfrak{M}'$, $\varphi \in \mathfrak{M}$. Then $f\varphi$ is a (Schwartz) distribution with rapid decrease (i.e. form $\mathcal{O}'_c$).*

Proof. We use the general form of generalized functions in $\mathfrak{M}'$ (see for instance [5]), the general form of distributions in $\mathcal{O}'_c$ (see for instance [13]) and the identity

$$(D^\alpha \psi_1)\,\psi_2 = \sum_{i=0}^{\alpha} (-1)^{|i|} \binom{\alpha}{i} D^{\alpha-i}(\psi_1 D^i \psi_2)\,; \qquad \binom{\alpha}{i} = \binom{\alpha_1}{i_1} \cdots \binom{\alpha_n}{i_n}$$

where ψ_1, ψ_2 are $\mathscr{C}^\infty$-functions.

From Lemma 9, using the convolution theorem for ultradistributions (see for instance [3]), follows that for $f \in \mathscr{C}'$, $\varphi \in \mathscr{C}$, $f * \varphi$ is a rapidly increasing function (i.e. in $\mathcal{O}'_M$). However, this gives exactly the Osterwalder-Schrader polynomial bound [12], p. 17, which assures the contractivity of the time-translations $T_t, t \geqq 0$ and hence the fact that $T_t = e^{-tH}$, where H is a selfadjoint operator. H will be the Hamiltonian. Now, returning to the proof of Theorem 4, observe that the analyticity of the Schwinger functions in the time variables can be proved as in [11] (see also [14] where similar results were independently proved). Indeed, first the analytic regularity is valid in the framework of ultradistributions satisfying the Cauchy-Riemann equations (see for instance [1]).

On the other hand the edge of the wedge theorem and the Malgrange-Zerner theorem are valid for ultradistributions, too. Following "distributional" proofs in [14] and [4] we see that we need only the following results which hold for ultradistributions (see also [9]).

Let G be an open set in $\mathbb{R}^n$. Then we define $\mathscr{D}_\omega(G)$ as in § 2. Let $\mathscr{D}'_\omega(G)$ the dual of $\mathscr{D}_\omega(G)$. The following properties hold

a) $\mathscr{D}'_\omega(G) * \mathscr{D}_\omega(G) \subset \mathscr{C}^\infty$

where $*$ means convolution and $\mathscr{C}^\infty$ is the class of all infinetely differentiable functions

b) $\mathscr{D}_\omega(G)$ is dense in $\mathscr{D}(G)$ (see [1]),

c) $\mathscr{D}_\omega(G)$ is nuclear (see Appendix).

This was Part B of the Osterwalder-Schrader proof.

Concerning Part C, let us neglect (as in [11]) the space variables. From (E0′) and Lemma 7 we conclude that there are integers α, β, γ, and δ such that for $\xi_k > 0$, $k = 1, 2, \ldots, n$

$$|S_n(\xi_1, \ldots, \xi_n)| \leqq (\alpha_n)^{\beta n} \prod_{i=1}^{n} (1+\xi_i)^\gamma \quad [1 + \Lambda_\delta(\xi_i)]\,.$$

The arguments of Osterwalder [11] go through and we get the following estimates for $S_n(\underline{\zeta})$:

$$|S_n(\underline{\zeta})| \leqq C_n(1+|\underline{\zeta}|)^a \left[1+\Lambda_\delta\left(\tfrac{1}{2}\min_k \xi_k\right)\right]\left[1+\left(\tfrac{1}{2}\min_k \xi_k\right)^{-1}\right]^{2\beta n}.$$

From the definition of Λ it follows now that (8) is valid for some constants a, b, c. This completes the proof of Theorem 4.

§ 5. Conclusions and Remarks

We have shown that the results of Osterwalder and Schrader on Euclidean Green's functions [11] can be extended to Jaffe fields. Here we formulated Jaffe fields in terms of ultradistributions. We remark that in the proof of Theorem 4, Part C, the following product has been found to characterize the singularities of the analytic functions $S_n(\underline{\zeta})$:

$$\left[1+\Lambda_\delta\left(\tfrac{1}{2}\min_k \xi_k\right)\right]\left[1+\left(\tfrac{1}{2}\min_k \xi_k\right)^{-1}\right].$$

The first factor comes from the part $\prod_{k=1}^n \|f_k\|^{(\mathscr{C})}$ of the bound in (E0′) and the second factor from the term $(n!)^L$ of this bound. Since the product of two Λ-functions is again a Λ-function (see for instance [3]) it is likely that in the framework of ultradistributions the growth $(n!)^L$ can be relaxed. This remark seems to agree with the remark in [6] that the *original* Osterwalder-Schrader theorems in [12] could be valid in the frame of hyper-functions (see also [2]). Nevertheless, the bound $(n!)^L$ seems to be good enough for the present status of Constructive Quantum Field Theory (see for instance [11]).

Appendix. Nuclearity of $\mathscr{D}_\omega$, $\mathfrak{M}_\omega$, and $\mathscr{C}_\omega$.

Let ω be a Jaffe indicatrix, $\exp(\omega(x^2)) = \Sigma a_{2k}|x|^{2k}$. Define: $\mathscr{D}_K^{(a_k)}$ ($K \subset \mathbb{R}^n$ compact) to be the set of all infinitely differentiable functions φ with compact support in K such that

$$\sup_{x\in\mathbb{R}^n}\{|\Delta^k\varphi(x)|\} \leqq C_h h^k / a_{2k}$$

for all $h > 0$. Let

$$\mathscr{D}^{(a_k)}(\mathbb{R}^n) = \lim_{\cup K_j = \mathbb{R}^n} \operatorname{ind}_{K_j \subset K_{j+1}} \mathscr{D}_{K_j}^{(a_k)}.$$

Then

Lemma. *The spaces $\mathscr{D}^{(a_k)}(\mathbb{R}^n)$ and $\mathscr{D}_\omega(\mathbb{R}^n)$ are equal*

Proof.

$$\begin{aligned}\sup_{\xi\in\mathbb{R}^n}\{e^{\lambda'\omega(\xi^2)}|\tilde{\varphi}(\xi)|\} &\leqq \sup_{\xi\in\mathbb{R}^n}\{e^{\omega(\lambda\xi^2)}|\tilde{\varphi}(\xi)|\} \qquad \text{(nuclearity)}\\ &= \sum_{k=0}^{\infty} a_{2k}\lambda^k \sup_{\xi\in\mathbb{R}^n}\{|\xi|^{2k}|\tilde{\varphi}(\xi)|\}\\ &\leqq \sum_{k=0}^{\infty} a_{2k}\lambda^k \int |\Delta^k\varphi(x)|\,dx\\ &\leqq C\sum_{k=0}^{\infty} a_{2k}\lambda^k C_h\frac{h^k}{a_{2k}}\end{aligned} \tag{9}$$

which is finite if we choose h suitably. Hence $\varphi\in\mathscr{D}_\omega$. Conversely, let $\varphi\in\mathscr{D}_\omega$. Then

$$\sup_{x\in\mathbb{R}^n}\{|\Delta^k\varphi(x)|\}\leqq\int|\xi|^{2k}|\tilde{\varphi}(\xi)|\,d\xi\,.$$

Therefore

$$\begin{aligned}\sum_{k=0}^{\infty} a_{2k}h^{-k}\sup\{|\Delta^k\varphi(x)|\} &\leqq \sum_{k=0}^{\infty} a_{2k}h^{-k}\int|\xi|^{2k}|\tilde{\varphi}(\xi)|\,d\xi\\ &=\int\Sigma a_{2k}h^{-k}|\xi|^{2k}|\tilde{\varphi}(\xi)|\,d\xi\\ &=\int\exp\left(\omega\left(\frac{\xi^2}{h}\right)\right)|\tilde{\varphi}(\xi)|\,d\xi\,.\end{aligned} \tag{10}$$

Since $\varphi\in\mathscr{D}_\omega$, this last integral is finite, so the terms in the infinite sum (10) are uniformly bounded, which completes the proof.

Let us now consider the function

$$\Omega(x^2)=\sup_p\{\log x^{2p}a_{2p}\}$$

on $\mathbb{R}^n$, called the *associated function* to the sequence $\{a_k\}$ (see [10]). Comparing the Paley-Wiener theorems for test functions ([1], p. 365 and [10], p. 82), we find that even $\mathscr{D}_\omega$ is equal to $\mathscr{D}_\Omega$ [with the help of the nuclearity Condition d)], so by Theorem 1.3.18 of [1]:

$$\begin{aligned}\omega(x)&\leqq a+b\,\Omega(x)\\ \Omega(x)&\leqq c+d\,\omega(x)\end{aligned}$$

which yields, choosing $n\geqq\dfrac{\log b+\log d}{\log 2}+1$

$$2^n\Omega(x)\leqq c'+2^n d\omega(x)\leqq c''+d\omega(A^n x)\leqq c+bd\Omega(A^n x)$$

or

$$2\Omega(x)\leqq c+\Omega(A^n x)$$

which is just the nuclearity Condition d). Using a theorem of Komatsu [10], we find that Ω satisfies d) if and only if:

$$a_k^{-1} \leqq C L^k \left(\sup_{0 \leqq l \leqq k} a_l a_{k-l} \right)^{-1}$$

for some constants C and L. This immediately gives:

$$a_{k+1}^{-1} \leqq C L^k a_k^{-1} \tag{11}$$

and with (11) and Theorem 8 of [16], we have proved:

Theorem. *Let ω be a Jaffe indicatrix. Then the spaces $\mathfrak{M}_\omega, \mathscr{C}_\omega$ are nuclear.*

The nuclearity of $\mathscr{D}_\omega$ is proved in [10], Theorem 2.6.

Remark. Condition (11) (stability under differential operators), which is sufficient for nuclearity, is much weaker than Condition d'). However, since d) and d') are equivalent, and d) is such a convenient inequality to deal with, while (11) is hard to express in terms of ω, we preferred, for simplicity's sake, the more restrictive one.

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Communicated by A. S. Wightman

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