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Statistical Mechanics of a One-Dimensional Lattice Gas

D. Ruelle
I.H.E.S., 91. Bures-sur-Yvette

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Abstract. We study the statistical mechanics of an infinite one-dimensional classical lattice gas. Extending a result of van Hove we show that, for a large class of interactions, such a system has no phase transition. The equilibrium state of the system is represented by a measure which is invariant under the effect of lattice translations. The dynamical system defined by this invariant measure is shown to be a K-system.

1. Introduction and Statement of Results

Let $\mathbb{Z}$ be the set of all integers $\gtrless 0$. We think of the elements of $\mathbb{Z}$ as the sites of a one-dimensional lattice, each site may be occupied by 0 or 1 particle. If n particles are present on the lattice, at positions $i_1 < \cdots < i_n$, we associate to them a "potential energy"

$$U(\{i_1, \ldots, i_n\}) = \sum_{k \geqq 1} \sum_{\{j_1, \ldots, j_k\} \subset \{i_1, \ldots, i_n\}} \Phi^k(j_1, \ldots, j_k) . \tag{1.1}$$

The "k-body potential" Φ^k is a real function of its arguments $j_1 < \cdots < j_k$ and is assumed to be translationally invariant i.e., if $l \in \mathbb{Z}$,

$$\Phi^k(j_1 + l, \ldots, j_k + l) = \Phi^k(j_1, \ldots, j_k) . \tag{1.2}$$

Let $S \subset \mathbb{Z}$ and K^S be the product of one copy of the set $K = \{0, 1\}$ for each point of S; K^S is the space of all configurations of occupied and empty sites in S; K^S is compact for the product of the discrete topologies of the sets $\{0, 1\}$. Let $\mathscr{C}(K^S)$ be the Banach space of real continuous functions on K^S with the uniform norm and $\mathscr{M}(K^S)$ its dual, i.e. the space of real measures on K^S.

If $S \subset T \subset \mathbb{Z}$ we may write

$$K^T = K^S \times K^{T \setminus S} \tag{1.3}$$

and there is a canonical mapping $\alpha_{TS} : \mathscr{C}(K^S) \to \mathscr{C}(K^T)$ such that

$$\alpha_{TS}\, \varphi(x_S, x_{T \setminus S}) = \varphi(x_S) . \tag{1.4}$$

We denote by α^*_{ST} the adjoint of α_{TS}:

$$\alpha^*_{ST}\mu(\varphi) = \mu(\alpha_{TS}\varphi) . \tag{1.5}$$

It will be convenient to use a functional notation for measures, writing $\mu(x)\,dx$ instead of $d\mu$. We have then

$$\alpha^*_{ST}\mu(x_S) = \int dx_{T\setminus S}\,\mu(x_S, x_{T\setminus S})\,. \tag{1.6}$$

Let $(a, b] = \{i \in \mathbb{Z} : a < i \leqq b\}$ be a finite interval of $\mathbb{Z}$. The *Gibbs measure* $\gamma_{ab} \in \mathscr{M}(K^{(a,b]})$ associates to each point $x = (x_{a+1}, \ldots, x_b)$ of $K^{(a,b]}$ the mass

$$\gamma_{ab}(x) = e^{-U(S(x))} \tag{1.7}$$

where[1]

$$S(x) = \{i \in (a, b] : x_i = 1\}\,. \tag{1.8}$$

The measure γ_{ab} is positive, has total mass

$$Z_{b-a} = \int \gamma_{ab}(x)\,dx = \sum_{x_{a+1}=0}^{1} \cdots \sum_{x_b=0}^{1} \gamma_{ab}(x) \tag{1.9}$$

and the corresponding normalized measure is

$$\bar{\gamma}_{ab} = Z_{b-a}^{-1}\,\gamma_{ab}\,. \tag{1.10}$$

Theorem 1. *Let $\mathscr{E}$ be the space of sequences $\Phi = (\Phi^k)_{k\geqq 1}$ such that*

$$\sum_{l>0} \sum_{0<i_1<\cdots<i_l} i_l\,|\Phi^{l+1}(0, i_1, \ldots, i_l)| < +\infty \tag{1.11}$$

if $\Phi \in \mathscr{E}$, then

(i) *the following limit exists and is finite*

$$P(\Phi) = \lim_{b-a\to\infty} \frac{1}{b-a} \log Z_{b-a} \tag{1.12}$$

it is continuously differentiable on any finite dimensional subspace of $\mathscr{E}$.

(ii) *for every finite $S \subset \mathbb{Z}$ there exists $\varrho_S \in \mathscr{M}(K^S)$ such that*

$$\lim_{a\to-\infty,\, b\to\infty} \alpha^*_{S,(a,b]}\,\bar{\gamma}_{ab} = \varrho_S\,. \tag{1.13}$$

There is a measure $\varrho \in \mathscr{M}(K^{\mathbb{Z}})$ such that

$$\varrho_S = \alpha^*_{S\mathbb{Z}}\,\varrho \tag{1.14}$$

for all finite $S \subset \mathbb{Z}$, and ϱ depends continuously on Φ on any finite dimensional subspace of $\mathscr{E}$ for the vague topology of measures[2].

This theorem expresses that a thermodynamic limit (infinite system limit) exists for the statistical mechanics of a one-dimensional lattice system if the condition (1.11) is satisfied. Furthermore the state of the infinite system, described by the measure ϱ, depends continuously on the temperature and chemical potential, which means that no *phase transi-*

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[1] It is customary to write in (1.7) instead of $U(S)$ the expression $\beta(-n\mu + U'(S))$ where β^{-1} is the *temperature*, μ is the *chemical potential* and U' is computed by replacing $\sum_{k\geqq 1}$ by $\sum_{k>1}$ in (1.1). For notational convenience we absorb here $-\mu$ as Φ^1 and β as multiplicative constant in the definition of U.

[2] I.e. the w^*-topology or the weak topology of $\mathscr{M}(K^{\mathbb{Z}})$ in duality with $\mathscr{C}(K^{\mathbb{Z}})$.

tion can occur[3]; the system remains a "*gas*". If $\Phi^{l+1} = 0$ for $l > 1$, then (1.11) becomes

$$\sum_{i>0} i\,|\Phi^2(0, i)| < +\infty\,. \tag{1.15}$$

This condition ensures that the energy of interaction of all particles at the left of a point of $\mathbb{Z}$ with all the particles at the right is bounded[4].

Given $S \subset \mathbb{Z}$, the translation $T^l: i \to i + l$ defines a *homeomorphism* of K^S onto K^{S+l}:

$$T^l(\ldots, x_{-1}, x_0, x_1, \ldots) = (\ldots, x_{-l-1}, x_{-l}, x_{-l+1}, \ldots) \tag{1.16}$$

and if $f \in \mathscr{C}(K^S)$, $\mu \in \mathscr{M}(K^S)$ we define[5] $T^l f \in \mathscr{C}(K^{S+l})$, $T^l \mu \in \mathscr{M}(K^{S+l})$:

$$T^l f(x) = f(T^{-l}x)\,, \qquad T^l \mu(x) = \mu(T^{-l}x) \tag{1.17}$$

so that

$$\mu(T^l f) = \int dx\, \mu(x) f(T^{-l}x) = \int dx\, \mu(T^l x) f(x) = T^{-l}\mu(f) \tag{1.18}$$

Since the measure ϱ is visibly T-invariant in $\mathscr{M}(K^{\mathbb{Z}})$, the triple $(K^{\mathbb{Z}}, \varrho, T)$ is a dynamical system[6].

Theorem 2. *The dynamical system* $(K^{\mathbb{Z}}, \varrho, T)$ *is a K-system.*

This implies that the measure ϱ is ergodic and satisfies a "*cluster property*" (see Sec. 2) as one expects for a gas.

2. Proof of Theorems 1 and 2

Let $\mathbb{N}^* = \{i \in \mathbb{Z} : i > 0\}$ and $K_+ = K^{N^*}$. For every integer $m \geqq 0$ we may write

$$K_+ = K^{(0,m]} \times T^m K_+\,. \tag{2.1}$$

In particular if $x \in K_+$; then $(0, x) \in K_+$, $(1, x) \in K_+$.

We let $F_\Phi \in \mathscr{C}(K_+)$ be given by

$$F_\Phi(x) = \exp\Big[-\sum_{l \geqq 0}\ \sum_{0 < i_1 < \cdots < i_l} x_{i_1} \ldots x_{i_l} \Phi^{l+1}(0, i_1, \ldots, i_l)\Big] \tag{2.2}$$

where $x = (x_1, \ldots, x_i, \ldots) \in K_+$, $x_i = 0$ or 1 for each $i > 0$. The continuity of F_Φ on K_+ is ensured by (1.11). A mapping $\mathscr{L}_\Phi$ of $\mathscr{C}(K_+)$ into itself is defined by

$$\mathscr{L}_\Phi f(x) = f(0, x) + F_\Phi(x) f(1, x) \tag{2.3}$$

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[3] This result was known when Φ has finite range, i.e. when there exists $L < +\infty$ such that $\Phi^{l+1}(0, i_1, \ldots, i_l) = 0$ for $i_l > L$ (hence for $l > L$). In that case $P(\Phi)$ is real analytic on finite dimensional subspaces of $\mathscr{E}$ (is this true also here ?). A generalization of this result exists to continuous systems with a "hard core", see VAN HOVE [5].

[4] If $\Phi^2 \leqq 0$ and (1.15) is violated, the existence of a phase transition has been conjectured by M. FISHER [2] and M. KAC (private communications). I am indebted to M. FISHER for correspondence on this point.

[5] We let formally $d(T^l x) = dx$.

[6] The notions of dynamical systems and of K-system are discussed in ARNOLD and AVEZ [1] and JACOBS [3].

its adjoint $\mathscr{L}_\Phi^*: \mathscr{M}(K_+) \to \mathscr{M}(K_+)$ is given by

$$\begin{cases} \mathscr{L}_\Phi^* \mu(0, x) = \mu(x) \\ \mathscr{L}_\Phi^* \mu(1, x) = F_\Phi \mu(x) . \end{cases} \tag{2.4}$$

Theorem 3. (i) *For every $\Phi \in \mathscr{E}$ there exist $\lambda_\Phi > 0$, $h_\Phi \in \mathscr{C}(K_+)$, $\nu_\Phi \in \mathscr{M}(K_+)$ such that $h_\Phi > 0$, $\nu_\Phi \geqq 0$, $\nu_\Phi(1) = \nu_\Phi(h_\Phi) = 1$ and*[7]

$$\mathscr{L}_\Phi h_\Phi = \lambda_\Phi h_\Phi \tag{2.5}$$

$$\mathscr{L}_\Phi^* \nu_\Phi = \lambda_\Phi \nu_\Phi . \tag{2.6}$$

(ii) *If $f \in \mathscr{C}(K_+)$ the following limit*

$$\lim_{n \to \infty} \|\lambda_\Phi^{-n} \mathscr{L}_\Phi^n f - \nu_\Phi(f)\, h_\Phi\| = 0 \tag{2.7}$$

holds uniformly for Φ in a bounded subset of a finite dimensional subspace of $\mathscr{E}$.

(iii) *If $\mu \in \mathscr{M}(K_+)$ the following limit*

$$\lim_{n \to \infty} \lambda_\Phi^{-n} \mathscr{L}_\Phi^{*n} \mu = \mu(h_\Phi)\, \nu_\Phi \tag{2.8}$$

holds for the vague topology of $\mathscr{M}(K_+)$.

(iv) *On any finite dimensional subspace of $\mathscr{E}$, λ_Φ is continuously differentiable, h_Φ is continuous for the uniform topology of $\mathscr{C}(K_+)$, ν_Φ is continuous for the vague topology of $\mathscr{M}(K_+)$.*

This theorem will be proved in Sec. 3., here we use it to establish the results announced in Sec. 1. For notational simplicity we shall often drop the index Φ from F, $\mathscr{L}$, $\mathscr{L}^*$, λ, h, ν.

Lemma. *Let us write*

$$L = \lambda^{-1} \mathscr{L} , \qquad L^* = \lambda^{-1} \mathscr{L}^* . \tag{2.9}$$

(i) *If $\mu \in \mathscr{M}(K_+)$, then*

$$\sum_{n_1=0}^{1} \cdots \sum_{n_l=0}^{1} L^{*l} \mu(n_1, \ldots, n_l, x) = L^l 1(x) \cdot \mu(x) . \tag{2.10}$$

(ii) *If $f \in \mathscr{C}(K_+)$, then*

$$\nu \cdot \alpha_{N^*, N^*+l}\, T^l f = L^{*l}(\nu \cdot f) . \tag{2.11}$$

[7] For every finite $S \subset \mathbb{N}^*$ let

$$\lim_{m \to \infty} \alpha^*_{S,(0,m]}\, \bar{\nu}_{0m} = \nu_S .$$

One can show that ν_Φ defined by Theorem 3 (i) is such that

$$\nu_S = \alpha^*_{S\mathbb{N}^*}\, \nu .$$

The measure ν_Φ describes thus the state of a system occupying the semi-infinite interval $(0, +\infty) = \mathbb{N}^*$.

We prove (i) by induction on l:

$$\begin{aligned}
&\sum_{n_1} \cdots \sum_{n_{l+1}} L^{*\,l+1} \mu(n_1, \ldots, n_{l+1}, x) \\
&= \sum_{n_{l+1}} L^l 1(n_{l+1}, x) \cdot L^* \mu(n_{l+1}, x) \\
&= L^l 1(0, x) \cdot L^* \mu(0, x) + L^l 1(1, x) \cdot L^* \mu(1, x) \qquad (2.12) \\
&= L^l 1(0, x) \cdot \lambda^{-1} \mu(x) + L^l 1(1, x) \cdot \lambda^{-1} F(x) \cdot \mu(x) \\
&= L^{l+1} 1(x) \cdot \mu(x) .
\end{aligned}$$

To prove (ii) it suffices to apply repeatedly the following identity

$$\begin{aligned}
&[\nu \cdot \alpha_{N^*, N^*+1}\, T f](n_1, x) = \nu(n_1, x) \cdot f(x) = L^* \nu(n_1, x) \cdot f(x) \\
&= \begin{Bmatrix} \lambda^{-1} \nu(x) & \\ \lambda^{-1} F(x)\, \nu(x) \end{Bmatrix} \cdot f(x) = [L^*(\nu \cdot f)](n_1, x) \qquad (2.13)
\end{aligned}$$

Let $\delta \in \mathscr{M}(K_+)$ be the unit mass at $x_0 = (0, \ldots, 0, \ldots)$. It is readily checked that

$$\gamma_{0m} = \alpha^*_{(0,m], \mathbf{N}^*} \mathscr{L}^{*m} \delta . \qquad (2.14)$$

By (1.6), (1.9) we have

$$Z_m = \int \mathscr{L}^{*m} \delta(x)\, dx = \mathscr{L}^{*m} \delta(1) = \delta(\mathscr{L}^m 1) \qquad (2.15)$$

and using (2.7),

$$\lim_{b-a \to \infty} \frac{Z_{b-a}}{\lambda^{b-a}} = \lim_{n \to \infty} \frac{\delta(\mathscr{L}^n 1)}{\lambda^n} = \nu(1) \cdot \delta(h) = h(x_0) > 0 \qquad (2.16)$$

which implies[8] (1.12) with $P(\Phi) = \log \lambda_\Phi$ and Theorem 1 (i) follows from Theorem 3 (iv).

We study now the limit (1.13) with $S = (0, m]$ (this is sufficient because we may by translation of $\mathbb{Z}$ map S into $(0, m]$ for some m). Let $f \in \mathscr{C}(K^{(0,m]})$, using (2.14), (2.16), part (i) of the Lemma and parts (ii), (iii) of Theorem 3 we get

$$\begin{aligned}
&\lim_{a \to -\infty, b \to \infty} \alpha^*_{(0,m],(a,b]}\, \bar{\gamma}_{ab}(f) \\
&= \lim_{l,n \to \infty} \alpha^*_{(0,m],(-l,m+n]}\, \bar{\gamma}_{-l,m+n}(f) \\
&= \lim_{l,n \to \infty} \alpha^*_{(l,l+m],(0,l+m+n]}\, \bar{\gamma}_{0,l+m+n}(T^l f) \\
&= \lim_{l,n \to \infty} Z^{-1}_{l+m+n}\, \alpha^*_{(l,l+m], \mathbf{N}^*} \mathscr{L}^{*\,l+m+n} \delta(T^l f) \qquad (2.17) \\
&= h(x_0)^{-1} \lim_{l,n \to \infty} \sum_{n_1=0}^{1} \cdots \sum_{n_l=0}^{1} \int dx\, L^{*\,l+m+n} \delta(n_1, \ldots, n_l, x) \\
&\quad \cdot \alpha_{\mathbf{N}^*, (0,m]} f(x) \\
&= h(x_0)^{-1} \lim_{l,n \to \infty} \int dx\, L^l 1(x) \cdot L^{*\,m+n} \delta(x) \cdot \alpha_{\mathbf{N}^*, (0,m]} f(x) \\
&= h(x_0)^{-1} \int dx\, \nu(1)\, h(x) \cdot \delta(h)\, \nu(x) \cdot \alpha_{\mathbf{N}^*, (0,m]} f(x) \\
&= \int dx\, h(x) \cdot \nu(x) \cdot \alpha_{\mathbf{N}^*, (0,m]} f(x) .
\end{aligned}$$

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[8] Actually (2.16) is a much stronger statement than (1.12).

This establishes the existence of the limit (1.13) and shows that the measure ϱ defined by (1.14) satisfies

$$\alpha^*_{\mathbb{N}^*\mathbb{Z}}\, \varrho = h \cdot \nu \,. \tag{2.18}$$

In view of Theorem 3 (iv), the r.h.s. of (2.17) is a continuous function of Φ on finite dimensional subspaces of $\mathscr{E}$. Because of the invariance of ϱ under T, the same is true of $\varrho(\alpha_{\mathbb{Z}S} f)$ for every finite $S \subset \mathbb{Z}$ and $f \in \mathscr{C}(K^S)$. Part (ii) of Theorem 1 follows then from the density of

$$\cup_S \alpha_{\mathbb{Z}S} \mathscr{C}(K^S)$$

in $\mathscr{C}(K^{\mathbb{Z}})$ for the uniform topology.

We come now to the study of the dynamical system $(K^{\mathbb{Z}}, \varrho, T)$. Let $\mathscr{B}_1$ be the algebra of all ϱ-measurable subsets of $K^{\mathbb{Z}}$ (mod. 0) and $\mathscr{B}_0$ be the subalgebra consisting of the sets of measure 0 or 1 (i.e. $\emptyset$ and $K^{\mathbb{Z}}$ (mod. 0)). The system $(K^{\mathbb{Z}}, \varrho, T)$ is a K-system if there exists a subalgebra $\mathscr{A}$ of $\mathscr{B}_1$ such that

(i) $\mathscr{A} \subset T^{-1}\mathscr{A}$.
(ii) The union of the $T^{-l}\mathscr{A}$ generates $\mathscr{B}_1$.
(iii) The intersection of the $T^l\mathscr{A}$ is $\mathscr{B}_0$.

We write

$$K^{\mathbb{Z}} = K^S \times K^{\mathbb{Z}\setminus S} \tag{2.19}$$

and define $\mathscr{A}$ to be the subalgebra of $\mathscr{B}_1$ generated by all the sets $X \times K^{\mathbb{Z}\setminus S}$ where $X \subset K^S$ and S is a finite subset of $\mathbb{N}^*$. The properties (i) and (ii) are then clearly satisfied. Let now $A \in \bigcap_{l \geqq 0} T^l\mathscr{A}$ and B be of the form $X \times K^{\mathbb{Z}\setminus S}$ with $X \subset K^S$, S finite $\subset \mathbb{N}^*$. For all $l \geqq 0$ the characteristic function of A may be written as $\alpha_{\mathbb{N}^*, \mathbb{N}^*+l}\, T^l f_l$, let also $f_B \in \mathscr{C}(K_+)$ be the characteristic function of B. Using part (ii) of the Lemma, we get

$$\begin{aligned}
\varrho(A \cap B) &= \int dx\, h(x) \cdot \nu(x) \cdot \alpha_{\mathbb{N}^*, \mathbb{N}^*+l}\, T^l f_l(x) \cdot f_B(x) \\
&= \int dx\, [L^{*l}(\nu \cdot f_l)](x) \cdot h(x) \cdot f_B(x) \\
&= \int dx\, \nu(x) \cdot f_l(x) \cdot [L^l(h \cdot f_B)](x) \,.
\end{aligned} \tag{2.20}$$

Given $\varepsilon > 0$, (2.7) shows that, for sufficiently large l,

$$\|L^l(h \cdot f_B) - \nu(h \cdot f_B)\, h\| < \varepsilon \,. \tag{2.21}$$

From (2.20) and (2.21) we find

$$\begin{aligned}
|\varrho(A \cap B) - \varrho(A)\,\varrho(B)| = |\textstyle\int dx\, \nu(x) \cdot f_l(x) \cdot [L^l(h \cdot f_B)(x) \\
- \nu(h \cdot f_B)\, h(x)]| < \varepsilon
\end{aligned} \tag{2.22}$$

and therefore

$$\varrho(A \cap B) = \varrho(A)\,\varrho(B) \,. \tag{2.23}$$

By translation, (2.23) remains true for any B of the form $X \times K^{\mathbb{Z}\setminus S}$ with $X \subset K^S$, S finite $\subset \mathbb{Z}$, and therefore for any $B \in \mathscr{B}_1$. In particular for

$B = A$, we obtain $\varrho(A) = \varrho(A)^2$ hence $\varrho(A) = 0$ or 1, proving the property (iii) of K-systems and therefore Theorem 2.

Let S be a finite subset of $\mathbb{Z}$ and define $f_S \in \mathscr{C}(K^{\mathbb{Z}})$ by $f_S(x) = 1$ if $i \in S \Rightarrow x_i = 1$, $f_S(x) = 0$ otherwise. The *correlation function* $\bar{\varrho}$ associated to ϱ is a function of finite subsets of $\mathbb{Z}$ defined by

$$\bar{\varrho}(S) = \varrho(f_S) \,. \tag{2.24}$$

Notice that by Theorem 1, $\varrho_\Phi(S)$ is a continuous function of Φ on finite dimensional subspaces of $\mathscr{E}$. We have also

$$\lim_{l\to\infty} \bar{\varrho}(S_1 \cup T^l S_2) = \bar{\varrho}(S_1) \cdot \bar{\varrho}(S_2) \tag{2.25}$$

a property known as *cluster property* and which should be possessed by the correlation function of a gas. The cluster property (2.25) is a consequence of *strong mixing*, which is a property of all K-systems[9]. The entropy of a K-system is > 0[10], this entropy is identical to the mean entropy in the sense of statistical mechanics (see [4]). The K-system property (iii) has here a simple physical interpretation: it is not possible to make the system look different "at finite distances" by imposing restrictions "infinitely far away" on the configurations of the system (absence of long-range order).

3. Proof of Theorem 3

In this section we establish a series of propositions which will result in a proof of Theorem 3.

For $m \geqq 0$ we let $\mathscr{C}_m = \alpha_{\mathbb{N}^*,(0,m]} \mathscr{C}(K^{(0,m]})$, i.e. $\mathscr{C}_m$ is the subspace of $\mathscr{C}(K_+)$ consisting of those f such that $f(x) = f(x')$ if $x_i = x_i'$ for $i \leqq m$.

Proposition 1. *Let $f \in \mathscr{C}_m$, $f \geqq 0$ and $x_i = x_i'$ for $i = 1, \ldots, k$. If $n \geqq 0$, $n \geqq m - k$, then*

$$A_k^{-1} \leqq \frac{\mathscr{L}^n f(x')}{\mathscr{L}^n f(x)} \leqq A_k \tag{3.1}$$

where

$$A_k = \exp\left[\sum_{l>0} \sum_{0<i_1<\cdots<i_l>k} (i_l - k)\, |\Phi^{l+1}(0, i_1, \ldots, i_l)|\right]. \tag{3.2}$$

If $k \geqq m$, then $f(x') = f(x)$ and (3.1) holds thus for $n = 0$. If $n > 0$, (2.3) yields

$$\frac{\mathscr{L}^n f(x')}{\mathscr{L}^n f(x)} = \frac{\mathscr{L}^{n-1} f(0, x') + F(x')\, \mathscr{L}^{n-1} f(1, x')}{\mathscr{L}^{n-1} f(0, x) + F(x)\, \mathscr{L}^{n-1} f(1, x)} \,. \tag{3.3}$$

Using induction on n we may assume that for $n_1 = 0, 1$, we have

$$A_{k+1}^{-1} \leqq \frac{\mathscr{L}^{n-1} f(n_1, x')}{\mathscr{L}^{n-1} f(n_1, x)} \leqq A_{k+1} \tag{3.4}$$

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[9] See [1] 11.4.

[10] See [1] 12.31.

and

$$\exp\left[-\sum_{l>0}\ \sum_{0<i_1<\cdots<i_l>k} |\Phi^{l+1}(0, i_1, \ldots, i_l)|\right] \leqq \frac{F(x')}{F(x)} \leqq \exp\left[\sum_{l>0}\ \sum_{0<i_1<\cdots<i_l>k} |\Phi^{l+1}(0, i_1, \ldots, i_l)|\right]. \tag{3.5}$$

Therefore

$$A_k^{-1} \leqq \frac{\mathscr{L}^{n-1} f(0, x')}{\mathscr{L}^{n-1} f(0, x)} \leqq A_k \tag{3.6}$$

$$A_k^{-1} \leqq \frac{F(x')\ \mathscr{L}^{n-1} f(0, x')}{F(x)\ \mathscr{L}^{n-1} f(0, x)} \leqq A_k \tag{3.7}$$

and (3.1) follows.

Notice that if we write

$$B = \exp\left[\sum_{l\geqq 0}\ \sum_{0<i_1<\cdots<i_l} |\Phi^{l+1}(0, i_1, \ldots, i_l)|\right] \tag{3.8}$$

then $B^{-1} \leqq F(x) \leqq B$.

Proposition 2. *There exist* $\nu \in \mathscr{M}(K_+)$ *and* λ *real such that* $\nu \geqq 0$, $\|\nu\| = 1$ *and*

$$\mathscr{L}^* \nu = \lambda \nu . \tag{3.9}$$

Furthermore $1 + B^{-1} \leqq \lambda \leqq 1 + B$ *where* B *is given by* (3.8).

The set $\{\mu \in \mathscr{M}(K_+) : \mu \geqq 0 \text{ and } \mu(1) = 1\}$ is convex, vaguely compact and mapped continuously into itself by

$$\mu \to [\mathscr{L}^* \mu(1)]^{-1} \mathscr{L}^* \mu . \tag{3.10}$$

By the theorem of SCHAUDER-TYCHONOV this mapping has a fixed point ν: (3.9) holds with $\lambda = \mathscr{L}^* \nu(1) = \nu(\mathscr{L}\, 1)$. Since $\mathscr{L}\, 1(x) = 1 + F(x)$ and $B^{-1} \leqq F(x) \leqq B$, we have $1 + B^{-1} \leqq \lambda \leqq 1 + B$.

Proposition 3. (i) *The closed hyperplane* $H = \{f \in \mathscr{C}(K_+) : \nu(f) = 1\}$ *is mapped into itself by* $L = \lambda^{-1} \mathscr{L}$.

(ii) *Let* $f \in \mathscr{C}_m$, $f \geqq 0$, $n \geqq m$, *then*

$$\sup_{x\in K_+} L^n f(x) \leqq A_0 \nu(f) \tag{3.11}$$

$$\inf_{x\in K_+} L^n f(x) \geqq A_0^{-1} \nu(f) . \tag{3.12}$$

(iii) *If* $f \in \mathscr{C}(K_+)$, *the sequence* $\|L^n f\|$ *is bounded by* $A_0 \|f\|$.

(iv) *A norm* $|||\cdot|||$ *on* $\mathscr{C}(K_+)$ *is defined by*

$$|||f||| = \nu(|f|) = \int dx\, \nu(x)\, |f(x)| \leqq \|f\| . \tag{3.13}$$

(v) $|||Lf||| \leqq |||f|||$ *for all* $f \in \mathscr{C}(K_+)$.

(vi) *If* $f \in \mathscr{C}_m$, $\nu(f) = 0$, *and* $n \geqq m$, *then*

$$|||L^n f||| \leqq (1 - A_0^{-1})\, |||f||| . \tag{3.14}$$

(i) follows from

$$\nu(Lf) = \lambda^{-1}\, \mathscr{L}^* \nu(f) = \nu(f) , \tag{3.15}$$

(ii) follows from (3.1) with $k = 0$:

$$\begin{aligned}\nu(f) = \nu(L^n f) &\leqq \sup_{x' \in K^+} L^n f(x') \\ &\leqq A_0 \inf_{x \in K_+} L^n f(x) \leqq A_0 \nu(L^n f) = A_0 \nu(f)\,.\end{aligned} \tag{3.16}$$

Using (3.11) with $m = 0$ we have

$$\|L^n f\| \leqq \|L^n |f|\| \leqq \|f\| \sup_{x \in K_+} L^n 1(x) \leqq A_0 \|f\| \tag{3.17}$$

which proves (iii).

It is clear that $|||\cdot|||$ is a semi-norm and that $|||f||| \leqq \|f\|$. We conclude the proof of (iv) by showing that if $f \geqq 0$, $f \neq 0$ then $|||f||| > 0$. We may indeed choose m and $f' \in \mathscr{C}_m$ such that $0 \leqq f' \leqq f$ and $f' \neq 0$, then $L^m f' \neq 0$ and (3.11) yields

$$|||f||| = \nu(f) \geqq \nu(f') \geqq A_0^{-1} \|L^m f'\| > 0\,. \tag{3.18}$$

To prove (v) we notice that

$$\begin{aligned}|||Lf||| &= \nu(|Lf|) = \lambda^{-1} \nu(|\mathscr{L} f|) \leqq \lambda^{-1} \nu(\mathscr{L}|f|) = \lambda^{-1} \mathscr{L}^* \nu(|f|) \\ &= \nu(|f|) = |||f|||\,.\end{aligned} \tag{3.19}$$

To prove (vi) let $f_\pm = 1/2\,(|f| \pm f)$, we have

$$|||f_+||| = \nu(f_+) = \nu(f_-) = |||f_-|||\,. \tag{3.20}$$

On the other hand by (3.12)

$$\inf_{x \in K_+} L^n f_\pm(x) \geqq A_0^{-1} |||f_\pm|||\,. \tag{3.21}$$

Therefore

$$\begin{aligned}|||L^n f||| &= \nu(|L^n(f_+ - f_-)|) \\ &= \nu(|L^n f_+ - A_0^{-1}\,|||f_+|||) - (L^n f_- - A_0^{-1}\,|||f_-|||)|) \\ &\leqq \nu(|L^n f_+ - A_0^{-1} |||f_+|||\,| + |L^n f_- - A_0^{-1} |||f_-|||\,|) \\ &= \nu(L^n(f_+ + f_-) - A_0^{-1}(|||f_+||| + |||f_-|||) \\ &= \nu(L^n |f| - A_0^{-1} |||f|||) = \nu(|f|) - A_0^{-1} |||f||| \\ &= (1 - A_0^{-1})\,|||f|||\end{aligned} \tag{3.22}$$

which proves (3.14).

Proposition 4. *Define*

$$\Sigma = \{f \in \mathscr{C}(K_+) : \nu(f) = 1\,, \quad f \geqq 0$$

and

$$A_k^{-1} \leqq \frac{f(x')}{f(x)} \leqq A_k \quad \text{if} \quad x_i' = x_i \quad \text{for} \quad i = 1, \ldots, k\}\,. \tag{3.23}$$

(i) $L\Sigma \subset \Sigma$.

(ii) *If* $f \in \Sigma$, *then* $\|f\| \leqq A_0$ *and if* $x_i = x_i'$ *for* $i = 1, \ldots, k$, *then*

$$|f(x') - f(x)| \leqq A_0(A_k - 1)\,. \tag{3.24}$$

(iii) *The set* Σ *is convex and compact in* $\mathscr{C}(K_+)$.

(iv) *If* $f, f' \in \Sigma$, *then*

$$|||f - f'||| \geqq B^{-k}(1 + B)^{-k}(\|f - f'\| - 2A_0(A_k - 1)) \tag{3.25}$$

for all k.

(i) follows from Prop. 3 (i) and the same argument as in the proof of Prop. 1.

If $f \in \Sigma$, then $\nu(f) = 1$ hence $\nu(f - 1) = 0$ and one can choose $\tilde{x}$ such that $f(\tilde{x}) \leqq 1$ hence $f(x) \leqq A_0 f(\tilde{x}) \leqq A_0$, proving $\|f\| \leqq A_0$. If $x_i = x_i'$ for $i = 1, \ldots, k$ we get

$$f(x') - f(x) \leqq f(x)\,(A_k - 1) \leqq A_0(A_k - 1) \tag{3.26}$$

and (3.24) follows by exchanging the roles of x and x'.

The set Σ is clearly convex and closed, since it is bounded and equicontinuous by (ii) the theorem of ASCOLI shows that it is compact, proving (iii).

Let $f, f' \in \Sigma$. We can choose $\tilde{x}$ such that $|f(\tilde{x}) - f'(\tilde{x})| = \|f - f'\|$. Denote by g the characteristic function of the set $\{x \in K_+ : x_i = \tilde{x}_i$ for $i = 1, \ldots, k\}$, using (ii) we obtain

$$|||f - f'||| = \nu(|f - f'|) \geqq (\|f - f'\| - 2A_0(A_k - 1)) \cdot \nu(g) \tag{3.27}$$

and (iv) follows from

$$\nu(g) = \nu(L^k g) = \frac{\nu(\mathscr{L}^k g)}{\lambda^k} \geqq \frac{B^{-k}}{(1 + B)^k}\,, \tag{3.28}$$

where we have used $F(x) \geqq B^{-1}$, $\lambda \leqq 1 + B$ (see Prop. 2.).

Proposition 5. (i) *There exists* $h \in H$ *such that* $Lh = h$ (*i.e.* $\mathscr{L}h = \lambda h$), $\nu(h) = 1$.

(ii) *If* $f \in H$, *then* $\lim_{n\to\infty} \|L^n f - h\| = 0$, *more generally if* $f \in \mathscr{C}(K_+)$, *then*

$$\lim_{n\to\infty} L^n f = \nu(f)\, h \tag{3.29}$$

in the uniform topology.

(iii) *If* $\mu \in \mathscr{M}(K_+)$ *the following limit exists in the vague topology*

$$\lim_{n\to\infty} \lambda^{-n} (\mathscr{L}^*)^n \mu = \mu(h) \cdot \nu\,. \tag{3.30}$$

By Prop. 4 (i), (iii) the convex compact set Σ is mapped into itself by L which has therefore a fixed point h by the theorem of SCHAUDER-TYCHONOV, proving (i).

Let $\bar{f} \in \Sigma$, in view of Prop. 4. (i), (ii), we can for each integer $n > 0$ choose $m(n)$ independent of N such that

$$\|(L^N \bar{f} - h) - g\| < \frac{1}{n!} \tag{3.31}$$

for some $g \in \mathscr{C}_{m(n)}$ with $\nu(g) = 0$. Then by Prop. 3. (v), (vi),

$$\begin{aligned} &|||(L^{N+m(n)} \tilde{f} - h)||| \leqq |||L^{m(n)} g||| + \frac{1}{n!} \\ &\leqq (1 - A_0^{-1}) \, |||g||| + \frac{1}{n!} \leqq (1 - A_0^{-1}) \, |||L^N \tilde{f} - h||| + \frac{2}{n!} \,. \end{aligned} \tag{3.32}$$

If we put $M(n) = \sum_{i=1}^{n} m(i)$, we get

$$\lim_{n\to\infty} |||L^{N+M(n)} \tilde{f} - h||| = 0 \tag{3.33}$$

uniformly in N, using then Prop. 4. (iv), we have thus

$$\lim_{n\to\infty} \|L^n \tilde{f} - h\| = 0 \tag{3.34}$$

when $\tilde{f} \in \Sigma$. This remains true if $\tilde{f} \in H$ and $\tilde{f}$ is a linear combination of elements of Σ, these linear combinations include the elements of $\mathscr{C}_m$ for all m and are thus dense in H. By Prop. 3 (iii), $\|L^n f\|$ is bounded for all $f \in \mathscr{C}(K_+)$, hence the theorem of Banach-Steinhaus shows that

$$\lim_{n\to\infty} \|L^n f - \nu(f) \cdot h\| = 0 \tag{3.35}$$

proving (ii).

If $\mu \in \mathscr{M}(K_+)$, then for every $f \in \mathscr{C}(K_+)$

$$\lim_{n\to\infty} \lambda^{-n} (\mathscr{L}^*)^n \mu(f) = \lim_{n\to\infty} \mu(L^n f) = \mu(\nu(f) \cdot h) = \mu(h) \, \nu(f) \tag{3.36}$$

proving (iii).

Proposition 6. *Let $\mathscr{F}$ be a finite dimensional subspace of $\mathscr{E}$ and B a bounded subset of $\mathscr{F}$.*

(i) *The limit* $\lim_{n\to\infty} \|L_\Phi^n f - \nu_\Phi(f) \cdot h_\Phi\| = 0$ *holds uniformly in $\Phi \in B$.*

(ii) *h_Φ is a continuous function of $\Phi \in \mathscr{F}$ for the uniform topology of $\mathscr{C}(K_+)$.*

(iii) *ν_Φ is a continuous function of $\Phi \in \mathscr{F}$ for the vague topology of $\mathscr{M}(K_+)$.*

(iv) *Let $\Phi, \Psi \in \mathscr{F}$, $\Phi(t) = \Phi + t\Psi$, $t \in \mathbb{R}$, then the function $t \to \lambda_{\Phi(t)}$ has a derivative*

$$\frac{d}{dt} \lambda_{\Phi(t)} = \nu_{\Phi(t)} (\mathscr{L}'_{\Phi(t),\Psi} h_{\Phi(t)}) \tag{3.37}$$

where $\mathscr{L}'_{\Phi,\Psi}$ is the bounded operator on $\mathscr{C}(K_+)$ defined by

$$\begin{aligned} \mathscr{L}'_{\Phi,\Psi} f(x) = \Big[- \sum_{l \geqq 0} \sum_{0 < i_1 < \cdots < i_l} x_{i_1} \ldots x_{i_l} \Psi^{l+1}(0, i_1, \ldots, i_l) \Big] \\ \cdot F_\Phi(x) f(1,x) \end{aligned} \tag{3.38}$$

and $\frac{d}{dt} \lambda_{\Phi(t)}$ is a continuous function of $\Phi \in \mathscr{F}$.

Let $\tilde{f} > 0$ satisfy, for all k and all $\Phi \in B$

$$A_k^{-1} \leqq \frac{\tilde{f}(x')}{\tilde{f}(x)} \leqq A_k \quad \text{if} \quad x'_i = x_i \quad \text{for} \quad i = 1, \ldots, k \,. \tag{3.39}$$

Then, $\nu_\Phi(\tilde{f})^{-1}\tilde{f} \in \Sigma$. Since A_k, B depend continuously on $\Phi \in \mathscr{F}$, the estimates in the proof of Prop. 5 (ii) can be made uniformly in $\Phi \in B$, hence

$$\lim_{n\to\infty} \|\nu_\Phi(\tilde{f})^{-1} L_\Phi^n \tilde{f} - h_\Phi\| = 0 \tag{3.40}$$

uniformly in $\Phi \in B$. Since $\nu_\Phi(\tilde{f}) < \|\tilde{f}\|$, (i) holds for $f = \tilde{f} > 0$ satisfying (3.39).

In particular $L_\Phi^n 1$ tends to h_Φ uniformly in $\Phi \in B$, and $\|L_\Phi^n 1\|^{-1} L_\Phi^n 1 = \|\mathscr{L}_\Phi^n 1\|^{-1}\mathscr{L}_\Phi^n 1$, which is continuous in $\Phi \in B$, tends uniformly in $\Phi \in B$ towards $\|h_\Phi\|^{-1} h_\Phi$ which is therefore continuous in $\Phi \in \mathscr{F}$.

We have the identity

$$t^{-1}(\lambda_{\Phi+t\Psi} - \lambda_\Phi)\,\nu_\Phi\left(\frac{h_{\Phi+t\Psi}}{\|h_{\Phi+t\Psi}\|}\right) = \nu_\Phi\left(t^{-1}[\mathscr{L}_{\Phi+t\Psi} - \mathscr{L}_\Phi]\frac{h_{\Phi+t\Psi}}{\|h_{\Phi+t\Psi}\|}\right) \tag{3.41}$$

and, in the norm of operators on $\mathscr{C}(K_+)$,

$$\lim_{t\to 0} \|t^{-1}(\mathscr{L}_{\Phi+t\Psi} - \mathscr{L}_\Phi) - \mathscr{L}'_{\Phi,\Psi}\| = 0\,. \tag{3.42}$$

Therefore

$$\lim_{t\to 0} t^{-1}(\lambda_{\Phi+t\Psi} - \lambda_\Phi) = \nu_\Phi(\mathscr{L}'_{\Phi,\Psi} h_\Phi) \tag{3.43}$$

which proves (3.37); λ_Φ is a continuous function of $\Phi \in \mathscr{F}$ because of the boundedness of $|\nu_\Phi(\mathscr{L}'_{\Phi,\Psi} h_\Phi)|$ for $\Phi \in B$ (use $h \in \Sigma$).

We may consider $L^n: f \to L_\Phi^n f$ as a bounded operator from $\mathscr{C}(K_+)$ to $\mathscr{C}(K_+ \times B)$. For each $f \in \mathscr{C}(K_+)$ the sequence $L_\Phi^n f$ is bounded in $\mathscr{C}(K_+ \times B)$ by Prop. 3 (iii). We have seen that (i) is satisfied for linear combinations of $\tilde{f} \geqq 0$ satisfying (3.39) for all k and all $\Phi \in B$, these include again the elements of $\mathscr{C}_m$ for all m and are thus dense in $\mathscr{C}(K_+)$. Applying the theorem of BANACH-STEINHAUS to the sequence L^n proves then (i).

Applying (i) to $f = 1$ yields (ii). More generally (i) shows that $\nu_{\Phi(f)} h_\Phi$ is continuous in $\Phi \in \mathscr{F}$, using then (ii) we see that $\nu_\Phi(f)$ is continuous in Φ for each $f \in K_+$, proving (iii). Finally the continuity of the derivative (3.37) follows from the continuity in $\Phi \in \mathscr{F}$ of ν_Φ (by (ii)), h_Φ (by (iii)) and $\mathscr{L}'_{\Phi,\Psi}$.

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D. RUELLE
Institut des Hautes Etudes Scientifiques
F 91 Bures-sur-Yvette