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Renormalization of Non Polynomial Lagrangians in Jaffe's Class

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Abstract. It is shown how a renormalized perturbation series can be defined for a theory with strictly local, non-polynomial, interacting Lagrangian

$$\mathscr{L}(x) = \sum_{r=0}^{\infty} t_r \frac{:A(x)^r:}{r!}$$

so as to preserve locality at every order.

1. Introduction

In Ref. [1] an inductive construction of the perturbation series for polynomial Lagrangians was given. It was shown that, given a Wick polynomial $\mathscr{L}(x)$ in a free field $A(x)$ one can construct (by induction on n) the chronological products

$$T(x_1, \ldots, x_n) = T(\mathscr{L}(x_1) \ldots \mathscr{L}(x_n))$$

which generate the perturbation series of a theory whose interaction Lagrangian reduces to $\mathscr{L}(x)$ in the first order. In this note we show how to extend this method to the case when $\mathscr{L}(x)$ is no longer a Wick polynomial but an entire function in the free field, denoted

$$\mathscr{L}(x) = \sum_{r=0}^{\infty} t_r \frac{:A(x)^r:}{r!}. \tag{1}$$

Such an entire function is still a strictly localized field in the sense of Jaffe [2] provided the coefficients t_r do not grow too fast; in fact we shall restrict ourselves to a special class of theories in which

$$|t_r| < E M^r r^{\lambda r} \tag{2}$$

where $0<\lambda<\frac{1}{2}$. Under such conditions the two point function

$$(\Omega, \mathscr{L}(x)\,\mathscr{L}(0)\,\Omega)=\int e^{-ipx}\,\varrho(p)\,dp$$

is given by

$$\varrho(p)=\sum_{r=0}^{\infty}\frac{|t_r|^2}{r!}\,\varrho_r(p)$$

where $\varrho_r(p)$ is the r-particle phase space and, for $r\geqq 2$, is bounded by

$$\frac{C^r p^{2(r-2)}}{(r!)^2}, \quad \text{hence} \quad \sum_{r=2}^{\infty}\frac{M'^r p^{2(r-2)}}{(r!)^{(3-2\lambda)}}\leqq E^2\exp\left(M''|p|^{\frac{1}{\frac{3}{2}-\lambda}}\right).$$

As a consequence $\varrho(p)$ can be integrated with test functions decreasing like $\exp\left(-A|p|^{\frac{1}{\sigma}}\right)$ where $\frac{1}{\sigma}>\frac{1}{\frac{3}{2}-\lambda}$, i.e. $\sigma<\frac{3}{2}-\lambda$.

Such functions may have Fourier transforms with compact support only if $\sigma>1$. Hence the condition $\lambda<\frac{1}{2}$ ensures the strict locality of $\mathscr{L}(x)$. In the inductive construction of the perturbation series, we wish to preserve this strict locality, i.e. to satisfy the causal factorization property of the chronological products. For detailed information regarding generalized functions, see [2–6].

2. Induction Hypothesis

Just as in the polynomial case, it is useful to introduce the local fields

$$\mathscr{L}^{(r)}(x)=\sum_{s=0}^{\infty}t_{r+s}\frac{:A(x)^s:}{s!} \tag{3}$$

and to define chronological products for all the $\mathscr{L}^{(r)}$. These will be denoted $T_r(X)$ where X stands for $(x_1,\ldots,x_\nu)\in\mathbb{R}^{4\nu}$ and $r=(r_1,\ldots,r_\nu)$ is a multi-integer. These notations are the same as in Ref. [1] and will not be further explained. In our induction hypothesis we assume that the $T_r(X)$ have been constructed for all X with $|X|\leqq n-1$ and satisfy

$$T_r(X)=\sum_s t_{r+s}(X)\frac{:A(X)^s:}{s!} \tag{4}$$

Here the summation runs over all positive multi-integers $s=(s_1,\ldots,s_\nu)$ $(X=1,\ldots,\nu)$; $:A(X)^s:$ means $:A(x_1)^{s_1}\ldots A(x_\nu)^{s_\nu}:$ and as usual $s!=\prod_{j=1}^{\nu}(s_j!)$. The $t_r(X)=(\Omega, T_r(X)\,\Omega)$ are translationally invariant generalized functions which satisfy

$$|\langle t_r, f\rangle|\leqq E_{|X|}M_{|X|}^{|r|}|r|^{\lambda|r|}\,\|f\|_{|X|}\,. \tag{5}$$

Here $M_{|X|}$ and $E_{|X|}$ are constants depending only on $|X|$ and

$$\|f\|_{|X|} = \sum_{\alpha} \sup (1+|x|)^P \, |\alpha|^{-\sigma|\alpha|} A^{-|\alpha|} |D^\alpha f(x)|$$
$$\equiv N(f; \sigma, A, P). \tag{6}$$

Here α runs over all multi-indices; The positive constants P, A, σ depend only on $|X|$ (and not on r or α) and

$$1 < \sigma < \tfrac{3}{2} - \lambda. \tag{7}$$

We also make the same assumption for the antichronological products (see [1]) denoted

$$\bar{T}_r(X) = \sum_r \bar{t}_r(X) \frac{:A(X)^r:}{r!}$$

The $\bar{t}_r(X)$ are supposed to satisfy exactly the same inequalities (5), (6) as the $t_r(X)$. In the following $T_r^{\#}(X)$ and $t_r^{\#}(x)$ stand for $T_r(X)$ or $\bar{T}(X)$ and for $t_r(X)$ or $\bar{t}_r(X)$.

3. Going from $n-1$ to n: First Step

Our first step in showing that the inductive construction extends the property (5) to the case $|X| = n$ is to show that

$$(\Omega, T_{r-s}^{\#}(X)\, T_s^{\#}(X')\Omega) \quad (\text{where } |X| + |X'| = n)$$

satisfies a condition of the type (5). More generally let

$$X = (x_1, \ldots, x_\nu), \qquad X' = (x'_1, \ldots, x'_\varkappa)$$

with ν and $\varkappa \leqq n-1$, $r = (r_1, \ldots, r_\nu)$, $s = (s_1, \ldots, s_\varkappa)$ two multi-indices. Then

$$(\Omega, T_r^{\#}(X)\, T_s^{\#}(X')\Omega)$$
$$= \sum_{l=0}^{\infty} \sum_{\substack{a,b \\ |a|=|b|=l}} t_{r+a}^{\#}(X)\, t_{s+b}(X') \frac{(\Omega, :A(X)^a: :A(X')^b: \Omega)}{a!\, b!}. \tag{8}$$

It is easy to see (by "separating the points" in $:A(X)^a:$ and $:A(X')^b:$) that $\frac{1}{a!\,b!} t_{r+a}^{\#}(X)\, t_{s+b}^{\#}(X')\ (\Omega, :A(X)^a: :A(X')^b: \Omega)$ is the sum of $l!$ Terms, each of which is of the form

$$\frac{1}{a!\,b!} t_{r+a}^{\#}(X)\, t_{s+b}^{\#}(X') \prod_{j=1}^{l} (\Omega, A(x_{u(j)})\, A(x'_{v(j)})\Omega). \tag{9}$$

Here u is a map from $(1, \ldots, l)$ into $(1, \ldots, \nu)$ taking exactly a_j times the value j for each $j = 1, \ldots, \nu$; v is a map from $(1, \ldots, l)$ into $(1, \ldots, \varkappa)$ taking

b_j times the value j for each $j=1,\ldots,\varkappa$. To study this quantity we use the variables

$$\xi_j = x_j - x_\nu \qquad (j=1,\ldots,\nu-1. \text{ Conventionally we also introduce } \xi_\nu = 0)$$

$$\xi'_j = x'_j - x'_\varkappa \qquad (j=1,\ldots,\varkappa-1;\ \xi'_\varkappa \equiv 0)$$

$$\eta = x_\nu - x'_\varkappa\,.$$

If we integrate (9) with a test function $f(\xi,\xi',\eta)$ we obtain

$$\frac{1}{a!\,b!}\int t^{\#}_{r+a}(\xi)\, t^{\#}_{s+b}(\xi')\, \psi(\xi,\xi')\, d\xi\, d\xi'\,, \tag{10}$$

$$\psi(\xi,\xi') = \int f(\xi,\xi',\eta)\, d\eta \exp -i\left[P\eta + \sum_{j=1}^{\nu} \xi_j p_{I_j} + \sum_{j=1}^{\varkappa} \xi'_j p_{J_j}\right]$$
$$d^4P\,\delta\left(P - \sum_{j=1}^{l} p_j\right) \prod_{j=1}^{l} \delta(p_j^2 - m^2)\,\theta(p_j^0)\, d^4 p_j\,. \tag{11}$$

Here

$$p_{I_j} = \sum_{u(t)=j} p_t\,, \qquad p_{J_j} = \sum_{v(t)=j} p_t$$

and we recall that $\xi_\nu = \xi'_\varkappa = 0$. We also denote

$$\hat{p} = (p_{I_1},\ldots,p_{I_{\nu-1}})\,, \qquad \check{p} = (p_{J_1},\ldots,p_{J_{\varkappa-1}})\,.$$

According to the induction hypothesis, there are positive constants C, P, Q, σ, L, F independent of r, s, a, b, u, v (i.e., depending only on n) such that the modulus of (10) is bounded by

$$\frac{F}{a!\,b!} L^{|r|+|s|+|a|+|b|}\, |r+a|^{\lambda|r+a|}\, |s+b|^{\lambda|s+b|}$$
$$\sum_{\alpha,\beta} C^{|\alpha|+|\beta|} \sup_{\xi,\xi'} (1+|\xi|)^P\, (1+|\xi'|)^Q\, |\alpha|^{-\sigma|\alpha|}\, |\beta|^{-\sigma|\beta|}\, |D^\alpha_\xi D^\beta_{\xi'} \psi(\xi,\xi')|\,. \tag{12}$$

But

$$D^\alpha_\xi D^\beta_{\xi'} \psi = \sum_{\substack{\gamma,\mu\\ \gamma\leqq\alpha\\ \mu\leqq\beta}} \frac{\alpha!}{\gamma!(\alpha-\gamma)!}\, \frac{\beta!}{\mu!(\beta-\mu)!}$$
$$\int B^l_{\gamma,\mu}(\xi,\xi',\eta) \left(-i\frac{\partial}{\partial\eta_0} + 1\right)^2 \left(-i\frac{\partial}{\partial\eta^0}\right)^M D^{\alpha-\gamma}_\xi D^{\beta-\mu}_{\xi'} f(\xi,\xi',\eta)\, d\eta$$

where $M = |\gamma| + |\mu| + 2l - 1$ and

$$B^l_{\gamma,\mu}(\xi,\xi',\eta) = \int d^4P \frac{(i\hat{p})^\gamma\,(i\check{p})^\mu}{(P^0+1)^2\,(P^0)^M}\, e^{-iP\eta - i\hat{p}.\xi - i\check{p}.\xi'}$$
$$\delta\left(P - \sum_{j=1}^{l} p_j\right) \prod_{j=1}^{l} \delta(p_j^2 - m^2)\,\theta(p_j^0)\, d^4 p_j\,.$$

It is straightforward to verify (see e.g. a similar calculation in [1]) that there is a universal constant A such that

$$|B^l_{\gamma,\mu}(\xi,\xi',\eta)| \leqq \frac{A^l}{(l!)^2} \qquad (l \geqq 1).$$

(The case $l=0$ is trivial.) The modulus of (10) is bounded by

$$\begin{aligned}
&\frac{F\,A^l}{(l!)^2\,a!\,b!}\,L^{|r|+|a|+|s|+|b|}\,|r+a|^{\lambda|r+a|}\,|s+b|^{\lambda|s+b|} \\
&\sum_{\alpha,\beta} C^{|\alpha|+|\beta|}\,|\alpha|^{-\sigma|\alpha|}\,|\beta|^{-\sigma|\beta|}\,\sup_{\xi,\xi'}(1+|\xi|)^P\,(1+|\xi'|)^Q \\
&\int d\eta\,\frac{\alpha!}{\gamma!(\alpha-\gamma)!}\;\frac{\beta!}{\mu!(\beta-\mu)!} \\
&\cdot\left|\left(-i\frac{\partial}{\partial\eta^0}+1\right)^2\left(-\frac{i\partial}{\partial\eta^0}\right)^{|\gamma|+|\mu|+2l-1} D_\xi^{\alpha-\gamma}D_{\xi'}^{\beta-\mu}f(\xi,\xi',\eta)\right|. \qquad (13)
\end{aligned}$$

We note the following inequalities:

1. if x is a positive real number

$$e^{-x}x^x \leqq \Gamma(x+1) \leqq 3x^x.$$

2. let $\alpha_1,\ldots,\alpha_N$ be integers $\geqq 0$ with $|\alpha| = \sum_{j=1}^N \alpha_j = \omega$. Then

$$\sum_{\substack{\alpha\\|\alpha|=\omega}} \frac{\omega!}{\alpha!} = N^\omega$$

so that

$$\sum_{\substack{\alpha\\|\alpha|=\omega}} 1 \leqq N^\omega\,; \quad \frac{\omega!}{\alpha!} < N^\omega\,;$$

$$\prod_{j=1}^N \alpha_j^{\sigma\alpha_j} \equiv \alpha^{\sigma\alpha} \leqq \omega^{\sigma\omega} \leqq N^{\sigma\omega}\alpha^{\sigma\alpha} \qquad (\text{all } \sigma>0).$$

By using these inequalities it is easy to verify that there are constants $B_{\nu+\varkappa}$, $L_{\nu+\varkappa}$, $C'_{\nu+\varkappa}$ depending on $\nu+\varkappa$ but independent of r, s, a, b, u, v, l, such that

$$\left|\frac{1}{a!\,b!}\langle t_{r+a}\otimes t_{s+b},\psi\rangle\right| \leqq F\,\frac{L_{\nu+\varkappa}^{|r|+|s|+1}\,B_{\nu+\varkappa}^{\,l}(2l+1)^{\sigma(2l+1)}(|r|+l)^{\lambda(|r|+l)}(|s|+l)^{\lambda(|s|+l)}}{(l!)^2\;a!\,b!}\,\|f\|_{\nu+\varkappa}$$

and

$$\|f\|_{\nu+\varkappa} = \sum_\gamma \sup_{\xi,\xi'\eta} C'^{|\alpha|}|\alpha|^{-\sigma|\alpha|}|D^\alpha f(\xi,\xi',\eta)|\,(1+|\xi,\xi',\eta|)^{P+Q+5}.$$

As a consequence we have (remembering that there are $l!$ terms such as (9))

$$|\int (\Omega, T_r^{\#}(X)\, T_s^{\#}(X')\Omega)\, f(\xi, \xi', \eta)\, d\xi\, d\xi'\, d\eta|$$
$$\leqq F \| f \|_{\nu+\varkappa} \sum_{l=0}^{\infty} L'^{|r|+|s|}_{\nu+\varkappa} B'^{l}_{\nu+\varkappa} e^{-(3-2\lambda-2\sigma) l \log l} |r|^{\lambda|r|} |s|^{\lambda|s|}$$

where F, $B'_{\nu+\varkappa}$ and $L'_{\nu+\varkappa}$ are constants independent of r and s (and l). If $3-2\lambda-2\sigma>0$, there are constants R and U depending on $\nu+\varkappa$ but independent of r and s such that

$$|\int (\Omega, T_r^{\#}(X)\, T_s^{\#}(X')\Omega)\, f(\xi, \xi', \eta)\, d\xi\, d\xi'\, d\eta|$$
$$\leqq F R^{|r|+|s|} (|r|+|s|)^{\lambda(|r|+|s|)} N(f, \sigma, U, P+Q+5)\,.$$

Moreover, by iterating this process we reach the following conclusion: Let $X=x_1, \ldots, x_n$ and let $I_1, \ldots, I_\nu$ be a partition of X into non empty subsets such that each of them contains at most $n-1$ points. Let $r=(r_1, \ldots, r_n)$ be a multi-integer and $r|I_1, \ldots, r|I_\nu$ be its restriction to $I_1, \ldots, I_\nu$ in the obvious sense. There exist constants F_n, K_n, U_n and P_n depending only on n, and not on r (or ν) such that

$$\begin{aligned}|\int (\Omega, T^{\#}_{r|I_1}(I_1)\, T^{\#}_{r|I_2}(I_2) \ldots T^{\#}_{r|I_\nu}(I_\nu)\Omega)\, f(\xi_1, \ldots, \xi_{n-1})\, d\xi_1 \ldots d\xi_{n-1}| \\ \leqq F_n K_n^{|r|}\, |r|^{\lambda|r|} \quad N(f, \sigma, U_n, P_n)\,.\end{aligned} \tag{14}$$

(Here $\xi_j = x_j - x_n$.)

Let $X=(x_1, \ldots, x_n)$, $Y=(x_1, \ldots, x_{n-1})$. According to the procedure of [1] we define

$$D(Y; x_n) = \sum_{\substack{I \cup J = Y \\ I \cap J = \emptyset \\ I \neq \emptyset}} (-1)^{|I|} [T(J, n), \bar{T}(I)] \tag{15}$$

$$D(Y; x_n) = \sum_r d_r(Y; n) \frac{:A(X)^r:}{r!} \tag{16}$$

$$d_r(Y; n) = \sum_{\substack{I \cup J = Y \\ I \cap J = \emptyset \\ I \neq \emptyset}} (-1)^{|I|} (\Omega, [T_{r|(J,n)}(J, n), \bar{T}_{r|I}(I)]\,\Omega) \tag{17}$$

with obvious notations. It follows that

$$|\int d_r(Y; n)\, f(\xi)\, d\xi| \leqq F_n K_n'^{|r|} |r|^{\lambda|r|} N(f, \sigma, U_n, P_n)\,. \tag{18}$$

We note that the preceding estimates actually serve to define the operator-valued distributions of the form

$$G(X) = \sum_r g_r(X) \frac{:A(X)^r:}{r!} \tag{19}$$

where the $g_r(X)$ are translationally invariant generalized functions satisfying:

$$|\int g_r(X) f(X)\, dX| \leqq BC^{|r|} |r|^{\lambda |r|} N(f, \sigma, A, P)$$

(where B, C, A, P, σ may depend on G but not on r, and $1 < \sigma < \frac{3}{2} - \lambda$). (Note that, in particular, we could take all but a finite number of the $g_r(X)$ to be identically 0.) We also denote

$$G_r(X) = \sum_s g_{r+s}(X) \frac{:A(X)^s:}{s!}, \tag{20}$$

$$G_r^{(N)}(X) = \sum_{|s| \leqq N} g_{r+s}(X) \frac{:A(X)^s:}{s!}. \tag{21}$$

Then estimates identical to the preceding show that, if $G, H, \ldots, K$ are p objects similar to G

$$\lim_{N_1, \ldots, N_p \to \infty} \int G^{(N_1)}(X_1)\, H^{(N_2)}(X_2) \ldots K^{(N_p)}(X_p)\, \Omega$$
$$f(X_1, \ldots, X_p)\, dX_1 \ldots dX_p$$

exists in the sense of the strong topology of the Hilbert space provided some $N(f, \sigma, A, P)$ is finite. As a consequence the $G_r(X)$ define operator valued generalized functions on a dense domain. On this domain they can be freely multiplied. The limiting procedure shows that these operators are generalized functions in the Jaffe class, and that the considerations of locality, supports etc. usual in the polynomial case also apply to them.

As a consequence by the same arguments as in [1], the support of $D_r(Y, n)$ (hence of $d_r(Y, n)$) is contained in $\Gamma^+ \cup \Gamma^-$,

$$\Gamma^+ = \{x_1, \ldots, x_n : x_j - x_n \in \bar{V}^+ \quad \text{for all } j\} = -\Gamma^-$$

and the next problem is to split $D(Y, n)$ into two pieces with supports Γ^+ and Γ^-, by splitting each $d_r(Y, n)$. In so doing we must be able to have each of the two pieces satisfy an inequality of the type (18). We follow exactly the same procedure as in the polynomial case (see [1]).

4. Second Step: The Splitting Operation

Let $\tau > 0$ be an arbitrarily small number. Then there exists a function Ш (indeed an infinity of such functions) with the following properties:

1. Ш is defined over $\mathbb{R}^N$ and is $\mathscr{C}^\infty$ everywhere except at the origin.
2. $Ш(\varrho\xi) = Ш(\xi)$ for every $\xi \neq 0$ in $\mathbb{R}^N$ and every $\varrho > 0$.
3. $0 \leqq Ш \leqq 1$; Ш takes the value 1 (resp. the value 0) in a neighbourhood of $\Gamma^+ - \{0\}$ (resp. in a neighbourhood of $\Gamma^- - \{0\}$).

4. There exists a constant C such that for all $\xi \neq 0$ and all α, $|D^\alpha \mathrm{III}(\xi)| \leq C|\xi|^{-|\alpha|}|\alpha|^{(\tau+1)|\alpha|}$.

To construct such a function, one first constructs it on the unit sphere and then extends it by homogeneity to the whole of $\mathbb{R}^N - \{0\}$.

According to (18), d_r is a continuous linear functional on the Banach space of $\mathscr{C}^\infty$ functions f such that $N(f, \sigma, U_n, P_n) < \infty$. We plan to show that, provided τ has been chosen small enough, there exist new constants U'_n, U''_n, R_n, S_n such that

(i) if f is $\mathscr{C}^\infty$ with $D^\alpha f(0) = 0$ for all α and $N(f, \sigma - \tau, U''_n, P_n) < \infty$, then $\mathrm{III}\, f$ is a $\mathscr{C}^\infty$ function such that

$$N(\mathrm{III}\, f, \sigma, U_n, P_n) \leq R_n N(f, \sigma - \tau, U''_n, P_n).$$

(ii) There exists an operator W on $\mathscr{C}^\infty$ functions f satisfying $N(f, \sigma - 2\tau, U'_n, P_n) < \infty$ such that Wf is again a $\mathscr{C}^\infty$ function and:

a) $N(Wf, \sigma - \tau, U''_n, P_n) \leq S_n N(f, \sigma - 2\tau, U'_n, P_n)$.

b) $D^\alpha(Wf)(0) = 0$ for all α (for any f).

c) If $D^\alpha f(0) = 0$ for all α then $Wf = f$.

From this it will follow that:

$$N(\mathrm{III}\, Wf, \sigma, U_n, P_n) \leq R_n S_n N(f, \sigma - 2\tau, U'_n, P_n)$$

which will enable us to define the advanced "function" $a_r(Y; n)$ by

$$\langle a_r, f \rangle = \langle d_r, \mathrm{III}\, Wf \rangle$$

with

$$|\langle a_r, f \rangle| \leq F'_n K''^{|r|}_n |r|^{\lambda |r|} N(f, \sigma - 2\tau, U'_n, P_n).$$

This will yield a set of $t_r(X)$ for $|X| = n$ again satisfying

$$|\langle t_r, f \rangle| \leq E_n M_n^{|r|} |r|^{\lambda |r|} N(f, \sigma - 2\tau, U_{n+1}, P_n). \tag{22}$$

Proof of (i). We assume that τ has been chosen so that $0 < \tau < \dfrac{\sigma - 1}{2}$. Suppose f is a $\mathscr{C}^\infty$ function with $D^\alpha f(0) = 0$ for all α and $N(f, \sigma - \tau, U''_n, P_n) \leq 1$, where $U''_n \leq 1$. (Note that $\sigma - \tau > \tau + 1 > 1$.) We have:

$$D^\alpha f(\xi) = \int_0^1 dt (1-t)^{\omega - 1} \sum_{\substack{\gamma \\ |\gamma| = \omega}} \frac{\omega}{\gamma!} \xi^\gamma D^{\alpha + \gamma} f(t\xi).$$

Hence, for $|\xi| < 1$,

$$|D^\alpha f(\xi)| \leq U''^{|\alpha| + \omega}_n (|\alpha| + \omega)^{(\sigma - \tau)(|\alpha| + \omega)} \frac{N^\omega}{\omega!} |\xi|^\omega. \tag{23}$$

For $\xi \neq 0$

$$|D^\alpha(\mathrm{III} f)(\xi)| = \left|\sum_{\gamma \leqq \alpha} \frac{\alpha!}{\gamma!(\alpha-\gamma)!} D^\gamma \mathrm{III}(\xi) D^{\alpha-\gamma} f(\xi)\right|$$
$$\leqq \sum_{\gamma \leqq \alpha} \frac{\alpha!}{\gamma!(\alpha-\gamma)!} C \frac{|\gamma|^{(1+\tau)|\gamma|}}{|\xi|^{|\gamma|}} |D^{\alpha-\gamma} f(\xi)| .$$

For $|\xi| \leqq 1$, by (23), this is smaller than

$$C|\alpha| (eN^2)^{|\alpha|} U_n''^{|\alpha|} |\alpha|^{\sigma|\alpha|} .$$

While, for $|\xi| \geqq 1$,

$$(1+|\xi|)^{P_n} |D^\alpha(\mathrm{III} f)(\xi)|$$
$$\leqq \sum_{\gamma \leqq \alpha} \frac{\alpha!}{\gamma!(\alpha-\gamma)!} C|\gamma|^{(1+\tau)|\gamma|} U_n''^{|\alpha-\gamma|} |\alpha-\gamma|^{(\sigma-\tau)|\alpha-\gamma|}$$
$$\leqq \sum_{\omega \leqq |\alpha|} CN^\omega \frac{U''^{|\alpha|-\omega}}{(|\alpha|-\omega)^{\tau(|\alpha|-\omega)} \omega^{(\sigma-\tau-1)\omega}} |\alpha|^{\sigma|\alpha|}$$
$$\leqq \frac{|\alpha| C(2N)^{|\alpha|}}{|\alpha|^{\tau|\alpha|}} |\alpha|^{\sigma|\alpha|} .$$

Finally, we see that there is a constant C_1, independent of α and U_n'', such that

$$\sup_\xi (1+|\xi|)^{P_n} |D^\alpha(\mathrm{III} f)(\xi)|$$
$$\leqq C_1^{(1+|\alpha|)} (U_n''^{|\alpha|} + |\alpha|^{-\tau|\alpha|}) |\alpha|^{\sigma|\alpha|} .$$

Hence it is obviously possible to choose U_n'' sufficiently small so that the series

$$\sum_\alpha C_1 (C_1 U_n^{-1})^{|\alpha|} (U_n''^{|\alpha|} + |\alpha|^{-\tau|\alpha|})$$

converges to a constant R_n. This proves (i).

Proof of (ii). To prove (ii) we choose a $\mathscr{C}^\infty$ function w over $\mathbb{R}^N$ such that $w(\xi) = 1$ if $|\xi| \leqq \frac{1}{2}$, $w(\xi) = 0$ if $|\xi| \geqq 1$ and, for all α

$$D^\alpha w(\xi) < K|\alpha|^{(1+\frac{\tau}{2})|\alpha|} \tag{24}$$

We define the operator W by

$$(Wf)(\xi) = f(\xi) - \sum_\alpha w(\varepsilon_{|\alpha|}^{-1} \xi) \frac{\xi^\alpha}{\alpha!} D^\alpha f(0) . \tag{25}$$

Here $\{\varepsilon_k\}$ is a decreasing sequence of positive real numbers < 1 satisfying:

$$\varepsilon_k = \varepsilon k^{-(\sigma-2\tau-1)} \quad \text{for all integers} \quad k > 0 . \tag{26}$$

Here ε is a real number >0.

$$D^\alpha[(1-W)f](\xi) = \sum_\beta \sum_{\substack{\gamma \\ \gamma \leqq \beta \\ \gamma \leqq \alpha}} \varepsilon_{|\beta|}^{-|\alpha|+|\gamma|} (D^{\alpha-\gamma} w)\left(\frac{\xi}{\varepsilon_{|\beta|}}\right) \frac{\xi^{\beta-\gamma}}{(\beta-\gamma)!} D^\beta f(0)$$

$$|D^\alpha(1-W)f(\xi)| \leqq \sum_\beta \sum_{\substack{\gamma \\ \gamma \leqq \beta \\ \gamma \leqq \alpha}} \varepsilon_{|\beta|}^{|\beta|-|\gamma|} K|\alpha-\gamma|^{(1+\frac{\tau}{2})|\alpha-\gamma|} N^{|\beta-\gamma|} |\beta-\gamma|^{-|\beta-\gamma|}$$

$$U_n'^{|\beta|} |\beta|^{(\sigma-2\tau)|\beta|} .$$

The sum of terms for which $|\beta| > |\alpha|$ is majorized by:

$$\sum_{\substack{\beta \\ |\beta| \leqq |\alpha|}} \sum_{\substack{\gamma \leqq \alpha \\ \gamma \leqq \beta}} \{\varepsilon(|\beta|-|\alpha|)^{-(\sigma-2\tau-1)}\}^{(|\beta|-|\alpha|)} K N^{|\beta|-|\gamma|}$$

$$\cdot |\alpha-\gamma|^{\frac{\tau}{2}|\alpha-\gamma|} (|\beta|-|\alpha|)^{(\sigma-2\tau-1)(|\beta|-|\alpha|)} U_n'^{|\beta|-|\alpha|}$$

$$\cdot U_n'^{|\alpha|} |\alpha|^{(\sigma-2\tau)|\alpha|} 2^{(\sigma-2\tau)|\beta|}$$

$$\leqq K|\alpha| U_n'^{|\alpha|} |\alpha|^{(\sigma-\frac{3}{2}\tau)|\alpha|} 2^{(\sigma-2\tau)|\alpha|}$$

$$\sum_{m=0}^{\infty} N^{2m} U_n'^m 2^{(\sigma-2\tau)m} \varepsilon^m .$$

If U_n' is small enough, this is bounded by

$$\text{const}\, U_n'^{|\alpha|} 2^{(\sigma-2\tau)|\alpha|} |\alpha|^{(\sigma-\frac{3}{2}\tau)|\alpha|+1} .$$

The sum of terms for which $|\alpha| \geqq |\beta|$ is majorized by

$$\sum_{\substack{\beta \\ |\beta| \leqq |\alpha|}} \sum_{\substack{\gamma \leqq \alpha \\ \gamma \leqq \beta}} K|\beta|^{(\sigma-2\tau-1)(|\alpha|-|\beta|)} |\alpha|^{(1+\frac{\tau}{2})|\alpha|} |\gamma|^{-\frac{\tau}{2}|\gamma|}$$

$$\cdot 2^{|\beta|} N^{|\beta|-|\gamma|} |\beta|^{(\sigma-2\tau-1)|\beta|} U_n'^{|\beta|} \varepsilon^{-|\alpha|}$$

$$\leqq \sum_{\substack{\beta \\ |\beta| < |\alpha|}} K|\beta| N^{2|\beta|} 2^{|\beta|} |\alpha|^{(\sigma-\frac{3}{2}\tau)|\alpha|} U_n'^{|\beta|} \varepsilon^{-|\alpha|} .$$

If $U_n' \leqq 1$, as we shall suppose, this is bounded by

$$K \frac{(2N)^{3|\alpha|} |\alpha|^{(\sigma-\tau)|\alpha|}}{|\alpha|^{\frac{\tau}{2}|\alpha|} \varepsilon^{|\alpha|}} .$$

Hence, for sufficiently small U_n', the series

$$\sum_\alpha U_n''^{-|\alpha|} |\alpha|^{-|\sigma-\tau||\alpha|} \sup_\xi (1+|\xi|)^{P_n} |D^\alpha(1-W)f(\xi)|$$

is majorized by a finite constant, independent of f. This proves (ii).

Remark 1. Note that the same splitting operation has been used to define the $a_r(Y; n)$ for *all* r. This operation depends on n.

Remark 2. If f is $\mathscr{C}^\infty$ and verifies: $D^\alpha f(0) = 0$ for all α, and $N(f, \sigma - \tau, U_n'', P_n) < \infty$, the quantity $\langle d_r, \text{Ш} f\rangle = \langle a_r, f\rangle$ does not depend on any particular choice of Ш. Indeed, let Ш′ be another auxiliary function having the properties 1 to 4 required from Ш. Let $\{f_k\}$ be a sequence of $\mathscr{C}^\infty$ functions such that $N(f - f_k, \sigma - \tau, U_n'', P_n) \to 0$ as $k \to \infty$, and that each f_k vanishes in a neighbourhood of 0. Then

$$\langle d_r, \text{Ш} f\rangle = \lim_{k\to\infty} \langle d_r, \text{Ш} f_k\rangle = \lim_{k\to\infty} \langle d_r, \text{Ш}' f_k\rangle = \langle d_r, \text{Ш}' f\rangle .$$

However, the splitting operation depends on the particular choice of the operator W, i.e. on the particular choice of the auxiliary function w and of the sequence $\varepsilon_{|\alpha|}$.

Remark 3: ambiguity of the definition of the a_r; Lorentz invariance. To simplify further consideration let $\sigma' = \sigma - 2\tau$ and let $\mathscr{C}'_{\sigma'}$ be the class of generalized functions G such that there exist constants K and V (depending on G) such that, for all f,

$$|\langle G, f\rangle| \leqq K N(f, \sigma', V, P_n) .$$

Let C be a generalized function in the class $\mathscr{C}'_{\sigma'}$ with

$$|\langle C, f\rangle| \leqq N(f, \sigma, U_n, P_n)$$

having support in $\Gamma^+ \cup \Gamma^-$. Suppose there are two pairs of generalized functions $F_1{}^\pm, F_2{}^\pm$ in the class $\mathscr{C}'_{\sigma'}$, with

$$\begin{cases} \text{support } F_j{}^\pm \subset \Gamma^\pm \\ F_j{}^+ - F_j{}^- = C, \quad j = 1, 2 . \end{cases}$$

Then $F_1{}^+ - F_2{}^+ = F_1{}^- - F_2{}^-$ is a generalized function in $\mathscr{C}'_{\sigma'}$, with support at the origin. The Fourier transform of a member G of $\mathscr{C}'_{\sigma'}$ with support at the origin is an entire function (over complex momentum space) $\tilde{G}$ such that

$$|\tilde{G}(p + iq)| < \text{const} \exp(B|p + iq|^{\frac{1}{\sigma'}})$$

and conversely (the constants here depend on G). The space $\mathscr{C}'_{\sigma'}$ is invariant under the real Lorentz group and the subspace of its elements having support at 0 is also Lorentz invariant. We denote this subspace by $\mathscr{C}'^{0}_{\sigma'}$.

Suppose C is invariant under the real Lorentz group and $C = F^+ - F^-$ with (support $F^\pm) \subset \Gamma^\pm$ and $F^\pm \in \mathscr{C}'_{\sigma'}$. Denote, for each generalized function G, and each $\Lambda \in L_+^\uparrow$,

$$\langle \Lambda G, f \rangle = \langle G, f_\Lambda \rangle \quad \text{where} \quad f_\Lambda(\xi) = f(\Lambda \xi)$$

(i.e. formally $(\Lambda G)(\xi) = G(\Lambda^{-1}\xi)$). Clearly the mapping $(\Lambda, G) \to \Lambda G$ is a continuous (and even $\mathscr{C}^\infty$) map of $L_+^\uparrow \times \mathscr{C}'_{\sigma'}$ into $\mathscr{C}'_{\sigma'}$ (the latter being equipped with the topology described previously, i.e. dual of a Fréchet). Let us denote

$$E(\Lambda) = \Lambda F^+ - F^+ = \Lambda F^- - F^- .$$

This is an element of $\mathscr{C}'^0_{\sigma'}$. It satisfies

$$M E(\Lambda) = E(M\Lambda) - E(M) \tag{27}$$

for all Λ and M in $L_+^\uparrow$. From this we wish to deduce the existence of an element E_0 of $\mathscr{C}'^0_{\sigma'}$ such that [1]:

$$E(\Lambda) = \Lambda E_0 - E_0 \tag{28}$$

Then, denoting $G^\pm = F^\pm - E_0$ we would find

$$C = G^+ - G^- , \qquad \Lambda G^\pm = G^\pm , \tag{29}$$

i.e. we would have obtained a Lorentz invariant splitting of C. However, before doing this we shall require C to possess a property common to all the $d_r(Y; n)$, namely that its Fourier transform should vanish in a real region containing all Jost points. From this it follows that the Fourier transforms $\tilde{F}^\pm$ of $F^\pm$ are two branches of the same analytic $H(p+iq)$, holomorphic in a domain which is invariant under the whole complex Lorentz group $L_+(\mathbb{C})$. Furthermore the Fourier transform $\tilde{E}(\Lambda)$ of $E(\Lambda)$ is the restriction to the reals of the function.

$$\tilde{E}(\Lambda, p+iq) = H(\Lambda, p+iq) - H(p+iq)$$

which is entire in $p+iq$ and in $\Lambda \in L_+(\mathbb{C})$. It can be shown that, for every complex Λ, this entire function of $p+iq$ is still of order σ'^{-1}. As a consequence, if we come back to ξ-space, we see that $E(\Lambda)$ can be extended to an entire function of $\Lambda \in L_+(\mathbb{C})$ with values in $\mathscr{C}'^0_{\sigma'}$. Note also that, if $G \in \mathscr{C}'^0_{\sigma'}$, ΛG is also in this subspace for every $\Lambda \in L_+(\mathbb{C})$ since, for real Λ, ΛG is the Fourier transform of $\tilde{G}(\Lambda p)$. Complexifying Λ and p in the latter expression again yields an entire function of order σ'^{-1} which

[1] The following considerations are adapted from a paper in preparation in collaboration with R. Stora whom we thank for permission to include them here.

depends holomorphically on Λ. Hence ΛG is an entire function of Λ with values in $\mathscr{C}'^{0}_{\sigma'}$. Let $\mathcal{O}$ denote one of the maximal compact subgroups of $L_+(\mathbb{C})$ (for example the subgroup of all complex Lorentz transformations Λ such that Λ_{0j} and Λ_{j0} are pure imaginary for $j=1,2,3$, all other $\Lambda_{\mu\nu}$ being real) and let $d\Lambda$ denote the invariant measure on $\mathcal{O}$ normalized so that $\int_{\mathcal{O}} d\Lambda = 1$.

Set

$$E_0 = -\int_{\mathcal{O}} E(M)\, dM\,. \tag{30}$$

Since (27) obviously extends to all Λ and M in $L_+(\mathbb{C})$ by analytic continuation,

$$\begin{aligned} &E(\Lambda) - \Lambda E_0 + E_0 \\ &\quad = \int_{\mathcal{O}} [E(\Lambda) + \Lambda E(M) - E(M)]\, dM \\ &\quad = \int_{\mathcal{O}} [-E(M) + E(\Lambda M)]\, dM = 0\,. \end{aligned}$$

The last integral vanishes because it is an entire function of Λ taking the value 0 when $\Lambda \in \mathcal{O}$ (by invariance of dM). Hence our problem is solved and, denoting $G^{\pm} = F^{\pm} - E_0$, there are constants K and V, depending only on U_n and P_n such that

$$|\langle G^+, f\rangle| \leqq K N(f, \sigma', V, P_n)\,.$$

This shows that, if the $T(X)$ have been defined in a Lorentz invariant way for $|X| \leqq n-1$, they can be defined in a Lorentz invariant way for $|X| = n$ (while still verifying inequalities of the type (22)).

Conclusion

It has been shown here that a renormalized perturbation series can be defined for a strictly localizable but non-polynomial Lagrangian. The requirements of locality and Lorentz invariance are fulfilled. The existence of an adiabatic limit for Green functions can be proved (for theories with non-zero masses) in the same way as in the polynomial case [1]. However, our treatment is preliminary since it does not touch on the question of minimality studied in Ref. [6]. Moreover the existence of a strong adiabatic limit remains to be proved.

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