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# Billiards and Bernoulli Schemes

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**Abstract.** Some two dimensional billiards are Bernoulli flows.

## 0. Introduction

"Billiards" will mean a point particle moving on a table with smooth convex obstacles and bouncing elastically against them. Upon collision with the boundary the particle is either elastically reflected (reflecting billiards) or disappears to reappear at the opposite side (periodic billiards).

The qualitative theory of the above motion leads, in a natural way, to consider the flow $S_t$ on the particle's phase space endowed with the Liouville measure.

Some simple questions can be answered if it is known that the flow $S_t$ is ergodic.

Recently Sinai has given a proof that $S_t$ is not only ergodic but, also, a $K$-flow. In this paper, making use of the results and techniques of Refs. [1–3], we prove that $S_t$ is a Bernoulli flow.

We mention, however, that the knowledge that $S_t$ is a $K$-flow or a $B$-flow is not sufficient to answer many questions of direct physical interest: consider, for instance, the average (with respect to the Liouville measure) of the cosine of the angle between the particle's velocity at time zero and its velocity at time $t$. How fast does it go to zero when $t$ tends to infinity? [4].

The fact that it goes to zero is implied by the $K$-property (actually mixing would suffice): the $B$-property does not teach more about this problem and it seems that much work has still to be done to obtain other relevant information [4].

The reader will be assumed familiar with the definitions and the ideas of the paper in Ref. [3] which is necessary to understand without pain the thread behind the lemmas of Section 4.

The basic notations are in Section 1.

The construction in Sections 2 and 3 are due to Sinai [1]: so Sections 2 and 3 could, in principle, be extracted from his paper or from [2].

We work with approximate foliations of contraction and dilatation rather than using the exact ones and this is the reason why Section 4 appears mere complicated than the part of Ref. [3] which deals with the same problem.

## § 1. Basic Notations

$Q$ will be a 2-dimensional torus which will be represented as the 2-dim. plane $R^2$ in which the points $q=(\alpha,\beta)$, $q'=(\alpha',\beta')$ are identified if there exist integers $m,n$ such that $\alpha-\alpha'=m$, $\beta-\beta'=n$.

Let $C_1, C_2, \ldots, C_s$ be $s$ disjoint open, convex, connected sets in $R^2$ with closures which are "pairwise disjoint on the torus $Q$": i.e., $\overline{C}_i$ is disjoint from all the non-trivial integral translates of $\overline{C}_j\,\forall i,j=1,2,\ldots s$, (if $A$ is a set in $R^2$ an integral translate $\tau_{m,n}A$ is the set

$$\tau_{n,m}A=\{q\,|\,q\in R^2, q=(\alpha+n,\beta+m),(\alpha,\beta)\in A\}\,,$$

where $n, m$ are integers).

The set $O=\bigcup_{j=1}^{s}\bigcup_{n=-\infty}^{+\infty}\bigcup_{m=-\infty}^{+\infty}\tau_{n,m}C_j$ will be called the "set of obstacles".

We assume that $\partial C_i$, $i=0,1,\ldots,s$, are $C^3$-smooth and have non-zero curvature at every point.

The billiards flow is defined on the set of pairs $(q,\theta), q\in Q\backslash O$, $0\leqq\theta\leqq 2\pi$: an element $x\in Q\backslash O$ will be thought of as an "arrow" through $q$ forming an angle $\theta$ with the 1-axis of $R^2$. Through every point $q\in Q\backslash O$ draw an oriented straight line parallel to the arrow $(q,\theta)$; when this line hits an obstacle reflect it on the obstacle and continue it by drawing a straight line in the new direction, and so on: this construction leads to an oriented broken line $r_{q,\theta}$. The curvilinear abscissa on $r_{q,\theta}$ will be measured starting from $q$. Define $S_t(q,\theta)=(q',\theta')$ where $q'$ is the point which, along $r_{q,\theta}$, has an abscissa $t$ and $\theta'$ is the angle of $r_{q,\theta}$ with the 1-axis at $q'$. In case of ambiguities in the choice of $\theta'$, we shall choose it so that the arrow $(q',\theta')$ points inwards with respect to the obstacle.

The flow $S_t$ conserves the normalized Lebesgue measure $\mu(dqd\theta)$ $=\dfrac{dqd\theta}{\text{normalization}}$ on $V$. Therefore, the triple $(V,S_t,\mu)$ is a dynamical system.

There is another dynamical system which can be associated with the billiards flow.

Let $M$ be the set of points $(q, \theta) \in V$ with $q \in \bigcup_{i=1}^{s} \partial C_i$ and $\theta$ such that the angle $\varphi$ between the outer normal and the oriented direction $\theta$ is between $\frac{\pi}{2}$ and $\frac{3\pi}{2}$: i.e., $M$ is the set of the "colliding arrows" in $V$.

A point $(q, \theta) \in M$ will be identified by the three numbers $(i, r, \varphi)$: the first being the name $i, i = 1, \ldots, s$, of the obstacle $\partial C_i$ to which $q$ belongs, the second is the abscissa of $q$ on $\partial C_i$ counted clockwise, and the third is the angle between the direction $\theta$ and the outer normal to $\partial C_i$. Therefore, the space $M$ can be thought of as a union of $s$ disjoint pieces $M_1, \ldots M_s$ with $M_i$ homeomorphic in a natural way to the cylinder $M_i = \left\{(r, \varphi) \middle| \frac{\pi}{2} \leqq \varphi \leqq \frac{3\pi}{2}, 0 \leqq r \leqq l_i\right\}$, where $l_i =$ length of $\partial C_i$ and $(0, \varphi) \equiv (l_i, \varphi)$.

If $x \in M$, define $\tau(x)$ to be the first negative time such that $S_{\tau(x)} x \in M$ and let $Tx$ be the point in $M$ such that $S_{-\tau(x)}(Tx) \equiv x$.

It can be shown that the mapping $T: M \to M$ conserves the measure $v(dr\,d\varphi) = \frac{-\cos\varphi\, dr\, d\varphi}{\text{normalization}}$; therefore, $(M, v, T)$ is a dynamical system which will be called the natural "section" of the flow $(V, \mu, S_t)$.

The following theorem holds: [1, 2],

Theorem (Sinai): $(V, \mu, S_t)$ is a $K$-flow and $(M, v, T)$ is a $K$-system.

Here we prove

**Theorem.** *$(V, \mu, S_t)$ is a B-flow and $(M, v, T)$ is a B-system.*

In Sections 2, 3 we present a self-contained construction of expanding and contracting foliations for the system $(M, v, T)$ and a proof of their local absolute continuity. The reader familiar with Sinai's work will not find any new ideas here. There is a technical difference with respect to Sinai's approach which is reflected in the fact that we never study the absolute continuity properties of the expanding or contracting fibers but we always deal with only approximate fibers.

In Section 4 we show how the results of Sections 2, 3 together with Sinai's theorem imply that $(M, v, T)$ is a $B$-shift. In Section 4 we also sketch the proof for $(V, \mu, S_t)$; the techniques used here are, of course, essentially the same as the ones in the Ref. [3].

In Section 5 we give a few concluding remarks concerning non-periodic billiards and open problems.

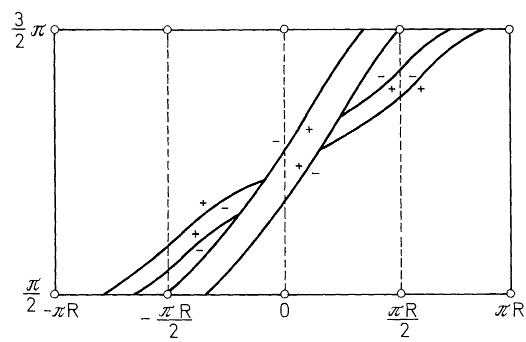
## § 2. The Expanding and Contracting Fibers

Observe that the mappings $T$ and $T^{-1}$ are not smooth. The singularity set $S_T$ for $T$ consists in the union of

$$\partial M = \left\{ x \,|\, x = (r(x), \varphi(x)) \in M_i, i = 1, \ldots, s, \varphi(x) = \frac{\pi}{2} \text{ or } \frac{3\pi}{2} \right\}$$

and $T^{-1}\partial M$. The singularity set $S_{T^{-1}}$ for $T^{-1}$ is, similarly, $S_{T^{-1}} = \partial M \cup T(\partial M)$.

It is easy to see that $S_T$ consists in several connected, pairwise disjoint pieces, each of which is the union of a finite or denumerable family of smooth lines.




In the picture we consider the case of a single circular obstacle with radius $R$ close to $\frac{1}{2}$; $S_T$ is the union of eight pieces and only parts of two of them are in the picture. Parts of the other six pieces can be obtained by translations of the ones drawn along the cylinder. In this case each family contains a denumerable number of smooth curves which accumulate at the eight marked points. The signs $+$ identify the sides of the singularity lines on which $T$ is continuous. In general, there will always be only finitely many accumulation points for the singularity lines and the function $\tau(x)$ will be unbounded in the neighborhood of such points (if any).

A similar picture holds for $T^{-1}$ (now the singularity lines will be decreasing rather than increasing).

*1. Definition.* If $x \in M$, we put $d_T(x)$ = distance of $x$ from $S_T$; $d_{T^{-1}}(x)$ = distance of $x$ from $S_{T^{-1}}$. If $x \in M_i$, we denote $k(r(x))$ the modulus of the curvature of the obstacle $\partial C_i$ at $r(x)$.

Because of the discontinuities of $T$ and $T^{-1}$ we cannot expect to prove the existence of expanding and contracting foliations in a topological sense. One can only hope that they exist in a measure-theoretical sense.

We shall construct a "natural" approximation procedure for the contracting or dilating fibers.

2. *Definition.* A curve $\gamma$ in $M_i$, $i=1,\dots,s$, will be a monotone function $\varphi=\varphi(r)$ defined for $r'_\gamma \leqq r \leqq r''_\gamma$. A curve $\gamma$ in $M_i$ will also be called a curve in $M$ and it will be identified with its equation $\varphi=\varphi(r)$.

3. *Definition.* If $\gamma$ is a curve in $M_i$ we put

$$p(\gamma) = -\int_\gamma \cos\varphi(r)\,dr$$

and we shall call $p(\gamma)=p$-length of $\gamma$.

4. *Definition.* If $x_0 \in M_i$ for some $i=1,\dots,s$ and $d_T(x_0)>0$, $d_{T^{-1}}(x_0)>0$, we define the 0-th order contracting fiber $\hat{\gamma}_c^{(0)}(x_0)$ and the 0-th order expanding fiber $\hat{\gamma}_e^{(0)}(x_0)$ through $x_0$ as the curves with respective equations:

$$\frac{d\varphi}{dr} = k(r)\,, \qquad \varphi(r(x_0)) = \varphi(x_0)\,, \qquad \text{for} \quad \hat{\gamma}_e^{(0)}(x_0)$$

$$\frac{d\varphi}{dr} = -k(r)\,, \qquad \varphi(r(x_0)) = \varphi(x_0)\,, \qquad \text{for} \quad \hat{\gamma}_c^{(0)}(x_0)$$

where the above differential equations are meant to define the functions $\varphi(r)$ in an interval $[r', r'']$ around $r(x_0)$ which is maximal in the sense that the values of $\varphi(r')$ and $\varphi(r'')$ are either $\frac{3\pi}{2}$ or $\frac{\pi}{2}$.

Clearly, $\hat{\gamma}_e^{(0)}(x_0)$ is, in the "arrow" picture, the set of arrows colliding with $\partial C_i$ and parallel to the arrow $x_0$; $\hat{\gamma}_c^{(0)}(x_0)$ is a set of arrows colliding with $C_i$ and aimed in such a way as to come out of the collision parallel.

5. *Definition.* Let $q>1$, and let $k$ be a non-negative integer. Define

$$M_q^{(k)+} = \left\{ x \,|\, x \in M;\ -\cos\varphi(T^{-h}x),\, d_T(T^{-h}x),\, d_{T^{-1}}(T^{-h}x) > \frac{1}{q(1+h^2)}\,, \right.$$
$$\left. \cdot\, h = 0, 1, \dots, k \right\}$$

$$M_q^{(k)-} = \left\{ x \,|\, x \in M,\ -\cos\varphi(T^{+h}x),\, d_T(T^{h}x),\, d_{T^{-1}}(T^{h}x) > \frac{1}{q(1+h^2)}\,, \right.$$
$$\left. \cdot\, h = 0, 1, \dots, k \right\}$$

$$M_q^{\pm} = \bigcap_{k=0}^{\infty} M_q^{(k)\pm}\,,\ M_q = M_q^{+} \cap M_q^{-}\,,\ M_q^{(k)} = M_q^{(k)+} \cap M_q^{(k)-}\,.$$

This definition allows, in a natural way, to represent the sets $M_q^{(k)\pm}$ as unions of complements of $(k+1)$ sets $M_{q,h}^{\pm}$, $(h=0,1,\dots,k)$: a more detailed study of the singularity lines of $T$, together with the Y-measure preserving property of $T$, allows to conclude that the Y-measure of $M_{q,h}^{\pm}$ does not exceed $b(\log h)/(1+h^2)$ for some $b>0$. Therefore the Borel-Cantelli lemma implies:

**6. Lemma.** $\nu\left(\bigcup_{q=1}^{\infty} M_q\right) = \lim_{q\to\infty} \nu(M_q) = 1.$

The construction and the properties of the contracting and expanding fibers are explained in the following series of lemmas. Their proof will, however, be postponed until the end of the section. We shall use the following further notation: $\frac{1}{R+} = \min_{\in M} k(r(x))$, $\frac{1}{R-} = \max_{x\in M} k(r(x))$, $\tau_0 = \min_{x\in M} |\tau(x)|$.

**7. Lemma.** *Given $k \geqq 0, q>0$, there exists $\Delta_q > 0$ such that if $x_0 \in M_q^{(k)+}$, the connected part of the set $T^k\gamma_c^{(0)}(T^{-k}x_0)$ which contains $x_0$ is a smooth curve at least in the interval $[r(x_0)-\Delta_q,\ r(x_0)+\Delta_q]$; its equation $\varphi=\varphi(r)$ is monotonically decreasing and its derivative is bounded away from 0 and $\infty$ $\left(\text{actually, } \frac{1}{R+} \leqq \left|\frac{d\varphi}{dr}\right| \leqq \frac{1}{R-} + \frac{1}{\tau_0}\right)$.*

8. *Definition.* Given $k \geqq 0$, $q>0$, $x_0 \in M_q^{(k)+}$, the connected part of the set $T^k\hat{\gamma}_c^{(0)}(T^{-k}x_0)$ which is above the interval $[r(x_0)-\Delta_q, r(x_0)+\Delta_q]$ and contains $x_0$ will be denoted by $\gamma_c^{(k)}(x_0)$.

**9. Lemma.** *Given $k \geqq 0$, $q>1$, $x \in M_q^{(k)+}$, there exists $\lambda>1$ such that*

$$P(T^l\gamma_c^{(k)}(x)) \leqq \lambda^{-l} p(\gamma_c^{(k)}(x)), \qquad l=0,1,\ldots,k$$

*$\left(\text{and } \lambda \text{ could be chosen } 1+\frac{\tau_0}{R+}\right)$. Furthermore, if $y\in\gamma_c^{(k)}(x)$ and $\varphi=\varphi(r)$ is the equation for $\gamma_c^{(k)}(x)$, the ratio $\left(\frac{d\varphi}{dr}\right)_y \Big/ \left(\frac{d\varphi}{dr}\right)_x$ is between $\exp \pm \tilde{\theta}_q p(\gamma_c^{(k)}(x))$, for a suitable $\tilde{\theta}_q>0$.*

**10. Corollary.** *If $q>1$, $k\geqq 0$, $x\in M_q^{(k)+}$, $y\in\gamma_c^{(k)}(x)$, there exists $C_q>0$ such that $\frac{\cos\varphi(T^{-l}y)}{\cos\varphi(T^{-l}x)}, \frac{\tau(T^{-l}y)}{\tau(T^{-l}x)}, \frac{k(T^{-l}y)}{k(T^{-l}x)}$ are between $\exp\pm\lambda^{-l}C_q p(x,y)$, for all $l=0,\ldots,k$, where $p(x,y)=p$-length of the* arc *of $\gamma_c^{(k)}(x)$ between $x$ and $y$. Furthermore, $\Delta_q$ in Lemma 7 can be assumed (and will be assumed) to be such that $y\in M_{2q}^{(k)+}$.*

The above lemmas, definitions, and corollaries have obvious analogues for the expanding fibers and we shall refer to these lemmas, corollaries, and definitions as Lemma 7′, Definition 8′, Lemma 9′, and Corollary 10′.

The next lemma establishes a relationship between the expansion rates of two expanding fibers connected by a contracting one and is the key to the proof of the uniform local absolute continuity of the approximate fibers.

**11. Lemma.** *Let $x_0 \in M_{q_0}^{(k)-}$ and let $x, y \in \gamma_e^{(k)}(x_0)$; assume also that $x, y \in M_{q'}^{(h)+}$. Then the automorphism $T^l, l=0,1,\ldots,k$, is smooth in a neighborhood of $x$ and $y$ on the fibers $\gamma_c^{(h)}(x)$, $\gamma_c^{(h)}(y)$. The ratio of the local expansion rates, under $T^l, l=0,\ldots,k$, of the p-lengths of $\gamma_c^{(h)}$ at $x$ and $y$ is between $\exp \pm \theta_{q',q_0}(p_e(x,y))$ where $\theta_{q',q_0}(\varepsilon)$ is a suitable function, monotonic and infinitesimal as $\varepsilon \to 0$; $p_e(x,y)$ is the p-length of the arc of $\gamma_e^{(k)}(x_0)$ between $x$ and $y$.*

*Furthermore, if $\left(\frac{d\varphi^c}{dr}\right)_i(x)$, $\left(\frac{d\varphi^c}{dr}\right)_i(y)$ denote the derivatives of the equations for $T^i\gamma_c^{(h)}(x)$ and $T^i\gamma_c^{(h)}(y)$ at $T^i x$ or $T^i y$, respectively, $i=0,\ldots,k$, then their ratio is between $\exp \pm \lambda^{-i}\tilde{\theta}_{q',q_0}, i=0,\ldots,k$, for a suitable choice of $\tilde{\theta}_{q',q_0}$.*

*Finally the ratio $\left(\frac{d\varphi^c}{dr}\right)_i(x) \Big/ \left(\frac{d\varphi^c}{dr}\right)_i(y)$ can also be bounded by $\exp \pm \theta_{q',q_0}(p_e(x,y))$ for $i=0,1,\ldots,k$.*

**12. Lemma.** *Let $x \in M_q^+ (M_q^-)$; then the function $\varphi_c^{(k)}(x)\,(\varphi_e^{(k)}(r))$ of the curve $\gamma_c^{(k)}(x)\,(\gamma_e^{(k)}(x))$ converges uniformly, together with its derivative, to a limit $\varphi_c(r)\,(\varphi_e(r))$ which defines a curve that will be denoted by $\gamma_c(x)$, $(\gamma_e(x))$ or $\gamma_c^{(\infty)}(x)$, $(\gamma_e^{(\infty)}(x))$.*

*Proof of the Above Lemmas.* The proof is made easier by using some geometrical properties described in (i) through (vi) below:

(i) Let $\gamma \subset M_i$ be a smooth curve and let $T$ be smooth on $\gamma$. Notice that the $p$-length of an infinitesimal arc $d\gamma$ between $x$ and $\tilde{x}$ is, to first order in $dr = r(\tilde{x}) - r(x)$, the length of an orthogonal section of the cone defined in the billiards plane by ideally continuing the arrows $x$ and $\tilde{x}$. The section of the cone which gives the $p$-length of $d\gamma$ (i.e., $p(d\gamma) = -\cos\varphi\, dr$) is the one close to $\partial C_i$ [within $(dr)^2$].

Clearly $p(Td\gamma)$ is the length of another orthogonal section of the same cone; this time the section has to be considered at the surface $\partial C$ of the obstacles $C_j$ on which $T\gamma$ collides.

Notice also that the variation $d\alpha$ of the angle that the arrows $x$ and $\tilde{x}$ make with a fixed direction in the billiards plane is related to $\varphi(\tilde{x}) - \varphi(x) = d\varphi$ by $d\alpha = k(r)\,dr - d\varphi$. Furthermore, the distance between the points on $\partial C_i$ and $\partial C_j$ associated with $x$ and $Tx$ is $-\tau(x)$.

If we denote $\varphi' = \varphi(Tx)$, $r' = r(Tx)$, $dr' = r(Tx) - r(T\tilde{x})$, $d\varphi' = \varphi(Tx) - \varphi(T\tilde{x})$, the above remarks and some elementary trigonometry lead immediately to the following relations:

$$p(Td\gamma) = -\cos\varphi'\, dr' = -\cos\varphi\, dr - \tau(x)\,(k(r)\,dr - d\varphi)$$
$$d\varphi' = -k(r')\,dr' + (d\varphi - k(r)\,dr)\,,$$

which are correct to first order in $dr$.

(ii) An apparently more complicated but very useful form of the differential relations derived in (i) is, with the same notation and assumption as in (i):

$$\frac{p(Td\gamma)}{p(d\gamma)} \equiv \frac{-\cos\varphi' dr'}{-\cos\varphi\, dr} = \left(1 + \left(k(r) - \frac{d\varphi}{dr}\right)\frac{\tau(r,\varphi)}{\cos\varphi}\right)$$

$$= \frac{1}{1 + \left(k(r') + \dfrac{d\varphi'}{dr'}\right)\dfrac{\tau(r,\varphi)}{\cos\varphi'}}$$

and

$$\frac{d\varphi'}{-\cos\varphi' dr'} = \frac{k(r')}{\cos\varphi'} + \cfrac{1}{\tau(r,\varphi) + \cfrac{1}{\cfrac{1}{-\cos\varphi}\left(\cfrac{d\varphi}{dr} - k(r)\right)}}$$

$$\frac{d\varphi}{+\cos\varphi\, dr} = \frac{k(r)}{\cos\varphi} + \cfrac{1}{\tau(r,\varphi) + \cfrac{1}{\cfrac{1}{\cos\varphi'}\left(\cfrac{d\varphi'}{dr'} + k(r')\right)}}.$$

(iii) It immediately follows from (ii) that if $\gamma$ is a decreasing curve $\left(\text{i.e., } \frac{d\varphi}{dr} \leqq 0\right)$, then $T\gamma$ is also decreasing and

$$\frac{1}{R_+} \leqq \left|\frac{d\varphi'}{dr'}\right| \leqq \frac{1}{R_-} + \frac{1}{\tau_0}.$$

Similarly, if $\gamma$ is an increasing curve, $\gamma' = T^{-1}\gamma$ is also increasing and

$$\frac{1}{R_+} \leqq \left|\frac{d\varphi'}{dr'}\right| \leqq \frac{1}{R_-} + \frac{1}{\tau_0}.$$

(iv) If $\bar{\gamma}$ is smooth and has equation $\varphi = \bar{\varphi}(\bar{r})$ and $\frac{d\bar{\varphi}}{dr} \leqq 0$ and $T, T^2, \ldots, T^k$ are smooth on $\bar{\gamma}$, then a repeated application of the formulas of (ii) above imply that the equation $\varphi = \varphi(r)$ of $T^k\bar{\gamma} = \gamma$ is given by (if $T^k(\bar{r}, \bar{\varphi}) = (r, \varphi)$)

$$\left.\frac{d\varphi}{-\cos\varphi\, dr}\right|_x = f_{(k)}\left(\frac{2k(x)}{\cos\varphi(x)}, \tau(T^{-1}x), \quad \frac{2k(T^{-1}x)}{\cos\varphi(T^{-1}x)}, \ldots, \tau(T^{-k}x),\right.$$

$$\left.\left(-\left.\frac{d\bar{\varphi}}{dr}\right|_{T^{-k}x} + k(T^{-k}x)\right)\frac{1}{\cos\varphi(T^{-k}x)}\right),$$

where for $h=1,2,\ldots,\infty$,

$$f_{(h)}(b_1,a_1,b_2,a_2,\ldots,a_{h-1},b_h)=\frac{b_1}{2}+\cfrac{1}{a_1+\cfrac{1}{b_2+\cfrac{1}{a_2+\cdots+\cfrac{1}{a_{h-1}+\cfrac{1}{b_h}}}}}$$

(v) The only property of $f_{(h)}$ that we shall really need is that if $h$ is large, then $f_h$ depends very little on the variable with large index provided the entries of the continued fractions are not too small; more precisely, we shall need the following statement: if $0<\sigma\leqq\min_{1\leqq i\leqq k-1}|a_ib_{i+1}|$ and $0<\sigma\leqq\min_{1\leqq i\leqq k-1}|a_ib_i|$, and if the entries $a_i,b_i=1,\ldots,k$ all have the same sign there is a constant $Q$ such that

$$\left|\frac{\partial\log f_{(k)}}{\partial a_i}\right|\leqq\frac{Q}{(1+\sigma)^{2i}}\frac{1}{|a_i|},\qquad i=1,\ldots,k-1$$

$$\left|\frac{\partial\log f_{(k)}}{\partial b_i}\right|\leqq\frac{Q}{(1+\sigma)^{2i}}\frac{1}{|b_i|},\qquad i=1,\ldots,k$$

[$Q$ could actually be chosen as $Q=2(1+\sigma)^2$].

In the case $a_i,b_i$ have the values of (iv) above the parameter $\sigma$ could be $\sigma=\dfrac{\tau_0}{R_+}$, see Lemma 9.

(vi) If $d(x)=\min(d_T(x),d_{T^{-1}}(x))$, then there exists $C>0$ such that

$$-\cos\varphi(x)\geqq Cd(x),\qquad -\cos\varphi(T^{\pm1}x)\geqq Cd(x),\qquad |\tau(x)|\leqq\frac{C}{d(x)}.$$

(vii) Now the proof of the lemmas proceeds as follows:

Let $x\in M_q^{(k)+}$ and put $d_i=c_i=\dfrac{1}{q(1+i^2)}$. Let $\hat\gamma_c^{(k)}=T^k\hat\gamma_c^{(0)}(T^{-k}x)$. $\hat\gamma^{(k)}$ consists of a union of smooth curves. Let $\hat{\hat\gamma}^{(k)}$ be the smooth curve which contains $x$. Let $\Delta>0$ be so small that the curve $\hat{\hat\gamma}^{(k)}$ is at least above all the points of the interval $I=[r(x)-\Delta,r(x)+\Delta]$.

Call $\tilde c_h=\min_{y\in I}|\cos\varphi(T^{-h}y)|$, $\tilde d_h=\min_{y\in I}d(T^{-h}y)$. The formula for the contraction coefficient $p(Td\gamma)/p(d\gamma)$ in (ii) above implies

$$p(T^{-h}\hat{\hat\gamma}^{(k)})\leqq\lambda_0^{-h}p(\hat{\hat\gamma}^{(k)}),\qquad h=0,\ldots,k,$$

where $\lambda_0 = \min_{x \in M} \left(1 + \frac{K(r(x))\,\tau(x)}{\cos\varphi(x)}\right) > \left(1 + \frac{\tau_0}{R_+}\right)$. (Notice that we use the fact that, by construction, the curves $T^{-h}\hat{\gamma}^{(k)}$ are decreasing.)

The bounds in (iii) on $d\varphi/dr$ imply that the length of $l(T^{-h}\hat{\gamma}^{(k)})$ is related to the $p$-length of $T^{-h}\hat{\gamma}^{(k)}$ by

$$l(T^{-h}\hat{\gamma}^{(k)}) = \int_{T^{-h}\hat{\gamma}^{(k)}} \sqrt{1 + \left(\frac{d\varphi}{dr}\right)^2}\, dr \leqq C' \int_{T^{-h}\hat{\gamma}^{(k)}} dr \leqq \frac{C'}{\tilde{C}_h} p(T^{-h}\hat{\gamma}^{(k)})$$

$$\leqq \frac{C'}{\tilde{c}_h} \lambda_0^{-h} p(\hat{\gamma}^{(k)}) \leqq \frac{2C'}{\tilde{c}_h} \lambda_0^{-h} \Delta$$

for some suitable $C' > 0$. Hence

$$\tilde{d}_{h+1} \geqq d_{h+1} - l(T^{-h-1}\hat{\gamma}^{(k)}) \geqq d_{h+1} - \frac{2C'}{\tilde{c}_{h+1}} \lambda_0^{-(h+1)} \Delta$$

and (vi) above implies, for a suitable $C'' > 0$:

$$\tilde{d}_{h+1} \geqq d_{h+1} - \frac{C''}{\tilde{c}_h} \lambda_0^{-h-1} \Delta\,.$$

Similarly, for some $C'' > 0$,

$$\tilde{c}_{h+1} \geqq c_{h+1} - \int_{T^{-h-1}\hat{\gamma}^{(k)}} |\sin\varphi| \left|\frac{d\varphi}{dr}\right| dr \geqq c_{h+1} - \frac{C'''}{\tilde{c}_h} \lambda_0^{-h-1} \Delta\,.$$

From the last two formulas it is easy to infer, by induction, that there is a $\Delta_q > 0$ such that if $\Delta < \Delta_q$, then $\tilde{c}_h > \frac{c_h}{2}$, $\tilde{d}_h > \frac{d_h}{2}$. Clearly, this implies Lemma 7 and Corollary 10 and the first statement of Lemma 9. The second statement of Lemma 9 can be proved using the technique of $(x)$ below (but is much simpler) and we skip its proof.

(viii) If $x_o \in M_q^+$, then Corollary 10 together with the continued fraction formula in iv) guarantee that, if $\varphi^{(k)}(r)$ is the equation of $\gamma_c^{(k)}(x_0)$, the functions $\frac{d\varphi^{(k)}}{dr}(r)$ form an equicontinous sequence defined on $r(x_0) - \Delta_q, r(x_0) + \Delta_q$. This implies, if one takes into account the measure preserving property of $T$, that $\frac{d\varphi^{(k)}}{dr}$ tends uniformly to a limit as $k$ tends to infinity. This proves Lemma 12.

(ix) Finally, we prove Lemma 11 and divide its proof into two steps:

Assume $x_0 \in M_{q_0}^{(k)-}$, and $x, y \in \gamma_e^{(k)}(x_0)$; assume also that $x, y \in M_{q'}^{(h)+}$. By Corollary 10 it also follows that $x, y \in M_{2q_0}^{(k)-}$.

The formula in (ii) above for the expansion coefficient of the $p$-length allows us to express the ratio:

$$\frac{p(T^l d\gamma_c)}{p(d\gamma_c)}\bigg|_x \bigg/ \frac{p(T^l d\gamma_c)}{p(d\gamma_c)_y}$$

as

$$\prod_{i=1}^{l} \frac{1+\left(k(r(T^i x)) - \left.\dfrac{d\varphi}{dr}\right|_{T^i x}\right)\dfrac{\tau(T^i x)}{\cos\varphi(T^i x)}}{1+\left(k(r(T^i y)) - \left.\dfrac{d\varphi}{dr}\right|_{T^i y}\right)\dfrac{\tau(T^i y)}{\cos\varphi(T^i y)}}$$

where $\left.\left(\dfrac{d\varphi}{dr}\right)\right|_{T^i z}$, $z\in x, y$, denote, respectively, the derivatives in $T^i z$ of the equations of $T^i\gamma_c^{(h)}(z)$.

Corollary 10 and the remark that $x, y\in M_{2q_0}^{(k)-}$ reduces the problem to that of estimating the ratio between $\dfrac{1}{-\cos\varphi(T^i y)}\left.\dfrac{d\varphi}{dr}\right|_{T^i x}$ and $\dfrac{1}{-\cos\varphi(T^i y)}\left.\dfrac{d\varphi}{dr}\right|_{T_i y}$, $i=0, 1, \ldots, k$. In the next point we call $\exp\alpha_i(x, y)$ this ratio.

(x) We shall now use that, for $z=x, y$, $i=0, \ldots, k$,

$$\frac{1}{-\cos\varphi(T^i z)}\left.\frac{d\varphi}{dr}\right|_{T^i z} = f_{(h+i+1)}\left(\frac{2k(T^i z)}{\cos\varphi(T^i z)}, \tau(T^{i-1}z), \ldots, \tau(T^{-h}z), \frac{2k(T^{-h}z)}{\cos\varphi(T^{-h}z)}\right).$$

Now (v) and Corollary 10 imply the existence of $Q_{q_0}>0$ such that $\exp\alpha_i(x, y)$ is between

$$\exp\pm\left(\sum_{s=0}^{i}\frac{Q_{q_0}\lambda^{-(i-s)}}{(1+\sigma)^{2s}}\, p_e(x, y) + \sum_{s=i+1}^{i+h}\frac{Q}{(1+\sigma)^{2s}}\, D_s(x, y)\right),$$

where, if $c_{i-s}(x, y)=\min_{z=x,y}|\cos\varphi(T^{i-s}z)|$:

$$D_s(x, y)=\frac{R_+}{R_-}\frac{|\cos\varphi(T^{i-s}x)-\cos\varphi(T^{i-s}y)|}{(c_{i-s})^2}+\frac{|k(T^{i-s}x)-k(t^{i-s}y)|}{c_{i-s}(x, y)\cdot 1/R_+} + \frac{|\tau(T^{i-s}x)-\tau(T^{i-s}y)|}{\tau_0}.$$

Since, by assumption, $x, y\in M_{q'}^{(h)+}$, $c_{i-s}(x, y)\geqq \dfrac{1}{q'(1+(i-s)^2)}$, $i\leqq s\leqq i+h$, and, by (vi), $\tau(T^{i-s}z)\leqq \dfrac{C}{d(T^{i-s}z)}\leqq Cq'(1+(i-s)^2)$. Therefore,

$$QD_s(x, y)\leqq \bar{Q}_{q'}(1+(i-s)^2)^2$$

for a suitable $\bar{Q}_{q'}$; and the above estimate for $|\alpha_i|$ can be bounded by

$$\exp \pm \left( \sum_{s=0}^{i} \frac{Q_{q0} \lambda_0^{-(i-s)}}{(1+\sigma)^{2s}} 2\Delta_{q0} + \sum_{s=i+1}^{\infty} \frac{\bar{Q}_{q'}}{(1+\sigma)^{2s}} (1+(i-s)^2)^2 \right).$$

Hence there exists a suitable $\tilde{\theta}_{q',q0}$ such that

$$|\alpha_i(x, y)| \leqq \lambda^{-i} \tilde{\theta}_{q',q0}.$$

This proves the second part of Lemma 11.

(xi) To obtain the first and the third parts one needs a more refined estimate of the right-hand side of

$$|\alpha_i(x, y)| \leqq \sum_{s=0}^{i} \frac{Q_{q0} \lambda^{-(i-s)}}{(1+\sigma)^{2s}} p_e(x, y) + \sum_{s=i+1}^{i+h} \frac{\bar{Q}_{q'}}{(1+\sigma)^{2i}} D_s(x, y).$$

Clearly, we need an estimate on $\sum_{i=0}^{k} \alpha_i(x, y)$ and the first sum in the above formula gives a contribution bounded by, for a suitable choice of $a_{q0} > 0$:

$$a_{q0} p_e(x, y).$$

So the main problem is the contribution from the second sum. It is easy to realize, using (ii), (vi), that if the arc of $\gamma_e^{(k)}(x_0)$ between $x, y$ is smoothly transformed by $T^{-1}, \ldots, T^{-h}$, then the distance between $T^{-s}x, T^{-s}y$ is

$$d(T^{-s}x, T^{-s}y) \leqq \mu_{q'}(s)\, p_e(x, y), \qquad s = 0, \ldots, h$$

and where $\mu_{q'}(s)$ is a suitable function of $s$ [which can be estimated to grow as $(s!)^5$ as $s \to \infty$!] and $\mu_{q'}(s)$ can also be chosen so that if $\mu_{q'}(s)\, p_e(x, y) < \frac{1}{(2q'(1+s)^2)^2}$, then $T^{-1}, \ldots, T^{-s}$ are smooth on the arc of $\gamma_e^{(k)}(x_0)$.

This remark, together with the estimate for $D_s(x, y)$ given in (x), completes the proof of the lemma [and we could use for $\theta_{q',q0}(\varepsilon)$] the function

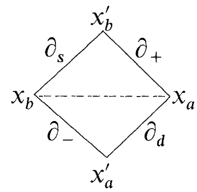
$$a_{q0}\varepsilon + \sum_{i=0}^{\infty} \sum_{s=0}^{\infty} \frac{\tilde{Q}_{q'}}{(1+\sigma)^{2s+2i}} \min\left((1+s^2)^2, \mu_{q'}(s)(1+s^2)^5 \varepsilon\right)$$

which tends to zero (very slowly) as $\varepsilon \to 0$.

## § 3. Local Absolute Continuity of the Approximate Foliations

The geometrical objects on which the local absolute continuity will be studied will be, as usual, the "quadrilaterals".

*13. Definition.* A domain $G \subset M$ will be called a "$k$-quadrilateral" if it is connected and if its boundary is the union of four curves $\partial_+, \partial_-, \partial_s, \partial_d$ such that $\partial_+ \cap \partial_- = \phi = \partial_s \cap \partial_d$ and such that $\partial_+, \partial_-$ are contained in some fibers $\gamma_c^{(k)}$ and $\partial_s, \partial_d$ are contained in some fibers $\gamma_e^{(k)}$, $(k = 0, \ldots, \infty)$. We call $x_b = \partial_- \cap \partial_s$, $x'_b = \partial_+ \cap \partial_s$, $x_a = \partial_+ \cap \partial_d$, $x'_a = \partial_- \cap \partial_d$.




*14. Definition.* Given $q > 1$, $\xi > 0$, $\infty \geqq k \geqq 0$, the family $\mathfrak{G}_q^{(k)}(\xi)$ consists in the $k$-quadrilaterals such that $x_a, x_b \in M_q$ and the straight segment $x_a x_b$ is contained in a cone around the $l$-axis with opening very small compared to $1/R_+$ (i.e., the angle between $x_a x_b$ and any tangent to $\partial_s, \partial_-$ is bounded away from zero); furthermore, we require that the diameter $D(G)$ of $G \in \mathfrak{G}_q^{(k)}(\xi)$ be positive and $\leqq \xi \Delta_q$.

The bounds on the derivatives in Lemmas 7, 7′, 12, 12′ imply that the family $\mathfrak{G}_q^{(k)}(\xi)$ covers, in the sense of Vitali, the Lebesgue set of $M_q$, $\forall k = 0, 1, \ldots, \infty$, $\forall \xi > 0$.

In the following we shall repeatedly use the remark in Corollaries 10, 10′ that if $x \in M_q^{(k)\pm}$ and $y \in \gamma_{c(e)}^{(k)}(x)$, then $y \in M_{2q}^{(k)\pm}$.

Notice that $\xi$ can be taken so small that, if $G \in \mathfrak{G}_q^{(k)}(\xi)$ and $y \in M_{4q}^{(k)\pm} \cap G$, the fiber $\gamma_{c(e)}^{(k)}(y)$ joins the opposite sides $\partial_s, \partial_e, (\partial_+, \partial_-)$ of $G$. We shall only consider those $\xi$ having this property.

*15. Definition.* Let $G \in \mathfrak{G}_q^{(k)}(\xi)$. Let $\partial_+^{(k)} = \partial_+ \cap M_{2q}^{(k)-}$, $\bar{x} \in M_q^{(k)} \cap G$, $\partial = \gamma_c^{(k)}(\bar{x}) \cap G$. Define $\Phi : \partial_+^{(k)} \to \partial$ as

$$\Phi(x) = \gamma_e^{(k)}(x) \cap \partial, \qquad \forall x \in \partial_+^{(k)}.$$

Similarly, if $\partial_s^{(k)} = \partial_s \cap M_{2q}^{(k)+}$. and $\bar{x} \in M_q^{(k)+} \cap G$, we define the mapping $\Psi : \partial_s^{(k)} \to \partial' = \gamma_e^{(k)}(\bar{x}) \cap G$ as

$$\Psi(x) = \gamma_e^{(k)}(x) \cap \partial'.$$

Notice that Definition 15 makes sense only if $\xi$ is small enough (see the above comment).

*16. Definition.* Let $G \in \mathfrak{G}_q^{(k)}(\xi)$ and let us define on $\partial_+, \partial_s$ the measures $m_+, m_s$ which measure the $p$-length of a subset of $\partial^+, \partial_s$, respectively. Similarly, if $\bar{x} \in M_q^{(k)\pm} \cap G$, then we introduce on

$$\partial = \gamma_c^{(k)}(\bar{x}) \cap G \; (\partial' = \gamma_e^{(k)}(\bar{x}) \cap G)$$

a measure $m(m')$ which measures the $p$-length of a subset of $\partial(\partial')$.

The main result of this section is contained in the following lemma:

**17. Lemma.** *Using the notations of the above definitions the mapping* $\Phi(\Psi)$ *takes the measure* $m(m')$ *into a measure absolutely continuous with respect to* $m_+(m_s)$ *and its Radon-Nykodim density* $f_c^{(k)}(x)\,(f_e^{(k)}(x))$ *is between* $\exp\pm\theta_q(D(G))$ *where* $\theta_q(\varepsilon)$ *is a suitable function infinitesimal as* $\varepsilon\to\infty$.

*Proof.* Consider first, for simplicity, the case when the obstacles are equal circles $(k(r(x))=\text{const}=1/R)$.

Observe that if $x\in\partial_+^{(k)}=\partial_+\cap M_{2q}^{(k)-}$ it also happens that $x\in M_{2q}^{(k)+}\cap M_{2q}^{(k)-}$ because $\partial^+\subset\gamma_c^{(k)}(x_a)$ and $x_a\in M_q$ by assumption. Similarly, if $y=\Phi x$, then $y\in M_{4q}^{(k)-}\cap M_{2q}^{(k)+}$, so we can assume that $\partial_+^{(k)}\cup\Phi\partial_+^{(k)}\subset M_{4q}^{(k)+}\cap M_{4q}^{(k)-}$.

The constant curvature assumption implies that $T^k\gamma_e^{(k)}(x)$ is a straight segment with slope independent on $x\in\partial_+^{(k)}$ (actually the slope is $1/R$, see Definition 4).

Notice also that $\partial_+^{(k)}$ is relatively open on $\partial_+$ and $\Phi$ is a diffeomorphism, so we can compute its Jacobian by considering an infinitesimal arc $d\gamma$ around $x$ in $\partial_+^{(k)}$ and

$$\left.\frac{m(\Phi d\gamma)}{m_+(d\gamma)}\right|_x\equiv\frac{P(\Phi d\gamma)}{p(d\gamma)}\equiv\left(\frac{\dfrac{p(\Phi d\gamma)}{p(T^k\Phi d\gamma)}}{\dfrac{p(d\gamma)}{p(T^k d\gamma)}}\right)\cdot\frac{p(T^k\Phi d\gamma)}{p(T^k d\gamma)}\,.$$

Lemma 11 implies that the product appearing in this formula is between $\exp\pm\theta_{4q,q}(D(G))$. Furthermore, the ratio outside the parentheses can be easily evaluated using the remark that $T^k\gamma_e^{(k)}(x)$ are straight parallel lines with slope $1/R$:

$$\frac{p(T^k d\gamma)}{p(T^k\Phi d\gamma)}=\frac{\cos\varphi(T^k\Phi x)}{\cos\varphi(T^k x)}\,\frac{1-R\left(\dfrac{d\varphi}{dr}\right)_k^c(\Phi x)}{1-R\left(\dfrac{d\varphi}{dr}\right)_k^c(x)}$$

where we are using the symbols of Lemma 11. So Lemma 17 follows from Lemma 11.

In the general case of non-constant curvature of the obstacles the last formula for $p(T^k\Phi d\gamma)/p(T^k d\gamma)$ is not so simple because $T^k\gamma_e^{(k)}(x)$ are not parallel straight lines; nevertheless, if $k(r)$ is smooth enough (i.e., if $\partial C_i$ are of class $C^3$, $i=1,2,\ldots,s$), then the lines of $T^k\gamma_e^{(k)}(x)$ describe, as $x$ varies in $\partial_+^{(k)}$, a very smooth foliation and the estimates of the above ratio do not substantially change (Corollary 10′ is needed).

*18. Definition.* Let $x \in M_q^{(k)} \cap G$, $G \in \mathfrak{G}_q^{(k)}(\xi)$. We can introduce natural coordinates $(\alpha, \beta)$ for $x$. They are defined as the $p$-length of the segment on $\partial_s$ or $\partial_+$, respectively, from $x'_b$ to the points $\gamma_c^{(k)}(x) \cap \partial_s$ or $\gamma_e^{(k)}(x) \cap \partial_+$.

Notice that, in the $(\alpha, \beta)$ coordinate, the set $M_q^{(k)} \cap G \subset \partial_s^{(k)} \times \partial_+^{(k)}$ (and, also, $M_q \cap G \subset \partial_s^{(k)} \times \partial_+^{(k)}, \forall k$).

*19. Definition.* Let $G \in \mathfrak{G}_q^{(k)}(\xi)$. Put

$$\hat{\partial}_+^{(k)} = M_{q/2}^{(k)-} \cap \partial_+ , \qquad \hat{\partial}_s^{(k)} = M_{q/2}^{(k)+} \cap \partial_s .$$

The define the set $\hat{G} \subset G$ which, in the sense of the $(\alpha, \beta)$ coordinates, is $\hat{G} = \hat{\partial}_s^{(k)} \times \hat{\partial}_+^{(k)}$. Clearly,

$$G \cap M_{q/4} \subset \hat{G} \subset M_q^{(k)} \cap G \subset \partial_s^{(k)} \times \partial_+^{(k)} \subset G .$$

We shall introduce on $\partial_s^{(k)} \times \partial_+^{(k)}$ the measure $v_k(d\alpha d\beta) = d\alpha d\beta$.

**20. Theorem.** *Let $G \in \mathfrak{G}^{(k)}(\xi)$ and use the above notations. Then the restriction of $v$ to $\hat{\partial}_s^{(k)} \times \hat{\partial}_+^{(k)}$ is absolutely continuous with respect to $v_{(k)}$ and vice versa; the Radon-Nykodim density $\varrho_{(k)}(x)$ of $v$ with respect to $v_{(k)}$ is bounded by a suitable constant $C_{q,\xi}$ $(\forall k = 0, 1, \ldots, < \infty)$ and is almost constant in the sense that there is a function $\overline{\theta}_q(\varepsilon)$, infinitesimal as $\varepsilon \to 0$, such that $\varrho_{(k)}(x)/\varrho_k(y)$ is between* $\exp \pm \overline{\theta}_q(D(G))$, $\forall x, y \in \hat{\partial}_s^{(k)} \times \hat{\partial}_+^{(k)}$.

*Proof.* Since on $\hat{\partial}^{(k)}$ and $\hat{\partial}_s^{(k)}$ the mappings $\Phi, \Psi$ of $\hat{\partial}_+^{(k)}, \hat{\partial}_s^{(k)}$ into $\gamma_c^{(k)}(\bar{x}) \cap G$, $\gamma_e^{(k)-}(x) \cap G$ with $\bar{x} \in M_q^{(k)} \cap G$, are diffeomorphisms it is easy to realize that $x \in \hat{\partial}_+^{(k)} \times \hat{\partial}_s^{(k)}$ (in the sense of the $(\alpha, \beta)$ coordinates)

$$\varrho_k(x) \equiv \left( \frac{-\cos\varphi \, d\varphi \, dr}{d\alpha \, d\beta} \right) = \frac{f_c^{(k)}(x) \, f_e^{(k)}(x)}{-\cos\varphi(x)} \left( \left( \frac{d\varphi}{dr} \right)_e - \left( \frac{d\varphi}{dr} \right)_c \right)$$

where $\left( \dfrac{d\varphi}{dr} \right)_{e(c)}$ are the derivatives in $x$ of the equation $\varphi = \varphi(r)$ of $\gamma_e^{(k)}(x)$ $(\gamma_c^{(k)}(x))$.

Therefore, Theorem 20 is a simple consequence of Lemma 17, Lemmas 9, 9′, and Corollaries 10, 10′.

## 4. Proof that $(M, v, T)$ and $(V, S_t, \mu)$ are Bernoulli Shifts

We shall first consider $(M, v, T)$. Let $\mathscr{P} = (P_1, \ldots, P_n)$ be a finite partition of $M$ into $s$ subsets with smooth boundary. We put $\partial \mathscr{P} = \bigcup_{i=1}^{n} \partial P_i$, $L = \sum_{i=1}^{n}$ (length of $\partial P_i) < \infty$. It will be enough to show that $\mathscr{P}$ is a very weak Bernoulli partition see, for instance, [3]. Given $1 > \varepsilon > 0$ let $q$ be so large that

$$v(M/M_{q/4}) < \varepsilon^2 .$$

Let us extract from the family $\mathfrak{G}_q^\infty(\xi)$ a denumerable family of disjoint sets $\{G_i\}$ which covers the Lebesgue set of $M_q$. Suppose also that $\xi$ has been chosen so small that $\exp \pm \bar{\theta}_q(\xi)$ in Theorem 20 is between $1 \pm \varepsilon^2$.

Let $n'_\varepsilon$ be so large that $\nu\left(\bigcup_{i=1}^{n'_\varepsilon} G_i \cap M_{q/4}\right) > 1 - \varepsilon^2$. Let $\mathfrak{B}_\varepsilon = (G_1, \ldots, G_{n_\varepsilon})$ be the subfamily of $\{G_i\}_{i=1}^{n'_\varepsilon}$ consisting in the sets in $\{G_i\}_1^{n'_\varepsilon}$ such that $\nu(G_i \cap M_{q/4}) > (1-\varepsilon)\,\nu(G_i)$. Clearly

$$\nu\Big(\bigcup_{g \in \mathfrak{B}_\varepsilon} g\Big) > 1 - 2\varepsilon\,.$$

Define also for further use the number $N_\varepsilon$: such that

$$(4L\Delta_q q) \sum_{h \geqq N_\varepsilon} \lambda^{-h}(1+h^2) < \tfrac{1}{4}\varepsilon^2 \min_{g \in \mathfrak{B}_\varepsilon} \nu(g)^2\,.$$

We may assume without loss of generality that $\nu\Big(M/\bigcup_{g \in \mathfrak{B}_\varepsilon} g\Big) > 0$ and call $G_0 = M\Big/\bigcup_{g \in \mathfrak{B}_\varepsilon} g$ and $\bar{\mathfrak{B}}_\varepsilon = \{G_0, G_1, \ldots, G_{n_\varepsilon}\}$. $\bar{\mathfrak{B}}_\varepsilon$ is a partition; $\nu(G_0) < 2\varepsilon$.

**Lemma 21.** *There is $N'_\varepsilon > N_\varepsilon$ such that given $N', N'' > N'_\varepsilon$, $\exists \mathfrak{A}' \subset V_{-N'}^{-N''}(T^{-1})^i \mathscr{P}$ with the properties*

$$\sum_{a \in \mathfrak{A}'} \nu(a) > 1 - \varepsilon \tag{1}$$

$$\left|\frac{\nu(a \cap g)}{\nu(a)} - \nu(g)\right| < \varepsilon \nu(g) \quad \begin{matrix} \forall g \in \bar{\mathfrak{B}}_\varepsilon \\ \forall g \in \bar{\mathfrak{B}}_\varepsilon \cap M_{q/4} \end{matrix} \tag{2}$$

*here $\bar{\mathfrak{B}}_\varepsilon \cap M_{q/4}$ means $\{G_0 \cap M_{q/4}, \ldots, G_{n_\varepsilon} \cap M_{q/4}\}$.*

*Proof.* This is an immediate consequence of the $K$-property of $(M, \nu, T)$.

Next construct the family $\Gamma_\varepsilon^{(k)} = (G_1^{(k)}, \ldots, G_{n_\varepsilon}^{(k)})$ of the quadrilaterals in $\mathfrak{G}_q^{(k)}(\xi)$ obtained by considering the vertices $x_a, x_b$ of a quadrilateral $G_i \in \mathfrak{B}_\varepsilon$ and by drawing $\gamma_c^{(k)}(x_a)$, $\gamma_e^{(k)}(x_a)$, $\gamma_c^{(k)}(x_b)$, $\gamma_e^{(k)}(x_b)$. $G_0^{(k)} = M\Big/\bigcup_{i=1}^{n_\varepsilon} G_i^{(k)}$, $\bar{\Gamma}_\varepsilon^{(k)} = (G_0^{(k)}, \ldots, G_{n_\varepsilon}^{(k)})$.

**Lemma 22.** *Given $\varepsilon > 0$ and $N'' \geqq N' \geqq N'_\varepsilon$ there is a $k_\varepsilon$ such that the family $\Gamma_\varepsilon^{(k)}$ consists in pairwise disjoints sets for $k \geqq k_\varepsilon$ and $\nu(G_0^{(k)}) < 2\varepsilon$.*

*Furthermore $\exists K_{\varepsilon,N''}$ such that if $k \geqq K_{\varepsilon,N''}$ there is a family $\mathfrak{A}'' \subset V_{-N'}^{-N''} T^{-i} \mathscr{P}$ such that*

1) $\sum_{a \in \mathfrak{A}''} \nu(a) > 1 - \varepsilon$,

2) $\left|\dfrac{\nu(a \cap g)}{\nu(a)} - \nu(g)\right| < \varepsilon \nu(g) \qquad \forall g \in \bar{\Gamma}_\varepsilon^{(k)} \cup (\bar{\Gamma}_\varepsilon^{(k)} \cap M_{q/4})$,

3) $\nu(g \cap M_{q/4}) > (1-\varepsilon)\,\nu(g) \qquad \forall g \in \Gamma_\varepsilon^{(k)}$,

4) $\dfrac{1}{2} < \dfrac{\nu(G_i^{(k)})}{\nu(G_i)} < 2 \qquad i = 0, 1, \ldots, n_\varepsilon$.

*Proof.* This lemma immediately follows from the preceding lemma, from the fact that the quadrilaterals in $\mathfrak{B}_\varepsilon$ are closed and disjoint and by Lemmas 12, 12′ that tell us that $\lim_{k\to\infty} G_i^{(k)} = G_i$, $i=0,1,\ldots,n_\varepsilon$, in measure.

*Definition 23.* Let $a$ be a set in $M$; We define $e\subset a$ to be the set of the points $x\in a\cap \bigcup_{i=1}^{n_\varepsilon} \hat{G}_i^{(k)}$ (cf., Definition 19 for $\hat{G}_i^{(k)}$) such that if $x\in e\cap \hat{G}_i^{(k)}$, for some $i$, then $\gamma_c^{(k)}(x)\cap \hat{G}_i^{(k)}\subset a$.

This definition makes sense because $\hat{G}_i^{(k)}\subset M_q^{(k)}$ and therefore there is a fiber $\gamma_c^{(k)}(x)$ through $x$.

Define $\hat{\Gamma}_\varepsilon^{(k)} = (\hat{G}_1^{(k)},\ldots,\hat{G}_{n_\varepsilon}^{(k)})$; then:

**Lemma 24.** *Let $N''\geqq N'\geqq N'_\varepsilon$. Then if $k>k_{\varepsilon,N''}$ (cf., Lemma 22) there is a family $\mathfrak{A}\subset V_{-N'}^{-N''}(T^{-1})^i\mathscr{P}$ such that*

1) $\sum_{a\in\mathfrak{A}} \nu(a) > 1-c\varepsilon$,

2) $\nu(e) > (1-c\varepsilon)\,\nu(a)\ \forall a\in\mathfrak{A}$,

3) $\left|\dfrac{\nu(e\cap g)}{\nu(e)} - \nu(g)\right| \leqq c\varepsilon\nu(g)\ \forall g\in\hat{\Gamma}_\varepsilon^{(k)}$

*for a suitable constant $c>0$. Provided $\varepsilon$ is small enough.*

*Proof.* Let $\mathfrak{A}''$ be the same family as in Lemma 22. If $A=\bigcup_{s=1}^{n_\varepsilon}\hat{G}_s^{(k)}$ and $x\in A\cap a$ but $x\notin e$ then there must exist $i, N'\leqq i\leqq N''$, such that $T^ix$ is within $4\lambda^{-i}(1+i^2)\,q\Delta_q$ from the boundary $\partial\mathscr{P}$ (because the fiber $\gamma_c^{(k)}(x)$ contracts as in Lemma 9 and $x\in M_q^{(k)}$).

Put $\tau=\min_{g\in\hat{\Gamma}_\varepsilon^{(k)}}\nu(g)$, the above remark and our choice of $N_\varepsilon$ imply

$$\nu\left(A\cap\bigcup_{a\in\mathfrak{A}''}(a/e)\right)\leqq 4L\Delta_q q\sum_{i=N_\varepsilon}^{\infty}\lambda^{-i}(1+i^2)<\varepsilon^2\tau^2$$

(here we have also used 4) in Lemma 22).

Hence if $\mathfrak{A}\subset\mathfrak{A}''$ is the family of the atoms $a\in\mathfrak{A}''$ such that

$$\nu(e)\equiv\nu(e\cap A)>\nu(a\cap A)\,(1-\varepsilon\tau)$$

it must be:

$$\sum_{a\in\mathfrak{A}}\nu(a\cap A)>\sum_{a\in\mathfrak{A}''}\nu(a\cap A)-\varepsilon\tau\,.$$

On the other hand 2) in Lemma 22 implies: (since $G_i^{(k)}\supset\hat{G}_i^{(k)}\supset G_i^{(k)}\cap M_{q/4}$):

$$\nu(a\cap G_0^{(k)})<(1+\varepsilon)\,2\varepsilon\nu(a)\quad\forall a\in\mathfrak{A}''$$

$$\nu\left(a\cap\bigcup_{i=1}^{n_\varepsilon}(G_i^{(k)}/\hat{G}_i^{(k)})\right)<4\varepsilon\nu(a)\quad\forall a\in\mathfrak{A}''\,.$$

Therefore for a suitable $C>0$ (one can take $C=10$):

$$\sum_{a\in\mathfrak{A}} \nu(a) > 1 - C\varepsilon$$

$$\nu(e) > (1-\varepsilon C)\,\nu(a)\,.$$

Let $\hat{G}_i^{(k)} \in \Gamma_\varepsilon^{(k)}$, and suppose $a \in \mathfrak{A} \subset \mathfrak{A}''$, then using the inequality $\nu(\hat{G}^{(k)}) > \tau$:

$$\nu(a\cap\hat{G}_i^{(k)}) - \nu(e\cap\hat{G}_i^{(k)}) = \nu((a/e)\cap\hat{G}_i^{(k)}) \leqq \nu((a/e)\cap A)$$
$$\leqq \varepsilon\tau\nu(A\cap a) \leqq 2\varepsilon\nu(\hat{G}_i^{(k)})\,\nu(a)\,.$$

So 3) follows from 2), 3) in Lemma 22 and the remark $G_i^{(k)} \cap M_{q/4} \subset \hat{G}_i^{(k)} \subset G_i^{(k)}$ (with some $c > C$).

We shall now reduce the problem of showing that $\mathscr{P}$ is a very weak Bernoulli partition to a theorem in [3]. Let us define a $c'\varepsilon$-measure preserving mapping $\zeta: a \to M$. This mapping will be defined on $e \subset a$ and will map $e\cap g$ onto $g$, $\forall g \in \hat{\Gamma}_\varepsilon^{(k)}$.

By our definition of $\hat{G}_i^{(k)}$ and of $e$ it follows that it is possible to map $e\cap\hat{G}_i^{(k)}$ onto $\hat{G}_i^{(k)}$ along dilating lines [in the $(\alpha,\beta)$ coordinates this map is along lines with $\beta$-const.] in such a way to preserve the $\nu_k$-measure: letting $i=1,2,\ldots,n_\varepsilon$ we define a mapping $\zeta: e \to \bigcup_{i=1}^{n_\varepsilon} \hat{G}_i^{(k)} \subset M$.

By our choice of $\xi$ at the beginning of this section, $\zeta$ is a map of $\hat{G}_i^{(k)} \cap e$ onto $\hat{G}_i^{(k)}$ which is $\varepsilon^2$-measure preserving as far as the measure $\nu$ is concerned (cf. Theorem 20) and therefore Lemma 24 implies that $\zeta$ is $c'\varepsilon$-measure preserving for some $c'>0$ [e.g. $c'=(c+\varepsilon)$].

We remark that $\zeta$ maps $e \subset \bigcup_{i=1}^{n_\varepsilon} \hat{G}_i^{(k)}$ onto $\bigcup_{i=1}^{n_\varepsilon} \hat{G}_i^{(k)}$ and $G_i^{(k)} \subset M_q^{(k)}$.

Therefore the distance between $T_x^l$ and $T^l\zeta(x)$ can be bounded, if $x \in e$ by:

$$d(T^l\zeta(x), T^l x) \leqq C_q \lambda_1^{-l}$$

for $0 \leqq l \leqq k$, as a consequence of Lemma 9′ with some $\lambda_1 > 1$.

Let now $F_s$ be the set of the points $x \in M$ such that (distance of $T^i x$ from $\partial\mathscr{P}) \geqq \dfrac{1}{s(1+i^2)}$, $i=0,1\ldots$. If $s$ is large enough $\nu(F_s) > 1-\varepsilon$. Therefore, since $\zeta$ is $c'\varepsilon$-measure preserving, $\nu(\zeta^{-1}F_s) > (1-c''\varepsilon)\,\nu(a)$ for a suitable $c''$ (e.g. $c''=c'+2$) and for all choices of the number $k \geqq k_{\varepsilon,N''}$ on which $\zeta$ depends (cf. Lemma 24).

Hence we conclude that there is a positive integer $i_0$, $k$-independent, such that $T^i\zeta(x)$ and $T^i x$ lie in the same set of $\mathscr{P}$ for $i_0 \leqq i \leqq k$ and for $x \in a$ and outside a set with measure $\nu(a)\,C''\varepsilon$, $\left(i_0 \text{ could be given by the condition } C_q\lambda_1^{-i} < \dfrac{1}{s(1+i^2)} \text{ for } i \geqq i_0\right)$.

An application of Lemma (1.3) of Ref. [3] immediately tells us that $\mathscr{P}$ is a very weak Bernoulli partition and, hence, $(M, \nu, T)$ is a $B$-shift.

It remains to deal with the dynamical system $(V, \mu, S_t)$. This system can be easily seen to be isomorphic to the flow generated by $(M, \nu, T)$ under the function $-\tau(T^{-1}x)$. The procedure for the proof of the isomorphism of $(V, \mu, S_t)$ is essentially identical to that in the Ref. [3] and we shall omit the details.

## 5. Concluding Remarks

The construction of the approximate fibers and the proof of their local absolute continuity is easily generalizable to a "true" billiards: i.e., to a billiards with reflecting boundary conditions (as long as the "table" is a rectangle). So it will be proven that this system is a $B$-flow as soon as it will be known that it is a $K$-flow: this theorem is proven in [2] and [5].

The periodic billiards is interesting for its connections with a simple "wind-tree" model: it is in fact clearly related to famous open problems such as the study of the mean square displacement of a particle which moves bouncing on a periodic array of scatterers.

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