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An Elementary Proof of Dyson's Power Counting Theorem*

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Abstract. For the case of Euclidean metric an elementary proof of the power counting theorem is given.

1. Introduction

As is well known, Dyson's power counting theorem plays an important part in the theory of renormalization [1]. For the case of Euclidean metric a rigorous proof of this theorem was obtained by WEINBERG as a by-product of his work on the high-energy behavior of Feynman integrals [2]. The purpose of the present paper is to give a short, direct proof of the power counting theorem which uses more elementary methods. We restrict ourselves to the Euclidean case. An extension to the case of Minkowski metric will be discussed in a forthcoming paper.

We will be concerned with integrals of the form

$$I(q\,\mu) = \int dk \frac{P(kq)}{\prod\limits_{j=1}^{n} (l_j^2 + \mu_j^2)} \tag{1.1}$$

where

$$\begin{aligned} &q = (q_1, \ldots, q_n) \qquad &&k = (k_1, \ldots, k_n) \\ &dk = dk_1 \ldots dk_n \qquad &&\mu = (\mu_1, \ldots, \mu_n) \qquad \mu_i \geqq 0 \end{aligned} \tag{1.2}$$

with q_i, k_i denoting Euclidean four vectors. P denotes a polynomial in the components of k_i and q_i, of degree g with respect to the k_i. The four vectors l_j are of the form

$$\begin{aligned} &l = K + q\,, \qquad &&K = Ck = K(k)\,, \\ &l = (l_1, \ldots, l_n)\,, \qquad &&K = (K_1, \ldots, K_n)\,, \qquad K_j \not\equiv 0\,. \end{aligned} \tag{1.3}$$

C denotes an $n \times m$ matrix.

For integrals of the form (1.1) the following version of the power counting theorem will be proved in Section 3.

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Theorem 1. *Let all masses* $\mu_j \neq 0$. *The integral* (1.1) *is absolutely convergent (a.c.) if*

(i) *the dimension*

$$d = q - 2n + 4m \tag{1.4}$$

of the integral is negative and if

(ii) *for any four vector* Q *the subintegrals*

$$I_j(Q\, q\, \mu) = \int_{H_j} dV \frac{P(kq)}{\prod_{i=1}^{n} (l_i^2 + \mu_i^2)} \tag{1.5}$$

are a.c. The integral (1.5) *extends over the hyperplane* H_j *defined by*

$$l_j = \sum C_{jj'} k_{j'} + q_j \equiv Q \tag{1.6}$$

with the volume element dV.

As a corollary of theorem 1 one obtains Weinberg's version of the power counting theorem.

Theorem 2. *The integral* (1.1) *is a.c. if* (1.1) *and any subintegral*

$$I(q\, \mu\, H) = \int_{H} dV \frac{P(kp)}{\prod_{j=1}^{n} (l_j^2 + \mu_j^2)} \tag{1.7}$$

have negative dimension. H *denotes a hyperplane in* R_{4m} *described by a set of linear equations*

$$\sum_{j=1}^{m} d_{ij} k_j = r_i\,. \tag{1.8}$$

The dimension of a rational integral is defined by $d = d' + d''$ *where* d' *is the number of integration variables and* d'' *the degree of the integrand with respect to the integration variables.*

To illustrate the method used in this paper we sketch the proof of theorem 1 for the case that the polynomial P is absent. Integral (1.1) then reads

$$I(q\mu) = \int \frac{dk}{\prod_{j=1}^{n} (l_j^2 + \mu_j^2)} \tag{1.9}$$

So far as convergence properties are concerned we may as well consider the integral

$$I = \int_{D} \frac{dk}{\prod_{j=1}^{n} K_j^2}\,. \tag{1.10}$$

D denotes the domain of all k satisfying

$$K_j^2 \geqq 1\,. \tag{1.11}$$

(1.10) is obtained from (1.9) by setting the external momenta q equal to zero and excluding the singularities at $K_j = 0$. Apparently the integrals (1.9) and (1.10) are either both convergent or both divergent (a detailed proof of this statement will be given in section 2, see Lemma 5b). Hence it suffices to prove theorem 1 for integrals of the form (1.10). By an appropriate reordering of the momenta K_j we may write (1.10) in the form

$$I = \int \frac{dk}{\prod_{j=1}^{c} (K_j^2)^{\lambda_j}} \tag{1.12}$$

such that

$$K_i^2 \not\equiv K_j^2 \quad \text{for} \quad i \neq j \, .$$

We now decompose (1.12) into

$$\begin{aligned} I &= \sum_{j=1}^{c} I_j \\ I_j &= \int_{D_j} \frac{dk}{\prod_{j=1}^{c} (K_j^2)^{\lambda_j}} \end{aligned} \tag{1.13}$$

where D_j is the domain of all k satisfying

$$K_\nu^2 \geqq K_j^2 \geqq 1 \, . \tag{1.14}$$

In order to estimate the term I_j we introduce new variables of integration $t = (t_1 \ldots t_m)$ by a linear transformation

$$t = A k \tag{1. 15}$$

satisfying

$$t_1 \equiv \frac{K_j}{c} \, , \qquad \det A = 1 \, . \tag{1.16}$$

The integral I_j then becomes

$$I_j = \int_U dt_1 \int_V \frac{dt_2 \ldots dtm}{\prod_{i=1}^{c} (K_i^2)^{\lambda_i}} \tag{1.17}$$

where the K_i are expressed in terms of the new variables

$$K_i = K_i(A^{-1}t) = \sum d_{i\,i'} t_{i'} \tag{1.18}$$

U is the region of all t_1 with

$$c^2 t_1^2 \geqq 1 \tag{1.19}$$

and V is the region of all $t_2 \ldots t_m$ with

$$K_i(A^{-1}t)^2 \geqq c^2 t_1^2 \, . \tag{1.20}$$

Substituting for t_i the expressions

$$t_i = c\,|t_1|\,t_i' \qquad i = 2, \ldots, m \tag{1.21}$$

in the inner integral we obtain

$$I_j = c^{4(m-1)-2n} \int_U d t_1\, |t_1|^{4(m-1)-2n}\, J(t_1')\,, \tag{1.22}$$

$$J(t_1') = \int_{V'} \frac{dt_2' \ldots dt_m'}{\prod_{i \neq j} (K_i'^2)^{\lambda_i}}\,, \quad K_i' = \sum d_{ii'}\, t_{i'}' \tag{1.23}$$

with

$$t_1' = \frac{t_1}{c\,|t_1|}\,. \tag{1.24}$$

The region V' consists of all $t_2', \ldots, t_m'$ satisfying

$$K_i(A^{-1}t') \geqq 1\,. \tag{1.25}$$

By hypothesis (ii) the integral $J(t_1')$ is convergent. According to (1.24) t_1' is bounded for all t_1, hence $J(t_1')$ is bounded (Section 2, Lemma 2). Thus

$$I_j \leqq A \int d t_1\, |t_1|^{4(m-1)-2n}\,. \tag{1.26}$$

As a consequence of (i) the integral over t_1 is convergent. This completes the proof of theorem 1 for $P = 1$.

In Section 2 some auxiliary Lemmas are derived. Section 3 contains the proof of theorems 1 and 2 for an arbitrary polynomial P.

2. Auxiliary Lemmas

It will be convenient to consider more general integrals of the form

$$I(q p \mu V) = \int_V dk \frac{P(kp)}{E(kq\mu)} \tag{2.1}$$

depending on additional four vectors p_i. In (2.1) the notation (1.3) is used and

$$p = (p_1 \ldots p_n)$$

$$E(kq\mu) = \prod_{j=1}^{n} e_j(kq_j\mu_j) \tag{2.2}$$

$$e_j(kq_j\mu_j) = l_j^2 + \mu_j^2\,, \qquad \mu_j \geqq 0\,.$$

We further introduce the notation

$$\bar{I}(q p \mu V) = \int dk \left|\frac{P(kp)}{E(kq\mu)}\right|. \tag{2.3}$$

The following Lemma states that the values of the external momenta q are irrelevant for the convergence of the integral I.

Lemma 1. *Let all masses $\mu_i \neq 0$. Necessary and sufficient for the absolute convergence (a.c.) of $I(q, p, \mu, V)$ is the a.c. of $I(0, p, \mu, V)$.*

Proof. The statement follows from the inequalities

$$\begin{aligned} \frac{e_j(k\, q_j\, \mu_j)}{e_j(k\, 0\, \mu_j)} &\leqq A_j\,, \\ \frac{e_j(k\, 0\, \mu_j)}{e_j(k\, q_j\, \mu_j)} &\leqq A_j \end{aligned} \tag{2.4}$$

where

$$A_j = 1 + \frac{|q_j|}{\mu_j} + \frac{q_j^2}{\mu^2{}_j}\,. \tag{2.5}$$

Using the same estimates we obtain

Lemma 2. *Let S be a set of vectors q_i which is bounded in R_{4m}. Assume $\mu_j \neq 0$. Then there exists a constant C independent of q_i such that*

$$\bar{I}(q, p, \mu, V) \leqq C\bar{I}(0, p, \mu, V)\,. \tag{2.6}$$

Lemma 3. *Let $Q(x_1 \ldots x_m, z_1 \ldots z_n)$ be a function of $x_1, \ldots, x_n$ and a polynomial in z_α of degree d_α*

$$Q(x_1 \ldots x_m\, z_1 \ldots z_n) = \sum_{i_1=0}^{d_1} \cdots \sum_{i_n=0}^{d_n} a_{i_1 \ldots i_n}(x_1 \ldots x_m)\, z_1^{i_1} \ldots z_n^{i_n}\,. \tag{2.7}$$

We use the notation

$$x = (x_1 \ldots x_m)\,, \qquad z = (z_1 \ldots z_n)$$

$$dx = dx_1 \ldots dx_m\,.$$

(a) *Let $z_i^{(0)}, \ldots, z_i^{(d_i)}$ be $d_i + 1$ different values for the variable z. If*

$$J(z) = \int_V dx\, Q(x, z) \tag{2.8}$$

is absolutely convergent (a.c.) at the $(d_1 + 1) \ldots (d_n + 1)$ points

$$z = z^{(i_1 \ldots i_n)} = (z_1^{(i_1)} \ldots z_n^{(i_n)})\,, \qquad i_\alpha = 0, \ldots, d_\alpha \tag{2.9}$$

then the integrals

$$\int dx\, a_{i_1 \ldots i_n}(x) \tag{2.10}$$

converge absolutely.

(b) *If the integral* (2.8) *diverges absolutely for a single value*

$$z = a = (a_1 \ldots a_n) \tag{2.11}$$

it diverges absolutely for almost any $(z_i \ldots z_n)$.

Proof. (a) The coefficients in (2.7) are given by [3]

$$a_{i_1 \ldots i_n}(x) = \sum_{j_1 \ldots j_n} c_{i_1 \ldots i_n j_1 \ldots j_n} Q(x, z^{(j_1 \ldots j_n)}) \tag{2.12}$$

where the $c_{i_1 \ldots i_n j_1 \ldots j_n}$ depend on the parameters (2.9) only and are determined by a generalization of Lagrange's interpolation formula to several variables. Hence

$$\int |a_{i_1 \ldots i_n}(x)|\, dx \leqq \sum_{j_1 \ldots j_n} |c_{i_1 \ldots i_n j_1 \ldots j_n}| \int |Q(x, z^{(j_1 \ldots j_n)})|\, dx\,.$$

(b) For $n = 1$ statement (a) implies that

$$\int_V Q(x, z)\, dx$$

is either a.c. everywhere or at d different points at most (d denotes the degree of Q in the variable z). Hence (b) is correct for $n = 1$. Assuming that (*b*) is correct for $n - 1$ we will prove it for n. The divergence of (2.8) at (2.11) implies that

$$\int_V Q(x, z, \ldots, z_{n-1}\, a_n)\, dx$$

is absolutely divergent (a.d.) everywhere except for

$$(z_1 \ldots z_{n-1}) \in D(a_n)$$

where D is a set of measure zero. Let now $(z_1 \ldots z_{n-1}) \notin D(a_n)$ then

$$\int_V Q(x, z_1 \ldots z_{n-1}\, a_n)\, dx$$

is a.d., hence

$$\int_V Q(x, z_1 \ldots z_{n-1}\, z_n)\, dx$$

is a.d. at z_n except for at most $d_n + 1$ different values

$$z_n = f_n^{(i)}(z_1 \ldots z_{n-1})\,, \qquad i = 0, \ldots, \alpha \leqq d_n\,.$$

Hence (2.8) is a.d. except for either $(z_1 \ldots z_{n-1}) \in D(a_n)$ or $z_n = f_n^{(i)}$ $(z_1 \ldots z_{n-1})$. This, however, is a set of measure zero.

The purpose of the following Lemmas 4 and 5 is to show that the mass terms in the denominators are not relevant for the convergence of the integrals (2.4) provided the zeros of the denominators are properly excluded from the domain of integration.

Lemma 4.

$$\bar{I}(q, p, \mu, D_q) \leqq \bar{I}(q, p, 0, D_q)\,, \tag{2.13}$$

$$\bar{I}(q, p, 0, D_q) \leqq C\bar{I}(q, p, \mu, D_q)\,. \tag{2.14}$$

C is independent of p and q. D_q denotes the region of all k satisfying

$$e_j(k, q_j, 0) \geqq r^2, \quad j = 1, \ldots, n \tag{2.15}$$

where r is a given positive number.

Proof. Eq. (2.13), (2.14) follow respectively from the inequalities

$$\frac{e_j(kq_j0)}{e_j(kq_j\mu_j)} \leqq 1, \tag{2.16}$$

$$\frac{e_j(kq_j\mu_j)}{e_j(kq_j0)} \leqq 1 + \frac{\mu_j^2}{r^2} \text{for} \quad e_i(kq_j0) \geqq r^2. \tag{2.17}$$

Lemma 5. *Let all masses $\mu_j \neq 0$.*

(a) *The integral*

$$I(qp\mu) = \int dk \frac{P(kp)}{E(kq\mu)} \tag{2.18}$$

is a.c. if and only if

$$I(qp\mu\, D_q) = \int\limits_{D_q} dk \frac{P(kp)}{E(kq\mu)} \tag{2.19}$$

is a.c.

(b) *The integral* (2.18) *is a.c. if and only if*

$$I(0p\,0D) = \int\limits_{D} dk \frac{P(kp)}{E(k00)} \tag{2.20}$$

is a.c. D denotes the set of all k satisfying

$$K_j^2 = e_j(k00) \geqq r^2, \quad j = 1, \ldots, n. \tag{2.21}$$

Proof. (a) It is obvious that the a.c. of $I(qp\mu)$ implies the a.c. of $I(qp\mu\, D_q)$. We will show that the a.c. of $I(qp\mu)$ follows from the a.c. of $I(qp\mu D_q)$ for m integration variables $k_1 \ldots k_m$. As hypothesis of induction we assume that the statement has been proved for all integrals of the type (2.18) with less than m variables of integration.

We first observe that the number of linearly independent forms among $K_1 \ldots K_n$ must be m. (If it were less than m the integral (2.19) would diverge.) The factors e_j of the denominator are now renumbered such that

$$l_1^2, \ldots, l_c^2 \tag{2.22}$$

represent all different quadratic forms among the l_j^2. Let s be any subset of $(1, \ldots, c)$ including $(1, \ldots, c)$ itself and the empty set. To every s we define X_s as the set of all k satisfying

$$\begin{aligned} l_j^2 &\leqq r^2 \quad \text{for} \quad j \in s \\ l_j^2 &\geqq r^2 \quad \text{for} \quad j \notin s. \end{aligned} \tag{2.23}$$

The integration domain R_{4m} of (2.18) is the union of all sets X_s

$$R_{4m} = \bigcup_s X_s \,. \tag{2.24}$$

The intersection $X_s \cap X_{s'}$ has measure zero provided $s \neq s'$. Corresponding to (2.24) we split the integral $I(qp\mu)$ into

$$I(qp\mu) = \sum_s I(qp\mu X_s) \,. \tag{2.25}$$

By an appropriate reordering of $e_1 \ldots e_n$ the region X_s becomes

$$X_s = Y_{s_1} \cap Y_{s_2}$$

where Y_{s_1} consists of all k satisfying

$$l_j^2 \leqq r^2, \qquad j = 1, \ldots, \alpha \tag{2.26}$$

and Y_{s_2} contains all k with

$$l_j^2 \geqq r^2, \qquad j = \alpha + 1, \ldots, c \,. \tag{2.27}$$

We can further arrange that

$$K_1, \ldots, K_a, K_{\alpha+1}, \ldots, K_b \,(a \leqq \alpha, b \leqq c) \tag{2.28}$$

form m independent linear forms of the k such that $K_{a+1}, \ldots, K_\alpha$ are linear combinations of $K_1 \ldots K_a$ and that $K_{b+1}, \ldots, K_n$ are linear combinations of (2.28). Let now Z_{1q} be the set of all $\bar{K} = (K_1 \ldots K_a)$ satisfying (2.26) for given q. Let $Z_{2q\bar{K}}$ be the set of all $\bar{\bar{K}} = (K_{\alpha+1} \ldots K_b)$ satisfying (2.27) for given q and $\bar{K}$. With (2.28) as integration variables $I(qp\mu X_s)$ takes the form

$$I(qp\mu X_s) = c \int_{Z_{1q}} d\bar{K} \frac{J(\bar{K}qp\mu)}{E^{(1)}(k(\bar{K}), q, \mu)} \,, \tag{2.29}$$

$$J(\bar{K}qp\mu) = \int_{Z_{2q\bar{K}}} d\bar{\bar{K}} \frac{P(k(\bar{K}\bar{\bar{K}}), p)}{E^{(2)}(k(\bar{K}\bar{\bar{K}}), q, \mu)} \,, \tag{2.30}$$

where

$$\begin{aligned} E^{(1)}(kq\mu) &= \prod_{j=1}^{\alpha} e_j(kq\mu_j) \,, \\ E^{(2)}(kq\mu) &= \prod_{j=\alpha+1}^{n} e_j(kq\mu_j) \,. \end{aligned} \tag{2.31}$$

For the integral (2.30) we will now derive an upper bound which is independent of $\bar{K}$. To this end we make use of the hypothesis that

$$\begin{aligned} I(qp\mu D_q) &= \int_{D_q} dk \frac{P(kp)}{E(kq\mu)} \\ &= \int_{Z'_{1q}} \frac{d\bar{K}}{E^{(1)}(k(\bar{K}), q, \mu)} \int_{Z_{2q\bar{K}}} d\bar{\bar{K}} \frac{P(k(\bar{K}\bar{\bar{K}}), p)}{E^{(2)}(k(\bar{K}\bar{\bar{K}}), q, \mu)} \end{aligned} \tag{2.32}$$

is a.c. In the second line Z'_{1q} denotes the region of all $\bar{K}$ satisfying

$$l_j^2 \geqq r^2, \quad j = 1, \ldots, \alpha .$$

Since (2.32) is a.c. the subintegral

$$\int_{Z_{2q\bar{K}}} d\bar{\bar{K}} \frac{P(k(\bar{K}\bar{\bar{K}}), p)}{E^{(2)}(k(\bar{K}\bar{\bar{K}}), q, \mu)} \tag{2.33}$$

is a.c. for $\bar{K} \in Z'_{1q}$ up to a set Σ of measure zero [4]. Applying the hypothesis of induction to (2.33) we find that

$$\int d\bar{\bar{K}} \frac{P(k(\bar{K}\bar{\bar{K}}), p)}{E^{(2)}(k(\bar{K}\bar{\bar{K}}), q, \mu)} \tag{2.34}$$

is a.c. in Z'_{1q} up to the set Σ. By Lemma 1 also

$$\int d\bar{\bar{K}} \frac{P(k(\bar{K}\bar{\bar{K}}), p)}{E^{(2)}(k(0\bar{\bar{K}}), q, \mu)}$$

is a.c. in Z'_{1q} up to the set Σ. According to Lemma 3b (2.35) must be a.c. for any $\bar{K}$. Applying Lemma 1 again we obtain a.c. of (2.34) for any $\bar{K}$. Furthermore (2.34) is majorized (Lemma 2) in Z_{1q} by a multiple of (2.35). Expanding P with respect to the components of $K_1, \ldots, K_a$ we find (Lemma 3a) that (2.35) is bounded in Z_{1q}. This proves that each term $I(qp\mu X_s)$ in (2.24) is a.c. and hence also $I(qp\mu)$.

(b) Statement (b) follows by combining (a) with Lemma 1 and Lemma 4.

Lemma 6. *Let*

$$P(kp) = \sum_{\alpha=1}^{a} P_\alpha(kp) \tag{2.35}$$

be the decomposition of the polynomial P into homogeneous parts P_α of degree α with respect to the k_i. Then the a.c. of

$$I(qp\mu) = \int dk \frac{P(kp)}{E(kq\mu)} \tag{2.36}$$

implies the a.c. of

$$I_\alpha(qp\mu) = \int dk \frac{P_\alpha(kp)}{E(kq\mu)} \tag{2.37}$$

Proof. According to Lemma 5b it is sufficient to prove the statement for

$$J = \int_D dk \frac{P(kp)}{E(k00)}, \tag{2.38}$$

$$J_\alpha = \int_D dk \frac{P_\alpha(kp)}{E(k00)} . \tag{2.39}$$

We have

$$J = \varrho^{4m-2\Sigma\lambda_i} \int\limits_{D'} dk \frac{\Sigma \varrho^\alpha P_\alpha(kp)}{E(k00)} \tag{2.40}$$

where D' is the set of all k satisfying

$$e_j(k00) \geqq \frac{r^2}{\varrho^2} .$$

According to Lemma 4 the a.c. of (2.40) implies that

$$\int\limits_{D'} dk \frac{\Sigma \varrho^\alpha P_\alpha(kp)}{E(k0\mu)} \tag{2.41}$$

is a.c. Hence (Lemma 5a)

$$\int\limits_{D} dk \frac{\Sigma \varrho^\alpha P_\alpha(kp)}{E(k0\mu)} \tag{2.42}$$

is a.c. Using again Lemma 4 we get that

$$\int\limits_{D} dk \frac{\Sigma \varrho^\alpha P_\alpha(kp)}{E(k00)} \tag{2.43}$$

is a.c. for all values of ϱ. Applying Lemma 3a we obtain the a.c. of

$$J_\alpha = \int\limits_{D} dk \frac{P_\alpha(kp)}{E(k00)} .$$

3. Power Counting Theorem for Euclidean Metric

After the preparations of the preceding section it is not difficult to prove the power counting theorem in the general form of Theorem 1. In the notation of section 2 the integrals (1.1) and (1.5) take the form

$$I(q\mu) = I(qq\mu R_{4m}) = \int dk \frac{P(kq)}{E(kq\mu)} , \tag{3.1}$$

$$I_j(Qq\mu) = \int\limits_{II_j} dV \frac{P(kq)}{E(kq\mu)} . \tag{3.2}$$

The problem is to prove the a.c. of (3.1) provided the dimension of (3.1) is negative and every subintegral (3.2) is a.c. Introducing new variables of integration $t = (t_1 \ldots t_m)$ by (1.15–16) we can write (3.2) in the more convenient form

$$I_j(Qq\mu) = \int dt_2 \ldots dt_m \frac{P(A^{-1}t, q)}{E(A^{-1}t, q, \mu)} \tag{3.3}$$

according to Lemma 5b it suffices to prove the a.c. of the integral

$$I(q) = \int\limits_{D} dk \frac{P(kq)}{E(k00)} \tag{3.4}$$

with D defined by (1.11). We reorder the e_j such that

$$K_1^2, \ldots, K_c^2, \quad c \leqq n,$$

represent all different quadratic forms among the K_j^2. In the following we formally split $I(q)$ into various parts and prove that each term is separately a.c. First we write

$$\begin{aligned} I(q) &= \sum_{j=1}^{c} I_j(q), \\ I_j(q) &= \int_{D_j} dk \frac{P(kq)}{E(k00)} \end{aligned} \tag{3.5}$$

where D_j is defined by (1.14). With the set (1.15—16) of integration variables I_j takes the form

$$I_j(q) = \int_U dt_1 \int_V dt_2 \ldots dt_m \frac{P(A^{-1}t, q)}{E(A^{-1}t, 0, 0)} \tag{3.6}$$

where U and V are defined by (1.19) or (1.20) respectively. Next we decompose the polynomial P according to

$$P(A^{-1}t, q) = \sum_{\alpha=1}^{g} T_\alpha(tq) \tag{3.7}$$

into parts T_α which are homogeneous in $t_2, \ldots, t_m$ of degree α. Furthermore

$$T_\alpha(t\,q) = \sum_{\beta_1, \ldots \beta_4 = 1}^{g-\alpha} T_{\alpha\beta_1\ldots\beta_4}(t\,q) \tag{3.8}$$

where $T_{\alpha\beta_1\ldots\beta_4}$ is homogeneous in $(t_1)_1, \ldots, (t_1)_4$ of degree $\beta_1, \ldots, \beta_4$ respectively. We thus obtain for the integral (3.3) the decomposition

$$\begin{aligned} I(q) &= \sum_{j\alpha\beta_1\ldots\beta_4} R_{j\alpha\beta_1\ldots\beta_4}(q) \\ R_{j\alpha\beta_1\ldots\beta_4}(q) &= \int_U dt_1 \int_V dt_2 \ldots dt_m \frac{T_{\alpha\beta_1\ldots\beta_4}(t\,q)}{E(A^{-1}t, 0, 0)}. \end{aligned} \tag{3.9}$$

Making the substitution (1.12) in the inner integral we obtain

$$R_{j\alpha\beta_1\ldots\beta_4}(q) = c^\sigma \int_U dt_1\, |t_1|^\sigma \int_{V'} dt_2' \ldots dt_m' \frac{T_{\alpha\beta_1\ldots\beta_4}(t'\,q)}{E(A^{-1}t', 0, 0)} \tag{3.10}$$

with (1.24) and V' defined by (1.25). The integer σ is

$$\begin{aligned} \sigma &= 4(m-1) + \alpha + \sum \beta_i - 2n \\ &\leqq 4m + q - 2n - 4 = d - 4. \end{aligned}$$

Hence we have

$$\sigma \leqq -5. \tag{3.11}$$

In order to prove the a.c. of the integral (3.10) we now derive an upper bound for the inner integral

$$J(t'\,q) = \int\limits_{V'} dt'_2 \ldots dt'_m \frac{T_{\alpha\beta_1\ldots\beta_4}(t'\,q)}{E(A^{-1}t', 0, 0)}\,. \tag{3.12}$$

Setting

$$T_{\alpha\beta_1\ldots\beta_4}(t'\,q) = (t'_1)_1^{\beta_1} \ldots (t'_1)_4^{\beta_4} S_{\alpha\beta_1\ldots\beta_4}(t'_2 \ldots t'_m\,q) \tag{3.13}$$

we obtain

$$|T_{\alpha\beta_1\ldots\beta_4}| \leqq C\,|S_{\alpha\beta_1\ldots\beta_4}|\,, \qquad C = c^{-(\beta_1+\cdots+\beta_4)}\,. \tag{3.14}$$

So we get the following estimates for the integral (3.12)

$$\begin{aligned} |J(t'_1\,q)| &\leqq C \int\limits_{V'} dt'_2 \ldots dt'_m \frac{|S_{\alpha\beta_1\ldots\beta_4}(t'_2\ldots t'_m\,q)|}{E(A^{-1}t', 0, 0)} \\ &\leqq C_1 \int\limits_{V'} dt'_2 \ldots dt'_m \frac{|S_{\alpha\beta_1\ldots\beta_4}(t'_2\ldots t'_m\,q)|}{E(A^{-1}t', 0, \mu)} \end{aligned} \tag{3.15}$$

(because of Lemma 4)

$$\leqq C'_1 \int dt'_2 \ldots dt'_m \frac{|S_{\alpha\beta_1\ldots\beta_4}(t'_2\ldots t'_m\,q)|}{E(A^{-1}t', 0, \mu)}\,.$$

The last integral (3.15) still depends on the four vector t'_1 which, however, is bounded. Applying Lemma 2 with respect to t'_1 we obtain

$$|J(t'_1\,q)| \leqq C_2 \int dt'_2 \ldots dt'_m \frac{|S_{\alpha\beta_1\ldots\beta_4}(t'_2\ldots t'_m\,q)|}{H_j(t'_2\ldots t'_m\,\mu)} \tag{3.16}$$

with C_2 independent of t_1. H_j denotes the value of E at $t'_1 = 0$.

$$H_j(t'_2\ldots t'_m\,\mu) = E(A^{-1}\,t', 0, \mu) \quad \text{for} \quad t'_1 = 0\,. \tag{3.17}$$

We have thus found an upper bound for $J(t'_1\,q)$ which is independent of t_1. It remains to prove that the integral

$$B(q\,\mu) = \int dt_2 \ldots dt_m \frac{|S_{\alpha\beta_1\ldots\beta_4}(t_2\ldots t_m\,q)|}{H_j(t_2\ldots t_m\,\mu)} \tag{3.18}$$

converges. To this end we start from hypothesis (ii) which states that the integral (3.3) is a.c. Using Lemma 1 we obtain a.c. for

$$\int dt_2 \ldots dt_m \frac{P(A^{-1}t, q)}{H_j(t_2\ldots t_m\,\mu)}\,. \tag{3.19}$$

By Lemma 6

$$\int dt_2 \ldots dt_m \frac{T_\alpha(t\,q)}{H_j(t_2\ldots t_m\,\mu)} \tag{3.20}$$

is a.c. Finally Lemma 3a implies the a.c. of

$$\int dt_2 \ldots dt_m \frac{S_{\alpha\beta_1\ldots\beta_4}(t_2\ldots t_m\,q)}{H_j(t_2\ldots t_m\,\mu)}\,. \tag{3.21}$$

We have thus proved that the right hand side of (3.18) converges. Inserting (3.12), (3.14) and (3.18) into (3.16) we obtain

$$|R_{j\,\alpha\,\beta_1\ldots\beta_4}(q)| \leqq C_2\, A\, B(q\,\mu) \tag{3.22}$$

with

$$A = |c|^{\sigma} \int_U |t|^{\sigma}\, dt\,.$$

Since $\sigma \leqq -5$ this integral converges. This completes the proof of the theorem.

For the proof of Theorem 2 we remark that the statement holds already if H is restricted to special hyperplanes of the form

$$l_{j_1} \equiv Q_{j_1}, \ldots, l_{j_\alpha} \equiv Q_{j_\alpha} \tag{3.23}$$

with $l_{j_1}, \ldots, l_{j_\alpha}$ linearly independent. In this form the theorem follows from Theorem 1 by induction with respect to α.

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