

Infinitely Divisible Representations and Positive Definite Functions on a Compact Group

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Abstract. A complete description of the infinitely divisible positive definite functions on a compact group is given. Their relation to infinitely divisible representations is also discussed.

§ 1. Introduction

In certain problems of quantum field theory it is necessary to define the notion of a continuous tensor product of Hilbert spaces. In his attempt to construct such tensor products Streater [3] has introduced the notion of an infinitely divisible group representation. These in turn are described by infinitely divisible positive definite functions on the group. If the group is abelian such positive definite functions are in one to one correspondence with the so called infinitely divisible distributions on the character group. The reader may find a detailed account of these in [1]. Here we shall give a complete description of the infinitely divisible positive definite functions on a compact group and describe some of the infinitely divisible representations.

§ 2. Infinitely Divisible Group Representations and Positive Definite Functions

Throughout this paper we shall denote by G a fixed compact metric group. By a representation of G we mean a continuous unitary representation of G in a complex separable Hilbert space with an inner product (\cdot, \cdot) . Let $T^{(k)}$, $k = 1, 2, \dots$ be an enumeration of all the inequivalent irreducible representations (up to equivalence). Suppose $T^{(k)}$ operates in a Hilbert space V_k of dimension d_k . Let $\varepsilon_1^k, \varepsilon_2^k, \dots, \varepsilon_{d_k}^k$ be a

fixed orthonormal basis in V_k and let

$$\psi_{rs}^{(k)}(g) = (T_g^{(k)} e_r^k, e_s^k), \quad 1 \leq r, s \leq d_k. \tag{2.1}$$

Integration in G will always be with respect to the normalised Haar measure in G . Under these assumptions it is well known (cf. [2]) that the functions $\psi_{rs}^{(k)}$ constitute an orthogonal basis for the Hilbert space $L_2(G)$ and

$$\int |\psi_{rs}^{(k)}(g)|^2 dg = \frac{1}{d_k}, \quad 1 \leq r, s \leq d_k, \quad k = 1, 2, \dots \tag{2.2}$$

Let \mathcal{S} denote the multiplicative group of complex numbers of modulus unity and D be the group of all continuous homomorphisms of G into \mathcal{S} . With the uniform topology, D becomes a discrete group with a finite or countable number of elements.

Definition 2.1. A representation U of G with cyclic vector x in a Hilbert space \mathcal{H} is said to be *infinitely divisible* if, for any positive integer q , there exists a representation $U^{1/q}$ with cyclic vector $x^{1/q}$ in a Hilbert space $\mathcal{H}^{1/q}$ such that U is unitarily equivalent to the representation $\underbrace{U^{1/q} \otimes \dots \otimes U^{1/q}}_q$ restricted to the cyclic subspace generated by

$x^{1/q} \otimes x^{1/q} \otimes \dots \otimes x^{1/q}$ by the action of the group. Under this equivalence x and $x^{1/q} \otimes \dots \otimes x^{1/q}$ are to correspond.

Definition 2.2. A function ϕ defined on G is said to be *positive definite* if and only if it is continuous and for every finite collection of points g_1, g_2, \dots, g_n in G and complex numbers a_1, a_2, \dots, a_n , the following inequality holds: $\sum a_i \bar{a}_j \phi(g_i g_j^{-1}) \geq 0$. Equivalently, a function ϕ is positive definite if and only if it is continuous and for every complex valued bounded measurable function f on G , $\int f(g) \overline{f(h)} \phi(gh^{-1}) dg dh \geq 0$.

Definition 2.3. A positive definite function ϕ is said to be *infinitely divisible* if, for any positive integer q , there exists a positive definite function ϕ_q such that $\phi(g) = [\phi_q(g)]^q$ for all $g \in G$.

We shall say that a positive definite function ϕ is *normalised* if $\phi(e) = 1$ where e is the identity element of G . Let \mathcal{P} be the space of all normalized positive definite functions on G with the uniform topology. If U is a representation of G in Hilbert space \mathcal{H} and $x \in \mathcal{H}$ is a unit vector, then the function $(U_g x, x)$ is in \mathcal{P} . Conversely every element of \mathcal{P} arises in this manner. The vector x can be assumed to be cyclic without loss of generality. It is an easy consequence of Definition 2.1 and 2.2 that every infinitely divisible positive definite function on G arises from an infinitely divisible representation and vice versa. Thus the problem of finding the infinitely divisible representations of G reduces to the problem of finding the infinitely divisible positive definite functions of G .

§ 3. Some Lemmas on Positive Definite Functions

Lemma 3.1. *A complex valued function ϕ defined on G belongs to \mathcal{P} if and only if it is of the form*

$$\phi(g) = \sum_k p_k \left\{ \sum_{1 \leq r, s \leq d_k} a_{rs}^{(k)} \psi_{rs}^{(k)}(g) \right\}$$

where p_1, p_2, \dots are non negative constants such that $\sum_k p_k = 1$, $((a_{rs}^{(k)}))$ is a positive definite constant matrix of order $d_k \times d_k$ and trace unity for every k , $\psi_{rs}^{(k)}$ are the functions defined by (2.1) and the convergence of the above series is uniform.

Proof. The function $\phi \in \mathcal{P}$ if and only if there exists a representation U and unit vector x such that $\phi(g) = (U_g x, x)$. U can be decomposed into a direct sum of primary representations. Each primary representation is a finite or countable direct sum of copies of some $T^{(k)}$. The lemma is an immediate consequence of these facts.

Lemma 3.2. *Suppose $\phi_n \in \mathcal{P}$, $n = 0, 1, 2, \dots$, and $\phi_n(g) \rightarrow \phi_0(g)$ as $n \rightarrow \infty$ for every g . Then $\phi_n \rightarrow \phi_0$ uniformly in G .*

Proof. By Lemma 3.1 we can write

$$\phi_n(g) = \sum_k p_{nk} \sum_{1 \leq r, s \leq d_k} a_{rs}^{(k)}(n) \psi_{rs}^{(k)}(g), \quad n = 0, 1, 2 \dots$$

where, for each fixed n , p_{nk} are non negative numbers such that $\sum_k p_{nk} = 1$ and for each fixed n and k , the matrix $((a_{rs}^{(k)}(n)))$ is positive definite and of trace unity. Since $|\phi_n(g)| \leq 1$ and $\phi_n(g) \rightarrow \phi_0(g)$ as $n \rightarrow \infty$ for every g , we have

$$\lim_{n \rightarrow \infty} \int \phi_n(g) \overline{\psi_{rs}^{(k)}(g)} dg = \int \phi_0(g) \overline{\psi_{rs}^{(k)}(g)} dg.$$

Since the functions $\psi_{rs}^{(k)}(g)$ are mutually orthogonal in $L_2(G)$ and have norm $d_k^{-\frac{1}{2}}$ (see (2.2)) we have

$$\lim_{n \rightarrow \infty} p_{nk} a_{rs}^{(k)}(n) = p_{0k} a_{rs}^{(k)}(0) \text{ for every } r, s, k.$$

Since $\sum_{r=1}^{d_k} a_{rr}^{(k)} = 1$, we obtain

$$\lim_{n \rightarrow \infty} p_{nk} = p_{0k}, \tag{3.1}$$

$$\lim_{n \rightarrow \infty} a_{rs}^{(k)}(n) = a_{rs}^{(k)}(0) \text{ if } p_{0k} \neq 0. \tag{3.2}$$

For each fixed n , $\{p_{n1}, p_{n2} \dots\}$ can be viewed upon as masses of a discrete probability distribution. Hence (3.1) implies that ¹

$$\lim_{n \rightarrow \infty} \sum_k |p_{nk} - p_{0k}| = 0. \tag{3.3}$$

Since $\sum_{r,s} a_{rs}^{(k)} \psi_{rs}^{(k)} \in \mathcal{P}$, its absolute value is not greater than unity. This together with (3.3) implies that $\phi_n \rightarrow \phi_0$ uniformly. This completes the proof of the lemma.

Lemma 3.3. *Let $\phi_n \in \mathcal{P}$ be a sequence such that $\text{Re } \phi_n$ (i.e. the real part of ϕ_n) converges uniformly. Then the sequence $\{\phi_n\}$ is conditionally compact in the uniform topology.*

Proof. Suppose $\phi \in \mathcal{P}$ and g, h are two points in G . Let U be a representation and x be a unit vector such that $\phi(g) \equiv (U_g x, x)$. We have

$$\begin{aligned} |\phi(g) - \phi(h)| &= |(U_g x, x) - (U_h x, x)| \\ &\leq \|U_g x - U_h x\| \\ &\leq \|U_{g^{-1}h} x - x\| \\ &\leq \{2[1 - \text{Re} \phi(g^{-1}h)]\}^{\frac{1}{2}}. \end{aligned}$$

If $\text{Re } \phi_n$ converges uniformly it follows from the above inequality that the sequence $\{\phi_n\}$ is uniformly equicontinuous. An application of Arzela-Ascoli theorem completes the proof of the lemma.

Lemma 3.4. *Let ψ be a continuous function defined on G and satisfying the inequality*

$$\int_{G \times G} f(g) \overline{f(h)} \psi(gh^{-1}) dg dh \geq 0$$

for every bounded measurable function f such that $\int f(g) dg = 0$. Then $\psi(g) - \int \psi(g) dg$ is positive definite.

Proof. Let f be a bounded measurable function such that $\int f(g) dg = m$. Let $\int \psi(g) dg = a$. Then the invariance of the Haar measure implies that

$$\begin{aligned} &\int f(g) \overline{f(h)} [\psi(gh^{-1}) - a] dg dh \\ &= \int (f(g) - m) \overline{(f(h) - \overline{m})} \psi(gh^{-1}) dg dh \geq 0. \end{aligned}$$

This implies that $\psi(g) - a$ is positive definite. The proof of the lemma is complete.

Lemma 3.5. *For every positive definite function ϕ on G ,*

$$\int \text{Im } \phi(g) dg = 0.$$

¹ For a proof of this the reader may refer to Lemma 9.1, page 206 in [1].

Proof. This is an immediate consequence of the fact that $\phi(g^{-1}) = \overline{\phi(g)}$ and the Haar measure is invariant under the mapping $g \rightarrow g^{-1}$.

Lemma 3.6. For any $\phi \in \mathcal{P}$, $g_1, g_2 \in G$,

$$[1 - \operatorname{Re} \phi(g_1 g_2)]^{\frac{1}{2}} \leq [1 - \operatorname{Re} \phi(g_1)]^{\frac{1}{2}} + [1 - \operatorname{Re} \phi(g_2)]^{\frac{1}{2}}.$$

Proof. Let U be a representation in a Hilbert space and x a unit vector such that $\phi(g) = (U_g x, x)$ for all $g \in G$. Then

$$\|U_g x - x\| = [2(1 - \operatorname{Re} \phi(g))]^{\frac{1}{2}}. \tag{3.4}$$

Now we have

$$\begin{aligned} \|U_{g_1 g_2} x - x\| &\leq \|U_{g_1 g_2} x - U_{g_1} x\| + \|U_{g_1} x - x\| \\ &\leq \|U_{g_2} x - x\| + \|U_{g_1} x - x\| \end{aligned} \tag{3.5}$$

(3.4) and (3.5) imply the required inequality. This completes the proof of the lemma.

Lemma 3.7. Let $H \subset G$ be a both open closed subgroup of G . Suppose ψ is a positive definite function defined on H . Let ϕ be defined by

$$\begin{aligned} \phi(g) &= \psi(g) \quad \text{if } g \in H, \\ &= 0 \quad \text{if } g \notin H. \end{aligned}$$

Then ϕ is a positive definite function on G .

Proof. The continuity of ϕ is obvious. Let now a_1, a_2, \dots, a_n be any n complex numbers and g_1, g_2, \dots, g_n any n points in G . Each element g_i belongs to some left coset of H in G . Let $H_i = H w_i$, $i = 1, 2, \dots, k$ be the distinct cosets to which g_1, g_2, \dots, g_n belong. Then

$$\begin{aligned} \sum_{1 \leq i, j \leq n} a_i \bar{a}_j \phi(g_i g_j^{-1}) &= \sum_i \left\{ \sum_{(r, s): g_r, g_s \in H_i} a_r \bar{a}_s \phi(g_r g_s^{-1}) \right\} \\ &= \sum_{i=1}^k \left\{ \sum_{(r, s): g_r, g_s \in H_i} a_r \bar{a}_s \phi((g_r w_i^{-1})(g_s w_i^{-1})^{-1}) \right\}. \end{aligned}$$

Since $g_r w_i^{-1} \in H$ for all r such that $g_r \in H w_i$ and $\phi = \psi$ on H , it follows that every term within brackets on the right hand side of the above equation is non negative. This completes the proof of the lemma.

§ 4. Representation of Infinitely Divisible Positive Definite Functions

Before proceeding to the statement of the main result we shall prove a lemma on the infinitely divisible elements of \mathcal{P} .

Lemma 4.1. Let $\phi \in \mathcal{P}$ be infinitely divisible and $\phi(g) \neq 0$ for every $g \in G$. Suppose $\phi = \phi_n^n$, $n = 1, 2, \dots$ where $\phi_n \in \mathcal{P}$ for all n . Then there exists a sequence $\chi_n \in D$ such that $\phi_n \chi_n$ converges uniformly to unity.

Proof. Since $\phi = \phi_n^n$ for all n , it follows that $n(|\phi_n| - 1)$ converges uniformly to $\log|\phi|$ as $n \rightarrow \infty$. By Lemma 3.1 we may assume that ϕ_n is given by

$$\phi_n(g) = \sum_{k=1}^{\infty} p_{nk} \sum_{1 \leq r, s \leq d_k} a_{rs}^{(k)}(n) \psi_{rs}^{(k)}(g).$$

Since $\int |\phi_n(g)|^2 dg \rightarrow 1$ as $n \rightarrow \infty$, we have from (2.2)

$$\lim_{n \rightarrow \infty} \sum_k p_{nk}^2 \sum_{1 \leq r, s \leq d_k} \frac{|a_{rs}^{(k)}(n)|^2}{d_k} = 1.$$

Since, for each k and n , $a_{rs}^{(k)}$ is a positive definite matrix of trace unity and $\sum_k p_{nk} = 1$, it follows that

$$\lim_{n \rightarrow \infty} \sum_k p_{nk}^2 / d_k = 1.$$

Hence

$$\lim_{n \rightarrow \infty} \sum_{k: d_k=1} p_{nk}^2 + \frac{1}{2} \sum_{k: d_k \geq 2} p_{nk}^2 = 1,$$

$$\lim_{n \rightarrow \infty} \sum_{k: d_k=1} p_{nk}^2 + \sum_{k: d_k \geq 2} p_{nk}^2 = 1.$$

Hence

$$\lim_{n \rightarrow \infty} \sum_{k: d_k=1} p_{nk}^2 = 1.$$

Thus

$$\lim_{n \rightarrow \infty} \left(\sup_{k: d_k=1} p_{nk} \right) \left(\sum_{k: d_k=1} p_{nk} \right) = 1.$$

In other words there exists a sequence k_n such that $\lim_{n \rightarrow \infty} p_{nk_n} = 1$, where $T^{(k_n)}$ is a one dimensional representation of G . Let χ_n be the corresponding homomorphism of G into \mathcal{S} . Then

$$\phi_n(g) \bar{\chi}_n(g) = p_{nk_n} + \gamma_n(g)$$

where $\gamma_n(g)$ is also positive definite and $\gamma_n(e) = 1 - p_{nk_n}$. Since $\gamma_n(e) \rightarrow 0$ as $n \rightarrow \infty$ and $|\gamma_n(g)| \leq \gamma_n(e)$, it follows that $\phi_n \bar{\chi}_n$ converges uniformly to unity. This completes the proof of the lemma.

Theorem 4.1. *Let ϕ be a normalised infinitely divisible positive definite function defined on G . Suppose $\phi(g) \neq 0$ for every $g \in G$. Then ϕ admits a representation*

$$\phi(g) = \chi(g) \exp[\gamma(g) - \gamma(e)] \quad \text{for all } g \in G, \tag{4.1}$$

where γ is a positive definite function on G and χ is an infinitely divisible element of the group of homomorphisms of G into \mathcal{S} .

Conversely every function ϕ of the form given by (4.1) is infinitely divisible and without zeros.

Proof. The converse part is trivial. In order to prove the first part choose a sequence $\phi_n \in \mathcal{P}$ such that $\phi_n^n = \phi$. By Lemma 4.1, there exists a $\chi_n \in D$ such that $\phi_n \chi_n \rightarrow 1$ uniformly as $n \rightarrow \infty$. Let $\psi_n = \phi_n \chi_n$. Then $\phi = \psi_n^n \bar{\chi}_n^n$. Let g be any point in G and A_g be the closed abelian subgroup generated by g . The functions ψ_n and ϕ restricted to A_g are Fourier transforms of probability measures μ_n and μ respectively on the character group \hat{A}_g of A_g . Since $\psi_n \rightarrow 1$ uniformly μ_n converges weakly to the measure degenerate at the identity of \hat{A}_g . Further μ_n^{*n} is a translate of μ in A_g . Hence by Theorem 5.1, page 89, [1], it follows that

$$\lim_{n \rightarrow \infty} |\psi_n(g)^n - \exp n[\psi_n(g) - 1]| = 0.$$

Hence

$$\lim_{n \rightarrow \infty} |\phi(g) - \bar{\chi}_n(g)^n \exp n[\psi_n(g) - 1]| = 0. \tag{4.2}$$

Since (4.2) holds for every g , Lemma 3.2 implies that the convergence is uniform. Since ϕ has no zeros we have

$$\lim_{n \rightarrow \infty} \sup_g |n[\operatorname{Re} \psi_n(g) - 1] - \log |\phi(g)|| = 0. \tag{4.3}$$

By Lemma 3.5,

$$n \int (\psi_n(g) - 1) dg = n \int (\operatorname{Re} \psi_n(g) - 1) dg = a_n, \text{ say.}$$

Let

$$a = \int \log |\phi(g)| dg.$$

By (4.3), $a_n \rightarrow a$ as $n \rightarrow \infty$. Since $\psi_n(g)$ is positive definite $n[\psi_n(g) - 1]$ satisfies the conditions of Lemma 3.4. Hence the function γ_n defined by

$$\gamma_n(g) = n[\psi_n(g) - 1] - a_n$$

is positive definite. By (4.3), $\operatorname{Re} \gamma_n(g)$ converges uniformly to $\log |\phi(g)| - a$. By Lemma 3.3, the sequence γ_n is conditionally compact in the uniform topology. Now (4.2) implies that $\bar{\chi}_n^n$ is also conditionally compact in the uniform topology. Since the group D is discrete, it follows that the set $\{\bar{\chi}_n^n\}$ is finite. Hence there exists a $\chi \in D$ such that $\chi = \bar{\chi}_n^n$ for infinitely many n . This χ is infinitely divisible. Now extracting a suitable convergent subsequence from γ_{n_i} and using (4.2) we conclude that

$$\phi(g) = \chi(g) \exp(\gamma(g) + a)$$

where γ is positive definite. Since $\phi(e) = \chi(e) = 1$, it follows that $a = -\gamma(e)$. This completes the proof of the theorem.

Now we shall proceed to analyse the nature of infinitely divisible positive definite functions with zeros.

Lemma 4.2. *Let $\phi \in \mathcal{P}$ be infinitely divisible. Then the set $\{g : \phi(g) \neq 0\}$ is a both open closed subgroup of G .*

Proof. Let $\{\phi_n\}$ be a sequence of elements on \mathcal{P} such that $\phi = \phi_n^n$. In particular $|\phi|^2 = |\phi_n|^{2n}$ and $|\phi(g)|^2 \neq 0$ for some g if and only if $\overline{\lim}_{n \rightarrow \infty} n(1 - |\phi_n(g)|^2) < \infty$. It follows from Lemma 3.6 that the set $\{g : \overline{\lim}_{n \rightarrow \infty} n(1 - |\phi_n(g)|^2) < \infty\}$ is actually a subgroup. Since $\phi(g)$ is continuous this subgroup is open. Since an open subgroup is closed the proof of the lemma is complete.

Theorem 4.2. *A normalised positive definite function ϕ on G is infinitely divisible if and only if it admits a representation*

$$\begin{aligned} \phi(g) &= \chi(g) \exp[\gamma(g) - \gamma(e)] \quad \text{if } g \in H \\ &= 0 \quad \text{otherwise} \end{aligned}$$

where H is a both open closed subgroup of G , χ is an infinitely divisible homomorphism of H into \mathcal{S} and γ is a positive definite function on H .

Proof. This is an immediate consequence of Theorem 4.1, Lemmas 4.2 and 3.7.

Remark 1. There are many compact groups which do not possess an infinitely divisible homomorphism into \mathcal{S} . Finite groups and compact connected semi simple Lie groups are such examples. In these cases the factor χ is absent in the representation of an infinitely divisible positive definite function.

For a compact connected group without infinitely divisible homomorphisms into \mathcal{S} every normalised infinitely divisible positive definite function is of the form $\exp[\gamma(g) - \gamma(e)]$ where γ is positive definite.

Remark 2. For any representation T with cyclic vector x in a Hilbert space \mathcal{H} we can construct an infinitely divisible representation as follows: let $e(\mathcal{H})$ be the Hilbert space $\mathbb{C} \oplus \mathcal{H} \oplus \mathcal{H} \otimes \mathcal{H} \oplus \dots \oplus \underbrace{\mathcal{H} \otimes \dots \otimes \mathcal{H}}_{\eta \text{ factors}} \oplus \dots$ and $e(x)$ be the vector

$$e^{-\frac{1}{2}\|x\|^2} \left\{ 1 \oplus x \oplus \frac{1}{\sqrt{2!}} (x \otimes x) \oplus \dots \oplus \frac{1}{\sqrt{n!}} (x \otimes \dots \otimes x) + \dots \right\}.$$

Let $e(T)$ denote the restriction of the representation $1 \oplus T \oplus T \otimes T \oplus \dots$ to the cyclic subspace generated by $e(x)$. Then $e(T)$ is an infinitely divisible representation with cyclic vector $e(x)$.

If U is an infinitely divisible representation with unit cyclic vector x such that $(U_g x, x) \neq 0$ for any g and $(U_g x, x) = \phi_n(g)^n$ for all $g \in G$ and $n = 1, 2, \dots$ where ϕ_n is a sequence of positive definite functions converging to unity as $n \rightarrow \infty$, then U is equivalent to $e(T)$ for some representation T . In the case of compact connected semisimple Lie groups every infinitely divisible representation U is equivalent to some $e(T)$.

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