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On Singular Integrals with Rough Kernels in Triebel-Lizorkin Weighted Spaces

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Abstract

Let $\Omega \in L^1(S^{n-1})$ have mean value zero and satisfy the condition

$$\sup_{\zeta' \in S^{n-1}} \int_{S^{n-1}} |\Omega(y')| (\ln|\zeta' \cdot y'|^{-1})^{(\ln(e+\ln|\zeta' \cdot y'|^{-1}))^{\beta}} \, d\sigma(y') < \infty \text{ for some } \beta > 0.$$

Under certain conditions on the measurable function h, we show that the singular integral

$$Tf(x) = \text{p. v.} \int_{\mathbb{R}^n} \frac{h(|y|)\Omega(y')}{|y|^n} f(x-y)\, dy$$

is bounded on the Triebel-Lizorkin weighted spaces $\dot{F}^{\alpha,w}_{p,q}(\mathbb{R}^n)$. We also study the Marcinkiewicz integral (with the same kernel Ω as above) in the L^p- weighted spaces.

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1 Introduction

In this note, we always assume that the kernel $\Omega \in L^1(S^{n-1})$ $(n \geq 2)$ satisfies the mean value zero property. Consider the singular integral Tf (with $h \equiv 1$) as defined in the abstract. Calderón and Zygmund [3] proved that T is a bounded operator on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$ if $\Omega \in L\log^+ L(S^{n-1})$. Afterward, Connett [6] and Ricci and Weiss [16] independently obtained the same result with the condition $\Omega \in H^1(S^{n-1})$, where $H^1(S^{n-1})$ is the Hardy

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space on the unit sphere. Later on, Fan and Pan [10] obtained the result for a more general class of operators.

Recently, Grafakos and Stefanov [13] proved that if $h \equiv 1$, and $\Omega \in L^1(S^{n-1})$ satisfies the condition

$$\textbf{(1)} \quad \sup_{\zeta' \in S^{n-1}} \int_{S^{n-1}} |\Omega(y')|(\log \frac{1}{|\zeta' \cdot y'|})^{1+\alpha}\, d\sigma(y') < \infty \text{ for some } \alpha > 1,$$

then

$$\|Tf\|_{L^p(\mathbb{R}^n)} \le C\|f\|_{L^p(\mathbb{R}^n)} \text{ for } |\frac{1}{2} - \frac{1}{p}| < \frac{\alpha}{2(2+\alpha)}.$$

Subsequently, this result was extended by Fan, Guo and Pan [11], where $\frac{\alpha}{2(2+\alpha)}$ is replaced by $\frac{\alpha}{2(1+\alpha)}$. Note that for every α satisfying $0 \le \alpha < 1$, Grafakos, Honzík, and Ryabogin [14] proved that there is an even integrable function Ω on S^{n-1} with mean value zero that satisfies a condition similar to condition (1) (where $\sup\limits_{\zeta' \in S^{n-1}}$ is replaced by $\mathrm{essup}_{\zeta' \in S^{n-1}}$) such that the singular integral

$$T_\Omega f(x) = \text{p. v.} \int_{\mathbb{R}^n} \frac{\Omega(y')}{|y|^n} f(x-y)\, dy$$

is unbounded on $L^p(\mathbb{R}^n)$ whenever $|\frac{1}{2} - \frac{1}{p}| > \frac{\alpha}{1+\alpha}$. In particular, there is a function Ω such that T_Ω is bounded on $L^p(\mathbb{R}^n)$ exactly when $p = 2$.

It may be possible that for $\alpha > 1$, T_Ω is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$, whenever Ω satisfies condition (1). However, this is still unknown at the present. The best result we can infer from [13] is that T_Ω is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$, if Ω satisfies condition (1) for all $\alpha > 0$. In fact, under the hypothesis that Ω satisfies condition (1) for all $\alpha > 0$, Jiecheng Chen and Chunjie Zhang [4] have obtained the boundedness of T_Ω on the homogeneous Triebel-Lizorkin space $\dot{F}^s_{p,q}(\mathbb{R}^n)$ for $1 < p, q < \infty$, $s \in \mathbb{R}$.

The purpose of this paper is to find an alternative condition on Ω so that the singular integral Tf (as defined in the abstract) is bounded on the homogeneous Triebel-Lizorkin weighted space $\dot{F}^{s,w}_{p,q}(\mathbb{R}^n)$ for $1 < p, q < \infty$, $s \in \mathbb{R}$, and for some appropriate weight w. It should be remarked that the proof in this paper follows some basic ideas in [5], which are different from those in [4]. In [4], the authors used the "vector-valued inequalities" approach, based on some ideas of Hofmann [15]. It is not obvious that we could obtain the boundedness of T on the homogeneous Triebel-Lizorkin weighted space $\dot{F}^{s,w}_{p,q}(\mathbb{R}^n)$ for $1 < p, q < \infty$, $s \in \mathbb{R}$, by applying their techniques. We state our results in section 3, and the proof will be given in section 4. Section 2 deals with some preliminary background and notations.

2 Background

2.1 $A_p(\mathbb{R}^n)$ weights.

Recall that $A_p(\mathbb{R}^n)$ $(p > 1)$ is the class of all weights w, which are non-negative and locally integrable, such that

$$\left(\frac{1}{|Q|}\int_Q w\right)\left(\frac{1}{|Q|}\int_Q w^{-1/(p-1)}\right)^{p-1} \le A < \infty.$$

Here $|Q|$ denotes the Lebesgue measure of the cube Q in $\mathbb{R}^n$. Note that A_p is the class of all weights $w \ge 0$ for which the Hardy-Littlewood maximal operator M is bounded on $L^p(w)$. A_1 is the class of weights $w \ge 0$ for which M satisfies a weak-type estimate on $L^1(w)$, i.e., $Mw(x) \le C\,w(x)$ a. e. for some positive constant C (see [12, 17] etc.).
Now let $\tilde{A}_p(\mathbb{R}^+)$ denote the class of all radial weights $w(x)$ such that
$w(x) = w(|x|) = v_1(|x|)\,v_2^{1-p}(|x|)$, where either $v_i \in A_1(\mathbb{R}^+)$ and is decreasing or $v_i^2 \in A_1(\mathbb{R}^+)$, $i = 1, 2$ (see [9]). By (8) in [9], the Hardy-Littlewood maximal function $Mf(x)$ is bounded on $L^p(w)$ for $w \in \tilde{A}_p(\mathbb{R}^+)$ and for all $p > 1$. Thus if $w \in \tilde{A}_p(\mathbb{R}^+)$, then $w \in A_p(\mathbb{R}^n)$ (see [17]). Moreover, by the properties of A_p weights and by the definition of $\tilde{A}_p(\mathbb{R}^+)$, we observe the following facts:
a) $w \in \tilde{A}_p(\mathbb{R}^+) \Longleftrightarrow w^{1-p'} \in \tilde{A}_{p'}(\mathbb{R}^+)$, $1 < p < \infty$,
b) $w \in \tilde{A}_p(\mathbb{R}^+) \Longrightarrow \exists\, \epsilon > 0 \ni w^{1+\epsilon} \in \tilde{A}_p(\mathbb{R}^+)$, $1 < p < \infty$,
c) $w \in \tilde{A}_p(\mathbb{R}^+) \Longrightarrow \exists\, \epsilon > 0 \ni w \in \tilde{A}_{p-\epsilon}(\mathbb{R}^+)$, $1 < p < \infty$, and
d) $w \in \tilde{A}_p(\mathbb{R}^+) \Longrightarrow w \in \tilde{A}_q(\mathbb{R}^+)$ for $1 < p < q < \infty$.

2.2 The Triebel-Lizorkin weighted space $\dot{F}^{\alpha,w}_{p,q}(\mathbb{R}^n)$.

Fix a radial Schwartz function $\Phi \in \mathscr{S}(\mathbb{R}^n)$ such that $\operatorname{supp}\hat{\Phi} \subset \left\{\xi \in \mathbb{R}^n : \frac{1}{2} \le |\xi| \le 2\right\}$, $\hat{\Phi}(\xi) \ge 0$, $\hat{\Phi}(\xi) \ge c > 0$, if $\frac{3}{5} \le |\xi| \le \frac{5}{3}$. Denote $\hat{\Phi}_t(\xi) = \hat{\Phi}(t\xi)$, $t \in \mathbb{R}$, so that $\Phi_t(x) = t^{-n}\Phi(x/t)$, $x \in \mathbb{R}^n$. For $1 < p, q < \infty$, $\alpha \in \mathbb{R}$, and $w(x) \in A_p(\mathbb{R}^n)$, the homogeneous Triebel-Lizorkin weighted space $\dot{F}^{\alpha,w}_{p,q}(\mathbb{R}^n)$ is the space of all tempered distributions $f \in \mathscr{S}'(\mathbb{R}^n)/\mathcal{P}(\mathbb{R}^n)$ with the norm defined by

$$\begin{aligned}\|f\|_{\dot{F}^{\alpha,w}_{p,q}(\mathbb{R}^n)} &\sim \left\{\int_{\mathbb{R}^n}\left(\int_0^\infty |t^{-\alpha}\Phi_t * f(x)|^q \frac{dt}{t}\right)^{p/q} w(x)\,dx\right\}^{1/p} \\ &\equiv \left\|\left(\int_0^\infty |t^{-\alpha}\Phi_t * f(x)|^q \frac{dt}{t}\right)^{1/q}\right\|_{L^p(w)} < \infty.\end{aligned}$$

The homogeneous Besov-Lipschitz weighted space $\dot{B}^{\alpha,w}_{p,q}(\mathbb{R}^n)$ is the space of all tempered distributions $f \in \mathscr{S}'(\mathbb{R}^n)/\mathcal{P}(\mathbb{R}^n)$ with the norm defined by

$$\|f\|_{\dot{B}^{\alpha,w}_{p,q}(\mathbb{R}^n)} \sim \left(\int_0^\infty (t^{-\alpha}\,\|\Phi_t * f(x)\|_{L^p(w)})^q \frac{dt}{t}\right)^{1/q} < \infty.$$

See [1, 2, 19] for more information on this subject. We will denote the homogeneous Triebel-Lizorkin unweighted space and the homogeneous Besov unweighted space by the symbols $\dot{F}^{\alpha}_{p,q}(\mathbb{R}^n)$ and $\dot{B}^{\alpha}_{p,q}(\mathbb{R}^n)$ respectively. Observe that by interpolation (see [19], p. 64, p. 244), we have

$$\left(\dot{F}^{\alpha_o}_{p,q_o}(\mathbb{R}^n),\ \dot{F}^{\alpha_1}_{p,q_1}(\mathbb{R}^n)\right)_{\theta,q} = \dot{B}^{\alpha}_{p,q}(\mathbb{R}^n).$$

Also, it is well known that the set

$$\mathcal{Z}(\mathbb{R}^n) = \left\{\phi \in \mathscr{S}(\mathbb{R}^n) : (D^\alpha \hat{\phi}) = 0 \text{ for every multi-index } \alpha\right\},$$

or equivalently the set

$$\mathcal{S}_\infty(\mathbb{R}^n) = \bigcap_{\alpha\in(\mathbb{N}\cup\{0\})^n} \left\{f \in \mathscr{S}(\mathbb{R}^n) : \int_{\mathbb{R}^n} x^\alpha f(x)dx = 0\right\}$$

is dense in both $\dot{F}^{\alpha}_{p,q}(\mathbb{R}^n)$ and $\dot{B}^{\alpha}_{p,q}(\mathbb{R}^n)$ for $\alpha \in \mathbb{R}$, $1 < p, q < \infty$ (see [19], p. 240).
Let $H^p_w(\mathbb{R}^n)$ denote the Hardy weighted space of all tempered distributions $f \in \mathscr{S}'(\mathbb{R}^n)$ for which

$$\|f\|_{H^p_w(\mathbb{R}^n)} = \|\sup_{t>0} |\psi_t * f|\|_{L^p(w)} < \infty,$$

where ψ is a fixed function in $\mathscr{S}(\mathbb{R}^n)$, $\displaystyle\int_{\mathbb{R}^n} \psi(x)\,dx = 1$, and $\psi_t(x) = t^{-n}\psi(x/t)$. By [1], we know that $\dot{F}^{0,w}_{p,2}(\mathbb{R}^n) = H^p_w(\mathbb{R}^n)$ (modulo polynomials), $w \in A_\infty(\mathbb{R}^n)$. Moreover, if $1 < p < \infty$ and $w \in A_p(\mathbb{R}^n)$, then $H^p_w(\mathbb{R}^n) = L^p(w)$ (see [1]). For a function $g(x,t)$, $x \in \mathbb{R}^n$, $t \in \mathbb{R}$, we define the mixed norm $\|g\|_{L^p(w,L^q(\mathbb{R}))}$ as

$$\|g\|_{L^p(w,L^q(\mathbb{R}))} = \left\|\left(\int_{\mathbb{R}} |g(x,t)|^q\,dt\right)^{1/q}\right\|_{L^p(w)} < \infty.$$

For the rest of this paper, the letter C will denote a positive constant which may vary at each occurrence, but it is independent of the essential variables.

3 Main Theorems

Let $\mathbb{R}^+$ denote the interval $(0,\infty)$. For $1 < p < \infty$, let p' stand for the conjugate of p, i.e., $\dfrac{1}{p} + \dfrac{1}{p'} = 1$. Let h be a measurable function on $[0,\infty)$. In the sequel, we assume that Ω satisfies either one of the following conditions:

(**2**) $\displaystyle\sup_{\zeta'\in S^{n-1}} \int_{S^{n-1}} |\Omega(y')|(\ln|\zeta'\cdot y'|^{-1})^{(\ln(e+\ln|\zeta'\cdot y'|^{-1}))^\beta}\,d\sigma(y') \le C_1 < \infty$ for some $\beta > 0$.

(**3**) $\displaystyle\sup_{\zeta'\in S^{n-1}} \sup_{\beta>0} \int_{S^{n-1}} |\Omega(y')|(\ln|\zeta'\cdot y'|^{-1})^\beta\,d\sigma(y') \le C_2 < \infty.$

For a Schwartz function $f \in \mathscr{S}(\mathbb{R}^n)$ $(n \ge 2)$, we define the singular integral Tf as

$$Tf(x) = \text{p. v.} \int_{\mathbb{R}^n} \frac{h(|y|)\Omega(y')}{|y|^n} f(x-y)\,dy.$$

Also we define the function $\mu_{\Omega,q}(f)$ by

$$\mu_{\Omega,q}(f)(x) = \left(\int_0^\infty |F_\Omega(x,t)|^q \frac{dt}{t^{q+1}}\right)^{1/q}, \text{ where } F_\Omega(x,t) = \int_{|y|\le t} \frac{h(|y|)\Omega(y')}{|y|^{n-1}} f(x-y)\,dy.$$

Observe that $\mu_{\Omega,2}(f)$ is the usual Marcinkiewicz integral. We have the following theorems.

Theorem 3.1. *Let $h \in C^1([0,\infty))$ be a measurable bounded function. Assume that either h is monotonic on $[0,\infty)$ or $h' \in L^1(\mathbb{R}^+)$. Let Ω satisfy the mean value zero property. Assume that either Ω satisfies either condition (2) or condition (3).*
If $w(|x|) \in \tilde{A}_{p/q}(\mathbb{R}^+)$, then $\|Tf\|_{\dot{F}^{\alpha,w}_{p,q}(\mathbb{R}^n)} \leq C\|f\|_{\dot{F}^{\alpha,w}_{p,q}(\mathbb{R}^n)}$ for $1 < q \leq p < \infty$, $\alpha \in \mathbb{R}$.
If $w(|x|)^{1-p'} \in \tilde{A}_{p'/q'}(\mathbb{R}^+)$, then $\|Tf\|_{\dot{F}^{\alpha,w}_{p,q}(\mathbb{R}^n)} \leq C\|f\|_{\dot{F}^{\alpha,w}_{p,q}(\mathbb{R}^n)}$ for $1 < p < q < \infty$, $\alpha \in \mathbb{R}$.
In particular, we have

$$\|Tf\|_{\dot{F}^{\alpha}_{p,q}(\mathbb{R}^n)} \leq C\|f\|_{\dot{F}^{\alpha}_{p,q}(\mathbb{R}^n)} \text{ for } 1 < p, q < \infty, \alpha \in \mathbb{R}, \text{ and}$$

$$\|Tf\|_{\dot{B}^{\alpha}_{p,q}(\mathbb{R}^n)} \leq C\|f\|_{\dot{B}^{\alpha}_{p,q}(\mathbb{R}^n)} \text{ for } 1 < p, q < \infty, \alpha \in \mathbb{R}.$$

Theorem 3.2. *Let h and Ω be given as in Theorem 1. If $w(|x|) \in \tilde{A}_{p/q}(\mathbb{R}^+)$, then*

$$\|\mu_{\Omega,q}(f)\|_{L^p(w)} \leq C\|f\|_{\dot{F}^{0,w}_{p,q}(\mathbb{R}^n)} \text{ for } 1 < q \leq p < \infty.$$

If $w(|x|)^{1-p'} \in \tilde{A}_{p'/q'}(\mathbb{R}^+)$, then $\|\mu_{\Omega,q}(f)\|_{L^p(w)} \leq C\|f\|_{\dot{F}^{0,w}_{p,q}(\mathbb{R}^n)}$ for $1 < p < q < \infty$.
In particular, $\|\mu_{\Omega,2}(f)\|_{L^p(w)} \leq C\|f\|_{L^p(w)}$ for $1 < 2 \leq p < \infty$ if $w \in \tilde{A}_{p/2}(\mathbb{R}^+)$ and for $1 < p < 2$ if $w^{-p'/p} \in \tilde{A}_{p'/2}(\mathbb{R}^+)$.

Remark 3.3. 1) Notice that the weights w appeared in Theorems 1 and 2 are radial weights.
2) See [7, 8] for the $L^p(w)-$ boundedness of the Marcinkiewicz integral under various conditions on the kernels Ω and the weights w.
3) Let $a > 0$. Let $w^{1+a} \in \tilde{A}_2(\mathbb{R}^n)$ if $p \geq 2$; otherwise, let w satisfy $w^{1+a} \in \tilde{A}_2(\mathbb{R}^n)$ and $w^2 \in \tilde{A}_1(\mathbb{R}^n)$ if $1 < p < 2$. Under these weights' conditions, the authors in [21] have obtained the boundedness of the fractional Marcinkiewicz integral from the space $\dot{F}^{\alpha,w}_{p,q}(\mathbb{R}^n)$ for a certain range of α. It is interesting to note that if $0 < \alpha < C_1$ (C_1 depends on p, q, and a), then Ω is only required to be integrable and to satisfy the cancellation condition (see Theorem 2 [21]). On the other hand, if $C_2 < \alpha < 0$ (C_2 depends on p, q, and a), then although the moment condition on Ω can be relaxed, Ω is imposed by a condition which is stronger than condition (2) in this paper (see (1.19) in [21]). Finally, when $\alpha = 0$, the authors in [21] obtained the results for the case of $\Omega \in L\log^+ L(S^{n-1})$. Observe that the condition that $\Omega \in L\log^+ L(S^{n-1})$ implies that

$$\sup_{\zeta' \in S^{n-1}} \int_{S^{n-1}} |\Omega(y')| \log \frac{1}{|\zeta' \cdot y'|} \, d\sigma(y') < \infty.$$

The interested readers can view [20, 21] for more information on this subject.

4 Proofs of Theorems

4.1 Proof of Theorem 1

It suffices to prove the theorem for $f \in \mathcal{S}_\infty(\mathbb{R}^n)$. We choose a real-valued, radial function $\phi \in \mathscr{S}(\mathbb{R}^n)$ such that $\operatorname{supp}\hat{\phi} \subset \left\{\xi \in \mathbb{R}^n : \frac{1}{2} \leq |\xi| \leq 2\right\}$, $\hat{\phi}(\xi) \geq 0$, $\hat{\phi}(\xi) \geq c > 0$, if $\frac{3}{5} \leq |\xi| \leq \frac{5}{3}$; and for all $\xi \neq 0$, $\displaystyle\int_{\mathbb{R}} |\hat{\phi}_{2^t}(\xi)|^2 \, dt = 1$, where $\hat{\phi}_{2^t}(\xi) = \hat{\phi}(2^t\xi)$, $t \in \mathbb{R}$. Note that $\phi_{2^t}(x) = 2^{-nt}\phi(2^{-t}x)$,

$x \in \mathbb{R}^n$. Denote $S_{2^t} f = \phi_{2^t} * f$. Then for $f \in \mathcal{S}_\infty(\mathbb{R}^n)$, $f = \int_{\mathbb{R}} S_{2^t}(S_{2^t} f)\,dt$. Also for $f \in \mathcal{S}_\infty(\mathbb{R}^n)$ and for each fixed $x \in \mathbb{R}^n$, we have

$$\begin{aligned} Tf(x) &= \int_{\mathbb{R}^n} \frac{h(|y|)\Omega(y')}{|y|^n} f(x-y)\,dy \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}^n} \frac{h(|y|)\Omega(y')}{|y|^n} \chi_{2^t}(|y|) f(x-y)\,dy\,dt \equiv \int_{\mathbb{R}} \sigma_{2^t} * f(x)\,dt, \end{aligned}$$

where

$$\sigma_{2^t} * f(x) = \int_{\mathbb{R}^n} \frac{h(|y|)\Omega(y')}{|y|^n} \chi_{2^t}(|y|) f(x-y)\,dy,$$

and $\chi_{2^t}(|y|) \equiv \chi_{[2^t,2^{t+1})}(|y|)$ is the characteristic function on the interval $[2^t, 2^{t+1})$, $t \in \mathbb{R}$. Note that the Fourier transform of the measures σ_{2^t} is

$$\hat{\sigma}_{2^t}(\xi) = \int_{\mathbb{R}^n} \frac{h(|y|)\Omega(y')}{|y|^n} e^{i\xi\cdot y} \chi_{2^t}(|y|)\,dy.$$

We have the following estimates for $\hat{\sigma}_{2^t}(\xi)$.

Lemma 4.1. *If Ω satisfies condition (2), then*

$$|\hat{\sigma}_{2^t}(\xi)| \le C\, min \left\{ |2^t\xi|,\ (\ln(e^2|2^t\xi|^{1/2}))^{-(\ln(e+\ln(e^2|2^t\xi|^{1/2})))^{\beta}} \right\}. \tag{4.1}$$

If Ω satisfies condition (3), then

$$|\hat{\sigma}_{2^t}(\xi)| \le C\, min \left\{ |2^t\xi|,\ |2^t\xi|^{-1/2} \right\}. \tag{4.2}$$

Proof. By the cancellation property of Ω, we have

$$\begin{aligned} |\hat{\sigma}_{2^t}(\xi)| &\le \|h\|_\infty \int_{2^t}^{2^{(t+1)}} \int_{S^{n-1}} |\Omega(y')(e^{i|\xi|r(\xi'\cdot y')} - 1)|\,d\sigma(y') \frac{dr}{r} \\ &\le C\,\|\Omega\|_{L^1(S^{n-1})} |2^{t+1}\xi| \le C\,|2^t\xi|. \end{aligned}$$

Fix $0 < \delta < 1$. This δ will be chosen later. We write

$$\begin{aligned} \hat{\sigma}_{2^t}(\xi) &= \int_{S^{n-1}} \Omega(y') K_\xi(y')\,d\sigma(y') \\ &= \int_A \Omega(y') K_\xi(y')\,d\sigma(y') + \int_B \Omega(y') K_\xi(y')\,d\sigma(y') \\ &\equiv J_1 + J_2, \text{ where} \end{aligned}$$

$K_\xi(y') = \int_1^2 h(2^t r) e^{i|2^t\xi|(\xi'\cdot y')r} \frac{dr}{r}$, $A = \left\{ y' \in S^{n-1} : |\xi' \cdot y'| \ge \frac{\delta}{e^2} \right\}$, and $B = S^{n-1} \setminus A$.
By the hypothesis of h, it follows that

$$|K_\xi(y')| \le C \min \left\{ 1,\ |2^t\xi|^{-1} |\xi' \cdot y'|^{-1} \right\}. \tag{4.3}$$

It is clear that inequality (4.3) implies $|J_1| \le \dfrac{C}{\delta|2^t\xi|}$. If Ω satisfies condition (2), then from inequality (4.3), we obtain $|J_2| \le C(\ln(e^2\delta^{-1}))^{-(\ln(e+\ln(e^2\delta^{-1})))^{\beta}}$ for some $\beta > 0$. Observe that on the set B,

$$(\ln(|\xi'\cdot y'|^{-1}))^{\alpha_\delta} > (\ln(e^2\delta^{-1}))^{\alpha_\delta} = e^2\delta^{-1} > \delta^{-1}, \text{ where } \alpha_\delta = \frac{\ln(e^2\delta^{-1})}{\ln(\ln(e^2\delta^{-1}))}.$$

So if Ω satisfies condition (3), then

$$|J_2| \le \delta\int_B |\Omega(y')|(\ln(|\xi'\cdot y'|^{-1}))^{\alpha_\delta}d\sigma(y') \le C\delta.$$

Thus we can obtain the estimates of $\hat{\sigma}_{2^t}(\xi)$ by choosing $\delta = |2^t\xi|^{-1/2}$. Lemma 1 is proved. □

For the remaining part of this article, we will prove for the case Ω satisfying condition (2). The proof of the remaining case is handled in the same manner. We write

$$\begin{aligned} Tf &= \int_{\mathbb{R}}(\sigma_{2^t}*f)\,dt = \int_{\mathbb{R}}\sigma_{2^t}*\left(\int_{\mathbb{R}}S_{2^{(t+s)}}S_{2^{(t+s)}}f\,ds\right)dt \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}}S_{2^{(t+s)}}(\sigma_{2^t}*S_{2^{(t+s)}}f)\,dt\,ds \equiv \int_{\mathbb{R}}T_sf\,ds, \end{aligned} \tag{4.4}$$

where

$$T_sf = \int_{\mathbb{R}}S_{2^{(t+s)}}(\sigma_{2^t}*S_{2^{(t+s)}}f)\,dt. \tag{4.5}$$

Observe that

$$\|f\|_{\dot{F}^{\alpha,w}_{p,q}(\mathbb{R}^n)} \sim \left\|\left(\int_0^\infty |t^{-\alpha}\phi_t*f|^q\,\frac{dt}{t}\right)^{1/q}\right\|_{L^p(w)} \sim \left\|\left(\int_{\mathbb{R}}|2^{-t\alpha}S_{2^t}f|^q\,dt\right)^{1/q}\right\|_{L^p(w)} \tag{4.6}$$

Thus for any function $g \in \dot{F}^{-\alpha,w^{-p'/p}}_{p',q'}(\mathbb{R}^n)$, we have

$$\begin{aligned} |\langle T_sf,g\rangle| &= \left|\int_{\mathbb{R}^n}\int_{\mathbb{R}}S_{2^{(t+s)}}(\sigma_{2^t}*S_{2^{(t+s)}}f)(x)\,g(x)\,dt\,dx\right| \\ &\le \int_{\mathbb{R}^n}\left|\int_{\mathbb{R}}(\sigma_{2^t}*S_{2^{(t+s)}}f)(x)\,\tilde{S}_{2^{(t+s)}}g(x)\,dt\right|dx \\ &\le \left\|\left(\int_{\mathbb{R}}|2^{-(t+s)\alpha}\sigma_{2^t}*S_{2^{(t+s)}}f|^q\,dt\right)^{1/q}\right\|_{L^p(w)} \\ &\quad\times\left\|\left(\int_{\mathbb{R}}|2^{(t+s)\alpha}\tilde{S}_{2^{(t+s)}}g|^{q'}\,dt\right)^{1/q'}\right\|_{L^p(w^{-p'/p})} \\ &\le C\|g\|_{\dot{F}^{-\alpha,w^{-p'/p}}_{p',q'}(\mathbb{R}^n)}\left\|\left(\int_{\mathbb{R}}|2^{-(t+s)\alpha}\sigma_{2^t}*S_{2^{(t+s)}}f|^q\,dt\right)^{1/q}\right\|_{L^p(w)}, \end{aligned}$$

where $\tilde{S}_{2^{(t+s)}}$ is the dual operator of $S_{2^{(t+s)}}$. That is, $\tilde{S}_{2^{(t+s)}}g(x) = S_{2^{(t+s)}}(\tilde{g})(-x)$, and $\tilde{g}(x) = g(-x)$. Taking the supremum over all $g \in \dot{F}^{-\alpha, w^{-p'/p}}_{p',q'}(\mathbb{R}^n)$ with $\|g\|_{\dot{F}^{-\alpha, w^{-p'/p}}_{p',q'}(\mathbb{R}^n)} \le 1$ yields

$$\|T_s f\|_{\dot{F}^{\alpha,w}_{p,q}(\mathbb{R}^n)} \le C \left\| \left(\int_{\mathbb{R}} |2^{-(t+s)\alpha} \sigma_{2^t} * S_{2^{(t+s)}} f|^q \, dt \right)^{1/q} \right\|_{L^p(w)}. \tag{4.7}$$

Substituting $p = q = 2$ and $w = 1$ in (4.7), we obtain

$$\|T_s f\|^2_{\dot{F}^{\alpha}_{2,2}(\mathbb{R}^n)} \le C \int_{\mathbb{R}} \int_{D_{t+s}} |2^{-(t+s)\alpha} \hat{\sigma}_{2^t}(\xi) \hat{\phi}(2^{(t+s)}\xi) \hat{f}(\xi)|^2 \, d\xi \, dt, \tag{4.8}$$

where $D_{t+s} = \left\{ \xi \in \mathbb{R}^n : \frac{1}{2} \le |2^{(t+s)}\xi| \le 2 \right\}$.
If $s \ge 0$, inequalities (4.1), (4.6) and (4.8) imply that

$$\begin{aligned} \|T_s f\|_{\dot{F}^{\alpha}_{2,2}(\mathbb{R}^n)} &\le C 2^{-s} \left(\int_{\mathbb{R}^n} \int_{\mathbb{R}} |2^{-(t+s)\alpha} \phi_{2^{(t+s)}} * f(x)|^2 \, dt \, dx \right)^{1/2} \\ &\le C 2^{-s} \|f\|_{\dot{F}^{\alpha}_{2,2}(\mathbb{R}^n)} \end{aligned} \tag{4.9}$$

If $s < 0$, by inequality (4.1) in Lemma 1, inequality (4.8) becomes

$$\|T_s f\|_{\dot{F}^{\alpha}_{2,2}(\mathbb{R}^n)} \le C 2^{-(\ln(c_1 + c_2|s|))^{1+\beta}} \|f\|_{\dot{F}^{\alpha}_{2,2}(\mathbb{R}^n)}, \tag{4.10}$$

where $c_1 = 2 - \dfrac{\ln 2}{2}$ and $c_2 = \dfrac{\ln 2}{2}$. In order to estimate the norm $\|T_s f\|_{\dot{F}^{\alpha,w}_{p,q}(\mathbb{R}^n)}$, we need the following lemma.

Lemma 4.2. *Denote* $L_t(f)(x) = \displaystyle\int_{\mathbb{R}^n} \frac{|\Omega(y')|}{|y|^n} f(x-y) \chi_{2^t}(|y|) dy$, *and denote* $\tilde{L}_t$ *the dual operator of* L_t, *i.e.,* $\tilde{L}_t(f)(x) = L_t(\tilde{f})(-x)$, *where* $\tilde{f}(x) = f(-x)$ *and* $t \in \mathbb{R}$. *Then*

$$|\sigma_{2^t} * S_{2^{(t+s)}} f(x)| \le C \|\Omega\|^{1/q'}_{L^1(S^{n-1})} (L_t(|S_{2^{(t+s)}} f|^q)(x))^{1/q} \tag{4.11}$$

$$\|\sigma_{2^t} * S_{2^{(t+s)}} f\|_{L^q(w)} \le C \|\Omega\|_{L^1(S^{n-1})} \|S_{2^{(t+s)}} f\|_{L^q(w)}, \text{ and} \tag{4.12}$$

$$\|\sup_{t \in \mathbb{R}} L_t(|f|)\|_{L^p(w)} \le C \|\Omega\|_{L^1(S^{n-1})} \|f\|_{L^p(w)} \text{ for } 1 < p < \infty,\ w \in \tilde{A}_p(\mathbb{R}^+). \tag{4.13}$$

Proof. By Hölder's inequality, we have

$$\begin{aligned} |\sigma_{2^t} * S_{2^{(t+s)}} f(x)| &= \left| \int_{\mathbb{R}^n} \frac{h(|y|)\Omega(y')}{|y|^n} \chi_{2^t}(|y|) S_{2^{(t+s)}} f(x-y) \, dy \right| \\ &\le \left(\int_{\mathbb{R}^n} \frac{|h(|y|)|^{q'}}{|y|^n} |\Omega(y')| \chi_{2^t}(|y|) \, dy \right)^{1/q'} \\ &\quad \times \left(\int_{\mathbb{R}^n} \frac{|\Omega(y')|}{|y|^n} |S_{2^{(t+s)}} f(x-y)|^q \chi_{2^t}(|y|) \, dy \right)^{1/q} \\ &\le C \|\Omega\|^{1/q'}_{L^1(S^{n-1})} (L_t(|S_{2^{(t+s)}} f|^q)(x))^{1/q}. \end{aligned}$$

This proves inequality (4.11). Moreover, we have

$$|\sigma_{2^t} * S_{2^{(t+s)}} f(x)| \leq \|h\|_\infty \int_{S^{n-1}} |\Omega(y')| \left(\int_{2^t}^{2^{(t+1)}} |S_{2^{(t+s)}} f(x - ry')| \frac{dr}{r} \right) d\sigma(y').$$

Observe that

$$\begin{aligned} \int_{2^t}^{2^{(t+1)}} |S_{2^{(t+s)}} f(x - ry')| \frac{dr}{r} &\leq 2 \sup_{r>0} \left\{ \frac{1}{r} \int_0^r |S_{2^{(t+s)}} f(x - \tau y')| d\tau \right\} \\ &\equiv 2 M_{y'} S_{2^{(t+s)}} f(x), \text{ for all } t \in \mathbb{R}. \end{aligned}$$

Here $M_{y'} S_{2^{(t+s)}} f(x)$ is the Hardy-Littlewood maximal function in the direction $y' \in S^{n-1}$. Thus

$$|\sigma_{2^t} * S_{2^{(t+s)}} f(x)| \leq C \|h\|_\infty \int_{S^{n-1}} |\Omega(y')| \, M_{y'} S_{2^{(t+s)}} f(x) \, d\sigma(y').$$

By Minskowski's inequality, it follows that

$$\begin{aligned} \|\sigma_{2^t} * S_{2^{(t+s)}} f\|_{L^q(w)} &\leq C \int_{S^{n-1}} |\Omega(y')| \, \|M_{y'} S_{2^{(t+s)}} f\|_{L^q(w)} \, d\sigma(y') \\ &\leq C \int_{S^{n-1}} |\Omega(y')| \, \|S_{2^{(t+s)}} f\|_{L^q(w)} \, d\sigma(y') \\ &\leq C \|\Omega\|_{L^1(S^{n-1})} \|S_{2^{(t+s)}} f\|_{L^q(w)}, \end{aligned}$$

where the second inequality follows from (8) in [9], and the bound C is independent of the direction vector $y' \in S^{n-1}$. Inequality (4.12) is proved.

It remains to prove inequality (4.13). Using the same techniques as in the proof of inequality (4.12), we obtain

$$\sup_{t \in \mathbb{R}} L_t(|f|)(x) \leq C \int_{S^{n-1}} |\Omega(y')| \, M_{y'} f(x) \, d\sigma(y').$$

Recall that by (8) in [9] , $M_{y'} f$ is bounded in $L^p(w)$ for $1 < p < \infty$, $w \in \tilde{A}_p(\mathbb{R}^+)$; and the bound is independent of the direction vector $y' \in S^{n-1}$. Hence, an application of Minskowski's inequality yields (4.13). Lemma 2 is proved. □

We now estimate the norm $\|T_s f\|_{\dot{F}^{\alpha,w}_{p,q}(\mathbb{R}^n)}$. When $p = q$, from inequalities (4.6), (4.7) and (4.12) we obtain

$$\begin{aligned} \|T_s f\|_{\dot{F}^{\alpha,w}_{q,q}(\mathbb{R}^n)} &\leq C \|\Omega\|_{L^1(S^{n-1})} \left(\int_{\mathbb{R}^n} \int_{\mathbb{R}} |2^{-(t+s)\alpha} S_{2^{(t+s)}} f(x)|^q \, dt \, w \, dx \right)^{1/q} \\ &\leq C \|\Omega\|_{L^1(S^{n-1})} \|f\|_{\dot{F}^{\alpha,w}_{q,q}(\mathbb{R}^n)} \leq C \|f\|_{\dot{F}^{\alpha,w}_{q,q}(\mathbb{R}^n)}. \end{aligned} \tag{4.14}$$

If $p > q$, inequality (4.7) implies that there exists a non-negative function $g \in L^{r'}(w^{1-r'})$

$(r = p/q)$ with unit norm such that $\|T_s f\|^q_{\dot{F}^{\alpha,w}_{p,q}(\mathbb{R}^n)}$

$$
\begin{aligned}
&\le C\int_{\mathbb{R}}\int_{\mathbb{R}^n} |2^{-(t+s)\alpha}\sigma_{2^t} * S_{2^{(t+s)}} f(x)|^q g(x)\,dx\,dt\\
&\le C\|\Omega\|^{q/q'}_{L^1(S^{n-1})}\int_{\mathbb{R}}\int_{\mathbb{R}^n} 2^{-(t+s)\alpha q} L_t(|S_{2^{(t+s)}} f|^q)(x)\,g(x)\,dx\,dt\\
&= C\|\Omega\|^{q/q'}_{L^1(S^{n-1})}\int_{\mathbb{R}}\int_{\mathbb{R}^n} |2^{-(t+s)\alpha} S_{2^{(t+s)}} f(x)|^q \tilde{L}_t g(x)\,dx\,dt\\
&\le C\|\Omega\|^{q/q'}_{L^1(S^{n-1})}\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}} |2^{-(t+s)\alpha} S_{2^{(t+s)}} f(x)|^q dt\right)\sup_{t\in\mathbb{R}} \tilde{L}_t g(x)\,dx\\
&\le C\|\Omega\|^{q/q'}_{L^1(S^{n-1})}\left(\int_{\mathbb{R}^n}\Big(\int_{\mathbb{R}} |2^{-(t+s)\alpha} S_{2^{(t+s)}} f(x)|^q dt\Big)^r w(|x|)\,dx\right)^{1/r}\\
&\quad\times\left(\int_{\mathbb{R}^n} |\sup_{t\in\mathbb{R}} \tilde{L}_t g(x)|^{r'} w^{1-r'}(|x|)\,dx\right)^{1/r'}\\
&\le C\|\Omega\|^{1+q/q'}_{L^1(S^{n-1})}\|f\|^q_{\dot{F}^{\alpha,w}_{p,q}(\mathbb{R}^n)}\|g\|_{L^{r'}(w^{1-r'})},
\end{aligned}
$$

where the second and the last inequalities follow from Lemma 2. Therefore,

$$\|T_s f\|_{\dot{F}^{\alpha,w}_{p,q}(\mathbb{R}^n)} \le C\|\Omega\|_{L^1(S^{n-1})}\|f\|_{\dot{F}^{\alpha,w}_{p,q}(\mathbb{R}^n)} \le C\|f\|_{\dot{F}^{\alpha,w}_{p,q}(\mathbb{R}^n)}$$

for $1 < q < p < \infty$, $\alpha \in \mathbb{R}$, $w \in \tilde{A}_{p/q}(\mathbb{R}^+)$, which together with inequality (4.14) yield

$$\|T_s f\|_{\dot{F}^{\alpha,w}_{p,q}(\mathbb{R}^n)} \le C\|f\|_{\dot{F}^{\alpha,w}_{p,q}(\mathbb{R}^n)} \text{ for } 1 < q \le p < \infty, \alpha \in \mathbb{R}, \text{ and } w \in \tilde{A}_{p/q}(\mathbb{R}^+). \tag{4.15}$$

Now set $q = 2$ and $w = 1$ in (4.15) and by applying duality, we obtain

$$\|T_s f\|_{\dot{F}^{\alpha}_{p,2}(\mathbb{R}^n)} \le C\|f\|_{\dot{F}^{\alpha}_{p,2}(\mathbb{R}^n)} \text{ for } 1 < p < \infty,\ \alpha \in \mathbb{R}. \tag{4.16}$$

Interpolating (4.9)-(4.16) and (4.10)-(4.16) (with $w = 1$) gives

$$\|T_s f\|_{\dot{F}^{\alpha}_{p,2}(\mathbb{R}^n)} \le C2^{-s\theta_1}\|f\|_{\dot{F}^{\alpha}_{p,2}(\mathbb{R}^n)} \tag{4.17}$$

for $0 < \theta_1 \le 1$, $s \ge 0$, $1 < p < \infty$, $\alpha \in \mathbb{R}$, and

$$\|T_s f\|_{\dot{F}^{\alpha}_{p,2}(\mathbb{R}^n)} \le C2^{-\delta_1(\ln(c_1+c_2|s|))^{1+\beta}}\|f\|_{\dot{F}^{\alpha}_{p,2}(\mathbb{R}^n)} \tag{4.18}$$

for $0 < \delta_1 \le 1$, $s < 0$, $1 < p < \infty$, $\alpha \in \mathbb{R}$.
Interpolating (4.15)-(4.17) and (4.15)-(4.18) (with $w = 1$) gives

$$\|T_s f\|_{\dot{F}^{\alpha}_{p,q}(\mathbb{R}^n)} \le C2^{-s\theta_2}\|f\|_{\dot{F}^{\alpha}_{p,q}(\mathbb{R}^n)} \tag{4.19}$$

for $0 < \theta_2 \le \theta_1 \le 1$, $s \ge 0$, $1 < q \le p < \infty$, $\alpha \in \mathbb{R}$, and

$$\|T_s f\|_{\dot{F}^{\alpha}_{p,q}(\mathbb{R}^n)} \le C2^{-\delta_2(\ln(c_1+c_2|s|))^{1+\beta}}\|f\|_{\dot{F}^{\alpha}_{p,q}(\mathbb{R}^n)} \tag{4.20}$$

for $0 < \delta_2 \le \delta_1 \le 1$, $s < 0$, $1 < q \le p < \infty$, $\alpha \in \mathbb{R}$. Since $p \ge q > 1$, $w \in \tilde{A}_{p/q}(\mathbb{R}^+) \Rightarrow w \in \tilde{A}_p(\mathbb{R}^+)$, and thus there exists an $\epsilon > 0$ such that $w^{1+\epsilon} \in \tilde{A}_p(\mathbb{R}^+)$. Hence inequality (4.15) implies that

$$\|T_s f\|_{\dot{F}_{p,q}^{\alpha,w^{1+\epsilon}}(\mathbb{R}^n)} \le C\|f\|_{\dot{F}_{p,q}^{\alpha,w^{1+\epsilon}}(\mathbb{R}^n)} \text{ for } 1 < q \le p < \infty,\ \alpha \in \mathbb{R}. \tag{4.21}$$

By interpolating (4.19)-(4.21) and (4.20)-(4.21) with the same p and q, but with change of measures (see [18]), we have

$$\|T_s f\|_{\dot{F}_{p,q}^{\alpha,w}(\mathbb{R}^n)} \le C 2^{-s\theta_3}\|f\|_{\dot{F}_{p,q}^{\alpha,w}(\mathbb{R}^n)} \tag{4.22}$$

for $s \ge 0$, $0 < \theta_3 = \dfrac{\theta_2\epsilon}{1+\epsilon} < 1$, $1 < q \le p < \infty$, $\alpha \in \mathbb{R}$, and

$$\|T_s f\|_{\dot{F}_{p,q}^{\alpha,w}(\mathbb{R}^n)} \le C 2^{-\delta_3(\ln(c_1+c_2|s|))^{1+\beta}}\|f\|_{\dot{F}_{p,q}^{\alpha,w}(\mathbb{R}^n)} \tag{4.23}$$

for $s < 0$, $0 < \delta_3 = \dfrac{\delta_2\epsilon}{1+\epsilon} < 1$, $1 < q \le p < \infty$, $\alpha \in \mathbb{R}$. It follows from (4.4), (4.22) and (4.23) that

$$\|Tf\|_{\dot{F}_{p,q}^{\alpha,w}(\mathbb{R}^n)} \le \int_{\mathbb{R}} \|T_s f\|_{\dot{F}_{p,q}^{\alpha,w}(\mathbb{R}^n)} ds \le C\|f\|_{\dot{F}_{p,q}^{\alpha,w}(\mathbb{R}^n)} \tag{4.24}$$

for $1 < q \le p < \infty$, $\alpha \in \mathbb{R}$, and $w \in \tilde{A}_{p/q}(\mathbb{R}^+)$.
We define the truncated singular integral $T^\epsilon f$ by

$$\begin{aligned} T^\epsilon f(x) &= \int_{|y|>\epsilon} \frac{h(|y|)\Omega(y')}{|y|^n} f(x-y)\,dy, \\ &\equiv \int_{\mathbb{R}^n} \frac{h_\epsilon(|y|)\Omega(y')}{|y|^n} f(x-y)\,dy \end{aligned}$$

where $h_\epsilon(|y|) = h(|y|)\chi_\epsilon(|y|)$, and $\chi_\epsilon(|y|)$ is the characteristic function defined on the set $\{y \in \mathbb{R}^n : |y| > \epsilon\}$. Note that $\|h_\epsilon\|_\infty \le \|h\|_\infty$ for all $\epsilon > 0$. Thus it follows from (4.24) that

$$\|T^\epsilon f\|_{\dot{F}_{p,q}^{\alpha,w}(\mathbb{R}^n)} \le C\|f\|_{\dot{F}_{p,q}^{\alpha,w}(\mathbb{R}^n)} \tag{4.25}$$

for $1 < q \le p < \infty$, $\alpha \in \mathbb{R}$, $w \in \tilde{A}_{p/q}(\mathbb{R}^+)$, and C is independent of $\epsilon > 0$. Now suppose $w^{1-p'} \in \tilde{A}_{p'/q'}(\mathbb{R}^+)$ with $1 < p < q < \infty$. An application of duality to inequality (4.25) yields

$$\|T^\epsilon f\|_{\dot{F}_{p,q}^{\alpha,w}(\mathbb{R}^n)} \le C\|f\|_{\dot{F}_{p,q}^{\alpha,w}(\mathbb{R}^n)} \text{ for } 1 < p < q < \infty,\ \alpha \in \mathbb{R},$$

and the constant C is again independent of $\epsilon > 0$. Passing to the limit as $\epsilon \longrightarrow 0$, we finally obtain

$$\|Tf\|_{\dot{F}_{p,q}^{\alpha,w}(\mathbb{R}^n)} \le C\|f\|_{\dot{F}_{p,q}^{\alpha,w}(\mathbb{R}^n)} \text{ for } 1 < p < q < \infty,\ \alpha \in \mathbb{R}, \text{ and } w^{1-p'} \in \tilde{A}_{p'/q'}(\mathbb{R}^+).$$

For the unweighted case, we simply set $w = 1$ to obtain the results for the Triebel-Lizorkin unweighted spaces. Moreover by interpolation (see [19]), we also get the results for the Besov unweighted spaces, finishing the proof of Theorem 1.

4.2 Proof of Theorem 2

Since the proof of this theorem is essentially similar to the proof of Theorem 1, we will only outline some necessary steps in order to obtain the following inequality:

$$\|\mu_{\Omega,q}(f)\|_{L^p(w)} \le C\|f\|_{\dot{F}^{0,w}_{p,q}(\mathbb{R}^n)} \text{ for } 1<q\le p<\infty \text{ if } w(|x|)\in \tilde{A}_{p/q}(\mathbb{R}^+),$$

and for $1<p<q<\infty$ if $w(|x|)^{1-p'}\in \tilde{A}_{p'/q'}(\mathbb{R}^+)$. Define the measures $\{\sigma_{2^t}\}_{t\in\mathbb{R}}$ by

$$\sigma_{2^t}*f(x)=2^{-t}\int_{|y|\le 2^t}\frac{h(|y|)\Omega(y')}{|y|^{n-1}}f(x-y)\,dy.$$

Then

$$\mu_{\Omega,q}(f)(x)\sim\left(\int_{\mathbb{R}}|\sigma_{2^t}*f(x)|^q\,dt\right)^{1/q}.$$

By a similar calculation as in the proof of Theorem 1, we obtain the same estimates for $\hat{\sigma}_{2^t}(\xi)$ as in Lemma 1. Moreover, we also have the following results

$$|\sigma_{2^t}*S_{2^{(t+s)}}f(x)|\le C\|\Omega\|^{1/q'}_{L^1(S^{n-1})}\left(N_t(|S_{2^{(t+s)}}f|^q)(x)\right)^{1/q}, \tag{4.26}$$

$$\|\sigma_{2^t}*S_{2^{(t+s)}}f\|_{L^q(w)}\le C\|\Omega\|_{L^1(S^{n-1})}\|S_{2^{(t+s)}}f\|_{L^q(w)}, \text{ and} \tag{4.27}$$

$$\|\sup_{t\in\mathbb{R}}N_t(|f|)\|_{L^p(w)}\le C\|\Omega\|_{L^1(S^{n-1})}\|f\|_{L^p(w)} \text{ for } 1<p<\infty,\ w\in\tilde{A}_p(\mathbb{R}^+). \tag{4.28}$$

Here $S_{2^{(t+s)}}f=\phi_{2^{(t+s)}}*f$, and

$$N_t(f)(x)=2^{-t}\int_{|y|\le 2^t}\frac{|\Omega(y')|}{|y|^{n-1}}f(x-y)\,dy.$$

The function ϕ is as in proof of Theorem 1, except for a slight modification that

$$\int_{\mathbb{R}}\hat{\phi}_{2^t}(\xi)\,dt=1$$

for all $\xi\ne 0$, instead of

$$\int_{\mathbb{R}}|\hat{\phi}_{2^t}(\xi)|^2\,dt=1.$$

Observe that

$$\sigma_{2^t}*f=\int_{\mathbb{R}}\sigma_{2^t}*S_{2^{(t+s)}}f\,ds.$$

By Minskowski's inequality, we have

$$\|\sigma_{2^t}*f\|_{L^q(\mathbb{R})}\le\int_{\mathbb{R}}\|\sigma_{2^t}*S_{2^{(t+s)}}f\|_{L^q(\mathbb{R})}ds\ \equiv\ \int_{\mathbb{R}}I_{q,s}f\,ds,$$

where

$$I_{q,s}f(x)=\left(\int_{\mathbb{R}}|\sigma_{2^t}*S_{2^{(t+s)}}f(x)|^q\,dt\right)^{1/q}.$$

By using similar arguments as in the proof of Theorem 1, we obtain

$$\|I_{q,s}f\|_{L^p(w)} \leq C\,2^{-\epsilon_1 s}\,\|f\|_{\dot{F}^{0,w}_{p,q}(\mathbb{R}^n)}$$

for some $\epsilon_1 > 0$, $s \geq 0$, $1 < q \leq p < \infty$, $w \in \tilde{A}_{p/q}(\mathbb{R}^+)$, and

$$\|I_{q,s}f\|_{L^p(w)} \leq C\,2^{-\epsilon_2(\ln(c_1+c_2|s|))^{1+\beta}}\,\|f\|_{\dot{F}^{0,w}_{p,q}(\mathbb{R}^n)}$$

for some $\epsilon_2 > 0$, $s < 0$, $1 < q \leq p < \infty$, $w \in \tilde{A}_{p/q}(\mathbb{R}^+)$. It follows that for $1 < q \leq p < \infty$,

$$\|\sigma_{2^t} * f\|_{L^p(w,L^q(\mathbb{R}))} \leq \int_{\mathbb{R}} \|I_{q,s}f\|_{L^p(w)}ds \leq C\,\|f\|_{\dot{F}^{0,w}_{p,q}(\mathbb{R}^n)}.$$

Thus

$$\|\mu_{\Omega_m,q}(f)\|_{L^p(w)} \leq C\,\|\sigma_{2^t} * f\|_{L^p(w,L^q(\mathbb{R}))} \leq C\,\|f\|_{\dot{F}^{0,w}_{p,q}(\mathbb{R}^n)}$$

for $1 < q \leq p < \infty$, $w \in \tilde{A}_{p/q}(\mathbb{R}^+)$, and an application of duality yields the remaining results.

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References

[1] H. Q. Bui, Weighted Besov and Triebel spaces: Interpolation by the real method. *Hiroshima Math. J.* **12** (1982), pp 581-605.

[2] H. Q. Bui, M. Paluszyński and M. H. Taibleson, A maximal function characterization of weighted Besov-Lipschitz and Triebel-Lizorkin spaces. *Studia Mathematica* **119**, no. 3 (1996), pp 219-246.

[3] A. P. Calderón and A. Zygmund, On singular integrals. *Amer. J. Math.* **78** (1956), pp 289-309.

[4] J. Chen and C. Zhang, Boundedness of rough singular integral operators on the Triebel-Lizorkin spaces. *Journal of Mathematical Analysis and Applications* **337** (2008), pp 1048-1052.

[5] J. Chen, D. Fan and Y. Ying, Singular integral operators on function spaces. *J. Math. Anal. Appl.* **276**, no. 2 (2002), pp 691-708.

[6] W. C. Connett, Singular integrals near L1. *Proc. Sympos. Pure Math. of the Amer. Math. Soc. (S. Wainger and G. Weiss, eds)* **35 I** (1979), pp 163-165.

[7] Y. Ding, D. Fan and Y. Pan, Weighted Boundedness for a Class of Rough Marcinkiewicz Integrals. *Indiana University Mathematics Journal* **48**, no. 3 (1999), pp 1037-1055.

[8] Y. Ding, Q. Xue and K. Yabuta, Boundedness of the Marcinkiewicz integrals with rough kernel associated to surfaces. *Tohoku Math. J.* **62** (2010), pp 233-262.

[9] J. Duoandikoetxea, Weighted norm inequalities for homogeneous singular integrals. *Trans. Amer. Math. Soc.* **336**, no. 2 (1992), pp 869-880.

[10] D. Fan and Y. Pan, Singular integral operators with rough kernels supported by subvarieties. *Amer. J. Math.* **119** (1997), pp 799-839.

[11] D. Fan, K. Guo and Y. Pan, A note of a rough singular integral operator. *Math. Ineq. and Appl.* **2** (1999), pp 73-81.

[12] G. Garcia and J. L. Rubio de Francia, Weighted norm inequalities and related topics. *North-Holland* (1985).

[13] L. Grafakos and A. Stefanov, L^p bounds for singular integrals and maximal singular integrals with rough kernel. *India. Univ. Math. J.* **47** (1998), pp 455-469.

[14] L. Grafakos, P. Honzík and D. Ryabogin, On the $p-$independence boundedness property of Calderón-Zygmund theory. *Journal für die Reine und Angewandte Mathematik (Crelles Journal)* **602** (2007), pp 227-234.

[15] S. Hofmann, Weighted norm inequalities and vector valued inequalities for certain rough operators. *Indiana Univ. Math. J.* **42** (1993), pp 1-14.

[16] F. Ricci and G. Weiss, A characterization of $H^1(S^{n-1})$. *Proc. Sympos. Pure Math. of Amer. Math. Soc. (S. Wainger and G. Weiss, eds)* **35 I** (1979), pp 289-294.

[17] E. M. Stein, Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals. *Princeton University Press* (1993). **87**, (1958), no. 1, pp 159-172.

[18] E. M. Stein and G. Weiss, Interpolation of operators with change of measures. *Trans. Amer. Math. Soc.* **87**, (1958), no. 1, pp 159-172.

[19] Hans Triebel, Theory of Function Spaces. *Monographs in Mathematics* **78** *Birkhäuser Verlag, Basel* (1983).

[20] 2425 Y. Sawano and K. Yabuta, Fractional Type Marcinkiewicz Integral Operators Associated to Surfaces. *http://www.esi.ac.at/preprints/ESI-Preprints.html*.

[21] 2426 Y. Sawano and K. Yabuta, Weighted Estimates for Fractional Type Marcinkiewicz Integral Operators Associated to Surfaces. *http://www.esi.ac.at/preprints/ESI-Preprints.html*.