Commun. math. Phys. 16, 48—80 (1970)

Renormalizable Models with Simple Symmetry Breaking

I. Symmetry Breaking by a Source Term

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Received September 30, 1969

Abstract. If to a Lagrangian density with invariance under a continuous group of linear transformations of the fields a term linear or bilinear in the fields is added, the symmetry is in general reduced and the currents associated with the original symmetry are only partially conserved. If the theory without the added term is renormalizable, the theory with that term also is, and the needed renormalization conditions are the essential content of the appropriate Ward-Takahashi-Kazes-Rivers identities. The case of symmetry breaking by a term linear in Bose fields (source term) is here analysed completely, in particular with respect to the nonsymmetric limit of vanishing source term, a particular Goldstone mode, and with respect to properties of the ground state energy density as a function of the strength of the source term. Induced and spontaneous breaking of a discrete symmetry are also treated.

Introduction

B. W. Lee [1] has discussed the sigma model [2, 3] from the point of view of renormalized perturbation theory, in order to have available a model that satisfies PCAC[1] and allows to calculate in a formal but consistent way the amplitudes for processes involving nonsoft pions.

We shall show here[2] that for such models the renormalized perturbation expansions can be very simply obtained if the relations, stemming from PCAC, between vertex functions of different numbers of arguments are exploited. These relations yield all the renormalization conditions required in Bogoliubov-Parasiuk-Hepp (BPH)[3] renormalization theory in terms of only that many parameters as the unrenormalized Lagrangian has. This technique also covers the Goldstone mode[4] obtained in the limit of vanishing source but, since it deals with renormalized quantities only,

[1] Ref. [4] discusses the sigma model and related models from the point of view of applications to pion physics.

[2] A short account was given in Ref. [5].

[3] See Ref. [6] and references given there.

[4] Ref. [7] gives a comprehensive presentation of the relevant material.

allows no direct conclusion concerning e.g. whether, switching off the source term while leaving the rest of the Lagrangian unchanged, the ground state for the latter would be the usual symmetric or the non-sysmmetric Goldstone one, although this question is meaningful.

The Goldstone situation can in an intuitively appealing way be illuminated in terms of the behaviour of the ground state energy density as a function of the source strength. The discussion hereto, familiar in the classical case, carries over with few changes to quantum field theory, whereby, however, a formal similarity to the theory of condensation of Yang and Lee [8] is noted.

In Section I, the well-known one-particle structure of Green's functions is presented concisely. In Section II, for Green's functions involving current operators that have simple commutators with the fields, Ward-Takahashi-Kazes- and Rivers-type identities are derived. In Section III, the formulas of the first two sections are written for the special case of a symmetric Lagrangian density with added term linear in Bose fields. These relations are used in Section IV to obtain the BPH renormalization conditions in terms of only the number of parameters that appear in the unrenormalized Lagrangian. The same is done, with some necessary precaution to avoid spurious infrared divergences, in Section V for the associated Goldstone mode, i.e. the limit theory with vanishing source and spontaneously broken symmetry, which may also be described directly in terms of a manifestly nonsymmetric Lagrangian. In Section VI, the relation between the theories with and without symmetry breaking source term is discussed and a comparison is made with the breaking of a discrete symmetry. The appendix contains the discussion of the properties of the ground state energy as a function of the source strength, which are obtained using results of Euclidean quantum field theory.

The calculation of the Green's functions involving a current operator will, because of the technique needed hereby, be included in the sequel paper, which deals with symmetry breaking by a term bilinear in fields.

I. One-Particle Structure of Green's Functions

We wish to consider the Poincaré-invariant theory of a multicomponent local hermitean field $A(x)$ described by the Lagrangian density

$$L = L(A, \partial A) . \tag{I.1}$$

To this end we consider, following Schwinger [9], the related theory described by the Lagrangian density

$$L' = L(A', \partial A') + J A' \tag{I.2}$$

where the components of $J(x)$, as far as they multiply Bose fields, are suitably smooth real functions that vanish sufficiently fast in infinity, and, as far as they multiply Fermi fields, elements of a suitably large Grassmann algebra. We will not discuss this latter case [10, 11] in any detail, however, since for Fermi fields the source terms merely serve to mechanize combinatorial manipulations while only in the Bose case we can, and will in Section III, interpret the source term realistically as a change in the physical Lagrangian density.

Let $\rangle$ denote the vacuum of the theory (I.1), and $\rangle'_{\text{in}}$ and $\rangle'_{\text{out}}$ the normalized ground states of the theory (I.2) at very early and very late times, respectively. Upon a suitable choice of the phase factor between these two states, the functional of J

$$G_{\text{disc}}\{J\} = {}_{\text{out}}'\langle\,|\,\rangle'_{\text{in}} \tag{I.3}$$

may be written as

$$\begin{aligned} G_{\text{disc}}\{J\} &= \langle(\exp[i \textstyle\int dx\, J(x)\, A(x)])_+\rangle \\ &= \sum_{n=0}^{\infty} (n!)^{-1}\, i^n \int \cdots \int dx_1 \ldots dx_n\, J(x_1) \ldots J(x_n)\, G_{\text{disc}}(x_1 \ldots x_n) \end{aligned} \tag{I.4}$$

such that it is the generating functional of the Green's functions

$$G_{\text{disc}}(x_1 \ldots x_n) = \langle(A(x_1) \ldots A(x_n))_+\rangle \tag{I.5}$$

of the source-free theory.

The $(\ )_+$-product, also called T^*-product, is for renormalizable theories of spin 0 and spin $\frac{1}{2}$ fields the ordinary T-product, but will in general differ from it by noncovariant "seagull" terms [12–14]. Green's functions can also be defined "axiomatically" i.e. without reference to a Lagrangian, and then are under certain technical assumptions [15, 16] Poincaré invariant generalized functions to the extent that all the later considerations of this section apply also to such theories.

Abbreviating throughout this paper functional derivatives with respect to the argument in curly brackets by subscripts, we have

$$\begin{aligned} &(-i)^n [\delta^n/\delta J(x_1) \ldots \delta J(x_n)]\, G_{\text{disc}}\{J\} \\ &\qquad \equiv (-i)^n\, G_{\text{disc}\, x_1 \ldots x_n}\{J\} = {}_{\text{out}}'\langle(A'(x_1) \ldots A'(x_n))_+\rangle'_{\text{in}}\,. \end{aligned} \tag{I.6}$$

By

$$\begin{aligned} G\{J\} &= \ln G_{\text{disc}}\{J\} \\ &= \sum_{n=1}^{\infty} (n!)^{-1}\, i^n \int \cdots \int dx_1 \ldots dx_n\, J(x_1) \ldots J(x_n)\, G(x_1 \ldots x_n) \end{aligned} \tag{I.7}$$

the generating functional of the connected parts $G(x_1 \ldots x_n)$ of the original Green's functions $G_{\text{disc}}(x_1 \ldots x_n)$ is defined.

We define the amputation operation, indicated by underlining, on a Green's function argument by the convolution with the (in the convolution sense) inverse-propagator matrix $G^{-1}(\ldots)$:

$$G\ldots(\ldots\underline{y}\ldots) \equiv \int dz\, G^{-1}(yz)\, G\ldots(\ldots z\ldots).$$

In view of the momentum-space analytic properties of these Green's functions, which are an essential element in their definition and are due to relativistic invariance and the nonnegativity of the energy spectrum, this operation has an unambiguous meaning[5] also outside of perturbation theory.

We now set, separating out the J-independent term,

$$-i\, G_x\{J\} - G(x) \equiv \mathscr{A}(x) \tag{I.8}$$

whereby J may be determinded as a functional of $\mathscr{A}$ e.g. as a Volterra series by iterative inversion of (I.8):

$$J(x) = -i \int dy\, G^{-1}(x\,y)\, \mathscr{A}(y) + \tfrac{1}{2} i \iint dy\, dz\, G(\underline{x}\,\underline{y}\,\underline{z})\, \mathscr{A}(y)\, \mathscr{A}(z) + \cdots. \tag{I.9}$$

We have

$$\delta \mathscr{A}(x)/\delta J(y) = -i\, G_{xy}\{J\} \tag{I.10a}$$

and therefrom

$$\delta J(x)/\delta \mathscr{A}(y) = i\, G^{-1}_{xy}\{J\} \tag{I.10b}$$

which is symmetric in x and y, such that we may express J as a functional derivative of a functional of $\mathscr{A}$,

$$-i\, J(x) = \Gamma_x\{\mathscr{A}\} \equiv [\delta/\delta\mathscr{A}(x)]\, \Gamma\{\mathscr{A}\}. \tag{I.11}$$

Explicitly, in this Legendre transformation [18]

$$\Gamma\{\mathscr{A}\} = -i \int dx\, J(x)\, [\mathscr{A}(x) + G(x)] + G\{J\} \tag{I.12}$$

where on the right hand side for J (I.9) is to be substituted. We have

$$\Gamma\{\mathscr{A}\} = \sum_{n=2}^{\infty} (n!)^{-1} \int \cdots \int dx_1 \ldots dx_n\, \mathscr{A}(x_1) \ldots \mathscr{A}(x_n)\, \Gamma(x_1 \ldots x_n) \tag{I.13}$$

with $\Gamma(x_1 \ldots x_n)$ the proper (amputated, connected, one-particle-irreducible[6]) n-point vertex function, and

$$\Gamma_{xy}\{\mathscr{A}\} = G^{-1}_{xy}\{J\} \tag{I.14}$$

such that in particular

$$\Gamma(x\,y) = -G^{-1}(xy) = -G(\underline{x}\,\underline{y}) \tag{I.15}$$

[5] However, if subtractions for the propagator are needed, certain conventions concerning these should be followed, cp. Ref. [17].

[6] See Ref. [17] and for a rigorous discussion, Ref. [19].

is the negative inverse matrix to the matrix of the propagators used before in amputation. The formulas

$$-i\,G_x\{J\} - G(x) = \mathscr{A}(x) = i \int dy\, G(x\,y)\, J(y) - i\, G'_x\{J\} \qquad \text{(I.16a)}$$

and

$$i\,\Gamma_x\{\mathscr{A}\} = J(x) = -i \int dy\, G^{-1}(x\,y)\, \mathscr{A}(y) + i\,\Gamma'_x\{\mathscr{A}\} \qquad \text{(I.16b)}$$

where $G'\{\ \}$ and $\Gamma'\{\ \}$ are obtained from the unprimed functionals by dropping terms lower than cubic, show the nonlinear tree-like structures that arise when the $G(\ldots)$ are expressed by the $\Gamma(\ldots)$ and, inversely, the $\Gamma(\ldots)$ by the $G(\ldots)$. We also note

$$-i\,G'_x\{J\} = \int dy\, G(x\,y)\, \Gamma'_y\{\mathscr{A}\} \qquad \text{(I.17)}$$

and emphasize that the vertex functions $\Gamma(\ldots)$ determine all connected Green's functions $G(\ldots)$ with the exception of $G(x)$.

Because of their relevance for some later points, we describe features of the irreducibility concept used here. One-particle irreducibility implies the absence of any poles in Fourier transforms attributable to stable-one-particle states that are created by applying the $A(x)$, the (supposed irreducible set of) fields used in (I.4) or, in our special case, the fields occuring explicitly in the Lagrangian, on the vacuum, but not of poles attributable to stable composite particles. On the other hand, also singularities other than poles are removed that are associated with the mass spectrum of the states obtained by applying the field operators singly on the vacuum. In perturbation theory, one-particle irreducibility implies the absense of any graphs that decompose into two, both having external lines before cutting, by cutting one line, which is the concept relevant in BPH renormalization theory[3]. It is also for the amputated and one-particle irreducible functions, in this sense rather than in others [20, 21] that PCAC leads to simple identities, as we will now see.

II. Ward-Takahashi-Kazes-Rivers Identities

We assume that in the theory envisaged in Section I S hermitean local vector[7] currents $j^\mu_\alpha(x)$, $\alpha = 1 \ldots S$ can be defined such that integrals of their zero components over sufficiently large space volumina transform the fields linearly homogeneously:

$$[j^0_\alpha(x), A(y)]\, \delta(x^0 - y^0) = i\,\tau_\alpha\, A(y)\, \delta(x - y) + \text{S.T.} \qquad \text{(II.1)}$$

where the τ_α are real square matrices acting on the suppressed component index of the field and S.T. ("Schwinger terms", different from equation to

[7] The considerations of this section can with corresponding changes in (II.1), immediately be extended to higher tensorial currents, such as energy-momentum tensor density.

equation) denotes as sum of local space derivative terms

$$\text{S.T.} = \sum_{n_1+n_2+n_3 \geqq 1} O_{n_1 n_2 n_3}(y)\, \partial^{n_1}_{x_1} \partial^{n_2}_{x_2} \partial^{n_3}_{x_3} \delta(x-y)$$

that need not, however, be really meaningful. For the scalar [7] divergences of the currents we write

$$\partial_\mu j^\mu_\alpha(x) \equiv d_\alpha(x)\,. \tag{II.2}$$

From (II.1) follows that multilinear local products of field operators, which do not involve time derivatives in the case the currents are not conserved, such as might be used as interpolating fields [22–24] for composite particles, transform (formally) also linearly homogeneously in analogy to the fields in (II.1) provided sufficiently many such combinations are introduced. Thus, we may take the following considerations to apply irrespective of whether the fields used are "fundamental" or other local ones.

We next assume that the Green's functions involving one current operator can be defined:

$$G^\mu_{\text{disc}\,\alpha}(x, y_1 \ldots y_n) \equiv \langle (j^\mu_\alpha(x)\, A(y_1) \ldots A(y_n))_+ \rangle \tag{II.3}$$

where the $(\ )_+$-product must involve [25] "seagull terms" whenever the Schwinger terms do not vanish for all equal-time commutators $[j^\mu, A]$ but, just like Schwinger terms, "seagull" terms need not be meaningful as the T-product in question need not be meaningful.

We make a corresponding assumption for the divergences, such that

$$G_{\text{disc}\,\alpha}(x, y_1 \ldots y_n) \equiv \langle (d_\alpha(x)\, A(y_1) \ldots A(y_n))_+ \rangle\,. \tag{II.4}$$

Then, from (II.1–4)

$$\begin{aligned} \partial^x_\mu G^\mu_{\text{disc}\,\alpha}(x, y_1 \ldots y_n) &= G_{\text{disc}\,\alpha}(x, y_1 \ldots y_n) \\ &+ i \sum_{\nu=1}^{n} (\tau_\alpha)_\nu\, G_{\text{disc}}(y_1 \ldots y_\nu \ldots y_n)\, \delta(x-y_\nu) + \Delta_{\text{disc}\,\alpha}(x, y_1 \ldots y_n) \end{aligned} \tag{II.5}$$

where the subscript of τ_α has an obvious meaning.

In (II.5), the Δ-term is formally a sum arising from the Schwinger terms in (II.1), the derivative of the "seagull" term connected with (II.3), and the "seagull" term connected with (II.4). This, necessarily covariant and meaningful, sum need not vanish, but has, in the case of a conserved current, be shown [26] to be of the form [8]

$$\begin{aligned}
&\Delta_{\mathrm{disc}\,\alpha}(x, y_1 \dots y_n)\\
&\quad = \partial^x_\varrho \Big\{ \sum_\nu G^{\varrho(1)}_{\mathrm{disc}\,\alpha}(x, y_1 \dots \hat{y}_\nu \dots y_n)\, \delta(x-y_\nu)\\
&\qquad + \sum_\nu \partial^{x\varrho} [G^{\varrho(2)}_{\mathrm{disc}\,\alpha}(x, y_1 \dots \hat{y}_\nu \dots y_n)\, \delta(x-y_\nu)]\\
&\qquad + \sum_{\nu<\lambda} G^{\varrho(3)}_{\mathrm{disc}\,\alpha}(x, y_1 \dots \hat{y}_\nu \dots \hat{y}_\lambda \dots y_n)\, \delta(x-y_\nu)\, \delta(x-y_\lambda) + \cdots \Big\}
\end{aligned} \tag{II.6}$$

such that is can be absorbed in the left hand side of (II.5) by a change of $G^\mu_{\mathrm{disc}\,\alpha}(\dots)$ by meaningful and covariant "seagull" terms.

For a nonconserved current, $\Delta_{\mathrm{disc}\,\alpha}$ need not be of the form (II.6). It is not possible to absorb non-divergence $\Delta^\alpha_{\mathrm{disc}}(x, y_1 \dots y_n)$ parts in $G^\alpha_{\mathrm{disc}}(x, y_1 \dots y_n)$ if for $G^\alpha_{\mathrm{disc}}(x, y_1 \dots y_n)$ a definition independent of (II.5) is given, i.e. directly in terms of the divergences, (II.4). This is in particular so in the case we will discuss in Section III. However, for a renormalizable model of that type, Sections IV and V, $\Delta_{\mathrm{disc}\,\alpha} = 0$. Moreover, recent results of Tung [27] imply the vanishing of Δ-terms in (II.5) for certain more general models in renormalized perturbation theory, the reason being "smoothness" of the current divergence operator in all these models, which also applies to our models of symmetry breaking by bilinear terms considered in the sequel paper. Therefore, we will omit the Δ-term in (II.5) in the following and have for $d_\alpha(x) = 0$ the Ward-Takahashi-Kazes [9] (WTK) identities.

We transscribe (II.5) into generating functionals. With

$$\sum_{n=0}^{\infty} (n!)^{-1}\, i^n \int \cdots \int dy_1 \dots dy_n\, J(y_1) \dots J(y_n) \cdot G^\mu_{\mathrm{disc}\,\alpha}(x, y_1 \dots y_n) \equiv G^\mu_{\mathrm{disc}\,\alpha}(x, J\} \tag{II.7a}$$

and

$$\sum_{n=0}^{\infty} (n!)^{-1}\, i^n \int \cdots \int dy_1 \dots dy_n\, J(y_1) \dots J(y_n) \cdot G_{\mathrm{disc}\,\alpha}(x, y_1 \dots y_n) \equiv G_{\mathrm{disc}\,\alpha}(x, J\} \tag{II.7b}$$

we find

$$\partial_\mu\, G^\mu_{\mathrm{disc}\,\alpha}(x, J\} = G_{\mathrm{disc}\,\alpha}(x, J\} + i\, J(x)\, \tau_\alpha\, G_{\mathrm{disc}\,x}\{J\}\,. \tag{II.8}$$

We introduce the connected Green's functions by their generating functionals

$$G^\mu_\alpha(x, J\} = G_{\mathrm{disc}}\{J\}^{-1}\, G^\mu_{\mathrm{disc}\,\alpha}(x, J\} \tag{II.9a}$$

and

$$G_\alpha(x, J\} = G_{\mathrm{disc}}\{J\}^{-1}\, G_{\mathrm{disc}\,\alpha}(x, J\} \tag{II.9b}$$

[8] Hatted arguments are to be omitted.

[9] See Ref. [28] and references given there.

and obtain

$$\partial_\mu G^\mu_\alpha(x, J\} = G_\alpha(x, J\} + i\, J(x)\, \tau_\alpha\, G_x\{J\}\,. \tag{II.10}$$

We integrate (II.10) over all space-time. If the boundary terms in infinity hereby vanish, a sufficient condition for which is the absence of non-vacuum zero-mass states in the theory, we find, multiplying the result by real numbers u_α and summing over α (using the summation convention and $u_\alpha \tau_\alpha \equiv u \cdot \tau$)

$$\int dx\, J(x)\, u \cdot \tau\, G_x\{J\} = i \int dx\, u_\alpha\, G_\alpha(x, J\}\,.$$

Upon replacing J by $\exp[\lambda u \cdot \tau^T] J$, with λ real, this becomes

$$(\partial/\partial\lambda)\, G\{e^{\lambda u \cdot \tau^T} J\} = i \int dx\, u_\alpha\, G_\alpha(x, e^{\lambda u \cdot \tau^T} J\}$$

whence

$$G\{e^{u \cdot \tau^T} J\} - G\{J\} = i \int_0^1 d\lambda \int dx\, u_\alpha\, G_\alpha(x, e^{\lambda u \cdot \tau^T} J\}\,. \tag{II.11}$$

In particular, if the currents are divergence-free, the Green's functions are invariant (i.e., do not change their values) under transformation on all arguments by the matrices $\exp[u \cdot \tau]$. In the cases of interest, these will form a representation of an S-parameter Lie group, the latter being an invariance group of the theory.

We now set

$$G^\mu_\alpha(x, J\} = G^\mu_\alpha(x,) + \int dy\, G^\mu_\alpha(x, \underline{y})\, [-i\, G_y\{J\} - G(y)] + G^{\mu i}_\alpha(x, J\} \tag{II.12a}$$

and

$$G_\alpha(x, J\} = G_\alpha(x,) + \int dy\, G_\alpha(x, \underline{y})\, [-i\, G_y\{J\} - G(y)] + G^i_\alpha(x, J\}\,, \tag{II.12b}$$

thereby defining Green's functions $G^{\mu i}_\alpha(x, \ldots)$ and $G^i_\alpha(x, \ldots)$ as those parts of the full ones that are one-particle irreducible (in the sense discussed in Section I) between x and the remaining arguments. For the generating functionals, we set

$$G^{\mu i}_\alpha(x, J\} = \Gamma^\mu_\alpha(x, \mathscr{A}\} \tag{II.13a}$$

and

$$G^i_\alpha(x, J\} = \Gamma_\alpha(x, \mathscr{A}\} \tag{II.13b}$$

thereby defining the proper vertex functions $\Gamma^\mu_\alpha(x, \ldots)$ and $\Gamma_\alpha(x, \ldots)$ with one argument corresponding to a current and a divergence operator, respectively. With (I.8), (II.12–13), (II.10) becomes

$$\begin{aligned}\partial_\mu\, G^\mu_\alpha(x, \,) &+ \int dy\, \partial^x_\mu\, G^\mu_\alpha(x, \underline{y})\, \mathscr{A}(y) + \partial_\mu\, \Gamma^\mu_\alpha(x, \mathscr{A}\} \\ &= G_\alpha(x, \,) + \int dy\, G_\alpha(x, \underline{y})\, \mathscr{A}(y) + \Gamma_\alpha(x, \mathscr{A}\} - i[\mathscr{A}(x) + G(x)]\, \tau^T_\alpha\, \Gamma_x\{\mathscr{A}\}\,.\end{aligned}$$

Here the $\mathscr{A}$-independent terms are trivially equal and, moreover, both vanish in a Poincaré-invariant theory. The terms linear in $\mathscr{A}$ cancel because of (I.15) and the special case of (II.10)

$$\partial_\mu^x G_\alpha^\mu(x, y) = G_\alpha(x, y) + i\,\tau_\alpha\, G(x)\, \delta(x-y)\,. \tag{II.14}$$

The higher terms are, with $\Gamma'\{\mathscr{A}\}$ from (I.16b)

$$\begin{aligned}\partial_\mu \Gamma_\alpha^\mu(x, \mathscr{A}\} &= \Gamma_\alpha(x, \mathscr{A}\} - i\,\mathscr{A}(x)\,\tau_\alpha^T\, \Gamma_x\{\mathscr{A}\} - i\,G(x)\,\tau_\alpha^T\, \Gamma_x'\{\mathscr{A}\} \\ &= \Gamma_\alpha(x, \mathscr{A}\} - i[\mathscr{A}(x) + G(x)]\,\tau_\alpha^T\, \Gamma_x\{\mathscr{A}\} - i\int dy\, G(x)\,\tau_\alpha^T\, G^{-1}(x\,y)\,\mathscr{A}(y)\,,\end{aligned} \tag{II.15}$$

which are, if $\Gamma_\alpha = 0$ and $G(x) = 0$, the identities of Rivers [29] (R) for the special case of only one current operator argument in the vertex functions.

Apart from the replacement of τ_α by $-\tau_\alpha^T$ and the presence of the last term, (II.15) has the same form as (II.10), and similar operations as performed on (II.10) lead, if again the boundary terms from space-time integration are assumed to vanish, to (in this equation, the generally infinite values for $\mathscr{A} = 0$ should be subtracted on both sides)

$$\begin{aligned}&\Gamma\{e^{u\cdot\tau}\mathscr{A} + (e^{u\cdot\tau} - 1)\,G\} - \Gamma\{\mathscr{A}\} \\ &\quad = i\int_0^1 d\lambda \int dx\, u_\alpha\, \Gamma_\alpha(x, e^{\lambda u\cdot\tau}\mathscr{A} + (e^{\lambda u\cdot\tau} - 1)\,G\} \\ &\qquad - \iint dx\, dy\, G(x)\, u\cdot\tau^T\, G^{-1}(x\,y) \\ &\qquad \cdot [(u\cdot\tau)^{-1}\,(e^{u\cdot\tau} - 1)\,\mathscr{A}(y) + (u\cdot\tau)^{-1}\,(e^{u\cdot\tau} - 1 - u\cdot\tau)\,G(y)]\,.\end{aligned} \tag{II.16}$$

In particular, if the divergences of all currents are linear in the fields and in addition $G(x) = 0$, then the vertex functions are invariant under the transformations contragredient to those under which the Green's functions are invariant, and in this case (I.11) gives

$$\mathscr{A}(x)\,\{e^{u\cdot\tau} J\} = e^{-u\cdot\tau^T}\,\mathscr{A}(x)\,\{J\}\,. \tag{II.17}$$

(II.16) shows that if the currents are conserved, the $\Gamma(\ldots)$ are not invariant under the linear transformations discussed here if $G(x) \neq 0$, even if the boundary terms in the space-time integration to obtain (II.16) from (II.15) vanish. Inversely, while for the $G(\ldots)$ noninvariance under the transformations considered in the case of conserved currents requires the nonvanishing of boundary terms and, more concretely, the contribution of states of one massless particle [30, 31] in the vacuum expectation values

$$\sum_n \langle j_\alpha^\mu(x)\,|\,n\rangle\, \langle n\,|\,\text{polynomial of the } A\rangle \tag{II.18}$$

at the indicated intermediate state, at the same time the integration of (II.15) may yield vanishing boundary terms provided $G(x) \neq 0$. The reason for this possibly different behaviour at zero momentum of $\Gamma^{\mu}_{\alpha}(\ldots)$ is, cp. (II.12a) and (II.13a), that Γ^{μ}_{α} is, apart from amputation, only the one-particle-irreducible part of G^{μ}_{α}. – The last observations are the essence of Jona-Lasinio's discussion [32] of spontaneously broken symmetry.

III. Green's Functions and WTKR Identities in the Presence of Source Term

If we drop the requirement that the source-free theory (I.1) be Poincaré invariant, but assume that asymptotic Poincaré invariance holds (i.e. at very late and very early times, at least formally achieved by letting the coefficients of the terms in the Lagrangian density then take constant, though not necessarily the same, values), then the Green's functions are uniquely defined only up to a common factor such as one was fixed for the theory (I.2). Denoting the Green's functions for the theory where a linear term KA is taken as part of the Lagrangian density

$$L^K = L(A, \partial A) + KA \tag{III.1}$$

by $G^K(\ldots)$, physically interpretable only for a pure Bose source, the choice of factor such that

$$G^K_{\text{disc}}\{J\} = G_{\text{disc}}\{K\}^{-1}\, G_{\text{disc}}\{J+K\} \tag{III.2}$$

is particularly convenient since it allows to let K become time- and even space-time-independent in a limiting process in which $G_{\text{disc}}\{J+K\}$ itself has no limit. The relation that is with $G^K\{0\}=0$ equivalent to (III.2)

$$\begin{aligned} G^K_x\{J\} &= G_x\{J+K\} \\ &= \sum_{n=0}^{\infty} (n!)^{-1} \int \cdots \int dy_1 \ldots dy_n\, K(y_1) \ldots K(y_n)\, G_{x y_1 \ldots y_n}\{J\} \end{aligned} \tag{III.3}$$

is usuable also for space-time independent K and yields a solution of the relevant functional differential equations [9] of the theory (III.1). We defer further discussion of the case of space-time independent source to the Appendix and here take simply (III.3) as our starting point with, however, particular attention to the possibility

$$\lim_{K\to 0} G^K_x\{J\} \neq G_x\{J\} \tag{III.4}$$

where, if this limit is not unique, $G_x\{J\}$ is even not expected to exist.

For the theory (III.1), from (III.2) and (I.8) we have immediately

$$\begin{aligned}\mathscr{A}^K(x)\{J\} &= -i\,G_x^K\{J\} - G^K(x)\\ &= -i\,G_x\{J+K\} + i\,G_x\{K\}\\ &= \mathscr{A}(x)\{J+K\} - \mathscr{A}(x)\{K\}\end{aligned} \tag{III.5}$$

with the now necessary indication of the source functions the $\mathscr{A}$ refer to. We will use the abbreviation

$$\mathscr{K}(x) \equiv \mathscr{A}(x)\{K\} = -i\,G_x\{K\} - G(x) \tag{III.6}$$

such that, from (I.11),

$$\Gamma_x\{\mathscr{K}\} = -i\,K(x)\,. \tag{III.7}$$

Now (I.11) and (III.5) give

$$\begin{aligned}\Gamma_x^K\{\mathscr{A}^K\{J\}\} &= -i\,J(x) = \Gamma_x^K\{\mathscr{A}\{J+K\} - \mathscr{K}\}\\ &= -i(J(x)+K(x)) + i\,K(x) = \Gamma_x\{\mathscr{A}\{J+K\}\} - \Gamma_x\{\mathscr{K}\}\,.\end{aligned}$$

Because of the arbitrariness of J and, consequently, of $\mathscr{A}\{J+K\}$, it follows that for all $\mathscr{A}$

$$\Gamma_x^K\{\mathscr{A}\} = \Gamma_x\{\mathscr{A}+\mathscr{K}\} - \Gamma_x\{\mathscr{K}\} \tag{III.8}$$

or

$$\Gamma^K\{\mathscr{A}\} = \Gamma\{\mathscr{A}+\mathscr{K}\} - \int dx\,\mathscr{A}(x)\,\Gamma_x\{\mathscr{K}\} - \Gamma\{\mathscr{K}\} \tag{III.9}$$

which indeed satisfies (I.12) for the theory (III.1). (III.8) means

$$\begin{aligned}\Gamma^K(x_1 \ldots x_n) = \sum_{l=0}^{\infty} (l!)^{-1} \int\cdots\int dy_1 \ldots dy_l\\ \cdot\,\mathscr{K}(y_1) \ldots \mathscr{K}(y_l)\,\Gamma(x_1 \ldots x_n\, y_1 \ldots y_l) \quad (n \geqq 2)\,,\end{aligned} \tag{III.10}$$

i.e. the vertex functions of the theory (III.1) are simple linear combinations of the infinitely many vertex functions of the theory (I.1). In particular,

$$\Gamma^K(x\,y) = \Gamma_{xy}\{\mathscr{K}\} \tag{III.11}$$

or, more explicitly, in view of (I.15),

$$\begin{aligned}(G^K)^{-1}(x\,y) = G^{-1}(x\,y) - \int dz\,\mathscr{K}(z)\,\Gamma(x\,y\,z)\\ - \tfrac{1}{2}\iint dz\,du\,\mathscr{K}(z)\,\mathscr{K}(u)\,\Gamma(x\,y\,z\,u) - \cdots\,.\end{aligned} \tag{III.12}$$

For use in the next sections, we write the WKTR identities for the theory (III.1) in the special case that the currents of the theory (I.1) are conserved. Since the additional term in (III.1) does not involve derivatives, it is natural to choose for that theory the current operators of the same functional form as for the ones of the theory (I.1). (III.2) gives

$$G^K_{\mathrm{disc}\,x_1\ldots x_n}\{J\} = G_{\mathrm{disc}}\{K\}^{-1}\, G_{\mathrm{disc}\,x_1\ldots x_n}\{J+K\} \tag{III.13}$$

such that forming the Green's functions involving one current operator by appropriate differentiations and limiting processes, involving space-like distances, on the right hand side of (III.13) will give on the left hand side the analogous construction, provided the prescription to be followed on the right hand side of (III.13) requires no explicit reference to $J+K$. This will be so for the familiar internal-symmetry currents having the property (II.1) in the usual renormalizable models [10] [27], in particular the one of Sections IV and V.

Thus

$$G^{K\mu}_{\mathrm{disc}\,\alpha}(x, J\} = G_{\mathrm{disc}}\{K\}^{-1}\, G^{\mu}_{\mathrm{disc}\,\alpha}(x, J+K\}$$

or, using (II.9a) and (III.2),

$$G^{K\mu}_{\alpha}(x, J\} = G^{\mu}_{\alpha}(x, J+K\}\,. \tag{III.14}$$

Since by assumption

$$G_\alpha(x, J+K\} = 0$$

(II.10) and (III.14) give

$$\partial_\mu\, G^{K\mu}_{\alpha}(x, J\} = i[J(x)+K(x)]\,\tau_\alpha\, G^K_x\{J\} \tag{III.15}$$

and thus, comparing with (II.10)

$$G^K_\alpha(x, J\} = i\,K(x)\,\tau_\alpha\, G^K_x\{J\} \tag{III.16a}$$

which means, for the theory (III.1),

$$\partial_\mu j^\mu_\alpha(x) = -K(x)\,\tau_\alpha\, A(x) \tag{III.16b}$$

and in particular, with (III.6)

$$G^K_\alpha(x,\ ) = \partial_\mu\, G^{K\mu}_\alpha(x,\ ) = -K(x)\,\tau_\alpha\,[\mathscr{K}(x)+G(x)] \tag{III.17}$$

Furthermore, from (III.15)

$$\partial^x_\mu\, G^{K\mu}_\alpha(x, y) = i\,G^K(x)\,\tau^T_\alpha\,\delta(x-y) - K(x)\,\tau_\alpha\, G^K(x\,y) \tag{III.18}$$

or, with (III.3) and (III.6)

$$\begin{aligned}\partial^x_\mu\, G^{K\mu}_\alpha(x, \underline{y}) = & -K(x)\,\tau_\alpha\,\delta(x-y)\\ & + i[\mathscr{K}(x)+G(x)]\,\tau^T_\alpha\,(G^K)^{-1}(x\,y)\,.\end{aligned} \tag{III.19}$$

[10] This holds also in quantum electrodynamics provided, instead of the more common external field, an external current is used.

(III.17) and (III.19) allow to write (III.15) as

$$\begin{aligned}
&\partial^x_\mu\{G^{K\mu}_\alpha(x,J\} - G^{K\mu}_\alpha(x,\ )\\
&\qquad - \int dy G^{K\mu}_\alpha(x,\underline{y})\,[-i\,G^K_y\{J\} - G^K(y)]\}\\
&\quad = i\,J(x)\,\tau_\alpha\,G^K_x\{J\}\\
&\qquad - \int dy[\mathscr{K}(x) + G(x)]\,\tau^T_\alpha (G^K)^{-1}(x\,y)\,[G^K_y\{J\} - i\,G^K(y)]\,.
\end{aligned}$$

Due to (II.12a) and (II.13a), this means

$$\partial_\mu\,\Gamma^{K\mu}_\alpha(x,\mathscr{A}\} = -i\,\mathscr{A}(x)\,\tau^T_\alpha\,\Gamma^K_x\{\mathscr{A}\} - i\,G^K(x)\,\tau^T_\alpha\,\Gamma^{K'}_x\{\mathscr{A}\} \tag{III.20}$$

such that, as comparison with (II.15) shows

$$\Gamma^K_\alpha(x,\mathscr{A}\} = 0 \tag{III.21}$$

as expected since, the divergence (III,16b) being linear in the field, removal of the source-independent and one-particle irreducible part from $G^K_\alpha(x,J\}$ leaves zero remainder. Integration of (III.20) yields with (I.16b) and the integrated form of (III.19), if in both integrations no boundary terms arise,

$$\int dx[\mathscr{A}(x) + G^K(x)]\,\tau^T_\alpha\,\Gamma^K_x\{\mathscr{A}\} = i\int dx\,K(x)\,\tau_\alpha\,\mathscr{A}(x)\,. \tag{III.22}$$

The formula (II.16), written for the theory (III.1), gives due to (III.21) simple transformation laws for the vertex functions Γ^K (of at least two arguments, those of less arguments vanishing identically according to (I.13)). The condition for applicability of (II.16) to (III.20) with (III.21), discussed at the end to Section II, gives: If the divergences of the currents are linear in the fields and the currents such that the corresponding "charges" (or, rather, integrals of the density over sufficiently large space volumina) transform the fields linearly homogeneously, and if in the matrix element (II.18) the mass spectrum of the intermediate states does not reach to zero after removal of the one-particle-reducible parts (in the sense of Section I), then invariance of the vertex functions and of the connected Green's functions with two or more arguments under the transformations induced by the "charges" is equivalent to the vanishing of the vacuum expectation values of the fields.

The restriction on the residual mass spectrum in the last statement is necessary in view of the possibility of "composite Goldstone particles" [7] i.e. massless particles associated with spontaneous symmetry breaking but not contained in the states obtained by applying single field operators on the vacuum, the case discussed in the sequel paper.

IV. Renormalization of a Model with Symmetry-Breaking Source Term

Following Lee [1], we consider the model of an $N+1$-tuplet of scalar hermitean fields $A_i(x)$, $i = 0 \ldots N$, with the Lagrangian density, in terms of unrenormalized fields and parameters (indicated by the subscript u) and written in ordinary rather than Wick products[11, 12]

$$L=\tfrac{1}{2}\,\partial_\mu A_{iu}\,\partial^\mu A_{iu}-\tfrac{1}{2}\,m_u^2\,A_{iu}A_{iu}-\tfrac{1}{8}\,g_u(A_{iu}A_{iu})^2+c_u\,A_{0u}\,. \qquad \text{(IV.1)}$$

We note that (IV.1) has the form (III.1) and thus the considerations of Section III apply to it, if for the currents we take the $\binom{N+1}{2}$ appropriate to the $O(N+1)$-invariant part of (IV.1); in addition, we may exploit Poincaré invariance.

From now on, we let Latin indices run from 1 to N, writing the zeroth component separately. We introduce the same amplitude renormalization factor Z_3 for all fields, leaving its specification till later, and introduce renormalized quantities

$$A_0=Z_3^{-\frac{1}{2}}\,A_{0u} \qquad \text{(IV.2a)}$$

$$A_i=Z_3^{-\frac{1}{2}}\,A_{iu} \qquad (i=1\,\ldots\,N)\,, \qquad \text{(IV.2b)}$$

$$c=Z_3^{\frac{1}{2}}\,c_u\,. \qquad \text{(IV.2c)}$$

(IV.1) yields $\binom{N}{2}$ conserved currents[13]

$$j^\mu_{ij}=A_{iu}\,\overleftrightarrow{\partial}^\mu A_{ju}=Z_3\,A_i\,\overleftrightarrow{\partial}^\mu A_j \qquad (i<j)\,, \qquad \text{(IV.3a)}$$

$$\partial_\mu j^\mu_{ij}=0 \qquad \text{(IV.3b)}$$

and N partially conserved currents

$$j^\mu_i=A_{0u}\,\overleftrightarrow{\partial}^\mu A_{iu}=Z_3\,A_0\,\overleftrightarrow{\partial}^\mu A_i\,, \qquad \text{(IV.4a)}$$

$$\partial_\mu j^\mu_i\equiv d_i=-c_u\,A_{iu}=-c\,A_i \qquad \text{(IV.4b)}$$

with the commutation relations with the fields

$$[j^0_{ij}(x),A_k(y)]\;\delta(x^0-y^0)=i(\delta_{ik}\,A_j(y)-\delta_{jk}\,A_i(y))\;\delta(x-y)+\text{S.T.}\,, \qquad \text{(IV.5a)}$$

$$[j^0_{ij}(x),A_0(y)]\;\delta(x^0-y^0)=\text{S.T.} \qquad \text{(IV.5b)}$$

$$[j^0_i(x),A_j(y)]\;\delta(x^0-y^0)=-i\,\delta_{ij}\,A_0(y)\;\delta(x-y)+\text{S.T.}\,, \qquad \text{(IV.6a)}$$

$$[j^0_i(x),A_0(y)]\;\delta(x^0-y^0)=i\,A_i(y)\;\delta(x-y)+\text{S.T.} \qquad \text{(IV.6b)}$$

---

[11] If one prefers, one may here use "symmetric" Wick contractions, which are, however, like "true" Wick contractions, not sufficient to give the field equations bona fide meaning. The transformation properties of the terms in the Lagrangian or in the field equations must, however, be explicit rather than clouded by, in this context, extraneous, conventions.

[12] $g^{\mu\nu}=\mathrm{diag}(1,\,-1,\,-1,\,-1)$.

[13] $\overleftrightarrow{\partial}=-\overleftarrow{\partial}+\overrightarrow{\partial}$.

where we need not discuss whether the Schwinger terms are zero, finite, or meaningless.

The relations (IV.3 b), (IV.5) are those of the remaining $O(N)$ symmetry, which in the following we assume not to be broken spontaneously, while (IV.4 b), (IV.6) yield for

$$G_{i_1\ldots i_l}(x_1 \ldots x_k, y_1 \ldots y_l) \equiv \langle (A_0(x_1) \ldots A_0(x_k) A_{i_1}(y_1) \ldots A_{i_l}(y_l))_+ \rangle$$

from (II.10) or (III.15) the identities

$$\begin{aligned} ic \int dz\, G_{i_1\ldots i_l j}&(x_1 \ldots x_k, y_1 \ldots y_l z) \\ &= \sum_{\lambda=1}^{l} \delta_{i_\lambda j}\, G_{i_1\ldots \hat{i}_\lambda \ldots i_l}(x_1 \ldots x_k y_\lambda, y_1 \ldots \hat{y}_\lambda \ldots y_l) \\ &\quad - \sum_{\kappa=1}^{k} G_{i_1\ldots i_l j}(x_1 \ldots \hat{x}_\kappa \ldots x_k, y_1 \ldots y_l x_\kappa) \\ &= F \int dz\, G_{i_1\ldots i_l j}(x_1 \ldots x_k, y_1 \ldots y_l \underline{z}) \end{aligned} \tag{IV.7}$$

with

$$F \equiv \langle A_0(x) \rangle, \tag{IV.8}$$

where the second equality in (IV.7), which is also directly obtained by integrating the equation preceding (III.20), is an immediate consequence of the first and its special case $k=0$, $l=1$. In (IV.7) it is assumed that the mass spectrum in the intermediate state indicated in (II.18) does not reach to zero, a condition that can be waived for the second equality in (IV.7), due to the amputation which allows a smooth transition to the zero-mass-state case, if we expressly exclude the occurrence of "composite Goldstone particles", cp. Section III. Under this same provision, (III.22) yields

$$\begin{aligned} F \int dz\, \Gamma_{i_1\ldots i_l j}&(x_1 \ldots x_k, y_1 \ldots y_l z) \\ &= \sum_{\lambda=1}^{l} \delta_{j i_\lambda}\, \Gamma_{i_1\ldots i_\lambda \ldots i_l}(x_1 \ldots x_k y_\lambda, y_1 \ldots \hat{y}_\lambda \ldots y_l) \\ &\quad - \sum_{\kappa=1}^{k} \Gamma_{i_1\ldots i_l j}(x_1 \ldots \hat{x}_\kappa \ldots x_k, y_1 \ldots y_l x_\kappa) \\ &\quad - ic\, \delta_{k0}\, \delta_{l1}\, \delta_{i_1 j} \end{aligned} \tag{IV.9}$$

with $\Gamma\ldots(\ldots)$ defined in analogy to G ...(...).

The second equality in (IV.7) in the off-shell form of the Adler self consistency condition [33] and, as remarked, is usable even if $c=0$ such that the first term in (IV.7) is meaningful, namely zero, only if the z integral exists i.e. there are no Goldstone particles. (IV.9) are the corresponding relations for vertex functions and, if one whishes, show how (IV.7) comes about if one thinks in terms of "pole diagrams" i.e. one-particle structure (in the sense of Section I).

We proceed to the construction of the renormalized perturbation theoretical expansions of the Green's functions in the manner of BPH [3]. First we determine the superficial divergence degrees of the vertex parts. In view of the last term in (IV.1), the other bare vertices obtained from (IV.1) may have σ-legs [14] disappear in the vacuum, and since such legs are disregarded in one-particle structure in the sense of Section I where all their effects are accounted for by the $G(x)$ term in (I.8), in addition to the bare vertices of the symmetric theory σ^4, $\sigma^2\pi\pi$, $\pi\pi\pi\pi$, π^2, σ^2 also the vertices σ^3 and $\sigma\pi\pi$ are elementary ones, and nonderivative ones like the first two usual ones they may be considered as being derived from.

Let N_4 and N_3 be the number of 4-and of 3-vertices, respectively, and E the one of external, I the one of internal lines, and L the number of loop integrations. Then

$$4N_4 + 3N_3 = 2I + E, \tag{IV.10a}$$

$$N_4 + N_3 = I - L + 1 \tag{IV.10b}$$

and the superficial divergence degree is

$$D \equiv 4L - 2I = 4 - E - N_3 . \tag{IV.10c}$$

Thus, $D=0$ for $E=4$ and $E=3$, since N_3 is odd if $E=3$. For $E=2$, $D=2$, while for $E=1$, $D=2$ because N_3 is odd, but these one-leg vertex parts play no other role than that noted above.

We need values for all superficially divergent vertex functions at arbitrary subtraction points in momentum space. The Lagrangian (IV.1) contains the parameters m_u^2, g_u, c_u such that we must have only three independent parameters in the renormalized perturbation expansion, and therefore there must be relations between the values of various vertex functions. The identities (IV.9) provide us with precisely all required relations and with no more, since the information contained in (IV.9) beyond the relation between the renormalization conditions for the superficially divergent vertex functions is not independent of the one contained in the general interrelation between the vertex functions in the form of their usual nonlinear integral equations [9].

The relevant information from (IV.9) is most easily extracted by choosing as subtraction points the ones of all momenta zero. We define

$$\begin{aligned}\int\cdots\int dx_1 \ldots dx_k\, dy_1 \ldots dy_l\, \Gamma_{i_1\ldots i_l}(x_1 \ldots x_k, y_1 \ldots y_l)\\ \cdot \exp[i p_1 x_1 + \cdots + i p_k x_k + i q_1 y_1 + \cdots + i q_l y_l]\\ \equiv (2\pi)^4\, \delta(p_1 + \cdots + q_l)\, \tilde{\Gamma}_{i_1\ldots i_l}(p_1 \ldots p_k, q_1 \ldots q_l)\,.\end{aligned}$$

[14] For conciseness, we call the A_0 field and associated particles σ-field and -particles, the A_i field and -particles π-field and pions.

Then (IV.9) gives

$$
\begin{aligned}
F\,\tilde{\Gamma}_{i_1\dots i_l j}(p_1\dots p_k, q_1\dots q_l 0) &= \sum_{\lambda=1}^{l} \delta_{i_\lambda j}\,\tilde{\Gamma}_{i_1\dots \hat{i}_\lambda \dots i_l}(p_1\dots p_k q_\lambda, q_1\dots \hat{q}_\lambda \dots q_l) \\
&\quad - \sum_{\kappa=1}^{k} \tilde{\Gamma}_{i_1\dots i_l j}(p_1\dots \hat{p}_\kappa \dots p_k, q_1\dots q_l p_\kappa) - ic\,\delta_{k0}\,\delta_{l1}\,\delta_{i_1 j}\,.
\end{aligned}
\tag{IV.11}
$$

We furthermore set, in view of the preserved $0(N)$ symmetry that also implies the vanishing of all Γ with an odd number of π-arguments,

$$
\tilde{\Gamma}_{i_1\dots i_{2l}}(0\dots 0, 0\dots 0) = \Gamma_{k,2l} \sum_{(2l-1)!!\ \text{pairings}} \delta_{i_{\varrho_1} i_{\varrho_2}} \cdots \delta_{i_{\varrho_{2l-1}} i_{\varrho_{2l}}}
$$

and obtain from (IV.11)

$$
F\,\Gamma_{k,2l+2} = \Gamma_{k+1,2l} - k\,\Gamma_{k-1,2l+2} - ic\,\delta_{k0}\,\delta_{l0} \qquad (k \geqq 0, l \geqq 0)\,. \tag{IV.12}
$$

To solve this, we introduce

$$
\Gamma(x, y) = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} (k!\,l!)^{-1}\, x^k y^l\, \Gamma_{k,2l} \tag{IV.13}
$$

for which (IV.12) takes the form

$$
F\,\Gamma_y = \Gamma_x - x\,\Gamma_y - ic
$$

with the general solution

$$
\Gamma(x, y) = icx + \sum_{l=0}^{\infty} (l!)^{-1}\, a_l (xF + y + \tfrac{1}{2}x^2)^l\,. \tag{IV.14}
$$

The properties

$$
\Gamma_{0,0} = \Gamma_{1,0} = 0 \tag{IV.15}
$$

yield

$$
a_0 = 0, \qquad a_1 = -icF^{-1}\,.
$$

We set

$$
\Gamma_{2,0} = -iM^2, \qquad \Gamma_{0,2} = -i\mu^2, \qquad \Gamma_{0,4} = -ig; \tag{IV.16}
$$

then $a_1 = -i\mu^2$, $a_2 = -ig$ such that

$$
c = F\mu^2\,, \tag{IV.17}
$$

$$
M^2 = \mu^2 + gF^2\,, \tag{IV.18}
$$

and

$$
\Gamma_{1,2} = -igF = -i[g(M^2 - \mu^2)]^{\frac{1}{2}}\,. \tag{IV.19}
$$

The $\Gamma_{k,2l}$ with $k + l \geqq 3$ require to dermine the a_r, $3 \leqq r \leqq k + l$; however, these are all obtainable from e.g. the vertex functions $\Gamma_{0,2r} = a_r$ which need no final subtraction. To calculate them, we must also determine the two negative inverse one-particle propagators $\tilde{\Gamma}(p(-p),\ ) \equiv \Gamma_\sigma(p^2)$ and $\tilde{\Gamma}_{ij}(\ , q(-q)) \equiv \delta_{ij}\,\Gamma_\pi(q^2)$. We set, for convenience, hereby fixing Z_3 in (IV.2),

$$(\partial/\partial q^2)\,\Gamma_\pi(q^2)|_{q^2=0} = i\,. \tag{IV.20}$$

Then the special case of (IV.9)

$$N^{-1} F\,\tilde{\Gamma}_{ii}(q,(-q)0) = \Gamma_\sigma(q^2) - \Gamma_\pi(q^2) \tag{IV.21}$$

yields

$$(\partial/\partial p^2)\,\Gamma_\sigma(p^2)|_{p^2=0} = i + N^{-1}\,F(\partial/\partial q^2)\,\tilde{\Gamma}_{ii}(q,(-q)\,0)|_{q^2=0} \tag{IV.22}$$

whereof the last term, which is of order g and higher, can be calculated without final subtraction. (IV.22) is the required renormalization condition for the σ-propagator upon the choice (IV.20) for the π-propagator. Of course, it is more convenient to calculate the σ-propagator directly from (IV.21) with (IV.19).

The system of renormalization conditions is now complete. Simple counting of orders on the basis of (IV.10) gives the result that in (IV.14)

$$a_r = g^r \sum_{L=1}^{\infty} \alpha_{r,L}\, g^{L-1} \quad (r \geqq 3) \tag{IV.23}$$

where the $\alpha_{r,L}$ are finite and only μ^2- and M^2-dependent, and L is the number of loop integrations whereby also self energy loops must be counted, while loops without external lines and attached to the rest of the diagram only through one vertex do not occur in the BPH scheme. The condition of one-particle-irreducibility of vertex diagrams is the reason why the sum in (IV.23) starts with $L=1$ rather than with $L=0$, which is the crux of the present procedure [15].

It remains to justify that the relations (IV.9) are exhausted by the relations (IV.17–19) and (IV.22), given the usual nonlinear integral equations, derived from the field equations, for vertex functions. This can be proven by adaption of Brandt's proof [34], in renormalized perturbation theory in quantum electrodynamics, of the WTK identities at zero momentum of the electromagnetic current operator argument; the analog of that operator is here the right hand side of the field equation for the pion field. The details are laborious and will not be given here; however, the calculation of Green's and vertex functions with one current operator argument in our model is of interest and will be given, because

[15] This method of calculation, where the WTKR relations are central, differs somewhat from the one of Ref. [1] where the PCAC condition was merely used as a check, the symmetry of the original Lagrangian, cp. Ref. [39], being invoked by the manner of regularization.

the necessary formalism is d oped there, in the sequel paper on symmetry breaking by terms bilinear in the fields.

Finally, we briefly describe the transition from the present renormalized functions to the more conventional ones.

One introduces

$$\Gamma_{\mathrm{ren}\,\sigma}(p^2) = Z_\sigma(m_\sigma^2, m_\pi^2, g_{\mathrm{ren}})\, \Gamma_\sigma(p^2)\,,$$
$$\Gamma_{\mathrm{ren}\,\pi}(q^2) = Z_\pi(m_\sigma^2, m_\pi^2, g_{\mathrm{ren}})\, \Gamma_\pi(q^2)$$

such that [16]

$$\mathrm{Im}\,\Gamma_{\mathrm{ren}\,\sigma}(m_\sigma^2) = 0\,, \tag{IV.24a}$$
$$\mathrm{Im}(\partial/\partial p^2)\,\Gamma_{\mathrm{ren}\,\sigma}(p^2)|_{p^2=m_\sigma^2} = 1\,, \tag{IV.24b}$$
$$\Gamma_{\mathrm{ren}\,\pi}(m_\pi^2) = 0\,, \tag{IV.25a}$$
$$(\partial/\partial q^2)\,\Gamma_{\mathrm{ren}\,\pi}(q^2)|_{q^2=m_\pi^2} = i \tag{IV.25b}$$

with

$$m_\sigma^2 = M^2 + \delta m_\sigma^2(m_\sigma^2, m_\pi^2, g_{\mathrm{ren}})\,,$$
$$m_\pi^2 = \mu^2 + \delta m_\pi^2(m_\sigma^2, m_\pi^2, g_{\mathrm{ren}})$$

where g_{ren} is some conventional replacement of g in (IV.16). These relations and the convention concerning g_{ren} allow to calculate Z_σ, Z_π, δm_σ^2 and δm_π^2 as expansions in g_{ren} with finite, m_σ^2- and m_π^2- dependent coefficients. Alternatively, one may construct the conventionally renormalized functions directly, using again (IV.11), a procedure resembling the one of Section V.

We remark that μ^2 and M^2 are positive [36] but otherwise arbitrary, except for $\mathrm{sign}\,g = \mathrm{sign}(M^2 - \mu^2)$ since F in (IV.16) must be real. However, $\mathrm{sign}(m_\sigma^2 - m_\pi^2) = \mathrm{sign}\,g$ or $= \mathrm{sign}\,g_{\mathrm{ren}}$ are required only for the perturbation theoretical expansion to be possible and thus need not hold rigorously. Note also that, while g_u in (IV.1) should be nonnegative [37], no rigorous result concerning the sign of g or g_{ren} is known.

The formula from (IV.17) and (IV.18),

$$c = \mu^2 [(M^2 - \mu^2)/g]^{\frac{1}{2}} \tag{IV.26}$$

shows that $c = 0$, i.e. the theory at least formally corresponding to a $O(N+1)$-symmetric Lagrangian, can be realized either by $\mu^2 \to M^2$, $F \to 0$, which yields an $O(N+1)$ symmetric renormalized theory, or by $\mu^2 \to 0$, $F \nrightarrow 0$, a merely $O(N)$-symmetric renormalized theory, within our approach. We defer a discussion of this alternative to Section VI and treat the second case in the next section.

[16] (IV.24) allows $m_\sigma^2 \geqq 4m_\pi^2$ i.e. the σ particle to be unstable. We are adopting the convention of Ref. [35].

V. Renormalization in the Case of Spontaneously Broken Symmetry

If in the formulas of Section IV one tries to perform the limes $\mu^2 \to 0$, $c \to 0$, with g and M^2 kept fixed, one encounters "infrared" (UR) divergences. To order g, Γ_π as calculated from the diagram of Fig. 1 imposing (IV.16)

Fig. 1

and (IV.20) with $\mu^2 = 0$, is easily found to be finite. Γ_σ is to order g simplest calculated from (IV.21), which requires to calculate $N^{-1}\,\tilde{\Gamma}_{ii}(p, (-p)\,0)$, to order $g^{\frac{3}{2}}$. The diagrams are shown in Fig. 2, a zero denoting the external line carrying zero momentum, the calculation to be done with

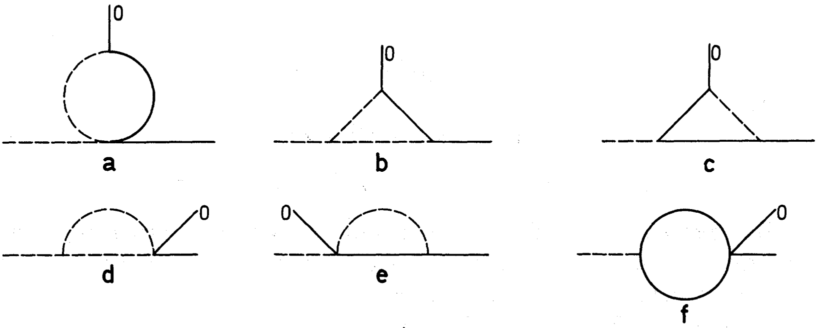

Fig. 2

one subtraction at zero momentum imposing (IV.19). Before subtraction, the Feynman integrals are all UR-convergent at any momentum different from zero. The subtracted term is, due to zero momentum, UR-divergent for the diagrams $2c$ and $2f$, and thus the renormalized result is UR-divergent, the reason being the absence of the diagram Fig. 3 which would

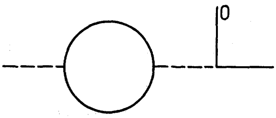

Fig. 3

have led to cancellation of the UR-divergence as we shall see later. Similarly, $\tilde{\Gamma}_{i_1 \ldots i_4}(\;, q_1 \ldots q_4)$ is UR-divergent in order g^2, since it is to be calculated by one subtraction at zero momentum from a set of diagrams from which those that would have led to UR-finiteness of the subtracted value at momenta zero are, as for Fig. 2, excluded by the condition that the diagrams be vertex diagrams. Thus, finite M^2 and g are for $\mu^2 = ($ in contradiction to finite renormalized Green's function, at least in perturbation theory.

The obvious remedy is to choose different subtraction points. While we could use the conventional renormalization discussed in connection with (IV.24–25), due to the form of (IV.11) the following choice of subtraction point for $\Gamma \ldots (p_1 \ldots p_k, q_1 \ldots q_l)$ is more convenient:

$$\begin{aligned} p_i^2 = q_j^2 &= -p_i p_j(k+l-1) = -p_i q_j(k+l-1) \\ &= -q_i q_j(k+l-1) = -\lambda^2 \qquad (\forall i, \forall j). \end{aligned} \tag{V.1}$$

This point (or manifold of points) is with $\lambda^2 > 0$ for $k+l \leqq 4$ consistent and real and realizable even with Euclidean (namely, pure space) momenta. For $k=0$, $l=2$, i.e. $\Gamma_\pi(q^2)$, we choose, however,

$$\Gamma_\pi(0) = 0, \tag{V.2a}$$

$$(\partial/\partial q^2)\, \Gamma_\pi(q^2)|_{q^2=0} = i. \tag{V.2b}$$

We set

$$\Gamma_\sigma(-\lambda^2) = -i(\bar{M}^2 + \lambda^2), \tag{V.3}$$

$$\tilde{\Gamma}_{i_1 \ldots i_4} \text{ (symmetry point)} = -i\bar{g}(\delta_{i_1 i_2}\, \delta_{i_3 i_4} + \cdots + \cdots), \tag{V.4}$$

the bar being introduced to avoid confusion with the earlier symbols. Renormalization now proceeds without any obstacle, some former exact identities, where all momenta were zero, now being replaced by corrected ones, due to the fact that on the left hand side of (IV.11) the momenta are not at the symmetry point if they are on the right hand side. E.g. (IV.18) becomes replaced by

$$\bar{M}^2(1 + \alpha_1 \bar{g} + \cdots) = F^2 \bar{g}(1 + \beta_1 \bar{g} + \cdots) \tag{V.5}$$

and (IV.19) by

$$\tilde{\Gamma}_{1,2} \text{ (symmetry point)} = -iF\bar{g}(1 + \gamma_1 \bar{g} + \cdots) \tag{V.6}$$

with finite $\alpha_1, \ldots, \beta_1, \ldots, \gamma_1, \ldots$.

There are now, for general momenta, no UR divergences in any diagram in spite of the zero pion momenta occurring in (IV.11) and (V.2). The reason is that, due to the $O(N)$ symmetry, the pions are only emitted or absorbed in pairs rather than singly. For this reason also the S-matrix is not UR-divergent, the state space [35] being the one of massless pions only.

Using results of many-particle structure analysis of Green's functions [38], one finds that the Adler self consistency conditions (IV.7), used for appropriate sums of subdiagrams within vertex diagrams, additionally improve on the UR-convergence of Feynman integrals for suitable sums of vertex diagrams. This is the cancellation mentioned before in connection with the three- and four-point vertices, where the missing terms would have completed the original Green's functions for which (IV.7) holds. While scattering amplitudes vanish due to (IV.7) if any pion momentum is zero, vertex functions remain singular if σ-momenta or certain sums of pion-momenta are zero; our renormalization convention was chosen just to avoid those singularities and to make best use of the remaining $O(N)$ symmetry.

The theory has two parameters, $\bar{M}^2$ and $\bar{g}$, the finite quantity F being given by (V.5), with λ^2 occurring merely as a dummy parameter in the sense that change of λ^2 can be compensated by obvious finite changes of $\bar{M}^2$ and $\bar{g}$, $\lambda^2=0$ being excluded. This two-parameter theory is the limit as $\mu^2 \to 0$ of the three-parameter theory of Section IV as can be made manifest by using also there subtractions at the symmetry points (V.1). Then $\mu^2 \to 0$ meets no difficulties, and $m_\pi^2 \to 0$ would meet none in the conventional renormalization (IV.24–25). (IV.18) shows that, due to the finiteness of F and the merely finite changes in the common normalization of the π- and σ-field, compared to our altered normalization convention, M^2 and g diverge (at least in perturbation theory) in the same manner when $\mu^2 \to 0$.

The Goldstone mode of the theory (IV.1) can be formulated in terms of a Lagrangian density that makes possible the perturbation theoretical construction in the usual elementary way. Writing

$$A_{0u} = \sigma + f\,, \tag{V.7a}$$

$$A_{iu} = \pi_i \tag{V.7b}$$

with

$$c_u = 0 \quad \text{and} \quad m_u^2 + \tfrac{1}{2} g_u f^2 = 0\,,$$

(IV.1) becomes, up to constant terms, in ordinary rather than Wick unrenormalized operator products [11]

$$L = \tfrac{1}{2}(\partial_\mu \sigma\, \partial^\mu \sigma + \partial_\mu \pi_i\, \partial^\mu \pi_i) - \tfrac{1}{8} g(\sigma^2 + \pi_i^2 + 2 f \sigma)^2$$

which, with $m_{\sigma u}^2 \equiv g_u f^2$, may also be written

$$\begin{aligned} L = {} & \tfrac{1}{2}(\partial_\mu \sigma\, \partial^\mu \sigma - m_{\sigma u}^2 \sigma^2 + \partial_\mu \pi_i\, \partial^\mu \pi_i) \\ & - \tfrac{1}{8} g_u(\sigma^2 + \pi_i^2)^2 - \tfrac{1}{2} g_u^{\frac{1}{2}} m_{\sigma u} \sigma(\sigma^2 + \pi_i^2)\,. \end{aligned} \tag{V.8}$$

Thus the bare pion mass is zero, and it is not difficult, exploiting conservation of the "axial" currents (IV.4a), to show that the pion self mass is zero in perturbation theory to all orders in g. E.g., the self mass from the sum of diagrams to order g, Fig. 4, is zero while the first diagram contributes to amplitude renormalization to this order. The vacuum expectation value of the σ field is zero only to lowest order but nonzeı to higher orders (prop. $g^{-\frac{1}{2}+n}$, $n \geqq 1$).

After introducing equal amplitude renormalization factors for tł σ and π fields, the theory (V.8) differs from the one analyzed so far in th section only by the shift (V.7a) of the σ-field, such that connected Green functions with at least two arguments and vertex functions are the sarr as before, the use of the same renormalization conditions being unde stood. The practical advantage of the BPH renormalization technique obvious from a comparison of Fig. 1 with Fig. 4.

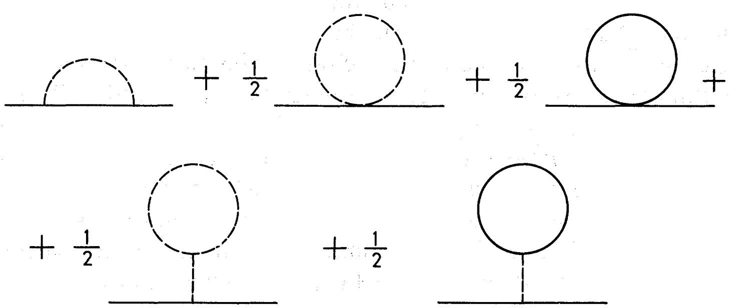

Fig. 4

For the sigma model [2, 3, 39] one obtains results analogous to thos for our simplified model. In the Goldstone case, the pions are massles and, chirality being no longer a good quantum number, the nucleon massive and merely isospin-degenerate. While pions can be emitte singly by nucleons, this leads to no more UR-complications than in ou model due to the pseudoscalar nature of the pions. Quite generally, fo PCAC models the Adler self consistency conditions lead to nonsingula scattering amplitudes for Goldstone particle momenta zero and thu to UR convergence of the S matrix, unless there are additional massles particles in the theory.

VI. Discussion

We have seen that symmetry breaking by a term linear in Bose field in a renormalizable theory can be dealt with easily with the help c the relevant WTKR identities. Thereby, however, as is characteristi for perturbation theoretical renormalization, one looses contact witl the Lagrangian itself, (IV.1) in our case. Rather than trying to study th expressions for unrenormalized parameters in terms of renormalize ones, it is here more meaningful to ask whether, in case the symmetry breaking term is switched off, to the remaining symmetric Lagrangiaı the symmetric solution or the nonsymmetric Goldstone one would be stable.

For an answer, we go back to (III.3) which gives

$$\begin{aligned} G_x\{J\} &= G_x^K\{J-K\} \\ &= \sum_{n=0}^{\infty} (n!)^{-1}(-1)^n \int\cdots\int dy_1 \ldots dy_n\, K(y_1)\ldots K(y_n)\, G^K_{y_1\ldots y_n}\{J\} \end{aligned} \tag{VI.1}$$

or, for vertex functions, according to (III.8) and (III.7)

$$\begin{aligned} \Gamma_x\{\mathscr{A}\} &= \Gamma_x^K\{\mathscr{A}-\mathscr{K}\} - \Gamma_x^K\{-\mathscr{K}\} = -i\,K(x) \\ &+ \sum_{n=0}^{\infty} (n!)^{-1}(-1)^n \int\cdots\int dy_1 \ldots dy_n\, \mathscr{K}(y_1)\ldots \mathscr{K}(y_n)\, \Gamma^K_{x y_1\ldots y_n}\{\mathscr{A}\}\,. \end{aligned} \tag{VI.2}$$

It is immediate to check that, on the basis of the identities (III.22) and the integral of (III.15), the closed expressions given in (VI.1–2) satisfy the corresponding identities with $K(x)\to 0$ and $G^K(x)\to G(x)$, which means that the new Green's and vertex functions have the expected symmetry provided the infinite sums in (VI.1–2) converge.

Now we have more generally

$$G_x^K\{J-zK\} = G_x^{(1-z)K}\{J\} \tag{VI.3}$$

and

$$\Gamma_x^K\{\mathscr{A}-z\mathscr{K}\} - \Gamma_x^K\{-z\mathscr{K}\} = \Gamma_x^{K_z}\{\mathscr{A}\} \tag{VI.4a}$$

where K_z is given, using (III.8) again, by

$$K_z(x) = i\,\Gamma_x\{(1-z)\mathscr{K}\} = i\,\Gamma_x^K\{-z\mathscr{K}\} + K(x)\,. \tag{VI.4b}$$

The considerations of the appendix suggest that, because of (VI.3), $G_x^K\{J-zK\}$ exists for infinitesimal J (i.e., its functional derivatives at $J=0$ exist), and that, because of (VI.4), $\Gamma^K\{\mathscr{A}-z\mathscr{K}\}$ exists for infinitesimal $\mathscr{A}$, in the non-Goldstone case for $0\leqq z\leqq 2$, but that these functionals have in the Goldstone case these properties only in the interval $0\leqq z<1$ for $G_x^K\{J-zK\}$ and only in the interval $0\leqq z<z_0<1$ for $\Gamma_x^K\{\mathscr{A}-z\mathscr{K}\}$, where z_0 is defined by $i\Gamma_x^K\{-z_0\mathscr{K}\}=-K(x)$, and should behave singularly at the approach $z\to 1-0$ and $z\to z_0-0$, respectively.

For space-time constant K and $\mathscr{K}$, $K_z\cdot\mathscr{K}$ is hereby a monotonic function of z and, in the model of Section IV, K_z a vector parallel to $\mathscr{K}$ as follows from (III.17). In the symmetric and Goldstone case, there is no reason for the functions (VI.3) and (VI.4) not to exist for arbitrary $z<0$, though the power series expansions in z should in the Goldstone case converge only for $|z|<1$ for $G_x^K\{J-zK\}$ and for $|z|<z_0$ for $\Gamma_x^K\{\mathscr{A}-z\mathscr{K}\}$.

It is in principle possible to carry out the infinite sums in (VI.1) and (VI.2) in all fixed orders of perturbation theory where, in view of (IV.18) and (IV.17), geometric series arise; this procedure is the inverse of the one Lee [1] applied to carry out the sums needed, in our notation, in (III.10). The result is, formally: The Green's function "G" and vertex functions "Γ" defined by (VI.1) and (VI.2) are the ones of a symmetric theory, i.e. (IV.1) with $c=0$, with renormalization conditions, in a notation analogous to the one of Section IV and written in terms of $\Gamma(x, y)$ of (IV.14), "Γ_2" $=\Gamma_{xx}(-F, 0)$, "Γ_3" $=0$, "Γ_4" $=\Gamma_{xxxx}(-F, 0)$, and analogously for "Γ_2'". In the one-loop approximation, which gives for each vertex function of $\geqq 5$ arguments the lowest nonvanishing order in g, we have by familiar techniques [17] [18, 40]

$$\Gamma(x, y)_{\text{ring}} = -i\, 2^{-7}\, \pi^{-3} (g z)^3 \cdot \int_0^1 du (1-u)^2 \left[N(\mu^2 + g u z)^{-1} + (\mu^2 + 3 g u z)^{-1}\right] + c_0 + c_1 z + c_2 z^2 + i c x \tag{VI.5}$$

where $z = y + \frac{1}{2}(x+F)^2$ and the c_i are such that (IV.15–16) are satisfied whence (IV.17)–(IV.19) follow. The higher approximations are increasingly complicated, and a discussion of the sums "Γ" to all orders, which would yield a necessary condition for the existence of the symmetric solution in the form [36]

$$i\,\text{``}\Gamma_2\text{''} = i\,\Gamma_{xx}(-F, 0) \geqq 0\,, \qquad i\,\text{``}\Gamma_2'\text{''} \leqq 0\,,$$

is of course outside the scope of this paper.

That the question about the implications of the Lagrangian with the symmetry-breaking term switched off can be meaningfully posed at all means that if the symmetric part of the Lagrangian (or, more properly, the symmetric part of the field equation) is written more carefully avoiding meaningless expressions, then that part can be taken as independent [11] of the symmetry-breaking term. (Or, in terms of regularization as employed by Lee [1, 39]: the regularization must conform with full symmetry.) This also holds for bilinear symmetry-breaking term, in the sense of the preceding, with respect to the field equations.

The symmetry breaking considered in Sections IV–V can also be described as a breaking of the reflection symmetry $O(1)$ with respect to the σ-field. Such treatment applies also to the case $N=0$ of the model (IV.1). (III.8) gives, because of $\Gamma_x\{\mathscr{A}\} = -\Gamma_x\{-\mathscr{A}\}$ for the model (IV.1) if $c_u = 0$,

$$\Gamma_x^K\{\mathscr{A} - \mathscr{K}\} + \Gamma_x^K\{-\mathscr{A} - \mathscr{K}\} = 2\Gamma_x^K\{-\mathscr{K}\} = 2i\,K(x) \tag{VI.6}$$

[17] A similar calculation appears already in Goldstone's original paper, Ref. [41].

which is equivalent to

$$\Gamma^K_x\{-\mathscr{K}\} = i\,K(x)\,, \tag{VI.7a}$$

$$\Gamma^K_{x_1\ldots x_{2n+1}}\{-\mathscr{K}\} = 0 \qquad (n \geqq 1)\,. \tag{VI.7b}$$

Taking $N=0$ for simplicity, we have for

$$\tilde{\Gamma}^K(\overbrace{0\cdots 0}^{l}) \equiv \Gamma_l\,, \qquad \langle A_0(x)\rangle = F\,, \qquad K(x) = c$$

from (VI.7)

$$-F\,\Gamma_2 + \tfrac{1}{2}F^2\,\Gamma_3 - \tfrac{1}{6}F^3\,\Gamma_4 + - \cdots = ic\,, \tag{VI.8a}$$

$$\Gamma_3 - F\,\Gamma_4 + - \cdots = 0\,, \qquad \Gamma_5 - + \cdots = 0\,, \ldots\,. \tag{VI.8b}$$

Our method requires to start the computation of F and of all Γ_l, $l \neq 2, 4$, from the solution of (VI.8) with all Γ_l, $l \geqq 5$ neglected. E.g., for $c=0$ we have to this lowest order the roots $F = \Gamma_3 = 0$ and the nontrivial ones

$$F = \pm(3M^2/g)^{\frac{1}{2}}, \qquad \Gamma_3 = \mp i(3M^2 g)^{\frac{1}{2}}, \qquad (\Gamma_2 = -iM^2,\ \Gamma_4 = -ig)\,.$$

Proceeding from these to higher approximations, i.e. to the inclusion of vertex diagrams with loops, we obtain the expansions

$$F = \pm(3M^2/g)^{\frac{1}{2}}\,(1 + \alpha_1 g + \cdots)\,, \tag{VI.9a}$$

$$\Gamma_3 = \mp i(3M^2 g)^{\frac{1}{2}}\,(1 + \beta_1 g + \cdots)\,, \tag{VI.9b}$$

$$\Gamma_l = (\pm 1)^{l-1}\,i\,M^{4-l}\,g^{l/2}(\gamma_{l0} + g\gamma_{l1} + \cdots) \qquad (l \geqq 5)\,, \tag{VI.9c}$$

whereby in one-loop approximation (VI.5) with $N=0$, $z = \frac{1}{2}(x+F)^2$ may be used [17]. There are, of course, no Goldstone particles in this case. The method can easily be extended to the case $c \neq 0$.

The difference compared with the procedure of Section IV is that now the transformation considered is not infinitesimally generated and genuine functional relations (VI.6) must be solved, which require the same type of summation to be carried out as discussed in connection with (VI.1–2). The great simplification by use of infinitesimal transformations in the case $N \geqq 1$ is obvious.

We remark in passing that from the form of (VI.9) one cannot conclude that $g > 0$ is necessary, but only that it is necessary in order for, in the spontaneous-symmetry-breaking case, an expansion in powers of g to be possible. For $N \geqq 1$, formulas (VI.9a–c) do not hold since, as we saw in Section V, M^2 does then not exist in perturbation theory.

We have not shown that the expansions of Sections IV and V satisfy the appropriate generalized unitarity equations [42]. However, the situation is not different here from the one in all perturbation theoretically renormalized theories, since e.g. the spontaneity of the symmetry breaking in the Goldstone case has no particular effect on the renormalized perturbation expansion as such, as put in evidence by the Lagrangian (V.8). The complication is, rather, that in the interesting cases unstable particles [18] [35] arise.

Acknowledgement. The author is indebted to the work of B. W. Lee, which was the starting point of this investigation. He acknowledges stimulating seminars at DESY by B. W. Lee and A. S. Wightman on symmetry breaking.

Appendix. Properties of the Ground State Energy Density

We first transscribe the one-particle-structure relations of Section I into relations in Euclidean quantum field theory [44] (EQFT). In the functions $G_{\text{disc}}(x_1 \dots x_n)$ and $G(x_1 \dots x_n)$ one replaces all x_i^0 by zx_i^0; the resulting distribution valued functions of z $G_{\text{disc}}^{(z)}(x_1 \dots x_n)$ and $G^{(z)}(x_1 \dots x_n)$ possess analytic continuations in z from the positive real axis into the lower complex half plane. We set

$$G_{\text{disc}}^{(z)}\{J\} = \sum_{n=0}^{\infty} (n!)^{-1} (iz)^n \int \cdots \int dx_1 \dots dx_n J(x_1) \dots J(x_n) G_{\text{disc}}^{(z)}(x_1 \dots x_n) \tag{A.1}$$

and

$$G^{(z)}\{J\} = \sum_{n=1}^{\infty} (n!)^{-1} (iz)^n \int \cdots \int dx_1 \dots dx_n J(x_1) \dots J(x_n) G^{(z)}(x_1 \dots x_n) \tag{A.2}$$

such that (I.7) becomes

$$G^{(z)}\{J\} = \ln G_{\text{disc}}^{(z)}\{J\} . \tag{A.3}$$

The functions $G_{\text{disc}}^{(-i)}(x_1 \dots x_n) \equiv S_{\text{disc}}(x_1 \dots x_n)$ and $G^{(-i)}(x_1 \dots x_n) \equiv S(x_1 \dots x_n)$ are the Euclidean Green's functions or Schwinger functions. They are Euclidean invariant and, for the model (IV.1), real and, if $c_u = 0$, the $S_{\text{disc}}(\dots)$ are positive-valued for the symmetric solution if there is one. Further properties are given elsewhere [19] [44].

In analogy to (I.8–14) we set $(S(x) = G(x))$

$$S_x\{J\} - S(x) = \alpha(x) \tag{A.4}$$

and have

$$\delta\alpha(x)/\delta J(y) = S_{xy}\{J\} \tag{A.5a}$$

$$\delta J(x)/\delta\alpha(y) = S_{xy}^{-1}\{J\} \tag{A.5b}$$

[18] V. Glaser, H. Epstein, Ref. [43], have verified the validity of the equations of Ref. [42] and related ones on the basis of a one-to-one correspondence between fields and stable particles, in renormalized perturbation theory.

[19] The notation of Ref. [44] is related to the one used here as follows:

$$\tau(x_1 \dots x_n) \to G_{\text{disc}}(x_1 \dots x_n), \quad T\{J\} \to G_{\text{disc}}\{J\},$$
$$S(x_1 \dots x_n) \to S_{\text{disc}}(x_1 \dots x_n), \quad S\{J\} \to S_{\text{disc}}\{J\},$$

G and S being used here for connected functions and functionals.

such that

$$J(x) = -\mathscr{H}_x\{\alpha\} \tag{A.6}$$

with

$$\mathscr{H}\{\alpha\} = -\int dx\, J(x)\,[\alpha(x) + S(x)] + S\{J\} \tag{A.7}$$

and

$$\mathscr{H}_{xy}\{\alpha\} = -S_{xy}^{-1}\{J\}\,. \tag{A.8}$$

$\mathscr{H}\{\alpha\}$ is the generating functional of the Euclidean vertex functions, i.e. the amputated one-particle-irreducible Schwinger functions $\mathscr{H}(x_1 \ldots x_n)$.

As has been discussed in detail elsewhere [44, 45], the Schwinger functions can, for theories such as (IV.1), be written

$$S_{\text{disc}}(x_1 \ldots x_n) = \int D(\varphi)\, \varphi(x_1) \ldots \varphi(x_n)$$

where $D(\varphi)$ is a normalized nonnegative measure on some function space [20]. Thus

$$S_{\text{disc}}\{J\} = \int D(\varphi) \exp\left[\int dx\, J(x)\varphi(x)\right] \tag{A.9}$$

and, as follows from the Hölder inequality and (A.3) for $z = -i$, $S\{J\}$ is for real J a convex functional: ($c + c' = 1$, $c \geqq 0$, $c' \geqq 0$)

$$S\{cJ + c'J'\} \leqq cS\{J\} + c'S\{J'\}\,. \tag{A.10}$$

In particular, $S_{xy}\{J\}$ is a positive-semidefinite function. Therefore, from (A.5) follows that the transformation from J to α is possible except on such manifolds (on J-space) that $S_{xy}\{J\}$ has eigenvalue $+\infty$ i.e. does not exist, and that the inverse transformation is possible except where $-\mathscr{H}_{xy}\{\alpha\}$ $(= S_{xy}^{-1}\{J\})$ has eigenvalues $+\infty$ i.e. does not exist.

Due to (A.4) and (A.10), (A.7) may be written

$$\mathscr{H}\{\alpha\} = \operatorname*{Inf}_J \left\{-\int dx\, J(x)\,[\alpha(x) + S(x)] + S\{J\}\right\} \tag{A.11}$$

with the infimum being global, and unique except for the manifold in α-space mentioned before. From (A.11) the global concavity of $\mathscr{H}\{\alpha\}$ follows:

$$\begin{aligned}
\mathscr{H}\{c\alpha + c'\alpha'\} &= \operatorname*{Inf}_J\left\{-\int dx\, J(x)[c\alpha(x) + c'\alpha'(x) + (c+c')S(x)] + (c+c')S\{J\}\right\} \\
&\geqq c \operatorname*{Inf}_J\left\{-\int dx\, J(x)\,[\alpha(x) + S(x)] + S\{J\}\right\} \\
&\quad + c' \operatorname*{Inf}_J\left\{-\int dx\, J(x)\,[\alpha'(x) + S(x)] + S\{J\}\right\} \\
&= c\,\mathscr{H}\{\alpha\} + c'\,\mathscr{H}\{\alpha'\}
\end{aligned}$$

[20] We are not referring here to a "Wiener history integral", Ref. [45], but are appealing to a generalized Bochner theorem, given the positive-definiteness of the Schwinger functions. This idea is due to J. Tarski.

with c and c' as in (A.10). Thus, (A.7) also yields

$$S\{J\} = \operatorname*{Sup}_{\alpha} \{\int dx\, \alpha(x)\, J(x) + \mathscr{H}\{\alpha\}\} + \int dx\, J(x)\, S(x)\,. \qquad \text{(A.12)}$$

(A.12) wins suggestive content in the transition to time-independent J and α. We have [45]

$$G^{(z)}_{\text{disc}}\{J\} = \left\langle \left(\exp\left[-iz \int_{-\infty}^{+\infty} dx^0\, H_J(x^0) \right] \right)_+ \right\rangle \qquad \text{(A.13a)}$$

where

$$H_J(x^0) = P^0 - \int d^3x\, J(x)\, A((0, \boldsymbol{x})) \qquad \text{(A.13b)}$$

is the instantaneous Hamilton operator to the theory (I.2), provided all components of $A(x)$ that appear in the source term are canonically independent variables. We write

$$j_T(x) \equiv \tfrac{1}{2}[1 + \operatorname{sign}(\tfrac{1}{2}T - |x^0|)]\, j(\boldsymbol{x})$$

then

$$S_{\text{disc}}\{j_T\} = \sum_{n=0}^{\infty} |\langle | n \rangle_j|^2 \exp(-T E_n\{j\}) \qquad \text{(A.14)}$$

where n numerates, with $n=0$ for the ground state, the eigenstates $|n\rangle_j$ and eigenvalues $E_n\{j\}$ of H_j. (A.14) leads us to expect

$$\lim_{T\to\infty} T^{-1} S\{j_T\} = -E_0\{j\} \qquad \text{(A.15)}$$

which we will now prove.

From

$$H_j |0\rangle_j = E_0\{j\}\, |0\rangle_j$$

follows under the same condition as for (A.13) in the familiar way

$$\begin{aligned} \delta E_0\{j\}/\delta j(\boldsymbol{x}) &= {}_j\langle 0 | \delta H_j/\delta j(\boldsymbol{x}) | 0\rangle_j \\ &= -\, {}_j\langle 0 | A((0, \boldsymbol{x})) | 0 \rangle_j = i\, G_x\{j_\infty\} \end{aligned} \qquad \text{(A.16)}$$

where (III.3) is used. However, due to $S^K_x\{J\} = S_x\{J+K\}$ for $K(x)$ x^0-independent, which can be derived as explained after (III.3) by performing the transition to Euclidean Green's functions, as defined at the beginning of this Appendix, directly in the defining integral equations provided the Lagrangian and Hamiltonian are time-independent, from (A.16) we also have

$$\delta E_0\{j\}/\delta j(\boldsymbol{x}) = -\, S_x\{j_\infty\}\,. \qquad \text{(A.17)}$$

Inserting here the expansion in powers of j_∞, one immediately obtains

$$\begin{aligned} E_0\{j\} &= -\bar{S}\{j\} \\ &\equiv -\sum_{n=1}^{\infty} (n!)^{-1} \int \cdots \int d^3x_1 \ldots d^3x_n\, j(\boldsymbol{x}_1) \ldots j(\boldsymbol{x}_n)\, \bar{S}(\boldsymbol{x}_1 \ldots \boldsymbol{x}_n) \end{aligned} \qquad \text{(A.18a)}$$

where

$$\bar{S}(x_1 \ldots x_n) \equiv \int \cdots \int dx_2^0 \ldots dx_n^0 \, S(x_1 \ldots x_n) . \tag{A.18 b}$$

That

$$\bar{S}\{j\} = \lim_{T \to \infty} T^{-1} S\{j_T\} . \tag{A.19}$$

is valid is, however, immediate due to the cluster property, with exponential decay[21] [46] also in "time" direction of $S(x_1 \ldots x_n)$. (A.18) with (A.19) gives (A.15). It is well known that the ground state energy is a concave function of any parameter on which the Hamiltonian depends linearly; here the concavity of $E_0\{j\}$ i.e. convexity of $\bar{S}\{j\}$ as a functional of j is an immediate consequence of (A.10) and (A.18).

At this point, we leave it to the reader to transsc̆ribe (A.12) for the case of "time"-independent but space-dependent J and α and immediately proceed to the case of space-time-constant J and α. Let Ω be a volume in three-space, χ_Ω its characteristic function, and $V(\Omega) = \int d^3x \, \chi_\Omega(x)$ its measure, and u real. Then

$$e_0(u) = \lim_{\Omega \to R^3} V(\Omega)^{-1} E_0\{\chi_\Omega u\} = -\bar{\bar{S}}(u) \tag{A.20}$$

exists, with

$$\bar{\bar{S}}(u) = \sum_{n=1}^{\infty} (n!)^{-1} u^n \bar{\bar{S}}_n , \tag{A.21 a}$$

$$\bar{\bar{S}}_n = \int \cdots \int dx_2 \ldots dx_n \, S(x_1 \ldots x_n) , \tag{4.21 b}$$

again due to the cluster property of $\bar{S}\{j\}$ with exponential decay[21] [46]. $e_0(u)$ is a concave function of u and is the ground state energy density for source strength $K(x) = u$ in (III.1). We note also that

$$\bar{\bar{S}}_n = i^{n-1} \int \cdots \int dx_2 \ldots dx_n \, G(x_1 \ldots x_n)$$

elementarily by "contour rotation", or from (A.16) and (III.3).

We consider the effect of $J(x) \to u$ and the concomitant $\alpha(x) \to v$ in the formulas (A.4–8). We have immediately

$$(\partial/\partial u) \bar{\bar{S}}(u) = v + S(x) , \tag{A.22}$$

$$u = \bar{\bar{S}}_2^{-1} v + 0(v^2) \tag{A.23}$$

with

$$\partial v/\partial u = (\partial^2/\partial u^2) \bar{\bar{S}}(u) , \tag{A.24}$$

$$u = -(\partial/\partial v) \bar{\bar{\mathscr{H}}}(v) , \tag{A.25}$$

$$\bar{\bar{\mathscr{H}}}(v) = -u(v + S(x)) + \bar{\bar{S}}(u) , \tag{A.26}$$

$$(\partial^2/\partial v_i \, \partial v_j) \bar{\bar{\mathscr{H}}}(v) = -[(\partial^2/\partial u \, \partial u) \bar{\bar{S}}(u)]_{ij}^{-1} , \tag{A.27}$$

$$\bar{\bar{\mathscr{H}}}(v) = -\tfrac{1}{2} v \, S_2^{-1} v + \sum_{n=3}^{\infty} (n!)^{-1} v^n \bar{\bar{\mathscr{H}}}_n , \tag{A.28 a}$$

$$\bar{\bar{\mathscr{H}}}_n = \int \cdots \int dx_2 \ldots dx_n \, \mathscr{H}(x_1 \ldots x_n) \, (n \geqq 3) . \tag{A.28 b}$$

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[21] We are assuming a mass gap, the Goldstone case being obtained as a limit situation.

Moreover, $\bar{\bar{\mathscr{H}}}(v)$ is concave in v, and

$$\bar{\bar{\mathscr{H}}}(v) = \operatorname*{Min}_{u} \{-u[v+S(x)] + \bar{\bar{S}}(u)\}\,, \tag{A.29}$$

$$-\bar{\bar{S}}(u) = e_0(u) = \operatorname{Min}\,[-uv - \bar{\bar{\mathscr{H}}}(v)] - uS(x)\,. \tag{A.30}$$

(A.30) shows that the ground state energy density is obtained from a minimum principle, the (generally, multicomponent) variable v being the "expectation value" of the field shifted by the number $S(x)$, the ground state expectation value of the field without source. At the minimum, v is, apart from the shift, the actual ground state expectation value of the field. Thus, the expression that is minimized is a pseudo-energy-density, formally an infinite power series in v with, in perturbation theory, finite coefficients. The lowest-order terms, from superficially divergent vertices, are just those occuring in the classical Hamiltonian density with, however, renormalized "masses" and coupling constants, the subtraction point being the one at zero momenta as in Section IV.

We try to understand the Goldstone phenomenon from the point of view of Eqs. (A.22–30). For easy visualization we take two-component u and v and consider a, without source, $O(2)$ invariant theory. $\bar{\bar{S}}(u)$ is a convex function of $\sqrt{u_0^2+v_1^2}$ and $\bar{\bar{\mathscr{H}}}(v)$ a concave function of v, the Legendre transform of $\bar{\bar{S}}(u)$, and set $S(x) = G(x)$ zero. If $\bar{\bar{S}}(u)$ is everywhere twice differentiable, so will be $\bar{\bar{\mathscr{H}}}(v)$, and, on the basis of (A.22) and (A.25), the one-to-one relation $u \leftrightarrow v$ is easily understood geometrically. The Goldstone case corresponds to a choice of parameters in the Lagrangian such that $\bar{\bar{S}}(u)$ is not twice differentiable at the origin:

$$\{[(\partial/\partial u_0)\;\; \bar{\bar{S}}(u)]^2 + [(\partial/\partial u_1)\,\bar{\bar{S}}(u)]^2\}^{\frac{1}{2}}|_{u=0} = \operatorname{tang}\Theta > 0\,.$$

Then to the point $u=0$ corresponds in the v-plane the circle $|v| = \operatorname{tang}\Theta$. $\bar{\bar{\mathscr{H}}}(v)$ is not defined inside this circle or, by (A.29), merely as the concave closure $\bar{\bar{\mathscr{H}}}(v) = 0$, $|v| \leqq \operatorname{tang}\Theta$. $\bar{\bar{\mathscr{H}}}(v)$ is concave and twice differentiable for $|v| > \operatorname{tang}\Theta$, and at $|v| = \operatorname{tang}\Theta$, on the basis of (III.9),

$$(\partial/\partial v_0)\,\bar{\bar{\mathscr{H}}}(v) = (\partial/\partial v_1)\,\bar{\bar{\mathscr{H}}}(v) = 0$$

but, at $v_0 = \operatorname{tang}\Theta$, $v_1 = 0$:

$$(\partial^2/\partial v_0^2)\,\bar{\bar{\mathscr{H}}}(v) = 0\,, \qquad (\partial^2/\partial v_1^2)\,\bar{\bar{\mathscr{H}}}(v) = 0\,,$$
$$(\partial^3/\partial v_1^3)\,\bar{\bar{\mathscr{H}}}(v) = 0\,, \qquad (\partial^3/\partial v_0\,\partial v_1^2)\,\bar{\bar{\mathscr{H}}}(v) = 0\,,$$
$$(\partial^4/\partial v_1^4)\,\bar{\bar{\mathscr{H}}}(v) = 0\,,$$

using the results of Section V supplemented by the assumption that in the Goldstone limit, $M^2 \to 0$ rather than $\to$ finite [36]. Because of (A.22), the direction of approach to $u=0$ in the u plane determines the limit point on the circle $|v| = \text{tang}\,\Theta$ that is hereby reached from outside the circle: $u_0/u_1 = v_0/v_1$. On the $\bar{\bar{S}}(u)$-surface at $u = 0$, the radial derivative

$$\lim_{u_0 \to +0} (\partial^n/\partial u_0^n)\, \bar{\bar{S}}(u)|_{u_1=0} = i^{n-1} \int \cdots \int dx_2 \ldots dx_n\, G(x_1 \ldots x_n,)$$

in the notation of Sections IV and V, does not exist for $n = 2$ if $M^2 \to 0$, and also not the higher tangential ones, e.g.

$$\lim_{u_0 \to +0} (\partial^2/\partial u_1^2)\, \bar{\bar{S}}(u)|_{u_1=0} = \infty$$

due to the masslessness of the Goldstone particle. Up to the absence of such particle, the case of breaking of a discrete symmetry disccussed in Section VI lies similar. To understand how the irregular behaviour of $\bar{\bar{S}}(u)$ may come about,[22] we use (A.9), (A.19) and (A.20) to write

$$\bar{\bar{S}}(u) = \lim_{\Omega \to R^4} [V(\Omega)^{-1} \ln \{\int D(\varphi) \exp[u \int dx\, \chi_\Omega(x)\, \varphi(x)]\}], \quad \text{(A.31)}$$

Ω being a volume in R^4, χ_Ω its characteristic function, and $V(\Omega) = \int dx\, \chi_\Omega(x)$. Certain results on regularization limites [45] suggest that the integral in (A.31), which is $S_{\text{disc}}\{u\chi_\Omega\}$ and is a power series in $u_0^2 + u_1^2$ with positive, for finite $V(\Omega)$ finite, coefficients, is in our model an entire function of $u_0^2 + u_1^2$, such that the quotient in (A.31) is for $V(\Omega) < \infty$ holomorphic in a strip along the real axis since on that axis $S_{\text{disc}}\{u\chi_\Omega\} \geqq 1$. If $V(\Omega)$ increases, the logarithmic singularities of the limitand off the real axis may approach that axis and, in particular, the origin $u = 0$ in such a way that the nonregular behaviour along that axis emerges for $\Omega = R^4$, similarly as such singularities approach the transition point in the activity plane in the theory of condensation of Yang and Lee [8].

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[22] The reader may have noticed that formulas (A.25–28) and most of the subsequent geometrical discussion were already given in Ref. [41] on the basis of "functional integration", and $\bar{\bar{\mathscr{H}}}(v)$ was there evaluated in one-loop approximation. In our treatment, relatively rigorously emerge the roles of $-\bar{\bar{\mathscr{H}}}(v) - uv$ as a convex pseudoenergy density and of $-\bar{\bar{S}}(u)$ as concave ground state energy density, and our concept and use of functional integration are quite different from Goldstone's.

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