Communications in Mathematical Analysis
Volume 21, Number 1, pp. 54-66 (2018) www.math-res-pub.org/cma
ISSN 1938-9787

Existence and Regularity for the Neumann Problem to the Poisson Equation and an Application to the Maxwell-Stokes Type Equation

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(Communicated by Jin Liang)

Abstract

In this paper, we consider the Neumann problem for the Laplace operator with a given data containing a divergence of a vector field. We demonstrate the existence and regularity of a weak solution. As an application, we consider the existence and regularity of a weak solution in regard to the Maxwell-Stokes type equation.

AMS Subject Classification:35J05, 35J25, 35A05, 35D10

Keywords: Neumann problem, Poisson equation, Maxwell-Stokes equation, Existence of a weak solution, regularity of a weak solution.

1 Introduction

In this paper, we consider the Neumann problem for the Laplace operator with data containing the divergence of a vector field of the form

$$\begin{cases} \Delta u = \operatorname{div} \boldsymbol{f} + g & \text{in } \Omega, \\ \frac{\partial u}{\partial \boldsymbol{n}} = \boldsymbol{f} \cdot \boldsymbol{n} + \psi & \text{on } \Gamma, \\ \int_\Omega u dx = 0, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with a boundary Γ, $\boldsymbol{n}$ denotes the outer unit normal vector to the boundary Γ, and $\boldsymbol{f}, g$ and ψ are given functions. We demonstrate the existence of a weak solution in the $W^{1,p}$ Sobolev and $C^{1,\alpha}$ Hölder spaces and obtain $W^{1,p}$ estimates and $C^{1,\alpha}$ Schauder estimates. To the best of our knowledge, this result has not been explicitly proved in the previous research, although a similar result was given for the Dirichlet problems by Gilbarg and Trudinger [10, Theorem 8.34].

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As an application of the above results, we consider the existence of a unique weak solution and $C^{1,\beta}$ regularity of the weak solution in regard to the Maxwell-Stokes problem containing a p-curl system.

In a bounded simply connected domain Ω in $\mathbb{R}^3$ without holes, Yin [18] considered the existence of a unique solution for the so-called p-curl system

$$\begin{cases} \operatorname{curl}[|\operatorname{curl}\boldsymbol{v}|^{p-2}\operatorname{curl}\boldsymbol{v}] = \boldsymbol{f} & \text{in } \Omega, \\ \operatorname{div}\boldsymbol{v} = 0 & \text{in } \Omega, \\ \boldsymbol{n}\times\boldsymbol{v} = \boldsymbol{0} & \text{on } \Gamma \end{cases} \tag{1.2}$$

where Γ denotes the $C^{2,\alpha}$ ($\alpha \in (0,1)$) boundary of Ω, $p > 1$, $\boldsymbol{n}$ the outer normal unit vector field to Γ, and $\boldsymbol{f}$ is a given vector field satisfying the compatibility condition $\operatorname{div}\boldsymbol{f} = 0$ in Ω. If $\boldsymbol{f}$ is a C^{α}-vector function, Yin [19] showed the optimal $C^{1+\beta}$-regularity for some $\beta \in (0,1)$ of a weak solution. See also Yin et al. [20].

The equation (1.2) is a steady-state approximation of Bean's critical state model for type II superconductors. For further physical background, see [20], Chapman [8] and Prigozhin [16].

Aramaki [3] extended the result of [19] on the $C^{1+\beta}$ regularity of a weak solution to a more general equation, in a simply connected domain without holes to the following system.

$$\begin{cases} \operatorname{curl}[S_t(x,|\operatorname{curl}\boldsymbol{v}|^2|)\operatorname{curl}\boldsymbol{v}] = \boldsymbol{f} & \text{in } \Omega, \\ \operatorname{div}\boldsymbol{v} = 0 & \text{in } \Omega, \\ \boldsymbol{n}\times\boldsymbol{v} = \boldsymbol{0} & \text{on } \partial\Omega \end{cases} \tag{1.3}$$

where the function $S(x,t) \in C^2(\Omega\times(0,\infty)) \cap C^0(\Omega\times[0,\infty))$ satisfies some structure conditions. Hereafter, we denote $\frac{\partial}{\partial t}S(x,t)$ by $S_t(x,t)$.

However, in a multi-connected domain, the systems (1.2) and (1.3) are not well posed. In fact, if the second Betti number is positive, for a weak solution $\boldsymbol{v}$ of (1.2) or (1.3), then $\boldsymbol{v}+\boldsymbol{z}$, where $\boldsymbol{z}$ satisfies $\operatorname{curl}\boldsymbol{z} = \boldsymbol{0}, \operatorname{div}\boldsymbol{z} = 0$ in Ω and $\boldsymbol{z}\times\boldsymbol{n} = \boldsymbol{0}$ on Γ, is also a weak solution of (1.2) or (1.3), respectively. Thus it is necessary to add some conditions to (1.2) and (1.3). Aramaki [6] showed the unique existence and optimal $C^{1+\beta}$-regularity of a weak solution to the system (1.3) with additive conditions.

In the author's previous paper Aramaki [4], we considered a system of quasilinear parabolic type equations involving p-curl system associated with the Maxwell equations in a multi-connected domain. We saw that the solution converges to a solution of the stationary problem as the time variable diverges to the infinity. The paper is a continuation of [3] and [4]. For this type of operators, see also Aramaki [5].

We must impose the compatibility condition

$$\operatorname{div}\boldsymbol{f} = 0 \text{ in } \Omega \tag{1.4}$$

for the existence of solution to (1.3). When (1.4) does not hold, we may consider the following equation with a potential.

$$\operatorname{curl}[S_t(x,|\operatorname{curl}\boldsymbol{u}|^2)\operatorname{curl}\boldsymbol{u}] = \boldsymbol{f}+\nabla\pi \text{ in } \Omega. \tag{1.5}$$

in a bounded multi-connected domain $\Omega \subset \mathbb{R}^3$.

In this paper, we consider the existence and regularity of a unique weak solution to (1.5) under some conditions in a bounded multi-connected domain in $\mathbb{R}^3$.

The paper is organized as follows. In section 2, we consider the Neumann problem for the Laplace operator of the form (1.1). In section 3, we specify some assumptions on the domain and function S for application to the Maxwell-Stokes problem. In section 4, we obtain the existence of a weak solution to the Maxwell-Stokes problem and its regularity.

2 Existence and regularity of weak solutions to the Poisson equation with the Neumann condition

In this section, we assume that Ω is a bounded domain in $\mathbb{R}^n$ ($n \geq 2$) with a C^1 boundary Γ and $1 < p < \infty$.

Hereafter, we use the notations $L^p(\Omega), W^{m,p}(\Omega)$ ($m \geq 0$, integer), $W^{s,p}(\Gamma)$ ($s \in \mathbb{R}$), $C^{1,\alpha}(\overline{\Omega})$ and so on, for the standard Sobolev spaces and Hölder spaces of functions. For any Banach space B, we denote B^n by the boldface character $\boldsymbol{B}$. We henceforth use this character to denote vectors and vector-valued functions, and we denote the standard inner product of vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ in $\mathbb{R}^n$ by $\boldsymbol{a} \cdot \boldsymbol{b}$.

We consider the following Neumann problem to the Poisson equation. For given $\boldsymbol{f} \in \boldsymbol{L}^p(\Omega)$ satisfying $\operatorname{div} \boldsymbol{f} \in L^p(\Omega)$, $g \in L^p(\Omega)$ and $\psi \in W^{-1/p,p}(\Gamma)$, find $u \in W^{1,p}(\Omega)$ such that

$$\begin{cases} \Delta u = \operatorname{div} \boldsymbol{f} + g & \text{in } \Omega, \\ \frac{\partial u}{\partial \boldsymbol{n}} = \boldsymbol{f} \cdot \boldsymbol{n} + \psi & \text{on } \Gamma, \\ \int_\Omega u dx = 0. \end{cases} \tag{2.1}$$

We note that if $\boldsymbol{f} \in \boldsymbol{L}^p(\Omega)$ and $\operatorname{div} \boldsymbol{f} \in L^p(\Omega)$, then $\boldsymbol{f} \cdot \boldsymbol{n} \in W^{-1/p,p}(\Gamma)$ is well defined (cf. Amrouche and Seloula [1]), and there exists a constant $C > 0$ depending only on p and Ω such that

$$\|\boldsymbol{f} \cdot \boldsymbol{n}\|_{W^{-1/p,p}(\Gamma)} \leq C(\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)} + \|\operatorname{div} \boldsymbol{f}\|_{L^p(\Omega)}).$$

Now we give a notion of a weak solution of (2.1).

Definition 2.1. *We say that $u \in W^{1,p}(\Omega)$ is a weak solution of* (2.1)*, if u satisfies*

$$\int_\Omega \nabla u \cdot \nabla \varphi dx = \int_\Omega \boldsymbol{f} \cdot \nabla \varphi dx - \int_\Omega g \varphi dx + \int_\Gamma \psi \varphi dS \tag{2.2}$$

for all $\varphi \in C^1(\overline{\Omega})$, where dS denotes the surface area of Γ and the last integral of (2.2) *means the duality between $W^{-1/p,p}(\Gamma)$ and $W^{1-1/p',p'}(\Gamma)$.*

The compatibility condition becomes

$$\int_\Omega g dx = \int_\Gamma \psi dS. \tag{2.3}$$

Remark 2.2. *Lieberman [13] considered a more general elliptic equation, but he only treated the case $p = 2$.*

We obtain the following existence result.

Proposition 2.3. *Let Ω be a bounded domain in $\mathbb{R}^n$ with a C^1 boundary Γ. For $\boldsymbol{f} \in \boldsymbol{L}^p(\Omega)$ satisfying* $\operatorname{div} \boldsymbol{f} \in L^p(\Omega)$, $g \in L^p(\Omega)$ *and* $\psi \in W^{-1/p,p}(\Gamma)$ *satisfying the compatibility condition* (2.3)*, the problem* (2.1) *has a unique weak solution* $u \in W^{1,p}(\Omega)$*, and there exists a constant* $C > 0$ *depending only on* n, p *and* Ω *such that*

$$\|u\|_{W^{1,p}(\Omega)} \le C(\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)} + \|g\|_{L^p(\Omega)} + \|\psi\|_{W^{-1/p,p}(\Gamma)}). \tag{2.4}$$

For the proof, we apply the following proposition given by [1, Theorem 4.2].

Proposition 2.4. *Let X and M be reflexive Banach spaces with the dual spaces X' and M', respectively, and let a be a continuous bilinear form defined on $X \times M$. Let $A \in \mathcal{L}(X, M')$ and $A' \in \mathcal{L}(M, X')$ be bounded linear operators defined by*

$$a(v,w) = \langle Av, w\rangle = \langle v, A'w\rangle \text{ for } v \in X, w \in M.$$

Put $V = \mathrm{Ker}A$. Then the following statements are equivalent.

(i) *There exists $\beta > 0$ such that*

$$\inf_{0 \neq w \in M} \sup_{0 \neq v \in X} \frac{a(v,w)}{\|v\|_X \|w\|_M} \ge \beta.$$

(ii) *$A : X/V \to M'$ is an isomorphism, and $1/\beta$ is a continuity constant of A^{-1}.*

(iii) *$A' : M \to X' \perp V := \{f \in X'; \langle f, v\rangle = 0 \text{ for all } v \in V\}$ is an isomorphism, and $1/\beta$ is a continuity constant of $(A')^{-1}$.*

Proof of Proposition 2.3. We rely on the following inequality (cf. Kozono and Yanagisawa [12, p. 3853] or Simader and Sohr [17, Theorem 1.3]). There exists a constant $c_0 > 0$ such that

$$\|\nabla u\|_{\boldsymbol{L}^p(\Omega)} \le c_0 \sup_{v \in W^{1,p'}(\Omega), \nabla v \neq \mathbf{0}} \frac{\int_\Omega \nabla u \cdot \nabla v dx}{\|\nabla v\|_{\boldsymbol{L}^{p'}(\Omega)}} \tag{2.5}$$

for any $u \in W^{1,p}(\Omega)$, where p' is the conjugate exponent of p, that is, $(1/p) + (1/p') = 1$. From now on, for any space B of functions defined in Ω, we denote

$$\dot{B} = \left\{ u \in B; \int_\Omega u dx = 0 \right\}.$$

Let $X = \dot{W}^{1,p}(\Omega)$ and $M = \dot{W}^{1,p'}(\Omega)$ which are closed subspaces of reflexive Banach spaces $W^{1,p}(\Omega)$ and $W^{1,p'}(\Omega)$, respectively. Therefore, X and M are also reflexive Banach spaces (cf. Brezis [7, Proposition III, 17]). For any $u \in W^{1,p}(\Omega)$, $v = u - \frac{1}{|\Omega|}\int_\Omega u dx \in \dot{W}^{1,p}(\Omega)$. So it can be easily seen that

$$\{\nabla u \,; u \in W^{1,p}(\Omega)\} = \{\nabla v \,; v \in \dot{W}^{1,p}(\Omega)\}.$$

Hence (2.5) also holds if we replace $W^{1,p}(\Omega)$ and $W^{1,p'}(\Omega)$ with $\dot{W}^{1,p}(\Omega)$ and $\dot{W}^{1,p'}(\Omega)$, respectively. That is,

$$\|\nabla u\|_{\boldsymbol{L}^p(\Omega)} \le c_0 \sup_{v \in \dot{W}^{1,p'}(\Omega), v \neq 0} \frac{\int_\Omega \nabla u \cdot \nabla v dx}{\|\nabla v\|_{\boldsymbol{L}^{p'}(\Omega)}} \tag{2.6}$$

for any $u \in \dot{W}^{1,p}(\Omega)$, If we define a continuous bilinear form on $X \times M$ by

$$a(u,v) = \int_\Omega \nabla u \cdot \nabla v dx \text{ for } (u,v) \in X \times M,$$

then it follows from (2.6) that (i) of Proposition 2.4 holds with $\beta = 1/c_0$. Define $A \in \mathcal{L}(X, M')$ by $a(u,v) = \langle Au, v\rangle$ for $u \in X$ and $v \in M$. From (2.6), we can see that $V = \mathrm{Ker}A = \{0\}$. Furthermore, we define a functional T by

$$\langle T, v\rangle = \int_\Omega \boldsymbol{f} \cdot \nabla v dx - \int_\Omega g v dx + \int_\Gamma \psi v dS$$

for $v \in \dot{W}^{1,p'}(\Omega)$. Since $\boldsymbol{f} \in \boldsymbol{L}^p(\Omega), g \in L^p(\Omega)$ and $\psi \in W^{-1/p,p}(\Gamma)$, we see that $T \in M'$. By Proposition 2.4 (ii), there exists a unique $u \in X = \dot{W}^{1,p}(\Omega)$ such that

$$a(u,v) = \langle T, v\rangle \text{ for all } v \in M = \dot{W}^{1,p'}(\Omega),$$

and

$$\|u\|_X \le c_0 \|T\|_{M'}. \tag{2.7}$$

For the Poincaré inequality, there exists a constant $C = C(n,p,\Omega)$ such that $\|v\|_{L^{p'}(\Omega)} \le C\|\nabla v\|_{\boldsymbol{L}^{p'}(\Omega)}$ for $v \in \dot{W}^{1,p'}(\Omega)$. Hence for $v \in \dot{W}^{1,p'}(\Omega)$, we have

$$\begin{aligned} |\langle T, v\rangle| &\le \|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)} \|\nabla v\|_{\boldsymbol{L}^{p'}(\Omega)} + \|g\|_{L^p(\Omega)} \|v\|_{L^{p'}(\Omega)} + \|\psi\|_{W^{-1/p,p}(\Gamma)} \|v\|_{W^{1-1/p',p'}(\Gamma)} \\ &\le C(\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)} + \|g\|_{L^p(\Omega)} + \|\psi\|_{W^{-1/p,p}(\Gamma)}) \|v\|_{\dot{W}^{1,p'}(\Omega)}. \end{aligned}$$

Therefore, we have

$$\|T\|_{M'} \le C(\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)} + \|g\|_{L^p(\Omega)} + \|\psi\|_{W^{-1/p,p}(\Gamma)}).$$

For any $v \in W^{1,p'}(\Omega)$, we have $v - c_v \in \dot{W}^{1,p'}(\Omega)$, where $c_v = \frac{1}{|\Omega|}\int_\Omega v dx$. Hence it follows from the compatibility condition (2.3) that

$$\begin{aligned} \int_\Omega \nabla u \cdot \nabla v dx &= \int_\Omega \boldsymbol{f} \cdot \nabla v dx - \int_\Omega g(v - c_v) dx + \int_\Gamma \psi(v - c_v) dS \\ &= \int_\Omega \boldsymbol{f} \cdot \nabla v dx - \int_\Omega g v dx + \int_\Gamma \psi v dS. \end{aligned}$$

for all $v \in W^{1,p'}(\Omega)$. Therefore, u is a unique weak solution of (2.1) and the estimate (2.4) holds. This completes the proof of Proposition 2.3.

For the regularity of the weak solution of (2.1), we want to prove the following estimate of a $C^{1,\alpha}$ weak solution.

Proposition 2.5. *Let Ω be a bounded domain with a $C^{1,\alpha}$ boundary Γ $(0 < \alpha < 1)$. Assume that $\boldsymbol{f} \in \boldsymbol{C}^\alpha(\overline{\Omega}), g \in L^\infty(\Omega)$ and $\psi \in C^\alpha(\overline{\Omega})$ satisfying the compatibility condition* (2.3)*. If $u \in C^{1,\alpha}(\overline{\Omega})$ is a weak solution of* (2.1)*, then there exists a constant $C > 0$ depending only on n, α and Ω such that*

$$\|u\|_{C^{1,\alpha}(\overline{\Omega})} \le C(\|\boldsymbol{f}\|_{\boldsymbol{C}^\alpha(\overline{\Omega})} + \|g\|_{L^\infty(\Omega)} + \|\psi\|_{C^\alpha(\overline{\Omega})}). \tag{2.8}$$

Proof. Since $C^{1,\alpha}(\overline{\Omega}) \subset W^{1,2}(\Omega)$, if we apply Lieberman [13, Theorem 5.54], then we have

$$||u||_{C^{1,\alpha}(\overline{\Omega})} \leq C(||u||_{C^0(\overline{\Omega})} + ||\boldsymbol{f}||_{\boldsymbol{C}^\alpha(\overline{\Omega})} + ||g||_{L^\infty(\Omega)} + ||\psi||_{C^\alpha(\overline{\Omega})}). \tag{2.9}$$

We must drop the first term on the right-hand side of (2.9). For this purpose, we use an argument similar to Nardi [14, Proof of Theorem 4.1]. Let us suppose that (2.8) is false. Then there exist $\{u_k\} \subset C^{1,\alpha}(\overline{\Omega}), \{g_k\} \subset L^\infty(\Omega)$ and $\{\psi_k\} \subset C^\alpha(\overline{\Omega})$ satisfying the compatibility condition $\int_\Omega g_k dx = \int_\Gamma \psi_k dS$, such that u_k is a weak solution of

$$\begin{cases} \Delta u_k = \operatorname{div} \boldsymbol{f}_k + g_k & \text{in } \Omega, \\ \frac{\partial u_k}{\partial \boldsymbol{n}} = \boldsymbol{f}_k \cdot \boldsymbol{n} + \psi_k & \text{on } \Gamma, \\ \int_\Omega u_k dx = 0, \end{cases}$$

$||u_k||_{C^{1,\alpha}(\overline{\Omega})} = 1$, and

$$||u_k||_{C^{1,\alpha}(\overline{\Omega})} \geq k(||\boldsymbol{f}_k||_{\boldsymbol{C}^\alpha(\overline{\Omega})} + ||g_k||_{L^\infty(\Omega)} + ||\psi_k||_{C^\alpha(\overline{\Omega})}).$$

Then $\boldsymbol{f}_k \to \boldsymbol{0}$ in $\boldsymbol{C}^\alpha(\overline{\Omega})$, $g_k \to 0$ in $L^\infty(\Omega)$ and $\psi_k \to 0$ in $C^\alpha(\overline{\Omega})$. Since $||u_k||_{C^{1,\alpha}(\overline{\Omega})} = 1$, there exists a constant $C(\Omega) > 0$ such that

$$|D^\beta u_k(x) - D^\beta u_k(y)| \leq C(\Omega)|x-y|^\alpha \text{ for } x, y \in \Omega, |\beta| \leq 1.$$

Iterating the Ascoli-Arzelà theorem, there exists a subsequence $\{u_{k_l}\}$ of $\{u_k\}$ such that $u_{k_l} \to u_0$ in $C^0(\overline{\Omega})$ and $D^\beta u_{k_l} \to u_\beta$ in $C^0(\overline{\Omega})$ for $|\beta| = 1$. This implies that $u_{k_l} \to u_0$ in $C^1(\overline{\Omega})$. Hence u_0 is a weak solution of

$$\begin{cases} \Delta u_0 = 0 & \text{in } \Omega, \\ \frac{\partial u_0}{\partial \boldsymbol{n}} = 0 & \text{on } \Gamma, \\ \int_\Omega u_0 dx = 0. \end{cases}$$

Thus we have $u_0 = 0$. From (2.9), we have

$$1 = ||u_{k_l}||_{C^{1,\alpha}(\overline{\Omega})} \leq C(||u_{k_l}||_{C^0(\overline{\Omega})} + ||\boldsymbol{f}_{k_l}||_{\boldsymbol{C}^\alpha(\overline{\Omega})} + ||g_{k_l}||_{L^\infty(\Omega)} + ||\psi_{k_l}||_{C^\alpha(\overline{\Omega})}) \to 0.$$

This leads to a contradiction. Therefore the estimate (2.8) holds. □

We present the main theorem in this section.

Theorem 2.6. *Let Ω be a bounded domain with a $C^{1,\alpha}$ boundary Γ $(0 < \alpha < 1)$. Assume that $\boldsymbol{f} \in \boldsymbol{C}^\alpha(\overline{\Omega})$ satisfying $\operatorname{div} \boldsymbol{f} \in L^p(\Omega)$, $g \in L^\infty(\Omega)$ and $\psi \in C^\alpha(\overline{\Omega})$ satisfying the compatibility condition* (2.3)*. Then a unique weak solution u of* (2.1) *belongs to $C^{1,\alpha}(\overline{\Omega})$, and there exists a constant $C > 0$ depending only on n, α and Ω such that* (2.8) *holds.*

Proof. Choose $\boldsymbol{f}_k \in \boldsymbol{C}^3(\overline{\Omega}), g_k \in C^3(\overline{\Omega})$ and $\psi_k \in C^3(\overline{\Omega})$ such that

$\boldsymbol{f}_k \to \boldsymbol{f}$ with $||\boldsymbol{f}_k||_{C^\alpha(\overline{\Omega})} \leq c||\boldsymbol{f}||_{\boldsymbol{C}^\alpha(\overline{\Omega})}$,

$g_k \to g$ in $L^1(\Omega)$ with $||g_k||_{L^\infty(\Omega)} \leq c||g||_{L^\infty(\Omega)}$,

$\psi_k \to \psi$ with $||\psi_k||_{C^\alpha(\overline{\Omega})} \leq c||\psi||_{C^\alpha(\overline{\Omega})}$.

Moreover, choose the $C^{2,\alpha}$ domain $\{\Omega_k\}$ exhausting Ω such that $\Gamma_k := \partial\Omega_k \to \Gamma$ and the surfaces Γ_k are uniformly in $C^{1,\alpha}$ (cf. [10, the proof of Theorem 8.34]). Since g_k and ψ_k might not satisfy the compatibility condition, we choose constants c_k such that

$$\int_\Omega g_k dx = \int_{\Gamma_k} (\psi_k + c_k) dS. \tag{2.10}$$

By the Lebesgue dominated convergence theorem, we can see that

$$\int_{\Omega_k} g_k dx \to \int_\Omega g dx, \quad \int_{\Gamma_k} \psi_k dS \to \int_\Gamma \psi dS.$$

Since $|\Gamma_k| \to |\Gamma|$, we have $c_k \to 0$ as $k \to \infty$. We consider the following smooth approximating Neumann problem

$$\begin{cases} \Delta u = \operatorname{div} \boldsymbol{f}_k + g_k & \text{in } \Omega_k, \\ \frac{\partial u}{\partial \boldsymbol{n}} = \boldsymbol{f}_k \cdot \boldsymbol{n} + \psi_k + c_k & \text{on } \Gamma_k, \\ \int_\Omega u_k dx = 0. \end{cases} \tag{2.11}$$

Now the compatibility condition (2.10) holds. Thus (2.11) is a regular problem with the Neumann boundary condition. Hence, it follows from [14, Theorem 3.1] that (2.11) has a unique solution $u_k \in C^{2,\alpha}(\overline{\Omega})$. Moreover using Proposition 2.5, there exists a constant $C > 0$ depending only on n, α and Ω such that

$$\begin{aligned} \|u_k\|_{C^{1,\alpha}(\overline{\Omega})} &\leq C(\|\boldsymbol{f}_k\|_{\boldsymbol{C}^\alpha(\overline{\Omega})} + \|g_k\|_{L^\infty(\Omega_k)} + \|\psi_k\|_{C^\alpha(\overline{\Omega_k})} + c_k|\Omega_k|) \\ &\leq Cc(\|\boldsymbol{f}\|_{\boldsymbol{C}^\alpha(\overline{\Omega})} + \|g\|_{L^\infty(\Omega)} + \|\psi\|_{C^\alpha(\overline{\Omega})}) + c_k C|\Omega|. \end{aligned}$$

Letting $k \to \infty$ in the weak form of (2.11), we obtain in the limit a unique weak solution u of (2.1) and $u \in C^{1,\alpha}(\overline{\Omega})$ which follows from [13, Theorem 5.54]. By Proposition 2.5, the solution u satisfies the estimate (2.8). □

3 Assumptions to an application

In this and next sections, we consider the Maxwell-Stokes problem in $\mathbb{R}^3$. Since we allow that the domain is multi-connected in $\mathbb{R}^3$, we assume that Ω has the following conditions as in [1] (cf. Amrouche and Seloula [2], Dautray and Lions [9] and Girault and Raviart [11]). Let $\Omega \subset \mathbb{R}^3$ be a bounded domain of class $C^{2,\alpha}$ with the boundary Γ and Ω be locally situated on one side of Γ.

(1) Γ has a finite number of connected components $\Gamma_0, \Gamma_1, \dots, \Gamma_m$ with Γ_0 denoting the boundary of the infinite connected component of $\mathbb{R}^3 \setminus \overline{\Omega}$.

(2) There exist n connected open surfaces Σ_j, $(j = 1, \dots, n)$, called cuts, contained in Ω such that

(a) Σ_j is an open subset of a smooth manifold $\mathcal{M}_j$.

(b) $\partial\Sigma_j \subset \Gamma$ $(j = 1, \dots, n)$, where $\partial\Sigma_j$ denotes the boundary of Σ_j, and Σ_j is non-tangential to Γ.

(c) $\overline{\Sigma_i} \cap \overline{\Sigma_j} = \emptyset\, (i \neq j)$.

(d) The open set $\dot{\Omega} = \Omega \setminus (\cup_{i=1}^n \Sigma_i)$ is simply connected and pseudo $C^{1,1}$ class.

The number n is called the first Betti number, which is equal to the number of handles of Ω, and m is called the second Betti number which is equal to the number of holes. We say that if $n = 0$, Ω is simply connected, and if $m = 0$, Ω has no holes.

Define two spaces by

$$\begin{aligned} \mathbb{K}^p_N(\Omega) &= \{\boldsymbol{v} \in \boldsymbol{L}^p(\Omega); \operatorname{div} \boldsymbol{v} = 0, \operatorname{curl} \boldsymbol{v} = \boldsymbol{0} \text{ in } \Omega, \boldsymbol{v} \times \boldsymbol{n} = \boldsymbol{0} \text{ on } \Gamma\}, \\ \mathbb{K}^p_T(\Omega) &= \{\boldsymbol{v} \in \boldsymbol{L}^p(\Omega); \operatorname{div} \boldsymbol{v} = 0, \operatorname{curl} \boldsymbol{v} = \boldsymbol{0} \text{ in } \Omega, \boldsymbol{v} \cdot \boldsymbol{n} = 0 \text{ on } \Gamma\}. \end{aligned}$$

It is well known that $\dim \mathbb{K}^p_T(\Omega) = n$ and $\dim \mathbb{K}^p_N(\Omega) = m$.

We assume that a function $S(x,t) \in C^2(\Omega \times (0,\infty)) \cap C(\Omega \times [0,\infty))$ satisfies the following structure conditions: there exist a constant $1 < p < \infty$ and positive constants $0 < \lambda \le \Lambda < \infty$ such that for all $x \in \Omega$

$$S(x,0) = 0 \text{ and } \lambda t^{(p-2)/2} \le S_t(x,t) \le \Lambda t^{(p-2)/2} \text{ for } t > 0. \tag{3.1a}$$

$$\lambda t^{(p-2)/2} \le S_t(x,t) + 2t S_{tt}(x,t) \le \Lambda t^{(p-2)/2} \text{ for } t > 0. \tag{3.1b}$$

$$\text{If } 1 < p < 2, S_{tt}(x,t) < 0, \text{ and if } p \ge 2, S_{tt}(x,t) \ge 0 \text{ for } t > 0. \tag{3.1c}$$

We note that from (3.1a), we have

$$\frac{2}{p}\lambda t^{p/2} \le S(x,t) \le \frac{2}{p}\Lambda t^{p/2} \text{ for } t \ge 0.$$

When $S(x,t) = t^{p/2}$, system (1.3) becomes (1.2), and by elementary calculations, we see that $S(x,t) = \nu(x) t^{p/2}$, where $\nu \in C^2(\Omega)$ and $0 < \nu_* \le \nu(x) \le \nu^* < \infty$, satisfies (3.1a)-(3.1c).

4 An application to the Maxwell-Stokes type equation

Since we allow that Ω is multi-connected, (1.3) is not well posed. In our previous paper [6], we considered the following equation.

$$\begin{cases} \operatorname{curl}[S_t(x, |\operatorname{curl} \boldsymbol{v}|^2) \operatorname{curl} \boldsymbol{v}] = \boldsymbol{f} & \text{in } \Omega, \\ \operatorname{div} \boldsymbol{v} = 0 & \text{in } \Omega, \\ \boldsymbol{v} \times \boldsymbol{n} = \boldsymbol{0} & \text{on } \Gamma, \\ \langle \boldsymbol{v} \cdot \boldsymbol{n}, 1 \rangle_{\Gamma_i} = 0 & \text{for } i = 1, 2, \ldots m, \end{cases} \tag{4.1}$$

where $\langle \cdot, \cdot \rangle_{\Gamma_i}$ denotes the duality between $W^{-1/p,p}(\Gamma_i)$ and $W^{1-1/p',p'}(\Gamma_i)$. We assume that $\boldsymbol{f} \in \boldsymbol{L}^{p'}(\Omega)$ satisfying $\operatorname{div} \boldsymbol{f} = 0$ in Ω and $\langle \boldsymbol{f} \cdot \boldsymbol{n}, 1 \rangle_{\Gamma_i} = 0$ for $i = 1, \ldots, m$, where the bracket $\langle \cdot, \cdot \rangle_{\Gamma_i}$ means the duality between $W^{-1/p',p'}(\Gamma_i)$ and $W^{1-1/p,p}(\Gamma_i)$. Though we use the same notation as the duality between $W^{-1/p,p}(\Gamma_i)$ and $W^{1-1/p',p'}(\Gamma_i)$, we should not be in confusion. We showed the existence and regularity of a weak solution to the system (4.1). To explain the result obtained precisely, we consider the space

$$\begin{aligned} \boldsymbol{V}^p(\Omega) = \{\boldsymbol{v} \in \boldsymbol{L}^p(\Omega); \operatorname{curl} \boldsymbol{v} \in \boldsymbol{L}^p(\Omega), \operatorname{div} \boldsymbol{v} = 0 \text{ in } \Omega, \\ \boldsymbol{v} \times \boldsymbol{n} = \boldsymbol{0} \text{ on } \Gamma, \langle \boldsymbol{v} \cdot \boldsymbol{n}, 1 \rangle_{\Gamma_i} = 0, i = 1, \ldots m\}. \end{aligned}$$

Then we have

Lemma 4.1. *Let $1 < p < \infty$. Then $\boldsymbol{V}^p(\Omega)$ is a closed subspace of $\boldsymbol{W}^{1,p}(\Omega)$, and we can regard $\boldsymbol{V}^p(\Omega)$ as a separable, reflexive Banach space with the norm*

$$\|\boldsymbol{v}\|_{\boldsymbol{V}^p(\Omega)} := \|\operatorname{curl} \boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)}$$

which is equivalent to $\|\boldsymbol{v}\|_{\boldsymbol{W}^{1,p}(\Omega)}$.

For the proof, see [6] and [1].

Moreover, we define a space

$$\boldsymbol{W}^{1,p}_{t0}(\Omega) = \{\boldsymbol{v} \in \boldsymbol{W}^{1,p}(\Omega); \boldsymbol{v} \times \boldsymbol{n} = \boldsymbol{0} \text{ on } \Gamma\}.$$

Definition 4.2. *For given $\boldsymbol{f} \in \boldsymbol{L}^{p'}(\Omega)$ satisfying $\operatorname{div}\boldsymbol{f} = 0$ in Ω, we say $\boldsymbol{v} \in \boldsymbol{V}^p(\Omega)$ is a weak solution of* (4.1)*, if $\boldsymbol{v}$ satisfies*

$$\int_\Omega S_t(x, |\operatorname{curl}\boldsymbol{v}|^2)\operatorname{curl}\boldsymbol{v} \cdot \operatorname{curl}\boldsymbol{w}\,dx = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{w}\,dx \text{ for all } \boldsymbol{w} \in \boldsymbol{W}^{1,p}_{t0}(\Omega).$$

In the previous paper, we obtained the following proposition [6, Proposition 3.5] and theorem [6, Theorem 2.2] on the existence and regularity of the weak solution. These results are used later.

Proposition 4.3. *Assume that $\boldsymbol{f} \in \boldsymbol{L}^{p'}(\Omega)$ satisfies $\operatorname{div}\boldsymbol{f} = 0$ in Ω and $\langle \boldsymbol{f} \cdot \boldsymbol{n}, 1\rangle_{\Gamma_i} = 0$ for $i = 1, \dots, m$. Then the system* (4.1) *has a unique weak solution $\boldsymbol{v} \in \boldsymbol{V}^p(\Omega)$, and there exists a constant $C > 0$ depending only on λ, p and Ω such that*

$$\|\boldsymbol{v}\|_{\boldsymbol{V}^p(\Omega)} \le C\|\boldsymbol{f}\|^{p'-1}_{\boldsymbol{L}^{p'}(\Omega)}.$$

Theorem 4.4. *Assume that Ω is a bounded domain in $\mathbb{R}^3$ with a $C^{2,\alpha}$ boundary Γ satisfying* (1) *and* (2) *for some $\alpha \in (0,1)$, and that a function $S(x,t)$ satisfies the conditions* (3.1a)-(3.1c)*. Moreover, $\boldsymbol{f}$ satisfies the condition that $\boldsymbol{f} \in \boldsymbol{C}^\alpha(\overline{\Omega})$ and*

$$\operatorname{div}\boldsymbol{f} = 0 \text{ in } \Omega \text{ and } \langle \boldsymbol{f} \cdot \boldsymbol{n}, 1\rangle_{\Gamma_i} = 0,\ i = 1, \dots m.. \tag{4.2}$$

Then the unique weak solution $\boldsymbol{v} \in \boldsymbol{V}^p(\Omega)$ of (4.1) *in the sense of Definition* 4.2 *belongs to $\boldsymbol{C}^{1,\beta}(\overline{\Omega})$ for some $\beta \in (0,1)$. Furthermore, there exists a constant $C > 0$ depending only on p, Ω and $\|\boldsymbol{f}\|_{\boldsymbol{C}^\alpha(\overline{\Omega})}$ such that*

$$\|\boldsymbol{v}\|_{\boldsymbol{C}^{1,\beta}(\overline{\Omega})} \le C.$$

When $\boldsymbol{f}$ does not satisfy the first equation of (4.2), the theorem is false. In this case, we may consider the following Maxwell-Stokes problem (cf. Pan [15]): to find $(\boldsymbol{u}, \pi) \in \boldsymbol{V}^p(\Omega) \times \dot{W}^{1,p'}(\Omega)$ such that

$$\begin{cases} \operatorname{curl}[S_t(x, |\operatorname{curl}\boldsymbol{u}|^2)\operatorname{curl}\boldsymbol{u}] = \boldsymbol{f} + \nabla\pi & \text{in } \Omega, \\ \operatorname{div}\boldsymbol{u} = 0 & \text{in } \Omega, \\ \boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{0} & \text{on } \Gamma, \\ \langle \boldsymbol{u} \cdot \boldsymbol{n}, 1\rangle_{\Gamma_i} = 0 & \text{for } i = 1, 2, \dots m, \\ \frac{\partial\pi}{\partial\boldsymbol{n}} = -\boldsymbol{f} \cdot \boldsymbol{n} & \text{on } \Gamma \end{cases} \tag{4.3}$$

Definition 4.5. *We say that $(\boldsymbol{u}, \pi) \in \boldsymbol{V}^p(\Omega) \times \dot{W}^{1,p'}(\Omega)$ is a weak solution of* (4.3)*, if $(\boldsymbol{u}, \pi)$ satisfies the following equality.*

$$\int_\Omega S_t(x, |\operatorname{curl}\boldsymbol{u}|^2)\operatorname{curl}\boldsymbol{u} \cdot \operatorname{curl}\boldsymbol{w}\,dx = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{w}\,dx + \langle \nabla\pi, \boldsymbol{w}\rangle_{\boldsymbol{W}^{1,p}_{t0}(\Omega)', \boldsymbol{W}^{1,p}_{t0}(\Omega)} \tag{4.4}$$

for all $\boldsymbol{w} \in \boldsymbol{W}^{1,p}_{t0}(\Omega)$.

First we obtain the following proposition on the existence of a weak solution.

Proposition 4.6. *Assume that* $\boldsymbol{f} \in \boldsymbol{L}^{p'}(\Omega)$ *and* $\operatorname{div}\boldsymbol{f} \in L^{p'}(\Omega)$. *Then the system* (4.3) *has a unique weak solution* $(\boldsymbol{u},\pi) \in \boldsymbol{V}^p(\Omega)\times \dot{W}^{1,p'}(\Omega)$*, and there exists a constant* $C > 0$ *depending only on* λ, p *and* Ω *such that*

$$\|\boldsymbol{u}\|_{\boldsymbol{V}^p(\Omega)} + \|\pi\|_{W^{1,p'}(\Omega)} \leq C(\|\boldsymbol{f}\|^{p'-1}_{\boldsymbol{L}^{p'}(\Omega)} + \|\boldsymbol{f}\|_{\boldsymbol{L}^{p'}(\Omega)}).$$

Proof. We first consider the Poisson equation with the Neumann boundary condition

$$\begin{cases} -\Delta\pi = \operatorname{div}\boldsymbol{f} & \text{in } \Omega, \\ \frac{\partial\pi}{\partial\boldsymbol{n}} = -\boldsymbol{f}\cdot\boldsymbol{n} & \text{on } \Gamma, \\ \int_\Omega \pi dx = 0. \end{cases} \tag{4.5}$$

Applying Proposition 2.3, (4.5) has a unique weak solution $\pi \in \dot{W}^{1,p'}(\Omega)$, and there exists a constant $C > 0$ depending only on p and Ω such that

$$\|\pi\|_{W^{1,p'}(\Omega)} \leq C(\|\boldsymbol{f}\|_{\boldsymbol{L}^{p'}(\Omega)} + \|\operatorname{div}\boldsymbol{f}\|_{L^{p'}(\Omega)}).$$

Since $\boldsymbol{f} + \nabla\pi \in \boldsymbol{L}^{p'}(\Omega)$, $\operatorname{div}(\boldsymbol{f}+\nabla\pi) = \operatorname{div}\boldsymbol{f} + \Delta\pi = 0$ in Ω and $(\boldsymbol{f}+\nabla\pi)\cdot\boldsymbol{n} = \boldsymbol{f}\cdot\boldsymbol{n} + \frac{\partial\pi}{\partial\boldsymbol{n}} = 0$ on Γ, the hypotheses of Proposition 4.3 replaced $\boldsymbol{f}$ with $\boldsymbol{f}+\nabla\pi$ hold. Hence (4.3) has a unique solution $\boldsymbol{u} \in \boldsymbol{V}^p(\Omega)$, and that there exists a constant $C > 0$ depending only on λ and p such that

$$\|\boldsymbol{u}\|_{\boldsymbol{V}^p(\Omega)} \leq C\|\boldsymbol{f}\|^{p'-1}_{\boldsymbol{L}^{p'}(\Omega)}.$$

The uniqueness of a solution $(\boldsymbol{u},\pi) \in \boldsymbol{V}^p(\Omega)\times\dot{W}^{1,p'}(\Omega)$ is now clear. □

Now we obtain the regularity of the weak solutions of (4.3).

Theorem 4.7. *Assume that* Ω *is a bounded domain in* $\mathbb{R}^3$ *with* $C^{2,\alpha}$ *boundary* Γ *satisfying* (1) and (2) for some $\alpha \in (0,1)$, and that a function $S(x,t)$ satisfies the conditions (3.1a)-(3.1c). Moreover, $\boldsymbol{f}$ satisfies the condition that $\boldsymbol{f} \in \boldsymbol{C}^\alpha(\overline{\Omega})$ and $\operatorname{div}\boldsymbol{f} \in \boldsymbol{L}^{p'}(\Omega)$. Then the weak solution $(\boldsymbol{u},\pi)$ of (4.4) belongs to $\boldsymbol{C}^{1,\beta}(\overline{\Omega})\times C^{1,\alpha}(\overline{\Omega})$ for some $\beta \in (0,1)$, and there exists a constant $C > 0$ depending only on p,α,Ω and $\|\boldsymbol{f}\|_{\boldsymbol{C}^\alpha(\overline{\Omega})}$ such that

$$\|\boldsymbol{u}\|_{\boldsymbol{C}^{1,\beta}(\overline{\Omega})} + \|\pi\|_{C^{1,\alpha}(\overline{\Omega})} \leq C. \tag{4.6}$$

Proof. By Proposition 2.6, the solution π of the Poisson equation with the Neumann boundary condition (2.1) is, in fact, in $C^{1,\alpha}(\overline{\Omega})$ and there exists a constant $C >$ depending only on α and Ω such that

$$\|\pi\|_{C^{1,\alpha}(\overline{\Omega})} \leq C\|\boldsymbol{f}\|_{\boldsymbol{C}^\alpha(\overline{\Omega})}.$$

Since $\operatorname{div}(\boldsymbol{f}+\nabla\pi) = 0$ in Ω and $(\boldsymbol{f}+\nabla\pi)\cdot\boldsymbol{n} = 0$ on Γ, it follows form Theorem 4.4 that $\boldsymbol{u}$ is in $\boldsymbol{C}^{1,\beta}(\overline{\Omega})$ for some $\beta \in (0,1)$, and the estimate (4.9) holds. □

Next we consider a special case where Ω has no holes. We note that in this case, we can adopt the Dirichlet boundary condition with respect to π. That is to say, we consider the following problem: to find $(\boldsymbol{u},\pi) \in \boldsymbol{V}^p(\Omega) \times W_0^{1,p'}(\Omega)$ such that

$$\begin{cases} \operatorname{curl}[S_t(x,|\operatorname{curl}\boldsymbol{u}|^2)\operatorname{curl}\boldsymbol{u}] = \boldsymbol{f} + \nabla\pi & \text{in } \Omega, \\ \operatorname{div}\boldsymbol{u} = 0 & \text{in } \Omega, \\ \boldsymbol{u}\times\boldsymbol{n} = \boldsymbol{0} & \text{on } \Gamma, \\ \pi = 0 & \text{on } \Gamma. \end{cases} \tag{4.7}$$

Since Ω has no holes, we can write

$$\boldsymbol{V}^p(\Omega) = \{\boldsymbol{v} \in \boldsymbol{L}^p(\Omega); \operatorname{curl}\boldsymbol{v} \in \boldsymbol{L}^p(\Omega), \operatorname{div}\boldsymbol{v} = 0 \text{ in } \Omega, \boldsymbol{v}\times\boldsymbol{n} = \boldsymbol{0} \text{ on } \Gamma\}.$$

For $\boldsymbol{f} \in \boldsymbol{L}^{p'}(\Omega)$, the following Dirichlet problem

$$\begin{cases} -\Delta\pi = \operatorname{div}\boldsymbol{f} & \text{in } \Omega, \\ \pi = 0 & \text{on } \Gamma \end{cases} \tag{4.8}$$

has a unique weak solution $\pi \in W_0^{1,p'}(\Omega)$ in the sense of

$$\int_\Omega \nabla\pi\cdot\nabla\varphi dx = -\int_\Omega \boldsymbol{f}\cdot\nabla\varphi dx \text{ for all } \varphi \in W_0^{1,p}(\Omega),$$

and there exists a constant $C > 0$ depending only on p and Ω such that

$$\|\pi\|_{W^{1,p'}(\Omega)} \le C\|\boldsymbol{f}\|_{\boldsymbol{L}^{p'}(\Omega)}.$$

These facts follows from the variational inequality (cf. [12, (2.16)])

$$\|\nabla\psi\|_{\boldsymbol{L}^{p'}(\Omega)} \le c_1 \sup_{0\ne\varphi\in W_0^{1,p}(\Omega)} \frac{\left|\int_\Omega \nabla\psi\cdot\nabla\varphi dx\right|}{\|\nabla\varphi\|_{\boldsymbol{L}^p(\Omega)}} \text{ for all } \psi \in W_0^{1,p'}(\Omega),$$

where c_1 is a positive constant depending only on p and Ω, and an application of Proposition 2.4 as in the proof of Proposition 2.3.

Furthermore, if $\boldsymbol{f} \in \boldsymbol{C}^\alpha(\overline{\Omega})$, it follows from [10, Theorem 8.34, 8.33] that the weak solutin π of (4.8) belongs to $C^{1,\alpha}(\overline{\Omega})$, and that there exists a constant $C > 0$ depending only on α and Ω such that

$$\|\pi\|_{C^{1,\alpha}(\overline{\Omega})} \le C(\|\pi\|_{C^0(\overline{\Omega})} + \|\boldsymbol{f}\|_{\boldsymbol{C}^\alpha(\overline{\Omega})}).$$

By the arguments as in the proof of Proposition 2.5, we can improve the estimate as follows.

$$\|\pi\|_{C^{1,\alpha}(\overline{\Omega})} \le C\|\boldsymbol{f}\|_{\boldsymbol{C}^\alpha(\overline{\Omega})}.$$

When $\boldsymbol{f} \in \boldsymbol{L}^{p'}(\Omega)$, since $\boldsymbol{f} + \nabla\pi \in \boldsymbol{L}^{p'}(\Omega)$ and $\operatorname{div}(\boldsymbol{f} + \nabla\pi) = \operatorname{div}\boldsymbol{f} + \Delta\pi = 0$ in Ω, and Ω has no holes, i.e., $\dim\mathbb{K}_N^p(\Omega) = m = 0$, it follows from Proposition 4.3 and Theorem 4.4 that we can demonstrate the following.

Proposition 4.8. *Let Ω be a bounded domain satisfying* (1) *and* (2) *with $m = 0$. Assume that $\boldsymbol{f} \in \boldsymbol{L}^{p'}(\Omega)$. Then the system* (4.7) *has a unique weak solution $(\boldsymbol{u},\pi) \in \boldsymbol{V}^p(\Omega) \times W_0^{1,p'}(\Omega)$, and there exists a constant $C > 0$ depending only on λ, p and Ω such that*

$$\|\boldsymbol{u}\|_{\boldsymbol{V}^p(\Omega)} + \|\pi\|_{W^{1,p'}(\Omega)} \le C(\|\boldsymbol{f}\|_{\boldsymbol{L}^{p'}(\Omega)}^{p'-1} + \|\boldsymbol{f}\|_{\boldsymbol{L}^{p'}(\Omega)}).$$

Now we also obtain the regularity of the weak solutions of (4.7), under the hypothesis $\boldsymbol{f} \in \boldsymbol{C}^{\alpha}(\overline{\Omega})$.

Theorem 4.9. *Assume that* Ω *is a bounded domain in* $\mathbb{R}^3$ *with* $C^{2,\alpha}$ *boundary* Γ *satisfying* (1) and (2) for some $\alpha \in (0,1)$ with $m = 0$, and that a function $S(x,t)$ satisfies the conditions (3.1a)-(3.1c). Moreover, if $\boldsymbol{f} \in \boldsymbol{C}^{\alpha}(\overline{\Omega})$, then the weak solution $(\boldsymbol{u},\pi)$ of (4.7) belongs to $\boldsymbol{C}^{1,\beta}(\overline{\Omega}) \times C^{1,\alpha}(\overline{\Omega})$ for some $\beta \in (0,1)$, and there exists a constant $C > 0$ depending only on α, Ω and $\|\boldsymbol{f}\|_{\boldsymbol{C}^{\alpha}(\overline{\Omega})}$ such that

$$\|\boldsymbol{u}\|_{\boldsymbol{C}^{1,\beta}(\overline{\Omega})} + \|\pi\|_{C^{1,\alpha}(\overline{\Omega})} \leq C. \tag{4.9}$$

Acknowledgments

The author thanks the referees for their careful reading of the manuscript and insightful comments.

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