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Oscillation Results for Fourth-Order Nonlinear Neutral Dynamic Equations

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Abstract

In this paper, the authors study the oscillatory and asymptotic properties of solutions of nonlinear fourth order neutral dynamic equations of the form

$$(r(t)(y(t)+p(t)y(\alpha(t)))^{\Delta^2})^{\Delta^2}+q(t)G(y(\beta(t)))-h(t)H(y(\gamma(t)))=0 \tag{H}$$

and

$$(r(t)(y(t)+p(t)y(\alpha(t)))^{\Delta^2})^{\Delta^2}+q(t)G(y(\beta(t)))-h(t)H(y(\gamma(t)))=f(t), \tag{NH}$$

where $\mathbb{T}$ is a time scale with $\sup\mathbb{T}=\infty$, $t\in[t_0,\infty)_{\mathbb{T}}$, and $t_0\geqslant 0$. They assume that $\int_{t_0}^{\infty}\frac{\sigma(t)}{r(t)}\Delta t<\infty$ and obtain results for various ranges of values of $p(t)$. Examples illustrating the results are included.

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1 Introduction

The study of dynamic equations on time scales goes back to seminal work of Stefan Hilger [8] and has received a lot of attention in recent years. Time scales were created to unify the study of continuous and discrete differential and difference equations. Many results concerning differential equations carry over quite easily to corresponding results for difference equations, while other results seem to be completely different from their continuous counterparts. The study of dynamic equations on time scales reveals such discrepancies, and allows us to avoid proving results twice, once for differential equations and once again for difference equations. The general idea is to prove a result for a dynamic equation where the domain of the unknown function is a time scale $\mathbb{T}$, which is a non-empty closed subset of the real numbers $\mathbb{R}$. In this way the results in this paper not only apply to the set of real numbers or set of integers, but also to more general time scales such as $\mathbb{T} = h\mathbb{N}$, $\mathbb{T} = q^{\mathbb{N}_0} = \{t : t = q^k,\ k \in \mathbb{N}_0\}$ with $q > 1$, $\mathbb{T} = \mathbb{N}_0^2 = \{t^2 : t \in \mathbb{N}_0\}$, $\mathbb{T} = \{\sqrt{n} : n \in \mathbb{N}_0\}$ e.t.c,. For basic notations on time scale calculus, we refer the reader to the monographs [1, 2] and the references cited therein.

In [12], the authors studied the oscillatory and asymptotic behavior of solutions of the fourth order nonlinear neutral dynamic equations

$$(r(t)(y(t) + p(t)y(\alpha(t)))^{\Delta^2})^{\Delta^2} + q(t)G(y(\beta(t))) = 0 \tag{1.1}$$

and

$$(r(t)(y(t) + p(t)y(\alpha(t)))^{\Delta^2})^{\Delta^2} + q(t)G(y(\beta(t))) = f(t) \tag{1.2}$$

for various ranges of $p(t)$ under the assumptions that $q(t) > 0$ and $\int_{t_0}^{\infty} \frac{t}{r(t)}\Delta t < \infty$. From their work it is apparent that it would be possible to obtain analogous results for the oscillation and asymptotic behavior of solutions of (1.1) and (1.2) in case $q(t) < 0$. It remains an open problem as to what happens if $q(t)$ is allowed to change signs. However, if $q(t) = q^+(t) - q^-(t)$, where $q^+(t) = \max\{0, q(t)\}$ and $q^-(t) = \max\{0, -q(t)\}$, then equations (1.1) and (1.2) take the forms

$$(r(t)(y(t) + p(t)y(\alpha(t)))^{\Delta^2})^{\Delta^2} + q^+(t)G(y(\beta(t))) - q^-(t)G(y(\gamma(t))) = 0 \tag{1.3}$$

and

$$(r(t)(y(t) + p(t)y(\alpha(t)))^{\Delta^2})^{\Delta^2} + q^+(t)G(y(\beta(t))) - q^-(t)G(y(\gamma(t))) = f(t), \tag{1.4}$$

respectively, which we see are in the form of (H) and (NH).

Our goal here is to study the oscillatory and asymptotic properties of solutions of the nonlinear fourth order neutral dynamic equations

$$(r(t)(y(t) + p(t)y(\alpha(t)))^{\Delta^2})^{\Delta^2} + q(t)G(y(\beta(t))) - h(t)H(y(\gamma(t))) = 0 \tag{H}$$

and

$$(r(t)(y(t) + p(t)y(\alpha(t)))^{\Delta^2})^{\Delta^2} + q(t)G(y(\beta(t))) - h(t)H(y(\gamma(t))) = f(t), \tag{NH}$$

on a time scale $\mathbb{T}$ such that $\sup \mathbb{T} = \infty$ and $t_0 \in \mathbb{T}$. We consider these equations under the assumption that

$$\int_{t_0}^{\infty} \frac{\sigma(t)}{r(t)} \Delta t < \infty \tag{H$_1$}$$

and for various ranges of values of $p(t)$. Here we extend the results of [12] to fourth order dynamic equations with positive and negative coefficients and generalize earlier work in [12]. Oscillation results for equations (H) and (NH) under the assumption that $\int_{t_0}^{\infty} \frac{\sigma(t)}{r(t)} \Delta t = \infty$ can be found in [7].

For equations (H) and (NH) we will use the notation that $[t_0, \infty)_{\mathbb{T}} = [t_0, \infty) \cap \mathbb{T}$ and assume that $r \in C_{rd}([t_0, \infty)_{\mathbb{T}}, (0, \infty))$, p, $f \in C_{rd}([t_0, \infty)_{\mathbb{T}}, \mathbb{R})$, q, $h \in C_{rd}([t_0, \infty)_{\mathbb{T}}, (0, \infty))$, G, $H \in C(\mathbb{R}, \mathbb{R})$ satisfy $uG(u) > 0$ and $uH(u) > 0$ for $u \neq 0$, G is nondecreasing, H is bounded, and $\alpha, \beta, \gamma \in C_{rd}(\mathbb{T}, \mathbb{T})$ are strictly increasing functions such that

$$\lim_{t\to\infty} \alpha(t) = \lim_{t\to\infty} \beta(t) = \lim_{t\to\infty} \gamma(t) = \infty, \quad \alpha(t),\ \beta(t),\ \gamma(t) \leqslant t,$$

and

$$(\alpha \circ \beta)(t) = (\beta \circ \alpha)(t) \quad \text{for all} \quad t \in [t_0, \infty)_{\mathbb{T}}.$$

The inverse of $\alpha(t)$ will be denoted by $\alpha^{-1}(t) \in C_{rd}(\mathbb{T}, \mathbb{T})$. Whenever we write $t \geq t_1$, we mean $t \in [t_1, \infty) \cap \mathbb{T}$.

Let $t_{-1} = \inf_{t \in [t_0, \infty)_{\mathbb{T}}} \{\alpha(t), \beta(t), \gamma(t)\}$. By a *solution* of (H) (or (NH)) we mean a function $y \in C_{rd}([t_{-1}, \infty)_{\mathbb{T}}, \mathbb{R})$ such that $y(t) + p(t)y(\alpha(t)) \in C^2_{rd}([t_0, \infty)_{\mathbb{T}}, \mathbb{R})$, $r(t)(y(t) + p(t)y(\alpha(t)))^{\Delta^2} \in C^2_{rd}([t_0, \infty)_{\mathbb{T}}, \mathbb{R})$, and such that (H) ((NH)) is satisfied on $[t_0, \infty)_{\mathbb{T}}$. A solution of (H) or (NH) is called *oscillatory* if it is neither eventually positive nor eventually negative, and it is *nonoscillatory* otherwise. In this paper we do not consider solutions that eventually vanish identically. An equation will be called oscillatory if all its solutions are oscillatory. We will need the following lemmas in the sequel.

Lemma 1.1. ([12, Lemma 3.1]) *Let (H_1) hold and $u(t)$ be a real-valued twice rd-continuously differentiable function on $[t_0, \infty]_{\mathbb{T}}$ such that $r(t)u^{\Delta^2}(t) \in C^2_{rd}([t_0, \infty)_{\mathbb{T}}, \mathbb{R})$ and $(r(t)u^{\Delta^2}(t))^{\Delta^2} \leq 0$ for large $t \in [t_0, \infty)_{\mathbb{T}}$. If $u(t) > 0$ eventually, then one of the following cases (a), (b), (c), or (d) holds for large t, and if $u(t) < 0$ eventually, then one of the cases (b), (c), (d), (e), or (f) holds for large t, where*

$$\begin{aligned}
&(a) \quad u^{\Delta}(t) > 0, \quad u^{\Delta^2}(t) > 0 \quad \textit{and} \quad (r(t)u^{\Delta^2}(t))^{\Delta} > 0,\\
&(b) \quad u^{\Delta}(t) > 0, \quad u^{\Delta^2}(t) < 0 \quad \textit{and} \quad (r(t)u^{\Delta^2}(t))^{\Delta} > 0,\\
&(c) \quad u^{\Delta}(t) > 0, \quad u^{\Delta^2}(t) < 0 \quad \textit{and} \quad (r(t)u^{\Delta^2}(t))^{\Delta} < 0,\\
&(d) \quad u^{\Delta}(t) < 0, \quad u^{\Delta^2}(t) > 0 \quad \textit{and} \quad (r(t)u^{\Delta^2}(t))^{\Delta} > 0,\\
&(e) \quad u^{\Delta}(t) < 0, \quad u^{\Delta^2}(t) < 0 \quad \textit{and} \quad (r(t)u^{\Delta^2}(t))^{\Delta} > 0,\\
&(f) \quad u^{\Delta}(t) < 0, \quad u^{\Delta^2}(t) < 0 \quad \textit{and} \quad (r(t)u^{\Delta^2}(t))^{\Delta} < 0.
\end{aligned}$$

Lemma 1.2. ([12, Lemma 3.2]) *Let (H_1) hold. Assume that $u(t)$ is a positive real valued rd-continuously Δ-differentiable function such that $r(t)u^{\Delta^2}(t)$ is twice continuously Δ-differentiable and $(r(t)u^{\Delta^2}(t))^{\Delta^2} \leqslant 0$ for large t. Then:*

(i) *If case (c) of Lemma 1.1 holds, then there exists a constant* $k \in (0,1)$ *such that the following inequalities hold for large t:*

(I_1) $u^{\Delta}(t) \geqslant -(r(t)u^{\Delta^2}(t))^{\Delta} R_1(t)$;

(I_2) $u^{\Delta}(t) \geqslant -r(t)u^{\Delta^2}(t)\int_t^{\infty} \frac{1}{r(s)}\Delta s$;

(I_3) $u(t) \geqslant ktu^{\Delta}(t)$;

(I_4) $u(t) \geqslant -k(r(t)u^{\Delta^2}(t))^{\Delta} tR_1(t)$;

where $R_1(t) = \int_t^{\infty} \frac{s-t}{r(s)}\Delta s$.

(ii) *If case (d) of Lemma 1.1 holds, then for large t,*

(I_5) $u(t) \geqslant r(t)u^{\Delta^2}(t)R_2(t)$,

where $R_2(t) = \int_t^{\infty} \frac{\sigma(s)-t}{r(s)}\Delta s$.

Remark 1.3. Since $R_1(t) < \int_t^{\infty} \frac{s}{r(s)}\Delta s$ and $R_2(t) < \int_t^{\infty} \frac{\sigma(s)}{r(s)}\Delta s$, then, in view of ($H_1$), $R_1(t)$, $R_2(t) \to 0$ as $t \to \infty$ and $\int_{t_0}^{\infty} \frac{1}{r(t)}\Delta t < \infty$. Clearly, $R_1(t) \leqslant R_2(t)$, and $R_1(t)$, $R_2(t)$ are monotone decreasing.

Lemma 1.4. ([12, Lemma 3.4]) *Let (H_1) and the hypotheses of Lemma 1.1 hold. If* $u(t) > 0$ *for large t, then there exists constants* $k_1 > 0$ *and* $k_2 > 0$ *such that* $k_1 R_2(t) \leqslant u(t) \leqslant k_2 t$ *for large t.*

Lemma 1.5. ([12, Lemma 3.5]) *Let F, H, P* : $[t_0,\infty)_{\mathbb{T}} \to \mathbb{R}$ *satisfy*

$$F(t) = H(t) + P(t)H(\alpha(t)) \quad \textit{for} \quad t \in [\hat{t},\infty)_{\mathbb{T}},$$

where $\hat{t} \in [t_0,\infty)_{\mathbb{T}}$ *is such that* $\alpha(t) \geqslant t_0$ *for all* $t \in [\hat{t},\infty)_{\mathbb{T}}$. *Assume that there exist constants* P_1, $P_2 \in \mathbb{R}$ *such that* $P(t)$ *is in one of the following ranges:*

$$(1)\ \ -\infty < P(t) \leqslant 0, \quad (2)\ \ 0 \leqslant P(t) \leqslant P_1 < 1, \quad (3)\ \ 1 < P_2 \leqslant P(t) < \infty.$$

If $H(t) > 0$ *for large* $t \in [t_0,\infty)_{\mathbb{T}}$, $\liminf_{t\to\infty} H(t) = 0$, *and* $\lim_{t\to\infty} F(t) = L \in \mathbb{R}$ *exists, then* $L = 0$.

Discussions of the oscillatory behavior of solutions of differential equations and difference equations for various ranges of values of $p(t)$ can be found in [6] and [13], respectively. Our final lemma is a very useful form of a chain rule for functions on time scales.

Lemma 1.6. ([1, Theorem 1.87]) *Assume* $g : \mathbb{R} \to \mathbb{R}$ *is continuous,* $g : \mathbb{T} \to \mathbb{R}$ *is* Δ*-differentiable on* $\mathbb{T}^k$, *and* $f : \mathbb{R} \to \mathbb{R}$ *is continuously differentiable. Then there exists c in the interval* $[t,\sigma(t)]$ *such that*

$$(f \circ g)^{\Delta}(t) = f'(g(c))g^{\Delta}(t).$$

2 Oscillation results for (H)

In this section, we study the asymptotic behavior of solutions of equation (H) under the assumption (H_1). We will make use of following conditions on the functions in equations (H) and (NH):

(H_2) $\int_{t_0}^{\infty} \frac{\sigma(s)}{r(s)} \int_s^{\infty} \sigma(t)h(t)\Delta t\Delta s < \infty$;

(H_3) there exists $\lambda > 0$ such that $G(u) + G(v) \geqslant \lambda G(u+v)$ for $u, v \in \mathbb{R}$ with $u, v > 0$;

(H_4) $G(u)G(v) = G(uv)$ for $u, v \in \mathbb{R}$;

(H_5) $\int^{\infty} Q(t)\Delta t = \infty$ where $Q(t) = \min\{q(t), q(\alpha(t))\}$;

(H_6) for some $l > 1$, $\int^{\infty} d(t)Q(t)G(R_2(\beta(t)))\Delta t = \infty$ where $d(t) = \min\{R_1^l(\sigma(t)), R_1^l(\sigma(\alpha(t)))\}$;

(H_7) $G(-u) = -G(u)$ for $u \in \mathbb{R}$;

(H_8) for some $l > 1$, $\int^{\infty} R_1^l(\sigma(t))q(t)G(R_2(\beta(t)))\Delta t = \infty$.

Remark 2.1. Notice that (H_4) implies (H_7), (H_6) implies

(H_6') $\int^{\infty} Q(t)G(R_2(\beta(t)))\Delta t = \infty$,

and (H_8) implies

(H_8') $\int^{\infty} q(t)G(R_2(\beta(t)))\Delta t = \infty$,

which in turn implies

$$\int_{t_0}^{\infty} q(t)\Delta t = \infty.$$

Theorem 2.2. *Assume that conditions (H_1)–(H_6) hold, and p_1, p_2, and p_3 are positive real numbers. If (i) $0 \leqslant p(t) \leqslant p_1 < 1$ or (ii) $1 < p_2 \leqslant p(t) \leqslant p_3 < \infty$ holds, then any solution of (H) is either oscillatory or converges to zero as $t \to \infty$.*

Proof. Let y be a nonoscillatory solution of (H) on $[t_0, \infty)_{\mathbb{T}}$, say y is an eventually positive solution. (The proof in case y is eventually negative is similar and will be omitted.) Then, there exists $t_1 \in [t_0, \infty)_{\mathbb{T}}$ such that $y(t)$, $y(\alpha(t))$, $y(\beta(t))$, $y(\gamma(t))$ and $y(\alpha(\beta(t)))$ are all positive for $t \geqslant t_1$. Set

$$z(t) = y(t) + p(t)y(\alpha(t)), \tag{2.1}$$

and

$$k(t) = \int_t^{\infty} \frac{\sigma(s) - t}{r(s)} \int_s^{\infty} (\sigma(\theta) - s)h(\theta)H(y(\gamma(\theta)))\Delta\theta\Delta s. \tag{2.2}$$

Notice that condition (H_2) and the fact that H is a bounded function imply that $k(t)$ exists for all t. Now if we let

$$w(t) = z(t) - k(t) = y(t) + p(t)y(\alpha(t)) - k(t), \tag{2.3}$$

then

$$(r(t)w^{\Delta^2}(t))^{\Delta^2} = -q(t)G(y(\beta(t))) \leqslant 0, \tag{2.4}$$

for all $t \in [t_1,\infty)_{\mathbb{T}}$. Clearly, $w(t)$, $w^{\Delta}(t)$, $(r(t)w^{\Delta^2}(t))$, and $(r(t)w^{\Delta^2}(t))^{\Delta}$ are monotonic on $[t_1,\infty)_{\mathbb{T}}$. In view of Lemma 1.1, we have to consider the two cases $w(t) > 0$ or $w(t) < 0$.

Suppose that $w(t) > 0$ for $t \geq t_2$ for some $t_2 > t_1$; then there exists $t_3 \in [t_2,\infty)_{\mathbb{T}}$ such that $w(\alpha(t))$, $w(\beta(t)) > 0$ for $t \in [t_3,\infty)_{\mathbb{T}}$. By Lemma 1.1, one of the cases (a), (b), (c) or (d) holds. If (a), (b) or (d) holds, then applying (H_3), (H_4), and (H_5) to equation (H) gives

$$\begin{aligned}
0 &= (r(t)w^{\Delta^2}(t))^{\Delta^2} + q(t)G(y(\beta(t))) + G(p)(r(\alpha(t))w^{\Delta^2}(\alpha(t)))^{\Delta^2} + G(p)q(\alpha(t))G(y(\beta(\alpha(t))))\\
&\geqslant (r(t)w^{\Delta^2}(t))^{\Delta^2} + G(p)(r(\alpha(t))w^{\Delta^2}(\alpha(t)))^{\Delta^2} + \lambda Q(t)G(y(\beta(t)) + py(\alpha(\beta(t))))\\
&\geqslant (r(t)w^{\Delta^2}(t))^{\Delta^2} + G(p)(r(\alpha(t))w^{\Delta^2}(\alpha(t)))^{\Delta^2} + \lambda Q(t)G(z(\beta(t)))
\end{aligned} \tag{2.5}$$

for $t \geqslant t_2 > t_1$, where we have used the fact that $z(t) \leqslant y(t) + py(\alpha(t))$. From (2.2), it follows that $k(t) > 0$ and $k^{\Delta}(t) < 0$. Hence, $w(\beta(t)) > 0$ for $t \geq t_3$ implies that $w(\beta(t)) < z(\beta(t))$ for $t \geq t_3$. From (2.5), we have

$$(r(t)w^{\Delta^2}(t))^{\Delta^2} + G(p)(r(\alpha(t))w^{\Delta^2}(\alpha(t)))^{\Delta^2} + \lambda Q(t)G(w(\beta(t))) \leqslant 0, \tag{2.6}$$

for $t \geqslant t_3 > t_2$. Applying Lemma 1.4 and (H_4) to (2.6) gives

$$(r(t)w^{\Delta^2}(t))^{\Delta^2} + G(p)(r(\alpha(t))w^{\Delta^2}(\alpha(t)))^{\Delta^2} + \lambda G(k_1)Q(t)G(R_2(\beta(t))) \leqslant 0,$$

for $t \geqslant t_4 > t_3$. Now $\lim_{t\to\infty}(r(t)w^{\Delta^2}(t))^{\Delta}$ exists, so integrating the above inequality implies

$$\lambda G(k_1)\int_{t_4}^{\infty} Q(t)G(R_2(\beta(t)))\Delta t < \infty,$$

which contradicts (H_6').

Next, suppose case (c) holds. By (I_4) and Lemma 1.4, we have

$$k(-r(t)w^{\Delta^2}(t))^{\Delta} tR_1(t) \leqslant w(t) \leqslant k_2 t. \tag{2.7}$$

for $t \geqslant t_3 > t_2$. Choose $f(x) = x^{1-l}$ with $l > 1$, which is continuous on $(0,\infty)$, and take $g(t) = (-r(t)w^{\Delta^2}(t))^{\Delta}$. Applying the chain rule (Lemma 1.6), using (2.4) and the fact that g is increasing, means there is a c in the real interval $[t,\sigma(t)]$ with $g(c) = L$, such that

$$\begin{aligned}
-[((-r(t)w^{\Delta^2}(t))^{\Delta})^{1-l}]^{\Delta} &= (l-1)L^{-l}(-r(t)w^{\Delta^2}(t))^{\Delta^2}\\
&= (l-1)L^{-l}q(t)G(y(\beta(t)))\\
&\geqslant (l-1)g^{-l}(\sigma(t))q(t)G(y(\beta(t))).
\end{aligned} \tag{2.8}$$

From (2.7), $kg(t)R_1(t) \leqslant k_2$ for $t \geqslant t_3$, so $kg(\sigma(t))R_1(\sigma(t)) \leqslant k_2$ for $t \geqslant t_3$. Thus, (2.8) becomes

$$-[((-r(t)w^{\Delta^2}(t))^{\Delta})^{1-l}]^{\Delta} \geqslant (l-1)L_1^l R_1^l(\sigma(t))q(t)G(y(\beta(t))), \tag{2.9}$$

where $L_1 = k/k_2$. Choose $t_4 \in [t_3,\infty)_{\mathbb{T}}$ such that $\alpha(t) \geq t_3$ for all $t \in [t_4,\infty)_{\mathbb{T}}$. Using (H$_3$), (H$_4$), and Lemma 1.4, we have

$$\begin{aligned}
- & \quad [((-r(t)w^{\Delta^2}(t))^{\Delta})^{1-l}]^{\Delta} - G(p)[((-r(\alpha(t))w^{\Delta^2}(\alpha(t)))^{\Delta})^{1-l}]^{\Delta} \\
\geq & \quad (l-1)L_1^l R_1^l(\sigma(t))q(t)G(y(\beta(t))) + G(p)(l-1)L_1^l R_1^l(\sigma(\alpha(t)))q(\alpha(t))G(y(\beta(\alpha(t)))) \\
\geq & \quad \lambda(l-1)L_1^l d(t)Q(t)G(z(\beta(t))) \\
\geq & \quad \lambda(l-1)L_1^l d(t)Q(t)G(w(\beta(t))) \\
\geq & \quad \lambda(l-1)L_1^l G(k_1)d(t)Q(t)G(R_2(\beta(t)))
\end{aligned}$$

for $t \geqslant t_4$. Therefore,

$$\int_{t_4}^{\infty} d(t)Q(t)G(R_2(\beta(t)))\Delta t < \infty,$$

which contradicts (H$_6$).

Now we suppose that $w(t) < 0$ for $t \geq t_2$. Then $z(t) - k(t) < 0$ implies $y(t) \leq z(t) = y(t) + p(t)y(\alpha(t)) < k(t)$. Thus, y is bounded. By Lemma 1.1, it follows that one of the cases (b), (c), (d), (e), or (f) holds for $t \geq t_2$. In cases (e) and (f), $\lim_{t\to\infty} w(t) = -\infty$ which contradicts the boundedness of y.

In cases (b) and (c), $w(t)$ is increasing and $w(t) < 0$, so $\lim_{t\to\infty} w(t)$ exists. Consequently,

$$\begin{aligned}
0 \geqslant \lim_{t\to\infty} w(t) &= \limsup_{t\to\infty}[z(t) - k(t)] \\
&\geq \limsup_{t\to\infty}[y(t) - k(t)] \\
&\geq \limsup_{t\to\infty} y(t) - \lim_{t\to\infty} k(t)
\end{aligned}$$

implying that $\lim_{t\to\infty} y(t) = 0$ since $\lim_{t\to\infty} k(t) = 0$.

Finally, let case (d) of Lemma 1.1 hold. Then $w(t) < 0$ is decreasing so $\lim_{t\to\infty} w(t) = L$ with $-\infty \leqslant L < 0$. Since $k(t) \to 0$, this implies $z(t)$ eventually becomes negative, which is a contradiction. This completes the proof of the theorem. □

The following corollary is immediate.

Corollary 2.3. *Under the conditions of Theorem 2.2, every unbounded solution of (H) oscillates.*

Our next theorem gives sufficient conditions for all unbounded solutions to oscillate.

Theorem 2.4. *Let $0 \leqslant p(t) \leqslant p < 1$. If (H_1), (H_2), (H_4) and (H_8) hold, then every unbounded solution of (H) oscillates.*

Proof. Let y be an unbounded nonoscillatory solution of (H), say $y(t)$, $y(\alpha(t))$, $y(\beta(t))$, $y(\gamma(t))$ and $y(\alpha(\alpha(t)))$ are all positive for $t \in [t_1,\infty)_{\mathbb{T}}$ for some $t_1 \geqslant t_0$. We set $z(t)$, $k(t)$ and $w(t)$ as in (2.1)–(2.3) to obtain (2.4) for $t \geq t_1$. Consequently, $w(t)$, $w^{\Delta}(t)$, $(r(t)w^{\Delta^2}(t))$, and $(r(t)w^{\Delta^2}(t))^{\Delta}$ are of constant signs on $[t_2,\infty)_{\mathbb{T}}$, $t_2 \geqslant t_1$.

Assume that $w(t) > 0$ for $t \geq t_2$. By Lemma 1.1, one of the cases (a), (b), (c), or (d) holds. First suppose (a) or (b) holds. Then $0 < w^{\Delta}(t) = z^{\Delta}(t) - k^{\Delta}(t)$. If $z(t)$ oscillates, then

$z^{\Delta}(t) \le 0$ at some arbitrarily large values of t which is a contradiction since $k^{\Delta}(t) < 0$ for all t. Thus, z is monotonic, and for the same reason we can not have $z^{\Delta}(t) \le 0$, so $z^{\Delta}(t) \ge 0$ for all large t, say $t \ge t_3 > t_2$. Hence, in these two cases,

$$\begin{aligned}(1-p)z(t) \le (1-p(t))z(t) < z(t) - p(t)z(\alpha(t))\\ = y(t) - p(t)p(\alpha(t))y(\alpha(\alpha(t))) < y(t),\end{aligned}$$

that is,

$$y(t) > (1-p)z(t) > (1-p)w(t) \tag{2.10}$$

for $t \geqslant t_2 > t_1$. Thus, (2.4) implies

$$G((1-p)w(\beta(t)))q(t) \leqslant -(r(t)w^{\Delta^2}(t))^{\Delta^2},$$

and applying Lemma 1.4 and (H_4) gives

$$G(k_1(1-p))G(R_2(\beta(t)))q(t) \leqslant -(r(t)w^{\Delta^2}(t))^{\Delta^2}. \tag{2.11}$$

Integrating (2.11) from t_3 to ∞, we have

$$\int_{t_2}^{\infty} q(t)G(R_2(\beta(t)))\Delta t < \infty,$$

which contradicts (H_8').

If case (d) holds, then $w^{\Delta}(t) < 0$ and w and z are bounded which can not happen if y is unbounded. If case (c) of Lemma 1.1 holds, we proceed as in the proof of Theorem 2.2 to obtain (2.9). From (2.9), (2.10) and Lemma 1.4, we have

$$-[((-r(t)w^{\Delta^2}(t))^{\Delta})^{1-l}]^{\Delta} \geqslant (l-1)L_1^l G((1-p)k_1)q(t)R_1^l(\sigma(t))G(R_2(\beta(t)))$$

for $t \geqslant t_3$. Integrating the last inequality from t_3 to ∞, we obtain

$$\int_{t_3}^{\infty} q(t)R_1^l(\sigma(t))G(R_2(\beta(t)))\Delta t < \infty,$$

contradicting (H_8).

Finally, we see that since y is unbounded, the case $w(t) < 0$ does not arise because $w(t) = z(t) - k(t) < 0$ implies $0 < z(t) < k(t)$ so again $z(t)$ is bounded. This completes the proof of the theorem. □

Our next two results are for the case where $p(t)$ is negative.

Theorem 2.5. *Let $-1 < p_4 \leqslant p(t) \leqslant 0$ and conditions (H_1), (H_2), (H_4), and (H_8) hold. Then any solution of (H) is either oscillatory or converges to zero as $t \to \infty$.*

Proof. Let y be a nonoscillatory solution of (H), say $y(t)$, $y(\alpha(t))$, $y(\beta(t))$, $y(\gamma(t))$ are positive for all $t \in [t_1, \infty)_{\mathbb{T}}$, $t_1 \geqslant t_0$. Setting $z(t)$, $k(t)$, and $w(t)$ as in (2.1), (2.2), and (2.3), we obtain (2.4) for $t \geqslant t_1$. Hence, $w(t)$ is monotonic for large $t \in [t_1, \infty)_{\mathbb{T}}$. Let $w(t) > 0$ for $t \geqslant t_2$, for some $t_2 \geqslant t_1$ and assume that one of the cases (a), (b), or (d) of Lemma 1.1 holds for $t \geqslant t_2$. By Lemma 1.4, we have $y(\beta(t)) \geqslant w(\beta(t)) \geqslant k_1 R_2(\beta(t))$ for $t \geqslant t_3 > t_2$, with which (2.4) yields

$$\int_{t_3}^{\infty} q(t)G(R_2(\beta(t)))\Delta t < \infty,$$

contradicting (H_8').

Next, we consider case (c). Proceeding as in the proof of Theorem 2.2, we obtain (2.9). Furthermore, $y(t) \geqslant w(t) \geqslant k_1 R_2(t)$ for $t \geqslant t_3$ by Lemma 1.4. Consequently, for $t \geqslant t_4 > t_3$,

$$-[((-r(t)w^{\Delta^2}(t))^{\Delta})^{1-l}]^{\Delta} \geqslant (l-1)L_1 G(k_1)q(t)R_1^l(\sigma(t))G(R_2(\beta(t))). \tag{2.12}$$

An integration of (2.12) gives

$$\int_{t_4}^{\infty} q(t)R_1^l(\sigma(t))G(R_2(\beta(t)))\Delta t < \infty,$$

contradicting (H_8).

Now suppose $w(t) < 0$ for $t \geqslant t_2$. We claim that y is bounded. If not, then there is an increasing sequence $\{\tau_n\}_{n=1}^{\infty} \subset [t_2,\infty)_{\mathbb{T}}$ such that $\tau_n \to \infty$, $y(\tau_n) \to \infty$ as $n \to \infty$, and $y(\tau_n) = \max\{y(t) : t_2 \leqslant t \leqslant \tau_n\}$. We choose τ_1 large enough so that $\alpha(\tau_1) \geqslant t_2$. Hence,

$$0 \geqslant w(\tau_n) \geqslant y(\tau_n) + p(\tau_n)y(\alpha(\tau_n)) - k(\tau_n) \geqslant (1+p_4)y(\tau_n) - k(\tau_n).$$

Since $k(\tau_n)$ is bounded and $1+p_4 > 0$, we have $w(\tau_n) > 0$ for large n, which is a contradiction, so our claim is true. Hence, $z(t)$ is bounded as is $w(t)$. Clearly, cases (e) and (f) of Lemma 1.1 cannot arise.

In cases (b) and (c), $-\infty < \lim_{t\to\infty} w(t) \leqslant 0$. Using the fact that $\lim_{t\to\infty} k(t) = 0$, we have $\lim_{t\to\infty} w(t) = \lim_{t\to\infty} z(t)$. Hence,

$$\begin{aligned}
0 \geq \lim_{t\to\infty} w(t) &= \lim_{t\to\infty} z(t) \\
&= \limsup_{t\to\infty}[y(t) + p(t)y(\alpha(t))] \\
&\geq \limsup_{t\to\infty} y(t) + \liminf_{t\to\infty}(p_4 y(\alpha(t)) \\
&= \limsup_{t\to\infty} y(t) + p_4 \limsup_{t\to\infty} y(\alpha(t)) \\
&= (1+p_4)\limsup_{t\to\infty} y(t),
\end{aligned}$$

which implies that $\limsup_{t\to\infty} y(t) = 0$, that is, $\lim_{t\to\infty} y(t) = 0$.

If case (d) holds, then $\lim_{t\to\infty}(r(t)w^{\Delta^2}(t))^{\Delta}$ exists and so (2.4) gives

$$\int_{t_2}^{\infty} q(t)G(y(\beta(t)))\Delta t < \infty. \tag{2.13}$$

If $\liminf_{t\to\infty} y(t) > 0$, then it follows from (2.13) that

$$\int_{t_2}^{\infty} q(t)\Delta t < \infty,$$

which contradicts Remark 2.1. Hence, $\liminf_{t\to\infty} y(t) = 0$. Using Lemma 1.5, we conclude that $\lim_{t\to\infty} w(t) = 0 = \lim_{t\to\infty} z(t)$. Proceeding as above, we may show that $\limsup_{t\to\infty} y(t) = 0$ and hence $\lim_{t\to\infty} y(t) = 0$. This completes the proof of the theorem. □

Theorem 2.6. *Assume there are constants p_5 and p_6 such that $-\infty < p_5 \leqslant p(t) \leqslant p_6 < -1$ and conditions (H_1), (H_2), (H_4), and (H_8) hold. Then any solution y of (H) is either oscillatory, or satisfies* $\liminf_{t\to\infty} |\, y(t)\,| = 0$, *or satisfies* $|\, y(t)\,| \to \infty$ *as* $t \to \infty$.

Proof. Proceeding as in the proof of Theorem 2.5 in cases (a), (b), (c), or (d) for $w(t) > 0$, we again obtain contradictions to (H_8). Next we consider case $w(t) < 0$ for $t \geqslant t_2$. Suppose case (b) or (d) holds. Since $\lim_{t\to\infty}(r(t)w^{\Delta^2}(t))^{\Delta}$ exists, (2.4) gives

$$\int_{t_2}^{\infty} q(t)G(y(\beta(t)))\Delta t < \infty. \tag{2.14}$$

If $\liminf_{t\to\infty} y(t) > 0$, then it follows that

$$\int_{t_2}^{\infty} q(t)\Delta t < \infty,$$

contradicting Remark 2.1. Hence, $\liminf_{t\to\infty} y(t) = 0$. If case (c) holds, then as in the proof of case (c) of Theorem 2.2, choose $f(x) = x^{1-l}$ and $g(t) = (-r(t)w^{\Delta^2}(t))^{\Delta}$. By Lemma 1.6, there exists c in the real interval $[t,\sigma(t)]$ with $g(c) = L$ such that

$$\begin{aligned} -[((-r(t)w^{\Delta^2}(t))^{\Delta})^{1-l}]^{\Delta} &= (l-1)L^{-l}(-r(t)w^{\Delta^2}(t))^{\Delta^2} \\ &= (l-1)L^{-l}q(t)G(y(\beta(t))). \end{aligned}$$

Integrating, we obtain

$$\int_{t_2}^{\infty} q(t)G(y(\beta(t)))\Delta t < \infty. \tag{2.15}$$

Hence, $\liminf_{t\to\infty} y(t) = 0$.

Finally, in cases (e) and (f), we have $w^{\Delta^2}(t) < 0$ for $t \geqslant t_2$, and integrating twice from t_3 to t, we obtain $w(t) \to -\infty$ as $t \to \infty$. From (2.3), $z(t) \to -\infty$ as $t \to \infty$, so $\lim_{t\to\infty} y(\alpha(t)) = \lim_{t\to\infty} y(t) = \infty$. This completes the proof of the theorem. □

3 Oscillatory results for (NH)

This section is concerned with the oscillatory and asymptotic behavior of solutions of equation (NH) for suitable forcing functions $f(t)$. We restrict our forcing functions to those that change signs. We will use the following conditions:

(H_9) There exists $F \in C^2_{rd}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$ such that $rF^{\Delta^2} \in C^2_{rd}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$, $(rF^{\Delta^2})^{\Delta^2} = f$, and

$$-\infty < \liminf_{t\to\infty} F(t) < 0 < \limsup_{t\to\infty} F(t) < \infty;$$

(H_{10}) There exists $F \in C^2_{rd}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$ such that $rF^{\Delta^2} \in C^2_{rd}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$, $(rF^{\Delta^2})^{\Delta^2} = f$,

$$\liminf_{t\to\infty} F(t) = -\infty, \quad \text{and} \quad \limsup_{t\to\infty} F(t) = +\infty.$$

Theorem 3.1. *Let either (i) $0 \leqslant p(t) \leqslant p_1 < 1$ or (ii) $1 < p_2 \leqslant p(t) \leqslant p_3 < \infty$, and assume that conditions (H_1)–(H_4) and (H_{10}) hold. If*

$$(H_{11})\ \limsup_{t\to\infty}\int_{t_0}^{t} d(s)Q(s)G(F(\beta(s)))\Delta s = +\infty$$
$$\text{and}\ \liminf_{t\to\infty}\int_{t_0}^{t} d(s)Q(s)G(F(\beta(s)))\Delta s = -\infty,$$

then every solution of (NH) oscillates.

Remark 3.2. Notice that condition (H_{11}) implies

$$(H'_{11})\ \limsup_{t\to\infty}\int_{t_0}^{t} Q(s)G(F(\beta(s)))\Delta s = +\infty \text{ and } \liminf_{t\to\infty}\int_{t_0}^{t} Q(s)G(F(\beta(s)))\Delta s = -\infty.$$

Proof of Theorem 3.1. Suppose that y is a nonoscillatory solution of (NH) on $[t_0,\infty)_{\mathbb{T}}$ so that $y(t)$, $y(\alpha(t))$, $y(\beta(t))$, $y(\gamma(t))$, and $y(\alpha(\beta(t)))$ are all positive on $[t_1,\infty)_{\mathbb{T}}$ for some $t_1 \geqslant t_0$. With z, k, and w as in (2.1)–(2.3), let

$$v(t) = w(t) - F(t) = z(t) - k(t) - F(t) \tag{3.1}$$

for $t \geqslant t_1$. Then (NH) becomes

$$(r(t)v^{\Delta^2}(t))^{\Delta^2} = -q(t)G(y(\beta(t))) \leqslant 0. \tag{3.2}$$

Thus, $v(t)$ is monotonic on $[t_2,\infty)_{\mathbb{T}}$, for some $t_2 > t_1$. If $v(t) > 0$ for $t \geqslant t_2$, then $z(t) > k(t) + F(t) > F(t)$. In view of (NH), ($H_3$), and ($H_4$), it is easy to see that

$$\begin{aligned}0 &= (r(t)v^{\Delta^2}(t))^{\Delta^2} + q(t)G(y(\beta(t))) + G(p)(r(\alpha(t))v^{\Delta^2}(\alpha(t)))^{\Delta^2} + G(p)q(\alpha(t))G(y(\beta(\alpha(t))))\\ &\geq (r(t)v^{\Delta^2}(t))^{\Delta^2} + G(p)(r(\alpha(t))v^{\Delta^2}(\alpha(t)))^{\Delta^2} + \lambda Q(t)G(z(\beta(t)))\\ &\geq (r(t)v^{\Delta^2}(t))^{\Delta^2} + G(p)(r(\alpha(t))v^{\Delta^2}(\alpha(t)))^{\Delta^2} + \lambda Q(t)G(F(\beta(t))),\end{aligned} \tag{3.3}$$

for $t \geqslant t_3 \geqslant t_2$. Let (a), (b) or (d) of Lemma 1.1 hold. Integrating (3.3), we obtain

$$\limsup_{t\to\infty}\int_{t_3}^{t} Q(t)G(F(\beta(t)))\Delta t < \infty$$

contradicting (H'_{11}).

Let case (c) of Lemma 1.1 hold. Then proceeding as in the proof of case (c) in Theorem 2.2, we obtain an inequality similar to (2.9) from which it follows that

$$\begin{aligned}-[((-r(t)v^{\Delta^2}(t))^{\Delta})^{1-l}]^{\Delta} &- G(p)[((-r(\alpha(t))v^{\Delta^2}(\alpha(t)))^{\Delta})^{1-l}]^{\Delta}\\ &\geqslant \lambda(l-1)L_1^l d(t)Q(t)G(z(\beta(t)))\\ &\geqslant \lambda(l-1)L_1^l d(t)Q(t)G(F(\beta(t)))\end{aligned}$$

for $t \geqslant t_3$. An integration shows

$$\limsup_{t\to\infty}\int_{t_0}^{t} d(s)Q(s)G(F(\beta(s)))\Delta s < +\infty$$

contradicting (H_{11}).

Therefore, $v(t) < 0$ for $t \geqslant t_2$ and one of the cases (b)–(f) of Lemma 1.1 holds. In each of these cases $z(t) \leq k(t) + F(t)$ which implies $\liminf_{t\to\infty} z(t) < 0$. This contradiction completes the proof of the theorem. □

Remark 3.3. We can drop condition (H_{11}) from the hypotheses of Theorem 3.1 and obtain that bounded solutions oscillate. In case $v(t) < 0$, the proof is the same. If $v(t) > 0$, then $z(t) > k(t) + F(t) > F(t)$ and condition (H_{10}) contradicts the boundedness of y.

Our next two results are for the case where $p(t) \leqslant 0$.

Theorem 3.4. *Let $-1 < p(t) \leqslant 0$ and conditions (H_1), (H_2), (H_7), and (H_{10}) hold. If*

$$(H_{12}) \quad \limsup_{t\to\infty} \int_{t_0}^{t} R_1^l(\sigma(s))q(s)G(F(\beta(s)))\Delta s = +\infty$$

$$\text{and} \quad \liminf_{t\to\infty} \int_{t_0}^{t} R_1^l(\sigma(s))q(s)G(F(\beta(s)))\Delta s = -\infty,$$

then any solution y of equation (NH) is either oscillatory or satisfies $\limsup_{t\to\infty} |y(t)| = \infty$.

Proof. Let y be a nonoscillatory solution of (NH), say $y(t)$, $y(\alpha(t))$, $y(\beta(t))$, and $y(\gamma(t))$ are all positive on $[t_1,\infty)_{\mathbb{T}}$, $t_1 \geqslant t_0$. Define $v(t)$ as in (3.1) so that we obtain (3.2). Consequently, $v(t)$ is monotonic on $[t_2,\infty)_{\mathbb{T}}$. Let $v(t) > 0$ for $t \geqslant t_2$. Then one of the cases (a)–(d) of Lemma 1.1 holds. Now, $v(t) > 0$ implies

$$y(t) > z(t) > k(t) + F(t) > F(t) \tag{3.4}$$

for $t \geqslant t_2 > t_1$. If any one of the cases (a), (b), or (d) holds, then using (3.4) in (3.2), we obtain

$$\limsup_{t\to\infty} \int_{t_3}^{t} q(s)G(F(\beta(s)))\Delta s < \infty$$

contradicting (H_{12}).

Assume that case (c) holds. Proceeding as in the proof of Theorem 2.2, we obtain

$$-[((-r(t)v^{\Delta^2}(t))^{\Delta})^{1-l}]^{\Delta} \geqslant (l-1)L_1^l R_1^l(\sigma(t))q(t)G(y(\beta(t))), \tag{3.5}$$

and using (3.4) and (3.5), this becomes

$$-[((-r(t)v^{\Delta^2}(t))^{\Delta})^{1-l}]^{\Delta} \geqslant (l-1)L_1^l R_1^l(\sigma(t))q(t)G(F(\beta(t))), \tag{3.6}$$

for $t \geqslant t_3 > t_2$. An integration yields a contradiction to (H_{12}).

We must have $v(t) < 0$ for $t \geqslant t_2$. Now, $z(t) - k(t) < F(t)$ which implies that $\liminf_{t\to\infty} z(t) = -\infty$ so $\limsup_{t\to\infty} y(t) = +\infty$, which completes the proof of the theorem. □

Theorem 3.5. *Let $-1 < p_4 \leqslant p(t) \leqslant 0$ and conditions (H_1), (H_2), (H_7), (H_9), and (H_{12}) hold. Then every unbounded solution of (NH) oscillates.*

Proof. Let y be a positive unbounded nonoscillatory solution of (NH) on $[t_0,\infty)_{\mathbb{T}}$. Proceeding as in the proof of Theorem 3.4, we have the required contradiction if $v(t) > 0$ for $t \geqslant t_2$.

Next, we suppose that $v(t) < 0$ for $t \geqslant t_2$. Since y is unbounded, there exists $\{\tau_n\}_{n=1}^{\infty} \subset [t_2,\infty)_{\mathbb{T}}$ such that $\tau_n \to \infty$, $y(\tau_n) \to \infty$ as $n \to \infty$, and

$$y(\tau_n) = \max\{y(t) : t_2 \leqslant t \leqslant \tau_n\}.$$

We may choose n large enough so that $\alpha(\tau_n) \geqslant t_2$. Hence,

$$z(\tau_n) \geqslant (1 + p_4)y(\tau_n).$$

By Lemma 1.1, one of the cases (b)–(f) holds. Now $z(t) = v(t) + k(t) + F(t)$ implies that $z(t) < k(t) + F(t)$, and so

$$\begin{aligned} \infty = (1+p)\limsup_{n\to\infty} y(\tau_n) &\leqslant \limsup_{n\to\infty}[k(\tau_n) + F(\tau_n)] \\ &\leq \lim_{t\to\infty} k(\tau_n) + \limsup_{n\to\infty} F(\tau_n) \\ &< \infty. \end{aligned}$$

This contradiction completes the proof of the theorem. □

The final theorem in this paper gives sufficient conditions for the equation (NH) to have a bounded positive solution.

Theorem 3.6. *Assume that* $1 < p_1 \leq p(t) \leq p_2 < \frac{1}{2}p_1^2 < \infty$ *and* (H_2) *hold. Suppose that* (H_9) *holds with* $\frac{-(p_1-1)}{16p_2} \leq F(t) \leq \frac{p_1-1}{8p_2}$. *In addition, assume that G and H are Lipschitz on* $\mathbb{R}$ *with Lipschitz constants* G_1 *and* H_1 *respectively. If*

$$\int_{t_0}^{\infty} \frac{\sigma(t)}{r(t)} \int_t^{\infty} \sigma(s)q(s)\Delta s\Delta t < \infty,$$

then (NH) admits a positive bounded solution.

Proof. By (H_2), we can choose $t_1 > t_0$ large enough so that

$$\int_{t_1}^{\infty} \frac{\sigma(t)}{r(t)} \int_t^{\infty} \sigma(s)h(s)\Delta s\Delta t < \min\left\{\frac{p_1-1}{4p_1H(1)}, \frac{p_1-1}{16p_2G(1)}\right\}.$$

Let $X = BC_{rd}([t_1,\infty)_{\mathbb{T}},\mathbb{R})$ be the Banach space of all bounded rd-continuous functions on $[t_1,\infty)_{\mathbb{T}}$ with the supremum norm

$$\|x\| = \sup\{|\, x(t)\, | : t \in [t_1,\infty)_{\mathbb{T}}\},$$

and let

$$S = \{x \in X : \frac{p_1-1}{8p_1p_2} \leqslant x(t) \leqslant 1,\ t \in [t_1,\infty)_{\mathbb{T}}\}.$$

Then, S is a closed, bounded, and convex subset of X. Take $t_2 \in [t_1, \infty)_{\mathbb{T}}$ so that $\alpha(t)$, $\beta(t)$, $\gamma(t) \geqslant t_1$ for all $t \in [t_2, \infty)_{\mathbb{T}}$. Define the mappings $A, B : S \to S$ by

$$Ax(t) = \begin{cases} Ax(t_2), & \text{for } t \in [t_1, t_2)_{\mathbb{T}}, \\ -\frac{x(\alpha^{-1}(t))}{p(\alpha^{-1}(t))} + \frac{2p_1^2 + p_1 - 1}{4p_1 p(\alpha^{-1}(t))}, & \text{for } t \in [t_2, \infty)_{\mathbb{T}}. \end{cases}$$

and

$$Bx(t) = \begin{cases} Bx(t_2), & \text{for } t \in [t_1, t_2)_{\mathbb{T}}, \\ \frac{F(\alpha^{-1}(t))}{p(\alpha^{-1}(t))} + \frac{k(\alpha^{-1}(t))}{p(\alpha^{-1}(t))} \\ -\frac{1}{p(\alpha^{-1}(t))} \int_{\alpha^{-1}(t)}^{\infty} \frac{\sigma(s) - \alpha^{-1}(t)}{r(s)} \int_s^{\infty} (\sigma(u) - s) q(u) G(x(\beta(u))) \Delta u) \Delta s, & \text{for } t \in [t_2, \infty)_{\mathbb{T}}. \end{cases}$$

For $x \in S$, we have

$$\begin{aligned} k(t) &= \int_t^{\infty} \frac{\sigma(s) - t}{r(s)} \int_s^{\infty} (\sigma(u) - s) h(u) H(x(\gamma(u))) \Delta u \Delta s \\ &\leqslant H(1) \int_t^{\infty} \frac{\sigma(s)}{r(s)} \int_s^{\infty} \sigma(u) h(u) \Delta u \Delta s \\ &< \frac{1}{4p_1}(p_1 - 1). \end{aligned}$$

For all $x, y \in S$ and all $t \in [t_2, \infty)_{\mathbb{T}}$, we have

$$Ax(t) + By(t) \leqslant \frac{1}{4p_1^2}(2p_1^2 + p_1 - 1) + \frac{1}{8p_1 p_2}(p_1 - 1) + \frac{1}{4p_1^2}(p_1 - 1) < 1$$

and

$$\begin{aligned} Ax(t) + By(t) &\geqslant -\frac{1}{p_1} + \frac{1}{4p_1 p_2}(2p_1^2 + p_1 - 1) - \frac{1}{16 p_1 p_2}(p_1 - 1) - \frac{1}{16 p_1 p_2}(p_1 - 1) \\ &\geq \frac{p_1 - 1}{8 p_1 p_2}. \end{aligned}$$

Thus, $Ax + By \in S$.

To show that A is a contraction mapping on S, first notice that

$$\begin{aligned} \|Ax - Ay\| &= \left\| -\frac{x(\alpha^{-1}(t))}{p(\alpha^{-1}(t))} + \frac{2p_1^2 + p_1 - 1}{4p_1 p(\alpha^{-1}(t))} + \frac{y(\alpha^{-1}(t))}{p(\alpha^{-1}(t))} - \frac{2p_1^2 + p_1 - 1}{4p_1 p(\alpha^{-1}(t))} \right\|. \\ &= \left\| -\frac{x(\alpha^{-1}(t))}{p(\alpha^{-1}(t))} + \frac{y(\alpha^{-1}(t))}{p(\alpha^{-1}(t))} \right\|. \\ &\leqslant \frac{1}{p_1} \left\| x(t) - y(t) \right\|. \end{aligned}$$

Since $p_1 > 1$, A is a contraction mapping.

To show that B is completely continuous on S, we need to show that B is continuous and maps bounded sets into relatively compact sets. In order to show that B is continuous, let $x,\ x_k = x_k(t) \in S$ be such that $\|x_k - x\| = \sup_{t \geqslant t_1}\{|x_k(t) - x(t)|\} \to 0$. Since S is closed, $x(t) \in S$. For $t \geqslant t_1$, we have

$$
\begin{aligned}
|(Bx_k) - (Bx)| &= \Bigg| \frac{1}{p(\alpha^{-1}(t))} \int_{\alpha^{-1}(t)}^{\infty} \frac{\sigma(s) - \alpha^{-1}(t)}{r(s)} \int_s^{\infty} (\sigma(u) - s)h(u)H(x_k(\gamma(u)))\Delta u\Delta s \\
&\quad - \frac{1}{p(\alpha^{-1}(t))} \int_{\alpha^{-1}(t)}^{\infty} \frac{\sigma(s) - \alpha^{-1}(t)}{r(s)} \int_s^{\infty} (\sigma(u) - s)q(u)G(x_k(\beta(u)))\Delta u\Delta s \\
&\quad - \frac{1}{p(\alpha^{-1}(t))} \int_{\alpha^{-1}(t)}^{\infty} \frac{\sigma(s) - \alpha^{-1}(t)}{r(s)} \int_s^{\infty} (\sigma(u) - s)h(u)H(x(\gamma(u)))\Delta u\Delta s \\
&\quad + \frac{1}{p(\alpha^{-1}(t))} \int_{\alpha^{-1}(t)}^{\infty} \frac{\sigma(s) - \alpha^{-1}(t)}{r(s)} \int_s^{\infty} (\sigma(u) - s)q(u)G(x(\beta(u)))\Delta u\Delta s \Bigg| \\
&= \Bigg| \frac{1}{p(\alpha^{-1}(t))} \int_{\alpha^{-1}(t)}^{\infty} \frac{\sigma(s) - \alpha^{-1}(t)}{r(s)} \int_s^{\infty} (\sigma(u) - s)h(u)(H(x_k(\gamma(u))) \\
&\qquad - H(x(\gamma(u))))\Delta u\Delta s \\
&\quad + \frac{1}{p(\alpha^{-1}(t))} \int_{\alpha^{-1}(t)}^{\infty} \frac{\sigma(s) - \alpha^{-1}(t)}{r(s)} \int_s^{\infty} (\sigma(u) - s)q(u)(G(x(\beta(u))) \\
&\qquad - G(x_k(\beta(u))))\Delta u\Delta s \Bigg| \\
&\leqslant \frac{1}{p_1} H_1 \|x_k - x\| \int_{\alpha^{-1}(t)}^{\infty} \frac{\sigma(s) - t}{r(s)} \int_s^{\infty} (\sigma(u) - s)h(u)\Delta u\Delta s \\
&\quad + \frac{1}{p_1} G_1 \|x_k - x\| \int_{\alpha^{-1}(t)}^{\infty} \frac{\sigma(s) - t}{r(s)} \int_s^{\infty} \sigma(u)q(u)\Delta u\Delta s \\
&\leqslant \frac{1}{4p_1^2}(p_1 - 1)\|x - x_k\| + \frac{1}{16p_1p_2}(p_1 - 1)\|x - x_k\|.
\end{aligned}
$$

Since for all $t \geqslant t_1$, $\{x_k(t)\}$ converges uniformly to $x(t)$ as $k \to \infty$, $\lim_{k\to\infty} | (Bx_k)(t) - (Bx)(t) | = 0$ for $t \geqslant t_1$. Thus, B is continuous.

To show that BS is relatively compact, it suffices to show that the family of functions $\{Bx : x \in S\}$ is uniformly bounded and equicontinuous on $[t_1, \infty)_{\mathbb{T}}$. The uniform boundedness is clear. To show that BS is equicontinuous, let $x \in S$ and $t'', t' \geqslant t_1$. Then

$$
\begin{aligned}
&|(Bx)(t'') - (Bx)(t')| \\
&\leqslant \left| \frac{F(\alpha^{-1}(t''))}{p(\alpha^{-1}(t''))} - \frac{F(\alpha^{-1}(t'))}{p(\alpha^{-1}(t'))} \right| + \left| \frac{k(\alpha^{-1}(t''))}{p(\alpha^{-1}(t''))} - \frac{k(\alpha^{-1}(t'))}{p(\alpha^{-1}(t'))} \right| \\
&\quad + \Bigg| \frac{1}{p(\alpha^{-1}((t'))} \int_{\alpha^{-1}(t')}^{\infty} \frac{\sigma(s) - \alpha^{-1}(t')}{r(s)} \int_s^{\infty} (\sigma(u) - s)q(u)G(x(\beta(u)))\Delta u\Delta s \\
&\quad - \frac{1}{p(\alpha^{-1}(t''))} \int_{\alpha^{-1}(t'')}^{\infty} \frac{\sigma(s) - \alpha^{-1}(t'')}{r(s)} \int_s^{\infty} (\sigma(u) - s)q(u)G(x(\beta(u)))\Delta u\Delta s \Bigg|
\end{aligned}
$$

so $|(Bx)(t'') - (Bx)(t')| \to 0$ as $t'' \to t'$. Therefore, $\{Bx : x \in S\}$ is uniformly bounded and equicontinuous on $[t_1, \infty)_{\mathbb{T}}$. Hence, BS is relatively compact. By Krasnosel'skii's fixed point theorem (see, for example, Lemma 3 in [9] or Lemma 2.4 in [7]), there exists $x \in S$ such that $Ax + Bx = x$. Thus, the theorem is proved. □

Remark 3.7. Results similar to Theorem 3.6 can be proved for other ranges of values for $p(t)$.

4 Examples

We conclude this paper with some examples of our main results.

Example 4.1. Let $\mathbb{T} = \mathbb{R}$ and consider the differential equation

$$\left(e^{\frac{t}{2}}\left(y(t) + \frac{1}{2}e^{\frac{-4t}{3}}y(t/3)\right)''\right)'' + \frac{1}{2}e^{\frac{9t}{2}}y^3(t) - 14e^{-t}(1+e^{-t})\frac{y(t/4)}{1+y^2(t/4)} = 0, \quad t \geq 0. \tag{4.1}$$

It is easy to verify that the hypotheses of Theorem 2.2 are satisfied. Here, $y(t) = e^{-2t}$ is a nonoscillatory solution of (4.1) that converges to 0 as $t \to \infty$.

Example 4.2. Let $\mathbb{T} = \mathbb{R}$ and consider the differential equation

$$\left(e^{\frac{t}{2}}\left(y(t) - \frac{1}{2}e^{-t}y(t/2)\right)''\right)'' + e^{\frac{9t}{2}}y^3(t) - \frac{11}{2}e^{-t}(1+e^{-t})\frac{y(t/4)}{1+y^2(t/4)} = 0, \quad t \geq 0. \tag{4.2}$$

It is easy to see that the hypotheses of Theorem 2.5 are satisfied. Here, $y(t) = e^{-2t}$ is a nonoscillatory solution of (4.2) that converges to 0 as $t \to \infty$.

Our next example is one of a difference equation.

Example 4.3. Let $\mathbb{T} = \mathbb{Z}$ and consider the difference equation

$$\begin{aligned}\Delta^2[e^n\Delta^2(y(n) + e^{-5n}y(n-2))] + e^{1/3}(e+1)^2(e^2+1)^2e^{5n/3}y^{\frac{1}{3}}(n-1)\\ - e^{-2}(e^{-4}+1)^2(e^{-3}+1)^2(1+e^{2n})e^{-4n}\frac{y(n)}{1+y^2(n)} = 0, \quad n \geq 2.\end{aligned} \tag{4.3}$$

Conditions (H_1)–(H_6) are satisfied so equation (4.3) satisfies the hypotheses of Theorem 2.2 and Corollary 2.3. Here we have $y(n) = (-1)^n e^n$ as an unbounded oscillatory solution.

Next, we have an example of a forced equation.

Example 4.4. Let $\mathbb{T} = \mathbb{R}$ and consider the equation

$$\begin{aligned}\left(e^t\left(y(t) + e^{-4t}y(t-\pi)\right)''\right)'' + 8e^{t+2\pi}y(t-2\pi)\\ - 50e^{-3t+\pi/2}\left(1 + e^{2t-3\pi}\cos^2 t\right)\frac{y(t-3\pi/2)}{1+y^2(t-3\pi/2)} = 6e^{2t}\cos t, \quad t \geq 2\pi.\end{aligned} \tag{4.4}$$

Conditions (H_1)–(H_4), (H_{10}), and (H_{11}) are satisfied with $F(t) = \frac{e^t}{25}(9\sin t - 12\cos t)$, so equation (4.4) satisfies the hypotheses of Theorem 3.1, and all solutions are oscillatory. Here $y(t) = e^t \sin t$ is such an oscillatory solution.

Our final example is on the time scale $\mathbb{T} = h\mathbb{Z}$.

Example 4.5. Let $\mathbb{T} = h\mathbb{Z}$ with h a quotient of odd positive integers and consider the equation

$$\Delta_h^2(e^t\Delta_h^2(y(t)-e^h(1+e^{-5t})y(t-h)))+2e^h\Big(\frac{e^h+1}{h}\Big)^2\Big(\frac{e^{2h}+1}{h}\Big)^2 e^{\frac{5t}{3}}y^{\frac{1}{3}}(t-3h)$$
$$-e^{2h}\Big(\frac{e^{-4h}+1}{h}\Big)^2\Big(\frac{e^{-3h}+1}{h}\Big)^2(1+e^{t-2h})e^{-4t}\frac{y(t-2h)}{1+|\,y(t-2h)\,|}=0, \quad t\geq 3h. \tag{4.5}$$

It is fairly easy to see that conditions (H_1), (H_2), and (H_4) hold and $-2e^h < p(t) < -e^h < -1$. In order to show that (H_8) holds, take $l = 1+\frac{1}{6} > 1$ and first note that

$$R_1(\sigma(t)) = R_1(t+h) = \sum_{s=t+h}^{\infty}\frac{s-t-h}{e^s} = 0+\frac{h}{e^{t+2h}}+\frac{2h}{e^{t+3h}}+\frac{3h}{e^{t+4h}}+\dots$$
$$\geqslant \frac{h}{e^{t+2h}}(1+\frac{1}{e^h}+\frac{1}{e^{2h}}+\dots) = \frac{h}{e^{t+2h}}(\frac{1}{1-\frac{1}{e^h}}) = \frac{h}{e^{t+h}(e^h-1)},$$

so

$$R_1^{\frac{7}{6}}(\sigma(t)) \geq (\frac{h}{e^h(e^h-1)})^{\frac{7}{6}}\frac{1}{e^{\frac{7t}{6}}}.$$

Also,

$$R_2(\beta(t)) = R_2(t-2h) = \sum_{s=t-2h}^{\infty}\frac{s-t+3h}{e^s} \geqslant \frac{h}{e^t}(\frac{1}{e^{-2h}}+\frac{1}{e^{-h}}+1+\frac{1}{e^h}+\frac{1}{e^{2h}}+...)$$
$$\geqslant \frac{h}{e^t}(1+\frac{1}{e^h}+\frac{1}{e^{2h}}+...) = \frac{he^h}{(e^h-1)e^t},$$

so

$$G(R_2(\beta(t))) = \Big(\frac{he^h}{(e^h-1)e^t}\Big)^{\frac{1}{3}} = \Big(\frac{he^h}{(e^h-1}\Big)^{\frac{1}{3}}\frac{1}{e^{\frac{t}{3}}}.$$

Then,

$$\int_{t_0}^{\infty}R_1^{\frac{7}{6}}(\sigma(t))q(t)G(R_2(\beta(t)))\Delta t = \sum_{t=t_0}^{\infty}R_1^{\frac{7}{6}}(\sigma(t))q(t)G(R_2(\beta(t)))$$
$$\geqslant \sum_{t=t_0}^{\infty}\Big((\frac{h}{e^h(e^h-1)})^{\frac{7}{6}}\frac{1}{e^{\frac{7t}{6}}}\Big)\Big(2e^h\Big(\frac{e^h+1}{h}\Big)^2\Big(\frac{e^{2h}+1}{h}\Big)^2e^{\frac{5t}{3}}\Big)\Big(\Big(\frac{he^h}{(e^h-1}\Big)^{\frac{1}{3}}\frac{1}{e^{\frac{t}{3}}}\Big)$$
$$=\Big(\frac{h}{e^h(e^h-1)}\Big)^{\frac{7}{6}}\Big(2e^h\Big(\frac{e^h+1}{h}\Big)^2\Big(\frac{e^{2h}+1}{h}\Big)^2\Big)\Big(\frac{he^h}{e^h-1}\Big)^{\frac{1}{3}}\sum_{t=t_0}^{\infty}e^{\frac{t}{6}} = \infty.$$

Hence, the hypotheses of Theorem 2.6 hold so any solution of (4.5) is either oscillatory, satisfies $\liminf_{t\to\infty}|\,y(t)\,| = 0$, or satisfies $|\,y(t)\,| \to \infty$ as $t \to \infty$. Here $y(t) = (-1)^t e^t$ is an oscillatory solution.

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