Commun. math. Phys. 10, 114—131 (1968)

A Classification of the Unitary Irreducible Representations of $SU(N, 1)$

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Received March 7, 1968

Abstract. All inequivalent continuous unitary irreducible representations of $SU(N, 1)$ ($N \geqq 2$) have been determined and classified. The matrix elements of the infinitesimal generators realized on a certain Hilbert space have been derived. Representations of the groups $\overline{SU(N, 1)}$, $SU(N, 1)/Z_{N+1}$, $\overline{U(N, 1)}$ and $U(N, 1)$ are classified in a similar manner.

Introduction

Besides the identity representation all continuous unitary irreducible representations (CUIR) of $SU(N, 1)$ are infinite-dimensional. In this paper we shall explicitly classify all the infinite-dimensional CUIR for $N \geqq 2$, and we shall further calculate the matrix elements of the corresponding infinitesimal operators, which are realized on a certain Hilbert space. The representations of $SU(1, 1)$ have previously been derived by BARGMANN [1]. The method emploid here is the same as the one previously used in [2] to calculate the CUIR of $SO_0(N, 1)$. The method is based on the following facts:

(i) In the decomposition of a UIR of $SU(N, 1)$ with regard to the maximal compact subgroup $U(N)$ each UIR of $U(N)$ occurs at most once [3, 4].

(ii) All UIR of $U(N)$ are known and they have been classified [5, 6].

(iii) There is a one to one correspondence between algebraically irreducible unitary representations of $su(N, 1)$ and CUIR of $\overline{SU(N, 1)}$, the universal covering group of $SU(N, 1)$ [7, 4].

(iv) A CUIR of $SU(N, 1)$ satisfies locally the commutation relation of $su(N, 1)$ and the conditions for unitarity.

(v) A representation of $SU(N, 1)$ satisfies global conditions so that the unit element is represented by the unit operator on a Hilbert space.

Some authors [8] have discussed problems similar to those considered here, but for various reasons no complete and explicit classification of the CUIR of $SU(N, 1)$ has previously been given.

The Algebras $u(N)$, $u(N, 1)$ and $su(N, 1)$

An element $g(\varepsilon)$ of $U(N)$ in the neighbourhood of the unit element can be written

$$g(\varepsilon) = 1 + \varepsilon_1^{i,j} x_{i,j} + \varepsilon_2^{i,j} y_{ij} + \varepsilon_3^{j} z_j + O(\varepsilon^2)\,. \tag{1}$$

A summation is performed over i from 1 to $j - 1$ and over j from 1 to N, $\varepsilon_1^{i,j}$, $\varepsilon_2^{i,j}$ and ε_3^{j} are real numbers and $x_{i,j}$, $y_{i,j}$ and z_j are matrices of the form

$$x_{i,j} = \begin{pmatrix} & & \vdots & \\ & & 1/2 & \\ \cdots & -1/2 & \cdots & \cdots \\ & & \vdots & \end{pmatrix}$$

$$y_{i,j} = \begin{pmatrix} & & \vdots & \\ & & i/2 & \\ \cdots & i/2 & \cdots & \cdots \\ & & \vdots & \end{pmatrix}$$

$$z_j = \begin{pmatrix} \cdot & & & \\ & \cdot & & \\ & & i & \\ & & & \ddots \end{pmatrix}$$

where all matrix elements of $x_{i,j}$ and $y_{i,j}$ except those in the places (i, j) and (j, i) are zero and only the matrix element in the place (j, j) is different from zero in z_j.

From these generators we construct

$$\begin{aligned} I_{i,j} &= x_{i,j} - i y_{i,j}\,, & i < j \\ I_{j,i} &= -x_{i,j} - i y_{i,j}\,, & i < j \\ I_{j,j} &= -i z_j & \end{aligned}$$

which span a basis for the algebra $u(N)$ and satisfy the commutation relations

$$[I_{i,j}, I_{k,l}] = \delta_{k,j} I_{i,l} - \delta_{i,l} I_{k,j}\,. \tag{2}$$

The operator

$$\sum_{i=1}^{N} I_{i,i}$$

forms an ideal corresponding to the invariant subalgebra $u(1)$. The complement of this operator corresponds to the $su(N)$ subalgebra.

An element in the neighbourhood of the unit in $U(N, 1)$ can be written in the same fashion as above (1) except that the summation over j goes from 1 to $N + 1$ and

$$x_{j,N+1} = \begin{pmatrix} & & \vdots \\ & & 1/2 \\ \cdots & 1/2 & \cdots \vdots \end{pmatrix}$$

$$y_{j,N+1} = \begin{pmatrix} & & \vdots \\ & & i/2 \\ \cdots & -i/2 & \cdots \vdots \end{pmatrix}$$

$$z_{N+1} = \begin{pmatrix} & \vdots \\ \cdots & i \end{pmatrix}.$$

We also form the linear combinations

$$
\begin{aligned}
I_{j,N+1} &= x_{j,N+1} - i y_{j,N+1} \\
I_{N+1,j} &= x_{j,N+1} + i y_{j,N+1} \\
I_{N+1,N+1} &= - i z_{N+1}
\end{aligned}
\tag{3}
$$

which together with the previously defined operators satisfy the commutation relations (2).

The operator

$$\sum_{i=1}^{N+1} I_{i,i}$$

spans a one dimensional ideal. The complement of this ideal is the $su(N, 1)$ subalgebra.

As the basis operators of the algebra $su(N, 1)$ we may choose

$$
J_{p,q} = \begin{cases} I_{p,q} & \text{when } p \neq q \\ I_{p,q} - \dfrac{1}{N+1} \displaystyle\sum_{k=1}^{N+1} I_{k,k} & \text{when } p = q \, . \end{cases}
\tag{4}
$$

These operators fulfill the same commutation relations as $I_{p,q}$ and furthermore they satisfy the equation

$$\sum_{p=1}^{N+1} J_{p,p} = 0 \, . \tag{5}$$

The Representations of $U(N)$

We will now state some wellknown results [5] on the representations of $U(N)$. Since the complex Lie algebra $u(N)$ is isomorphic to the complex Lie algebra $gl(N, R)$ the UIR of $U(N)$ may be classified similarly to the way in which the finite dimensional irreducible representations of $GL(N, R)$ were classified by GELFAND and TSEITLIN [6].

Therefore, every UIR of the group $U(N)$ is specified by N integers $m_{1,N} \geqq m_{2,N} \geqq \cdots \geqq m_{N-1,N} \geqq m_{N,N}$. Consider all possible arrays of integers of the form

$$
\alpha = \begin{pmatrix}
m_{1,N} & & m_{2,N} & & m_{3,N} & \cdots & & m_{N-1,N} & & m_{N,N} \\
& m_{1,N-1} & & m_{2,N-1} & & \cdots & m_{N-2,N-1} & & m_{N-1,N-1} & \\
& & m_{1,N-1} & & & \cdots & & m_{N-2,N-2} & & \\
& & & m_{1,3} & & m_{2,3} & & m_{3,3} & & \\
& & & & m_{1,2} & & m_{2,2} & & & \\
& & & & & m_{1,1} & & & &
\end{pmatrix}
$$

where the integers $m_{p,q}$ satisfy the conditions

$$m_{p,q+1} \geqq m_{p,q} \geqq m_{p+1,q+1} \, .$$

To each array α we assign a vector $\xi(\alpha)$. The irreducible representations of the Lie algebra $u(N)$, which are related to representations of the group $U(N)$, act on the finite dimensional vector space R spanned by the vectors $\xi(\alpha)$. Let the infinitesimal generator $I_{j,k}$ be represented by the operator $D_{j,k}$ acting on R. To specify the matrix form of $D_{j,k}$ in the basis $\{\xi(\alpha)\}$ it is sufficient to give the form of the operators $D_{k+1,k}$, $D_{k,k}$ and $D_{k,k+1}$ as the others may be obtained from them by means of the commutation relations.

Let α_k^{+i} denote the array which is obtained from the array α by changing $m_{i,k}$ to $m_{i,k}+1$ and let α_k^{-i} denote the array which is obtained from the array α by changing $m_{i,k}$ to $m_{i,k}-1$. Then we have

$$D_{k,k+1}\,\xi(\alpha) = \sum_{i=1}^{k} a_k^i(\alpha)\,\xi(\alpha_k^{+i})$$

$$D_{k,k}\quad \xi(\alpha) = \left(\sum_{i=1}^{k} m_{i,k} - \sum_{i=1}^{k-1} m_{i,k-1}\right)\xi(\alpha)$$

$$D_{k+1,k}\,\xi(\alpha) = \sum_{i=1}^{k} b_k^i(\alpha)\,\xi(\alpha_k^{-i})$$

where the numbers $a_k^i(\alpha)$ and $b_k^i(\alpha)$ are given by the formulae

$$a_k^j(\alpha) = b_k^j(\alpha_k^{+j}) = \left[-\frac{\prod_{i=1}^{k+1}(l_{i,k+1}-l_{j,k})\prod_{i=1}^{k-1}(l_{i,k-1}-l_{j,k}-1)}{\prod_{i\neq j}(l_{i,k}-l_{j,k}-1)\,(l_{i,k}-l_{j,k})}\right]^{1/2}$$

where the notation

$$l_{i,j} = m_{i,j} - i$$

has been introduced.

To get all the Hilbert space representations of $u(N)$ we must generalize the scheme above and let $m_{p,q}$ also take on non-integral values with the restriction that $m_{i,j} - m_{p,q}$ must always be integers. Hence the numbers $m_{i,j}$ shall have the same integer remainder.

The Hilbert Space of the Representations

When we derive the representations of $SU(N, 1)$ we will introduce the requirement (5) last. In this way we obtain the representations of $U(N, 1)$ as an intermediate result.

The maximal compact subgroup of $SU(N, 1)$ is $U(N)$. An infinitesimal generator of the invariant $U(1)$ subgroup is

$$K = -(N+1)\,J_{N+1,N+1} = \sum_{r=1}^{N} I_{r,r} - N I_{N+1,N+1}\,.$$

And we find

$$\exp(\mathrm{it}K) = 1 \qquad \text{for} \quad t = \ldots, 0, 2\pi, 4\pi, \ldots$$

The maximal compact subgroup of $U(N, 1)$ is $U(N) \times U(1)$. As the infinitesimal generators of the two invariant $U(1)$ subgroups we may choose

$$\sum_{r=1}^{N} I_{r,r} \quad \text{and} \quad \sum_{r=1}^{N+1} I_{r,r}.$$

When an irreducible representation of $SU(N, 1)$ is restricted to the maximal subgroup $U(N)$ the irreducible parts have the multiplicity zero or one [3, 4]. The corresponding statement holds for $U(N, 1)$, since $U(N, 1)$ is the direct product of two groups $U(N, 1) \approx SU(N, 1)/Z_{N+1} \times U(1)$, where the $U(1)$ group enters in the maximal compact subgroup. This $U(1)$ subgroup is represented by one fixed representation in a representation of $U(N, 1)$. In the construction below we may therefore ignore this subgroup.

As the Hilbert space H for representations of $U(N, 1)$ (and $SU(N, 1)$) we choose

$$H = \sum_{(l_{2,N}, l_{2,N}, \ldots) \in \Gamma} \oplus\, H(l_{1,N}, l_{2,N}, \ldots) = \sum_{\alpha} \oplus\, \xi(\alpha)$$

where $H(l_{1,N}, l_{2,N}, \ldots)$ is the Hilbert space of the representation of $U(N)$ which is labelled by $(l_{1,N}, l_{2,N}, \ldots)$. Γ indicates which representations of $U(N)$ that appear in the representation of $U(N, 1)$, and $\xi(\alpha)$ is a vector in the Hilbert space of such a representation.

Without much extra labour we can generalize the above scheme and get Hilbert space representations of the algebras $u(N, 1)$ and $su(N, 1)$. This can be done by summing over representations to $u(N)$ instead of representations to $U(N)$, which means that we let the numbers $l_{i,N}$ take also non-integral values.

We are now going to find the possible form of the matrix elements of the representations $D_{N,N+1}$, $D_{N+1,N}$ and $D_{N+1,N+1}$ of the operators $I_{N,N+1}$, $I_{N+1,N}$ and $I_{N+1,N+1}$.

Restrictions on the Matrix Elements from the Commutation Relations

We now exploit the full content of the commutation relations. This can be done in much the same way as in [2]. In the present case it is somewhat simpler as we here can use commutation relations of the first order more often. We will not repeat this derivation here but just state the appropriate order in which the commutation relations should be exploited to get hold of all conditions with a minimum of calculations.

We also summarize these conditions. The relations

$$
\begin{aligned}
&[J_{N,N+1}, J_{i,j}] = 0\,, \quad \text{if } i \text{ and } j < N\\
&[J_{N+1,N+1}, J_{i,j}] = 0\,, \quad \text{if } i \text{ and } j \leqq N\\
&[J_{N,N+1}, J_{N,N}] = -J_{N,N+1}\\
&[J_{N,N+1}, J_{N,N-1}] = 0\\
&[J_{N,N+1}, [J_{N,N+1}, J_{N-1,N}]] = 0\\
&[J_{N+1,N}, [J_{N+1,N}, J_{N,N-1}]] = 0\\
&[J_{N,N+1}, [J_{N+1,N}, J_{N,N-1}]] = J_{N,N-1}\\
&[J_{N+1,N}, [J_{N,N+1}, J_{N-1,N}]] = J_{N-1,N}\\
&[J_{N+1,N+1}, J_{N,N+1}] = -J_{N,N+1}\\
&[J_{N,N+1}, J_{N+1,N}] = J_{N,N} - J_{N+1,N+1}
\end{aligned}
$$

imply that

$$
\begin{aligned}
D_{N,N+1} \quad \xi(\alpha) &= \sum_{j=1}^{N} \varphi_j(\alpha)\, \xi(\alpha_N^{+j})\\
D_{N+1,N} \quad \xi(\alpha) &= \sum_{j=1}^{N} \psi_j(\alpha)\, \xi(\alpha_N^{-j})\\
D_{N+1,N+1}\, \xi(\alpha) &= \left(\chi_{N+1} - \sum_{j=1}^{N} l_{j,N}\right) \xi(\alpha)
\end{aligned}
$$

where χ_{N+1} is a constant, and where the matrix elements $\varphi_j(\alpha)$ and $\psi_j(\alpha)$ satisfy the equation

$$
\begin{aligned}
&\sum_{j=1}^{N} [\varphi_j(\alpha_N^{-j})\, \psi_j(\alpha) - \varphi_j(\alpha)\, \psi_j(\alpha_N^{+j})]\\
&\qquad = -\chi_{N+1} + 2 \sum_{j=1}^{N} l_{j,N} - \sum_{j=1}^{N-1} l_{j,N-1} + N\,.
\end{aligned}
\tag{6}
$$

The relations further imply that a set of functions $f_j(l_{j,N})$, which only depend on the respective $l_{j,N}$, can be defined for all j through the relations

$$
\begin{aligned}
&\prod_{i=1}^{N-1} |l_{i,N-1} - l_{j,N} - 1|\, f_j(l_{j,N})\\
&\qquad = \varphi_j(\alpha)\, \psi_j(\alpha_N^{+j}) \prod_{i \neq j} (l_{i,N} - l_{j,N})\, (l_{i,N} - l_{j,N} - 1)
\end{aligned}
$$

when $l_{j-1,N} \neq l_{j,N} + 1$ and $l_{j-1,N-1} \neq l_{j,N} + 1$, and

$$f_j(l_{j,N}) = 0$$

when $l_{j-1,N} = l_{j,N} + 1$.

Before we proceed it is conveneant to define some notations. Let $l_{i,N,\min}$ and $l_{i,N,\max}$ denote the minimal and the maximal values respectively of $l_{i,N}$ in a representation. In the calculations the $l_{i,N}$: s take their maximal values for certain values of i and their minimal values for other i:s. We let a number i_0 determine the conditions for each particular $l_{i,N}$. Let $l_{i,N,\mathrm{extr}}$ denote $l_{i,N,\min}$ for i less than or equal to i_0, and $l_{i,N,\max}$ for i greater than i_0. We further define:

$$
\begin{aligned}
V &= \{r \mid l_{r,N,\min} \neq l_{r,N,\max}\} \\
T(i_0) &= \{r \mid l_{r,N,\mathrm{extr}} \neq l_{r+1,N,\mathrm{extr}} + 1 \text{ and } r < N\} \\
T'(i_0) &= T(i_0) \cup \{N\} \\
Q(i_0) &= T'(i_0) \cap V \\
R(i_0) &= \{r \mid r \in V \text{ and, } l_{r,N,\mathrm{extr}} \neq l_{r-1,N,\mathrm{extr}} - 1 \text{ or } r = N\} \\
U(i_0) &= \{r \mid r \in T'(i_0) \text{ and } r \in V\} \\
Y(i_0) &= \{r \mid r \in V \text{ and } l_{r,N,\mathrm{extr}} = l_{r-1,N,\mathrm{extr}} - 1\} \\
W(i_0) &= \{r \mid r \in V \text{ and } r \notin T'(i_0)\} \\
J(i_0) &= \{r \mid r \in V \text{ and } r \leqq i_0, \text{ or } l_{r,N} \neq l_{r,N,\mathrm{extr}} \text{ and } l_{r-1,N} \neq l_{r,N} + 1\} \\
K(i_0) &= \{r \mid r \in V \text{ and } r \leqq i_0, \text{ or } l_{r,N} \neq l_{r,N,\mathrm{extr}} \text{ and } l_{r,N} \neq l_{r+1,N} + 1\} \\
L &= \{r \mid l_{r,N} \neq l_{r+1,N} + 1 \text{ and } r < N\} \\
M &= \{r \mid l_{r,N} \neq l_{r+1,N} + 1 \text{ or } r = N\}\,.
\end{aligned}
$$

When a set S or $S(i_0)$ is defined we let

$$
\begin{aligned}
S(i_0, +) &= \{r \mid r \in S \text{ and } r > i_0\} \\
S(i_0, -) &= \{r \mid r \in S \text{ and } r \leqq i_0\} \\
F(i_0) &= U(i_0, -) \cup T'(i_0, +) \\
G(i_0) &= U(i_0, +) \cup T'(i_0, -)\,.
\end{aligned}
$$

Finally let $n(i_0)$ and λ be the number of elements in $T(i_0)$ and L respectively.

We proceede to find all solutions f_i and χ_{N+1} of the Eq. (6). The details in the calculations will vary from case to case. One may distinguish between a large number of different subcases. The large number of sets was defined to compensate for this difficulty. The main feature in the calculation is the same in all these cases, and the sets enable a rather unified derivation. The strategy we follow is first to show that the f_i:s are polynomials, then that they are the same polynomial for different values of i and then find its properties and finally to calculate χ_{N+1}.

The Eq. (6) can be written on the standard form

$$\sum_{i\in J(i_0)} (-1)^i \frac{\prod\limits_{r\in L} (l_{i,N} - l_{r,N-1} + 1)\, f_i(l_{i,N})}{\prod\limits_{\substack{r\in M\\ r\neq i}} (l_{i,N} - l_{r,N} + 1) \prod\limits_{r\neq i} (l_{i,N} - l_{r,N})}$$

$$- \sum_{i\in K(i_0)} (-1)^i \frac{\prod\limits_{r\in L} (l_{i,N} - l_{r,N-1})\, f_i(l_{i,N} - 1)}{\prod\limits_{\substack{r\in M\\ r\neq i}} (l_{i,N} - l_{r,N}) \prod\limits_{r\neq i} (l_{i,N} - l_{r,N} - 1)} \tag{7}$$

$$= -\chi_{N+1} + \sum_{r=1}^{N} l_{r,N} + \sum_{r\in M} l_{r,N} - \sum_{r\in L} l_{r,N-1} + N\,.$$

Identifying terms with the same dependence on the $l_{r,N-1}$ yields

$$\sum_{i\in J(i_0)} (-1)^i \frac{(l_{i,N} + 1)^\lambda f_i(l_{i,N})}{\prod\limits_{\substack{r\in M(i_0)\\ r\neq i}} (l_{i,N} - l_{r,N} + 1) \prod\limits_{r\neq i} (l_{i,N} - l_{r,N})}$$

$$- \sum_{i\in K(i_0)} (-1)^i \frac{l_{i,N}^\lambda f_i(l_{i,N} - 1)}{\prod\limits_{\substack{r\in M(i_0)\\ r\neq i}} (l_{i,N} - l_{r,N}) \prod\limits_{r\neq i} (l_{i,N} - l_{r,N} - 1)} \tag{8}$$

$$= -\chi_{N+1} + \sum_{r=1}^{N} l_{r,N} + \sum_{r\in M(i_0)} l_{r,N} + N\,,$$

$$\sum_{i\in J(i_0)} (-1)^i \frac{(l_{i,N} + 1)^{\lambda-1} f_i(l_{i,N})}{\prod\limits_{\substack{r\in M(i_0)\\ r\neq i}} (l_{i,N} - l_{r,N} + 1) \prod\limits_{r\neq i} (l_{i,N} - l_{r,N})}$$

$$- \sum_{i\in K(i_0)} (-1)^i \frac{l_{i,N}^{\lambda-1} f_i(l_{i,N} - 1)}{\prod\limits_{\substack{r\in M(i_0)\\ r\neq i}} (l_{i,N} - l_{r,N} + 1) \prod\limits_{r\neq i} (l_{i,N} - l_{r,N} - 1)} = 1\,, \tag{9}$$

if $\lambda \geqq 1$
and

$$\sum_{i\in J(i_0)} (-1)^i \frac{(l_{i,N} + 1)^\nu f_i(l_{i,N})}{\prod\limits_{\substack{r\in M(i_0)\\ r\neq i}} (l_{i,N} - l_{r,N} + 1) \prod\limits_{r\neq i} (l_{i,N} - l_{r,N})}$$

$$- \sum_{i\in K(i_0)} (-1)^i \frac{l_{i,N}^\nu f_i(l_{i,N})}{\prod\limits_{\substack{r\in M(i_0)\\ r\neq i}} (l_{i,N} - l_{r,N}) \prod\limits_{r\neq i} (l_{i,N} - l_{r,N} - 1)} = 0 \tag{10}$$

$$\text{for } \nu = 0, \ldots, \lambda - 2 \text{ if } \lambda \geqq 2\,.$$

We next choose i_0 and further we let all the $l_{i,N}$:s for i different from j take their extremal values $l_{i,N,\text{extr}}$.

By starting with the extremal value on $l_{j,N}$ and then shifting it by one unit at the time we get a series of equations from which we can calculate $f_j(l_{j,N})$. Thus for $n(i_0) \geqq 2, j \in Q(i_0, -)$ and $l_{j,N,\max} \leqq l_{j,N} < l_{j,N,\max}$. We find

$$(-1)^{j+1} \frac{f_j(l_{j,N})}{\prod\limits_{\substack{r \in T'(i_0) \\ r \neq j}} (l_{j,N} - l_{r,N,\mathrm{extr}} + 1) \prod\limits_{r \neq j} (l_{j,N} - l_{r,N,\mathrm{extr}})}$$

$$= \sum_{l = l_{j,N,\min}}^{l_{j,N}} \sum_{\substack{i \in V(i_0,-) \\ i \neq j}} (-1)^i \frac{f_i(l_{i,N,\min})}{\prod\limits_{\substack{r \in T'(i_0) \\ r \neq i \\ r \neq j}} (l_{i,N,\min} - l_{r,N,\mathrm{extr}} + 1) \prod\limits_{\substack{r \neq i \\ r \neq j}} (l_{i,N,\min} - l_{r,N,\mathrm{extr}})}$$

$$\cdot \frac{1}{(l_{i,N,\min} - l + 1)(l_{i,N,\min} - l)}$$

$$- \sum_{l = l_{j,N,\min}}^{l_{j,N}} \sum_{i \in V(i_0,+)} (-1)^i \frac{f_i(l_{i,N,\max} - 1)}{\prod\limits_{\substack{r \in T'(i_0) \\ r \neq i \\ r \neq j}} (l_{i,N,\max} - l_{r,N,\mathrm{extr}}) \prod\limits_{\substack{r \neq i \\ r \neq j}} (l_{i,N,\max} - l_{r,N,\mathrm{extr}} - 1) \cdot (l_{i,N,\max} - l - 1) \cdot (l_{i,N,\max} - l)}$$

$$= \sum_{\substack{i \in V(i_0,-) \\ i \neq j}} (-1)^i \frac{f_i(l_{i,N,\min})(l_{j,N} - l_{j,N,\min} + 1)}{\prod\limits_{\substack{r \in T'(i_0) \\ r \neq i}} (l_{i,N,\min} - l_{r,N,\mathrm{extr}} + 1) \prod\limits_{\substack{r \neq i \\ r \neq j}} (l_{i,N,\min} - l_{r,N,\mathrm{extr}})(l_{i,N,\min} - l_{j,N})}$$

$$- \sum_{i \in V(i_0,+)} (-1)^i \frac{f_i(l_{i,N,\max} - 1)(l_{j,N} - l_{j,N,\min} + 1)}{\prod\limits_{\substack{r \in T'(i_0) \\ r \neq i}} (l_{i,N,\max} - l_{r,N,\mathrm{extr}}) \prod\limits_{\substack{r \neq i \\ r \neq j}} (l_{i,N,\max} - l_{r,N,\mathrm{extr}} - 1)(l_{i,N,\max} - l_{j,N} - 1)}$$

so that

$$(-1)^j f_j(l_{j,N})$$

$$= \prod_{r \in T'(i_0)} (l_{j,N} - l_{r,N,\mathrm{extr}} + 1) \prod_{r \notin V'(i_0,-)} (l_{j,N} - l_{r,N,\mathrm{extr}})$$

$$\cdot \sum_{i \in V(i_0,-)} (-1)^i \frac{f_i(l_{i,N,\min}) \prod\limits_{\substack{r \in V(i_0,-) \\ r \neq i}} (l_{j,N} - l_{r,N,\min})}{\prod\limits_{\substack{r \in T'(i_0) \\ r \neq i}} (l_{i,N,\min} - l_{r,N,\mathrm{extr}} + 1) \prod\limits_{r \neq i} (l_{i,N,\min} - l_{r,N,\mathrm{extr}})}$$

$$- \prod_{r \in G(i_0)} (l_{j,N} - l_{r,N,\mathrm{extr}} + 1) \prod_{r=1}^{N} (l_{j,N} - l_{r,N,\mathrm{extr}}) \tag{11}$$

$$\cdot \sum_{i \in V(i_0+)} (-1)^i \frac{f_i(l_{i,N,\max} - 1) \prod\limits_{\substack{r \in V(i_0,+) \\ r \neq i}} (l_{j,N} - l_{r,N,\max} + 1)}{\prod\limits_{\substack{r \in T'(i_0) \\ r \neq i}} (l_{i,N,\max} - l_{r,N,\mathrm{extr}}) \prod\limits_{r \neq i} (l_{i,N,\max} - l_{r,N,\mathrm{extr}} - 1)}$$

$$\equiv g_j^{(1)}(l_{j,N}).$$

Similarly when $n(i_0) \geqq 2, j \in V(i_0, +)$ and $l_{j,N,\min} < l_{j,N} \leqq l_{j,N,\max}$

$$
\begin{aligned}
(-1)^j f_j(l_{j,N}) = & \prod_{r \in F(i_0)} (l_{j,N} - l_{r,N,\mathrm{extr}} + 1) \prod_{r=1}^{N} (l_{j,N} - l_{r,N,\mathrm{extr}}) \\
& \cdot \sum_{i \in V(i_0,-)} (-1)^i \frac{f_i(l_{i,N,\min}) \prod\limits_{\substack{r \in V(i_0,-) \\ r \neq i}} (l_{j,N} - l_{r,N,\min} + 1)}{\prod\limits_{\substack{r \in T'(i_0) \\ r \neq i}} (l_{i,N,\max} - l_{r,N,\mathrm{extr}} + 1) \prod\limits_{r \neq i} (l_{i,N,\max} - l_{r,N,\mathrm{extr}})} \\
& - \prod_{r \in T'(i_0)} (l_{j,N} - l_{r,N,\mathrm{extr}} + 1) \prod_{r \notin V(i_0,+)} (l_{j,N} - l_{r,N,\mathrm{extr}}) \\
& \cdot \sum_{i \in V(i_0,+)} (-1)^i \frac{f_i(l_{i,N,\max} - 1) \prod\limits_{\substack{r \in V(i_0,+) \\ r \neq i}} (l_{j,N} - l_{r,N,\max})}{\prod\limits_{\substack{r \in T'(i_0) \\ r \neq i}} (l_{i,N,\max} - l_{r,N,\mathrm{extr}}) \prod\limits_{r \neq i} (l_{i,N,\max} - l_{r,N,\mathrm{extr}} - 1)} \\
\equiv & \; g_j^{(2)}(l_{j,N}) \, .
\end{aligned}
\tag{12}
$$

When $n(i_0) = 1, j \in Q(i_0)$ and when $n(i_0) \geqq 1, j \in W(i_0)$ we can derive the same expression as befor on $f_j(l_{j,N})$ with similar calculations.

When $n(i_0) = 0$ there are three possibilities for the set V; it can be $\{1\}$, $\{N\}$ or $\{1, N\}$. We find when $1 \in W(i_0)$ for some i_0

$$
\begin{aligned}
f_1(l_{1,N}) = & \; g_1^{(1)}(l_{1,N}) \\
& + (l_{1,N} - l_{N,N,\max} + 1) \prod_{r=1}^{N} (l_{1,N} - l_{r,N,\mathrm{extr}})
\end{aligned}
\tag{13}
$$

and when $N \in Y(i_0)$ for some i_0

$$
\begin{aligned}
f_N(l_{N,N}) = & \; g_N^{(2)}(l_{N,N}) \\
& + (l_{N,N} - l_{N,N,\max} + 1) \prod_{r=1}^{N} (l_{N,N} - l_{r,N,\mathrm{extr}}) \, .
\end{aligned}
\tag{14}
$$

The Eqs. (11)–(14) show that f_j are polynomials. We want to show that it is the same polynomial independently of j. All f_j where j lies on the same side of i_0 must, according to the equations, have the same form. By taking i_0 on the both sides of an element in V we can show that the form is the same on both sides of i_0. This is possible except in the case when $V = \{1, N\}$ and neither $l_{1,N,\max}$ nor $l_{N,N,\min}$ exists. This case will be handled separately. By the Eq. (9) we can determine the degree of the polynominal. As (9) holds for an infinite set of arguments the overall degree of the fraction in the left member is one. The degree in the denominator is $N + \lambda - 1$. So the degree of f is $N + 1$. It also follows immediately from the Eq. (9) that the coefficient in front of the term of the higest power is $(-1)^j$. Therefore we can write

$$
(-1)^j f_j(l_{j,N}) = \prod_{r=1}^{N+1} (l_{j,N} - l_{r,N+1})
$$

when $j \in V(i_0, -)$ and $l_{j,N,\min} - 1 \leqq l_{j,N,\max}$ and when $j \in V(i_0, +)$ and $l_{j,N,\min} \leqq l_{j,N} \leqq l_{j,N,\max}$. By the construction $f_j(l_{j,N})$ has to be zero for $l_{j,N,\max}$ and $l_{j,N,\min} - 1$ for $j \in V$. By inspection of the Eqs. (11)–(14) we find that $f_j(l_{j,N})$ is zero for all $l_{r,N}$ when V does not contain r and $l_{r,N,\min} - 1$ when $r \in T'(i_0)$ for some i_0. This puts restrictions on the number $l_{r,N,\max}$ and $l_{r,N,\min} - 1$ as at most $N + 1$ of these can be different.

In the case when $V = \{1, N\}$ and neither $l_{1,N,\max}$ nor $l_{N,N,\min}$ exists we can establish that f_1 and f_N have the same form in the following way. That they are polynomials of degree $N + 1$, that they have $N - 1$ roots in common and that the coefficient in front of the term with the degree $N + 1$ is -1 and $(-1)^N$ respectively follows as before. Let the remaining roots be u, v and x, y respectively. From Eq. (10) with both $l_{1,N}$ and $l_{N,N}$ free follows then that

$$(l_{1,N} + l_{N,N} - 1)(u + v - x - y) - 2uv + 2xy = 0$$

shall hold identically in $l_{1,N}$ and $l_{N,N}$. Therefore the remaining roots must be the same, so that f_1 and f_N have the same form even in this case.

From Eq. (8) we can easily evaluate χ_{N+1} by putting a very large $l_{1,N}$ (or if $l_{1,N,\max}$ exists, a negative $l_{N,N}$ with a very large absolute value) into Eq. (8). This yields

$$\chi_{N+1} = \sum_{r=1}^{N+1} l_{r,N+1} + N + 1\,.$$

Conditions for Unitarity, Irreducibility and Inequivalence

The unitarity of the representations and the Eq. (3) requires that χ_{N+1} is real and

$$\psi_j(\alpha) = -\overline{\varphi_j(\alpha_N^{-j})}\,.$$

We can change the phases of the vectors $\xi(\alpha)$ by multiplying them by a factor $\prod_{r=1}^{N} \omega_r(l_{r,N})$ of modulus one so that φ_j become positive or zero on Γ. We have, therefore,

$$D_{N,N+1}\xi(\alpha) = \sum_{j=1}^{N} \varphi_j(\alpha)\,\xi(\alpha_N^{+j})$$

$$D_{N+1,N}\xi(\alpha) = \sum_{j=1}^{N} \psi_j(\alpha)\,\xi(\alpha_N^{-j})$$

$$D_{N+1,N+1}\xi(\alpha) = \left(\sum_{j=1}^{N+1} l_{j,N+1} - \sum_{j=1}^{N} l_{j,N} + N + 1\right)\xi(\alpha)$$

where

$$\varphi_j(\alpha) = \left| \frac{\prod\limits_{r=1}^{N+1} (l_{j,N} - l_{r,N+1}) \prod\limits_{r=1}^{N-1} (l_{j,N} - l_{r,N-1} + 1)}{\prod\limits_{r \neq j} (l_{j,N} - l_{r,N} + 1)(l_{j,N} - l_{r,N})} \right|^{1/2}$$

and

$$\psi_j(\alpha) = - \overline{\varphi_j(\alpha_N^{-j})} .$$

For conveneance the expression for $\varphi_j(\alpha)$ has been extended. Owing to this the denominator may become zero for certain choice of α. When this occur $\varphi_j(\alpha)$ is zero.

That the representations obtained in this way are irreducible follows immediately from the fact that the representations of $u(N)$ occur at most once, and that we can not devide Γ into two parts so that all matrix elements between vectors in the different parts are zero.

To determine equivalence conditions it is suitable to establish the conditions which must be satisfied by a unitary operator that transforms one Gelfand-Tseitlin base into another Gelfand-Tseitlin base and is consistent with the phase convention. Clearly it must not mix the irreducible spaces of the $u(N)$-subalgebra since they correspond to inequivalent representations. This then implies that the unitary transformation irrespective of an irrelevant scalar phase-factor, reduces to a direct sum of unitary transformations in the irreducible spaces of $u(N)$. We can now proceed to $u(N-1)$, $u(N-2)$, ... and repeat the argument. So we find that irrespective of an irrelevant scalar phase factor the unit operator is the only unitary transformation that transfers one Gelfand-Tseitlin base into another Gelfand-Tseitlin base. Therefore two representations are equivalent if and only if all the matrix elements are the same in the two representations.

The Representations of $u(N, 1)$

Let us now summarize the results for the representations of $u(N, 1)$. We distinguish between nine kinds of representations distributed on three classes. For each class we give the requirements on all representation in the class, and thereafter we list the different cases. In each case we first write the notation of the representations $D(j, x; l_{1,N+1}, \ldots, l_{N+1,N+1})$ where the integer j indicates the class and together with the letter x the kind of the representation. The numbers $l_{1\,N+1}, \ldots, l_{N+1,N+1}$ then indicate the particular representation. The notation is followed by a prescription for the numbers $l_{i,N+1}$ which ends with a period. Thereafter follows a prescription for the range of the numbers $l_{i,N}$. To clearify the conditions we show in a diagram for a typical representation of the particular kind the values of $l_{i,N+1}$ and the range of the values $l_{j,N}$.

For every different choice of the numbers $l_{i,N+1}$ that satisfies the conditions below $u(N, 1)$ has an inequivalent unitary irreducible representation.

Class 1. $l_{N,N}$ bounded from below

The numbers $l_{i,N+1}$ are ordered so that $l_{1,N+1} > l_{2,N+1} > \cdots > l_{N,N+1}$, and $l_{1,N+1} - l_{i,N+1}$ are integer for $i = 2, \ldots, N$.

$D(1, s;\ l_{1,N+1}, \ldots, l_{N+1,N+1})$, where $l_{1,N+1} - 1_{N+1,N+1}$ is integer, $l_{N,N+1} < l_{N+1,N+1} < l_{1,N+1}$ and $l_{i,N+1} = l_{N,N+1} + N - i$ when $l_{i,N+1} < l_{N+1,N+1}$.

$l_{1,N} \geqq l_{1,N+1} + 1$; $l_{i,N} = l_{i,N+1} + 1$ when $l_{i,N+1} < l_{N+1,N+1}$; $l_{i-1,N+1} \geqq l_{i,N} \geqq l_{i,N+1+1}$ when $l_{i,N+1} > l_{N+1,N+1}$.

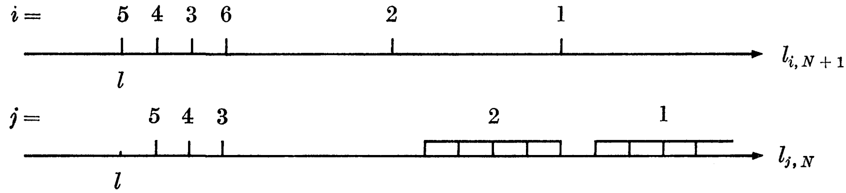

Fig. 1, s. The representations of $u(5)$ in the representation $D(1, s; l + 13, l + 8, l + 2, l + 1, l, l + 3)$ of $u(5, 1)$

$D(1, e;\ l_{1,N+1}, \ldots, l_{N+1,N+1})$, where $l_{N+1,N+1}$ is real, $l_{N+1,N+1} < l_{1,N+1} + 1$, $l_{i,N+1} = l_{N,N+1} + N - i$ when $N \geqq i \geqq N + l_{N,N+1} - l_{N+1,N+1}$.

$l_{1,N} \geqq l_{1,N+1} + 1$; $l_{i-1,N+1} \geqq l_{i,N} \geqq l_{i,N+1} + 1$ for $i = 2, \ldots, N$.

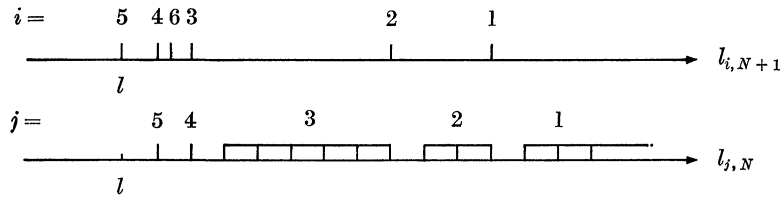

Fig. 1, e. The representations of $u(5)$ in the representation $D(1, e; l + 11, l + 8, l + 2, l + 1, l, l + 1.4)$ of $u(5, 1)$

Class 2. $l_{1,N}$ bounded from above

The numbers $l_{i,N+1}$ are ordered so that $l_{1,N+1} > l_{2,N+1} > \ldots > l_{N,N+1}$, and $l_{N+1} - l_{i,N+1}$ is integer for $i = 2, \ldots, N$.

$D(2, s;\ l_{1,N+1}, \ldots, l_{N+1,N+1})$, where $l_{1,N+1} - l_{N+1,N+1}$ is integer, $l_{N,N+1} < l_{N+1,N+1} < l_{1,N+1}$ and $l_{i,N+1} = l_{1,N+1} + 1 - i$ when $l_{i,N+1} > l_{N+1,N+1}$.

$l_{N,N} \leqq l_{N,N+1}$; $l_{i,N+1} \geqq l_{i,N} \geqq l_{i+1,N+1} + 1$ when $l_{i,N+1} < l_{N+1,N+1}$; $l_{i,N} = l_{i,N+1}$ when $l_{i,N+1} > l_{N+1,N+1}$.

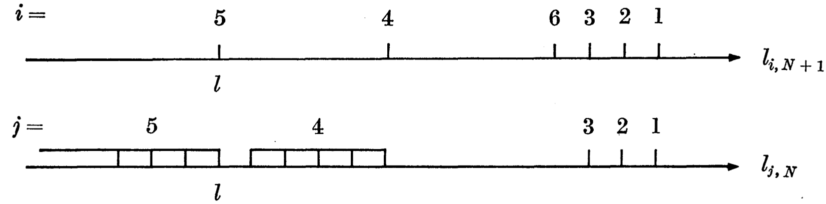

Fig. 2, s. The representations of $u(5)$ in the representation $D(2, s; l + 13, l + 12, l + 11, l + 5, l, l + 10)$ of $u(5, 1)$

$D(2, e; l_{1,N+1}, \ldots, l_{N+1,N+1})$, where $l_{N+1,N+1}$ is real, $l_{2,N+1} > l_{N+1,N+1} > l_{N,N+1} - 1$, $l_{i,N+1} = l_{1,N+1} + 1 - i$ when $1 \leqq i \leqq 1 + l_{1,N+1} - l_{N+1,N+1}$.

$l_{N,N} \leqq l_{N,N+1}$; $l_{i,N+1} \geqq l_{i,N} \geqq l_{i+1,N+1} + 1$ for $i = 1, \ldots, N - 1$.

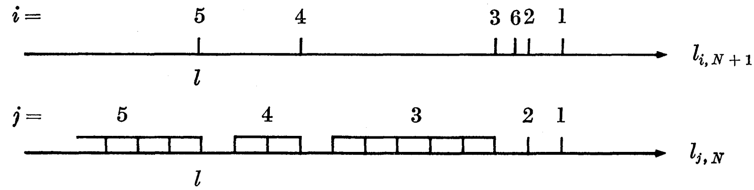

Fig. 2, e. The representations of $u(5)$ in the representation $D(2, e; l + 11, l + 10, l + 9, l + 3, l, l + 9.6)$ of $u(5, 1)$

Class 3. Neither is $l_{1,N}$ bounded from above nor $l_{N,N}$ from below

The numbers $l_{i,N+1}$ are ordered so that

$l_{1,N+1} > l_{2,N+1} > \cdots > l_{N-1,N+1}$, and $l_{1,N+1} - l_{i,N+1}$ is integer for $i = 2, \ldots, N - 1$.

$D(3, s; l_{1,N+1}, \ldots, l_{N+1,N+1})$, where $l_{1,N+1} - l_{N,N+1}$ and $l_{1,N+1} - l_{N+1,N+1}$ are integral, $l_{N-1,N+1} < l_{N+1,N+1} < l_{N,N+1} < l_{1,N+1}$ and $l_{i,N+1} = l - i$ for some l when $l_{N+1,N+1} < l_{i,N+1} < l_{N,N+1}$.

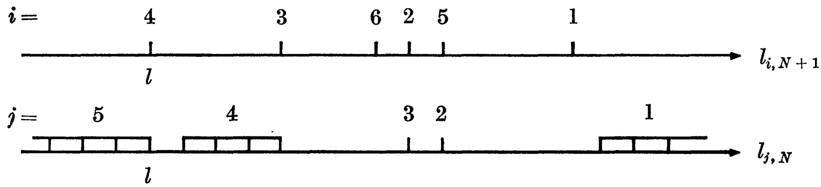

Fig. 3, s. The representations of $u(5)$ in the representation $D(3, s; l + 13, l + 8, l + 4, l, l + 9, l + 7)$ of $u(5, 1)$

$l_{1,N} \geqq l_{1,N+1} + 1$; $l_{i-1,N+1} \geqq l_{i,N} \geqq l_{i,N+1} + 1$ when $l_{1,N+1} > l_{i,N+1} > l_{N,N+1}$ or $l_{N+1,N+1} > l_{i,N+1} > l_{N-1,N+1}$.

$l_{i,N} = l_{i,N+1} + 1$ and $l_{i+1,N} = l_{i,N+1}$ when $l_{N+1,N+1} < l_{i,N+1} < l_{N,N+1}$; $l_{N,N} \leqq l_{N-1,N+1}$.

$D(3,\ e;\ l_{1,N+1}, \ldots, l_{N+1,N+1})$ where $l_{N,N+1}$ and $l_{N+1,N+1}$ are real, $l_{N-1,N+1} - 1 < l_{N+1,N+1} \leqq l_{N,N+1} < l_{1,N+1} + 1$, $l_{i,N+1} = l - i$ for some l when $l_{N+1,N+1} \leqq l_{i,N+1} \leqq l_{N,N+1}$, $|l_{j,N+1} - l_{N,N+1}| < 1$ and $|l_{k,N+1} - l_{N+1,N+1}| < 1$ for some $j \leqq N-1$ and some $k \leqq N-1$.

$l_{1,N} \geqq l_{1,N+1} + 1$; $l_{i-1,N+1} \geqq l_{i,N} \geqq l_{i,N+1} + 1$ for $i = 2, \ldots, N-1$; $l_{N,N} \leqq l_{N-1,N+1}$.

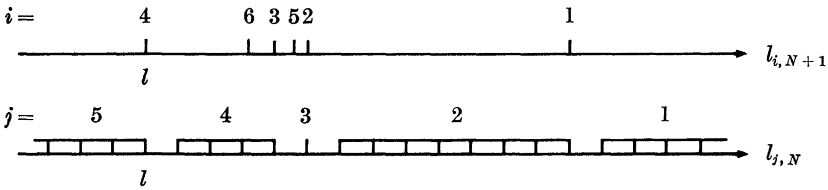

Fig. 3, e. The representations of $u(5)$ in the representation $D(3, e;\ l + 13,\ l + 5,\ l + 4,\ l,\ l + 4.6,\ l + 3.2)$ od $u(5, 1)$

$D(3, m; l_{1,N+1}, \ldots, l_{N+1,N+1})$, where $l_{1,N+1} - l_{N,N+1}$ is integral, $l_{1,N+1} > l_{N,N+1}$, $l_{N+1,N+1}$ is real, $l_{1,N+1} + 1 > l_{N+1,N+1} > -1 + \min\{l_{N-1,N+1}, l_{N,N+1}\}$, $l_{i,N+1} = l - i$ for some l when $l_{N+1,N+1} \leqq l_{i,N+1} < l_{N,N+1}$ or $l_{N,N+1} < l_{i,N+1} \leqq l_{N+1,N+1}$, $|l_{j,N+1} - l_{N+1,N+1}| < 1$ for some $j \leqq N$ or there is no $l_{i,N+1}$ such that $l_{N+1,N+1} < l_{i,N+1} < l_{N,N+1}$ or $l_{N,N+1} < l_{i,N+1} < l_{N+1,N+1}$.

$l_{1,N} \geqq l_{1,N+1} + 1$; $l_{i-1,N+1} \geqq l_{i,N} \geqq l_{i,N+1} + 1$ when $l_{i,N+1} > l_{N,N+1}$ or $l_{i-1,N+1} < l_{N,N+1}$; $l_{N,N+1} \geqq l_{i,N} \geqq l_{i,N+1} + 1$ when $l_{i-1,N+1} > l_{N,N+1} > l_{i,N+1}$; $l_{N,N} < \min\{l_{N-1,N+1}, l_{N,N+1}\}$.

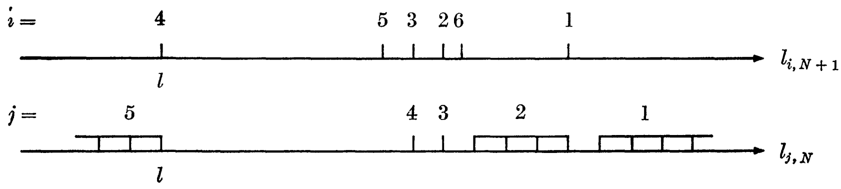

Fig. 3, m. The representations of $u(5)$ in the representation $D(3, m;\ l + 13,\ l + 9,\ l + 8,\ l,\ l + 7,\ l + 9.6)$ of $u(5, 1)$

$D(3,\ d;\ l_{1,N+1}, \ldots, l_{N+1,N+1})$, where $l_{N,N+1}$ and $l_{N+1,N+1}$ are real, $l_{N,N+1} \geqq l_{N+1,N+1}$, the relation $l_{N,N+1} \geqq l_{1,N+1} + j \geqq l_{N+1,N+1}$ is not fulfilled for any integer j; $|l_{N,N+1} - l_{i,N+1}| > 1$ for $i = 1, \ldots, N-1$.

$l_{1,N} \geqq l_{1,N+1} + 1$; $l_{i-1,N+1} \geqq l_{i,N} \geqq l_{i,N+1} + 1$ for $i = 2, \ldots, N-1$; $l_{N,N} \leqq l_{N-1,N+1}$.

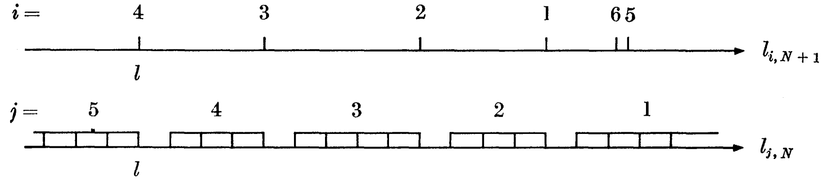

Fig. 3, d. The representations of $u(5)$ in the representation $D(3, d;\ l + 13,\ l + 9,\ l + 4,\ l,\ l + 15.6,\ l + 15.2)$ of $u(5, 1)$

$D(3,\ c;\ l_{1,N+1}, \ldots, l_{N+1,N+1})$, where $l_{N,N+1}$ and $l_{N+1,N+1}$ are conjugated complex, the imaginary part of $l_{N,N+1}$ is positive.

$l_{1,N} \geqq l_{1,N+1} + 1$; $l_{i-1,N+1} \geqq l_{i,N} \geqq l_{i,N+1} + 1$, for $i = 2, \ldots, N-1$; $l_{N,N} \leqq l_{N-1,N+1}$.

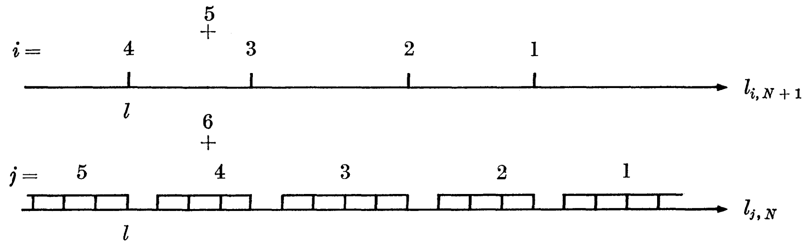

Fig. 3, c. The representations of $u(5)$ in the representation $D(3, c;\ l + 13,\ l + 9,\ l + 4,\ l,\ l + 2.6 + i\,1.8,\ l + 2.6 - i\,1.8)$ of $u(5, 1)$

The Representations of $su(N, 1)$

The generator $\sum_{k=1}^{N+1} I_{k,k}$ that appear in the connection (4) between generators of $u(N, 1)$ and $su(N, 1)$ is represented by the scalar operator $\sum_{r=1}^{N+1} l_{r,N+1} + \frac{(N+1)(N+2)}{2}$. So the infinitesimal generator

$$J_{k,k} = I_{k,k} - \frac{1}{N+1} \sum_{r=1}^{N+1} I_{r,r}$$

is represented as

$$\left(D_{k,k} - \frac{1}{N+1} \sum_{r=1}^{N+1} D_{r,r}\right) \xi(\alpha)$$

$$= \left(\sum_{r=1}^{k} l_{r,k} - \sum l_{r,k-1} - \frac{1}{N+1} \sum_{r=1}^{N+1} l_{r,N+1} + \frac{N}{2}\right) \xi(\alpha)\,.$$

The eigenvalue is invariant under the transformation

$$l_{p,q} \to l_{p,q} + \lambda$$

where λ is a real number. The same applies to the matrix elements of the representation of the generators

$$J_{k,k+1} = I_{k,k+1}$$

and

$$J_{k+1,k} = I_{k+1,k}\,.$$

Transformations with different values on λ therefore give rise to the same representation of $su(N, 1)$. A way to fix the niveaue on the numbers $l_{p,q}$ in a representation of $su(N, 1)$ could be to require that

$$l_{N-1,N+1} = 0\,.$$

For every representation of $u(N, 1)$ that satisfies this requirement there is a representation of $su(N, 1)$.

The Representations of $SU(N, 1)$

From results by Gårding, Harish-Chandra, Nelson and Dixmier [7], it is known [4], that there is a one-to-one correspondence between continuous unitary irreducible representations of $\overline{SU(N, 1)}$ and the algebraically irreducible unitary representations of $su(N, 1)$. The previous classification is therefore applicable both to the group $\overline{SU(N, 1)}$ and to the algebra $su(N, 1)$. Among these representations those are representations also of the group $SU(N, 1)$ that satisfy the global conditions of this group. One such condition is that the unit element of the group is represented by the unit operator, which means that

$$\exp\left(it\left((N+1)D_{k,k} - \sum_{r=1}^{N+1} D_{r,r}\right)\right) = 1 \quad \text{for} \quad t = 0, 2\pi, \ldots$$

and for all k. This involves that

$$(N+1)\left(\sum_{r=1}^{k} l_{r,k} - \sum_{r=1}^{k-1} l_{r,k-1}\right) - \sum_{r=1}^{N+1} l_{r,N+1} - \frac{N(N+1)}{2} \quad \text{is integer.}$$

Another condition is that $U(N)$ forms the maximal compact subgroup, which implies that $l_{r,N}$ are integers. These two conditions imply that $\sum_{r=1}^{N+1} l_{r,N+1}$ is integral. This means that we have to require that $l_{N+1,N+1}$ is integral for representations of the classes 1 and 2, and that $l_{N,N+1} + l_{N+1,N+1}$ is integral for representations of the class 3. We must also fix the niveaue on the integers $l_{i,j}$ which is done by requiring that

$$0 \leqq \sum_{r=1}^{N+1} l_{r,N+1} \leqq N\,. \tag{15}$$

For every representation of $u(N, 1)$ that satisfies these requirements there is a representation of $SU(N, 1)$ and besides the trivial one these are all the representations of $SU(N, 1)$. We do not once again list the requirements for the different representations as they are almost the same as in the $u(N, 1)$-case. Let us just notice that the set of representations of class 1 and 2 is discrete and that there is one continuous parameter that classifies the representations only for the $D(3, e; \ldots)$, $D(3, d; \ldots)$ and $D(3, c; \ldots)$ cases.

The Representations of $SU(N, 1)/Z_{N+1}$

Representations of $SU(N, 1)$ that represents the center Z_{N+1} trivially are also representations of the factor group $SU(N, 1)/Z_{N+1}$. This occurs if we replace the requirement (15) for the representations of $SU(N, 1)$ by

$$\sum_{r=1}^{N+1} m_{r, N+1} = 0 .$$

In this way we get all the representations of $SU(N, 1)/Z_{N+1}$ besides the trivial one.

The Representations of $U(N, 1)$

Representations of $u(N, 1)$ that fulfill global conditions analogous to those of the representation of $SU(N, 1)$ are also representations of $U(N, 1)$. The nivaue on the integers $l_{i,j}$ is now essential. Thus, we get all the representations of $U(N, 1)$ besides the trivial one by applying all the restrictions of the $SU(N, 1)$ case except (15) to representations of $u(N, 1)$.

I would like to thank Dr. A. Kihlberg and Professor J. Nilsson for helpful discussions.

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