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ON TIME-SPACE DEPENDENT CONSERVATION LAWS OF NONLINEAR EVOLUTION DIFFERENTIAL EQUATIONS

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Abstract

In this paper, we propose an alternative direct algebraic method of constructing, for nonlinear evolution partial differential equations, conservation laws that depend not only on dependent variables and its derivatives but also explicitly on independent variables. As illustration, the fifth order Korteweg de Vries (fKdV) and modified $(n+1)$-dimensional Zakharov-Kuznetvov (ZK) equations are probed.

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1 Introduction

The concept of conservation laws is substantially used in the field of partial differential equations (PDEs) ([8, 9] and references therein). Indeed, the investigation of conservation laws can lead to find some qualitative properties of PDEs such as integrability, stability,

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existence of global solutions and even the linearizability conditions [9, 10, 12, 13]. The literature prospers in various approaches to construct local conservation laws. For instance, we can cite the direct method [5, 1, 2], the Noether method [6, 3], the characteristic method [7, 11] and the variational derivative method [7, 14]. All these approaches allow, in most cases, to compute polynomial conservation laws that only depend on dependent variables and its derivatives, and not explicitly on independent variables.

In this paper, an alternative study of time-space dependent conservation laws for some classes of nonlinear evolution PDEs is performed by judiciously exploiting known algebraic methods [8, 15]. It generalizes, to any dimensional space, a previous algorithmic scheme [16] elaborated to construct spatio-temporal dependent conservation laws for n-order $(1+1)$-dimensional Korteweg de Vries (KdV) equations. Finally, we probe, in this framework, the fifth order Korteweg de Vries (fKdV) [4] and modified $(n+1)$-dimensional Zakharov-Kuznetvov (ZK) equations [8, 17].

2 Preliminaries: basic definitions, theorems and notations

This section, mainly based on [7, 8], addresses relevant definitions, theorems and notations playing a central role in studying conservation laws for nonlinear PDEs. Consider a system of s-order PDEs

$$F_\nu(x,u^{(s)}) = 0, \quad \nu = 1,\cdots,m, \tag{2.1}$$

where $x=(x^1,\dots,x^n)$ and $u=(u^1,\dots,u^m)$ and $X\times U^{(s)}$, the space whose coordinates are denoted by $(x,u^{(s)})$, (encompassing the independent variables x, the dependent variables u and their derivatives up to order s, $u^{(s)}$).

Definition 2.1. ***(Differential function)*** A function f defined on $X\times U^{(s)}$ is called s-order differential function if it is locally analytic, i.e., locally expandable in a Taylor series with respect to all arguments.

Definition 2.2. ***(Total derivative operator)*** Let f defined on $X\times U^{(s)}$ be an s-order differential function. The total derivative of f with respect to x^i is defined by:

$$D_{x^i} f = \frac{\partial f}{\partial x^i} + \sum_{j=1}^{m}\sum_{k_1=0}^{s_1^j}\cdots\sum_{k_n=0}^{s_n^j} u^j_{k_1x^1\cdots(k_i+1)x^i\cdots k_nx^n}\frac{\partial f}{\partial u^j_{k_1x^1\cdots k_nx^n}},$$

where

$$u^j_{k_1x^1\cdots k_nx^n} = \frac{\partial^{k_1+\cdots+k_n}u^j}{(\partial x^1)^{k_1}\cdots(\partial x^n)^{k_n}},$$

integer s_i^j is the maximal order of derivation of the variable u^j with respect to x^i in the differential function f.

Definition 2.3. ***(Maximal rank condition)*** The system (2.1) is said to be of maximal rank if the Jacobian matrix

$$J_F(x,u^{(s)}) = \left(\frac{\partial F_\nu}{\partial x}, \frac{\partial F_\nu}{\partial u^{(s)}}\right)$$

is of rank m whenever u is a solution of (2.1).

Definition 2.4. ***(Invariant scaling or dilatation group for a PDE)*** The system of s-order PDEs

$$F_\nu(x, u^{(s)}) = 0, \quad \nu = 1, \cdots, m$$

is said to be invariant under a dilatation group if there exist a nonzero parameter λ and vector constants $(a^1, \cdots, a^n) \in \mathbb{R}_\star^n$, and $(b^1, \cdots, b^m) \in \mathbb{R}_\star^m$ such that

$$F_\nu(\tilde{x}, \tilde{u}^{(s)}) = 0, \quad \nu = 1, \cdots, m$$

with $\tilde{x} = (\lambda^{a_1} x^1, \ldots, \lambda^{a_n} x^n)$ and $\tilde{u} = (\lambda^{b_1} u^1, \ldots, \lambda^{b_m} u^m)$.
Moreover, one can attribute weights to the variables and total derivatives as follows:

$$w(x^i) = a_i, \quad w(u^j) = b_j, \quad w(D_{x^i}) = -a_i.$$

Definition 2.5. ***(Rank of a monomial)*** Let f defined on $X \times U^{(s)}$ be an s-order polynomial differential function.

1. An s'-order monomial M is a term of f expressed as

$$M = c \prod_{k=0}^{s'} \prod_{i=1}^{n} \prod_{j=1}^{m} (x^i)^{\alpha_i} \left(D_{x^i}^k u^j \right)^{\beta_{kij}} \quad \text{with } s', \alpha_i, \beta_{kij} \in \mathbb{N}; c \in \mathbb{R}_\star.$$

2. The rank of the monomial M is the real number

$$\sum_{k=0}^{s'} \sum_{i=1}^{n} \sum_{j=1}^{m} \left[\alpha_i w(x^i) + \beta_{kij} \left(w(u^j) + k w(D_{x^i}) \right) \right]$$

3. f is said to be uniform in rank if all its monomials have the same rank.

Proposition 2.6. *The differential functions defining a system of polynomial PDEs invariant under a dilation group are uniform in rank.*

Definition 2.7. ***(Total divergence)*** A total divergence of an n-dimensional differential function $F = (F^1, \cdots, F^n)$ is defined by:

$$DivF = D_{x^1} F^1 + \cdots + D_{x^n} F^n.$$

Definition 2.8. ***(Conservation law for a PDE)*** Let

$$F_\nu(t, x, u^{(s)}) = 0, \quad \nu = 1, \cdots, m \tag{2.2}$$

be a system of $(n+1)$-dimensional s-order differential equations. A conservation law of (2.2) is the PDE

$$D_t \rho + DivJ = 0, \tag{2.3}$$

where ρ is called conserved density whose the associated conserved flux is the vector differential function $J = (J^1, \cdots, J^n)$.

Definition 2.9. ***(Zeroth-Euler operator)*** Let f defined on $X \times U^{(s)}$ be an s-order differential function.

1. The zeroth-Euler operator (also called the variational derivative) of f is given by

$$\frac{\delta}{\delta u}f = (\frac{\delta}{\delta u^1}f, \cdots, \frac{\delta}{\delta u^m}f),$$

where for $j = 1, \cdots, m$

$$\frac{\delta}{\delta u^j}f = \sum_{k_1=0}^{s_1^j} \cdots \sum_{k_n=0}^{s_n^j} (-D_{x^1})^{k_1} \cdots (-D_{x^n})^{k_n} \frac{\partial f}{\partial u^j_{k_1x^1\cdots k_nx^n}}.$$

2. f is said to be exact if there exists an $(s-1)$-order differential vector function $C = \left(C^1(x, u^{(s-1)}), \cdots, C^n(x, u^{(s-1)})\right)$ such that $f = DivC$.

Theorem 2.10. (Exactness theorem) *A differential function f is exact if and only if $\frac{\delta}{\delta u^j}f = 0$, $j = 1, \ldots, m$.*

Definition 2.11. ***(Divergence and divergence-equivalent terms)*** A term f is a divergence if there exists a vector C such that $f = DivC$. Two or more terms are divergence-equivalent if there exists a linear combination of these terms which is a divergence.

Theorem 2.12. (Characterization of divergence-equivalent terms) *When the zeroth-Euler operator is applied to a set of divergence-equivalent terms, their images under the zeroth-Euler operator are linearly dependent.*

Definition 2.13. ***(Higher-Euler operator)*** Let f defined on $X \times U^{(s)}$ be an s-order differential function. The $(i_1, \ldots, i_n)$-higher-Euler operator (also called the higher variational derivative) of f is given by

$$\frac{\delta^{(i_1,\ldots,i_n)}}{\delta u}f = \left(\frac{\delta^{(i_1,\ldots,i_n)}}{\delta u^1}f, \cdots, \frac{\delta^{(i_1,\ldots,i_n)}}{\delta u^m}f\right),$$

where for $j = 1, \cdots, m$

$$\frac{\delta^{(i_1,\ldots,i_n)}}{\delta u^j}f = \sum_{k_1=i_1}^{s_1^j} \cdots \sum_{k_n=i_n}^{s_n^j} \begin{pmatrix} k_1 \\ i_1 \end{pmatrix} \cdots \begin{pmatrix} k_n \\ i_n \end{pmatrix} (-D_{x^1})^{k_1-i_1} \cdots (-D_{x^n})^{k_n-i_n} \frac{\partial f}{\partial u^j_{k_1x^1\cdots k_nx^n}}.$$

Definition 2.14. ***(Homotopy operator)*** Let f defined on $X \times U^{(s)}$ be an s-order exact differential function with n independent variables $x = (x^1, \ldots, x^n)$. The n-dimensional homotopy operator is an n-component vector,
$\left(H_u^{(x^1)}f, \ldots, H_u^{(x^n)}f\right)$, where for $i = 1, \ldots, n$

$$H_u^{(x^i)}f = \int_0^1 \sum_{j=1}^m \left(I_{u^j}^{(x^i)}f\right)[\lambda u]\frac{d\lambda}{\lambda}$$

with the integrands $I_{u^j}^{(x^i)}f$ defined as

$$I_{u^j}^{(x^i)}f = \sum_{k_1=0}^{s_1^j} \cdots \sum_{k_i=0}^{s_i^j-1} \cdots \sum_{k_n=0}^{s_n^j} \frac{1+k_i}{1+k_1+\cdots+k_n} D_{x^1}^{k_1} \cdots D_{x^2}^{k_n} \left(u^j \frac{\delta^{(k_1,\ldots,k_i+1,\ldots,k_n)}}{\delta u^j}f\right).$$

The notation $\left(I_{u^j}^{(x^i)} f\right)[\lambda u]$ means that in $I_{u^j}^{(x^i)} f$, all components of the function u as well as their derivatives are multiplied by λ.

Theorem 2.15. *Let f defined on $X \times U^{(s)}$ be an s-order exact differential function with n independent variables $x = (x^1, \ldots, x^n)$. Then*

$$Div^{-1} f = \left(H_u^{(x^1)} f, \ldots, H_u^{(x^n)} f\right).$$

In one, two and three independent variables, the homotopy operator takes the following forms.

- ***One-dimensional homotopy operator,*** $x = x^1$, $H_u f$:

$$H_u f = \int_0^1 \sum_{j=1}^{m} (I_{u^j} f)[\lambda u] \frac{d\lambda}{\lambda},$$

where

$$I_{u^j} f = \sum_{k=1}^{s_1^j} \left(\sum_{i=0}^{k-1} u_{ix}^j (-D_x)^{k-i-1} \right) \frac{\partial f}{\partial u_{kx}^j} \quad \text{and} \quad D_x^{-1} f = H_u f.$$

- ***Two-dimensional homotopy operator,*** $x = (x^1, x^2)$, $\left(H_u^{(x^1)} f, H_u^{(x^2)} f\right)$:

$$H_u^{(x^i)} f = \int_0^1 \sum_{j=1}^{m} \left(I_{u^j}^{(x^i)} f\right)[\lambda, u] \frac{d\lambda}{\lambda}, \quad i = 1, 2,$$

with the integrands $I_{u^j}^{(x^i)} f$ defined as

$$\begin{aligned} I_{u^j}^{(x^i)} f \;=\; & \sum_{k_i=1}^{s_i^j} \sum_{k_p=0}^{s_p^j} \sum_{l_i=0}^{k_i-1} \sum_{l_p=0}^{k_p} u_{l_i x^i l_p x^p}^j \frac{\begin{pmatrix} l_i + l_p \\ l_i \end{pmatrix} \begin{pmatrix} k_i + k_p - l_i - l_p - 1 \\ k_i - l_i - 1 \end{pmatrix}}{\begin{pmatrix} k_i + k_p \\ k_i \end{pmatrix}} \\ & (-D_{x^i})^{k_i - l_i - 1} (-D_{x^p})^{k_p - l_p} \frac{\partial f}{\partial u_{k_i x^i k_p x^p}^j}, \end{aligned}$$

where

$$p \in \{1, 2\} \setminus \{i\} \quad \text{and} \quad Div^{-1} f = \left(H_u^{(x^1)} f, H_u^{(x^2)} f\right).$$

- ***Three-dimensional homotopy operator,*** $x = (x^1, x^2, x^3)$, $\left(H_u^{(x^1)} f, H_u^{(x^2)} f, H_u^{(x^3)} f\right)$:

$$H_u^{(x^i)} f = \int_0^1 \sum_{j=1}^{m} \left(I_{u^j}^{(x^i)} f\right)[\lambda u] \tfrac{d\lambda}{\lambda}, \; i = 1, 2, 3$$

with the integrands $I_{u^j}^{(x^i)} f$ defined as

$$
\begin{aligned}
I_{u^j}^{(x^i)} f \quad = \quad & \sum_{k_i=1}^{s_i^j} \sum_{k_p=0}^{s_p^j} \sum_{k_q=0}^{s_q^j} \sum_{l_i=0}^{k_i-1} \sum_{l_p=0}^{k_p} \sum_{l_q=0}^{k_q} u^j_{l_i x^i l_p x^p l_q x^q} \frac{\begin{pmatrix} l_i+l_p+l_q \\ l_i \end{pmatrix} \begin{pmatrix} l_p+l_q \\ l_p \end{pmatrix}}{\begin{pmatrix} k_i+k_p+k_q \\ k_i \end{pmatrix}} \\
& \frac{\begin{pmatrix} k_i+k_p+k_q-l_i-l_p-l_q-1 \\ k_i-l_i-1 \end{pmatrix} \begin{pmatrix} k_p+k_q-l_p-l_q \\ k_p-l_p \end{pmatrix}}{\begin{pmatrix} k_p+k_q \\ k_p \end{pmatrix}} \\
& (-D_{x^i})^{k_i-l_i-1} (-D_{x^p})^{k_p-l_p} (-D_{x^q})^{k_q-l_q} \frac{\partial f}{\partial u^j_{k_i x^i k_p x^p k_q x^q}},
\end{aligned}
$$

where

$$p,q \in \{1,2,3\} \setminus \{i\},\ q+1 \equiv i\,[3],\ p \neq q \quad \text{and} \quad Div^{-1} f = \left(H_u^{(x^1)} f, H_u^{(x^2)} f, H_u^{(x^3)} f \right).$$

3 Main results

We consider in this section a system of s-order $(n+1)$-dimensional evolution PDEs

$$(u^\nu)_t = F_\nu(t,x,u^{(s)}) \quad \nu = 1,\ldots,m, \tag{3.1}$$

where $x = (x^1,\ldots,x^n)$ and $u = (u^1,\ldots,u^m)$. Assuming that (3.1) is of maximal rank, we propose to find its conservation laws.

3.1 Construction of conservation laws

The approach consists of three successive steps: i) determination of a scaling symmetry group; ii) construction of a density and iii) calculation of a flux.

A scaling symmetry group can be obtained using linear algebra, which avoids having to solve a system of over determined PDEs. This is done by solving an algebraic system of weights for independent and dependent variables and total derivatives appearing in the PDEs, assuming that each equation of the system of PDEs is uniform in rank and taking into account the fact that the values of these ranks are not necessary the same for distinct equations. In other words, one makes the hypothesis that the p_ν monomials of the function F_ν have the same rank and one forms a set of $p_\nu - 1$ equations whose unknowns are the weights of variables and total derivatives present in the function F_ν. The final algebraic system of weight equations results from $\sum_{\nu=1}^m (p_\nu - 1)$ equations of m different determined sets.

Considering a system of s-order $(n+1)$-dimensional evolution PDEs

$$(u^\nu)_t = F_\nu(t,x,u^{(s)},\widetilde{u}^{(s)}) \quad \nu = 1,\ldots,m, \tag{3.2}$$

of maximal rank and $m+q \le \sum_{\nu=1}^m (p_\nu - 1) - 1$, where $x = (x^1,\ldots,x^n)$, $u = (u^1,\ldots,u^m)$ and $\widetilde{u} = (\widetilde{u}^1,\ldots,\widetilde{u}^q)$, then one can check that there exists a scaling symmetry group if $w(u^j)$, $w(\widetilde{u}^l)$ and $w(D_t)$ are expressible as functions of some $w(D_{x^i})$.

- In the case when the system (3.1) has a scaling symmetry group, since the scaling parameter λ is arbitrary, in general the solutions of the weight system can be of the form:

$$\begin{array}{rcl} w(u^j) & = & h_j\left(w(D_{x^{i_1}}),\ldots,w(D_{x^{i_k}})\right), \quad j=1,\ldots,m \\ w(D_t) & = & g_0\left(w(D_{x^{i_1}}),\ldots,w(D_{x^{i_k}})\right), \\ w(D_{x^l}) & = & g_l\left(w(D_{x^{i_1}}),\ldots,w(D_{x^{i_k}})\right), \quad l\in\{1,\ldots,n\}\setminus\{i_1,\ldots,i_k\}. \end{array} \tag{3.3}$$

  Setting $w(D_{x^{i_1}})=r_{i_1},\ldots,w(D_{x^{i_k}})=r_{i_k}$, the relations (3.3) give:

$$\begin{array}{rcl} w(u^j) & = & h_j\left(r_{i_1},\ldots,r_{i_k}\right)\equiv s_j, \quad j=1,\ldots,m \\ w(D_t) & = & g_0\left(r_{i_1},\ldots,r_{i_k}\right)\equiv r_0, \\ w(D_{x^l}) & = & g_l\left(r_{i_1},\ldots,r_{i_k}\right)\equiv r_l, \quad l\in\{1,\ldots,n\}\setminus\{i_1,\ldots,i_k\}. \end{array}$$

  Therefore, the weights of independent variables are $w(t)=-r_0$, $w(x_i)=-r_i$ and a scaling symmetry group of the system (3.2) is:

$$(t,x,u)\mapsto(\lambda^{-r_0}t,\lambda^{-r_1}x^1,\ldots,\lambda^{-r_n}x^n,\lambda^{s_1}u^1,\ldots,\lambda^{s_m}u^m).$$

- When the system (3.1) does not possess a scaling symmetry group, sometimes identified by the solution of the weight system under the form $w(u^j)=w(D_t)=w(D_{x^i})=0$, one can transform it into another PDEs system of the form

$$(u^\nu)_t=F_\nu(t,x,u^{(s)},\widetilde{u}), \quad \nu=1,\ldots,m, \tag{3.4}$$

  where $\widetilde{u}=(\widetilde{u}^1,\ldots,\widetilde{u}^q)$ is a vector of new dependent variables, which now has a scaling symmetry group. Such a transformation can be performed by one or both of the following actions:

  (i) conversion of some arbitrary monomials coefficients of considered PDEs system into new dependent variables;

  (ii) suitable identification of monomials in the PDEs system and their multiplication by new dependent variables.

  By the same arguments as before, the solutions of the weight system obtained from the system (3.4) are generally of the form:

$$\begin{array}{rcl} w(u^j) & = & h_j\left(w(D_{x^{i_1}}),\ldots,w(D_{x^{i_k}})\right), \quad j=1,\ldots,m \\ w(\widetilde{u}^l) & = & f_l\left(w(D_{x^{i_1}}),\ldots,w(D_{x^{i_k}})\right), \quad l=1,\ldots,q \\ w(D_t) & = & g_0\left(w(D_{x^{i_1}}),\ldots,w(D_{x^{i_k}})\right), \\ w(D_{x^l}) & = & g_l\left(w(D_{x^{i_1}}),\ldots,w(D_{x^{i_k}})\right), \quad l\in\{1,\ldots,n\}\setminus\{i_1,\ldots,i_k\}. \end{array} \tag{3.5}$$

  Putting $w(D_{x^{i_1}})=r_{i_1},\ldots,w(D_{x^{i_k}})=r_{i_k}$, (3.5) gives:

$$\begin{array}{rcl} w(u^j) & = & h_j\left(r_{i_1},\ldots,r_{i_k}\right)\equiv s_j, \quad j=1,\ldots,m \\ w(\widetilde{u}^l) & = & f_l\left(r_{i_1},\ldots,r_{i_k}\right)\equiv e_l, \quad l=1,\ldots,q \\ w(D_t) & = & g_0\left(r_{i_1},\ldots,r_{i_k}\right)\equiv r_0, \\ w(D_{x^l}) & = & g_l\left(r_{i_1},\ldots,r_{i_k}\right)\equiv r_l, \quad l\in\{1,\ldots,n\}\setminus\{i_1,\ldots,i_k\}. \end{array}$$

Therefore, the weights of independent variables are $w(t) = -r_0$, $w(x_i) = -r_i$ and a scaling symmetry group of the system (3.4) is given by:

$$(t,x,u,\widetilde{u}) \mapsto (\lambda^{-r_0}t, \lambda^{-r_1}x^1, \ldots, \lambda^{-r_n}x^n, \lambda^{s_1}u^1, \ldots, \lambda^{s_m}u^m, \lambda^{e_1}\widetilde{u}^1, \ldots, \lambda^{e_q}\widetilde{u}^q).$$

In the sequel, we consider the PDEs system (3.4) and treate the new dependent variables $\widetilde{u}^1, \ldots, \widetilde{u}^q$ as weighted constants. Once a scaling symmetry group of equation (3.4) is defined, one fixes the value r of the rank for the density ρ which can be constructed step by step as follows:

(i) Use the independent and dependent variables, and their partial derivatives to form a set Q of rank r monomials by: first constituting a set Q_0 such that

$$\begin{aligned} Q_0 \quad = \quad & \{M_\mu, \mu \in \Lambda \subset \mathbb{N},\, M_\mu = \prod_{j=1}^{m} (u^j)^{\beta_{\mu j}},\ \beta_{\mu j} \in \mathbb{N}, \\ & 0 < |\mathrm{rank} M_\mu| \leq |r| + w_{\max}\}, \end{aligned} \tag{3.6}$$

where $w_{\max} = \max\{|w(u^j)|, |w(\widetilde{u}^l)|, |w(D_{x^i})|, |w(D_t)|\}$. Second, for each M_μ, $\mu \in \Lambda$, build a set $Q_{0\mu}$ such that

$$\begin{aligned} Q_{0\mu} \quad = \quad & \{M_{0\mu},\, M_{0\mu} = t^{\alpha_{\mu 0}} \left(\prod_{i=1}^{n} (x^i)^{\alpha_{\mu i}}\right) \left(\prod_{l=1}^{q} (\widetilde{u}^l)^{\widetilde{\beta}_{\mu l}}\right) \left(\prod_{i=1}^{n} (D_{x^i})^{\eta_{\mu i}}\right) M_\mu, \\ & \mathrm{rank} M_{0\mu} = r,\ \alpha_{\mu 0}, \alpha_{\mu i}, \widetilde{\beta}_{\mu l}, \eta_{\mu i} \in \mathbb{N}\}. \end{aligned} \tag{3.7}$$

Finally, the required set can be written as

$$Q = \bigcup_{\mu \in \Lambda} Q_{0\mu}.$$

Remark 3.1. :

- All monomials defining $M_{0\mu}$ are taken separately in $Q_{0\mu}$ without their coefficients;
- Take $Q_{0\mu} = \emptyset$ when there do not exist positive integers $\alpha_{\mu 0}, \alpha_{\mu i}, \widetilde{\beta}_{\mu l}, \eta_{\mu i}$ such that $rank M_{0\mu} = r$.
- Take $Q_{0\mu} = \{M_\mu\}$ when $rank M_\mu = r$.

(ii) Set Q' the subset of all divergence terms of Q. Then denote by $P = Q \setminus \complement_Q^{Q'}$ the set of non divergence terms.

(iii) Using Theorem 2.12, identify in P all subsets of divergence-equivalent terms and form a set R' of elements of all such subsets, at the rate of one and only one element per subset. Then denote by $R = P \setminus \complement_P^{R'}$ the useful set for the density computation.

(iv) Linearly combine the monomials of R to form a rank r candidate density ρ. By the conservation law (2.3), $D_t\rho = -\mathrm{Div}J$, $D_t\rho$ is a divergence with respect to the space variables. Therefore, the undetermined coefficients of ρ are found by solving the linear algebraic system formed by setting to zero the coefficients of monomials in

$$\frac{\delta}{\delta u^j}\left(D_t\rho|_{u_t = F(t,x,u^{(s)},\widetilde{u})}\right) = 0, \quad j = 1, \ldots, m. \tag{3.8}$$

Provided a density ρ, the corresponding flux J can be computed by using the homotopy operator as follows:

$$J = (J^1, \ldots, J^n) = \mathrm{Div}^{-1}\left(-D_t\,\rho|_{u_t=F}\right). \tag{3.9}$$

Example 3.2. Consider the $(1+1)$-dimensional fifth order KdV equation given by [4]

$$u_t = -\gamma u_{5x} - \beta u_{3x} - \alpha u^p u_x \equiv F(t,x,u^{(5)}). \tag{3.10}$$

The uniformity condition for the rank applied to the function F leads to the weight system

$$\begin{aligned} w(D_t)+w(u) &= 5w(D_x)+w(u) \\ &= 3w(D_x)+w(u) \\ &= w(D_x)+(p+1)w(u) \end{aligned} \tag{3.11}$$

whose solution is $w(u) = w(D_t) = w(D_x) = 0$ indicating that (3.10) does not possess a scaling symmetry group. In order to get round this situation, we substitute the parameter β by a new dependent variable $\widetilde{u}(t,x)$, i.e. $\beta \leftrightarrow \widetilde{u}(t,x)$. Equation (3.10) then becomes

$$u_t = -\gamma u_{5x} - \widetilde{u} u_{3x} - \alpha u^p u_x \equiv F(t,x,u^{(5)},\widetilde{u}), \tag{3.12}$$

with the corresponding weight system

$$\begin{aligned} w(D_t)+w(u) &= 5w(D_x)+w(u) \\ &= 3w(D_x)+w(u)+w(\widetilde{u}) \\ &= w(D_x)+(p+1)w(u) \end{aligned} \tag{3.13}$$

yielding the solution

$$w(D_t) = 5w(D_x),\ w(u) = \frac{4}{p}w(D_x),\ w(\widetilde{u}) = 2w(D_x). \tag{3.14}$$

Setting $w(D_x) = 1$ leads to

$$w(x) = -1, \quad w(D_t) = 5 = -w(t), \quad w(u) = \frac{4}{p}, \quad w(\widetilde{u}) = 2 \tag{3.15}$$

from which we deduce the one parameter dilatation group of equation (3.12) as

$$(t,x,u,\widetilde{u}) \longmapsto \left(\lambda^{-5}t, \lambda^{-1}x, \lambda^{\frac{4}{p}}u, \lambda^2\widetilde{u}\right). \tag{3.16}$$

Let us now compute the relevant quantities.

- **Conserved density ρ_1 of rank $r = 8$ and its associated flux J^1 for (3.12) with $p = 1$**
  We have

  $$w(u) = 4,\ Q_0 = \{u^2, u\},\ Q_{01} = \{u^2\},\ Q_{02} = \{u_{4x}, \widetilde{u}^2 u, \widetilde{u} u_{2x}\}.$$
  $$Q = \{u^2, u_{4x}, \widetilde{u}^2 u, \widetilde{u} u_{2x}\},\ R = P = \{u^2, \widetilde{u}^2 u\}.$$

  From R, we form a candidate density $\rho_1 = c_1 u^2 + c_2 \widetilde{u}^2 u$. Condition (3.8) is satisfied for any constants c_1 and c_2. Seeking $c_1 = c_2 = 1$ gives $\rho_1 = u^2 + \widetilde{u}^2 u$. Let $E_1 = -D_t\rho_1|_{u_t=F(t,x,u^{(5)},\widetilde{u})}$. Applying the one dimensional homotopy operator to the differential function E_1, we obtain the flux

  $$J_1 = 2\gamma(uu_{4x} - u_x u_{3x} + \frac{1}{2}u_{2x}^2) + 2\widetilde{u}(uu_{2x} - \frac{1}{2}u_x^2) + \frac{2}{3}\alpha u^3 + \gamma\widetilde{u}^2 u_{4x}\widetilde{u}^3 u_{2x} + \frac{1}{2}\widetilde{u}^2\alpha u^2.$$

- **Conserved density ρ_2 of rank $r=8$ and its associated flux J^2 for (3.12) with $p=2$**
  We get

  $w(u)=2,\ Q_0=\{u^4,u^3,u^2,u\},\ Q=\{u^4,u^2u_{2x},\widetilde{u}u^3,\widetilde{u}^2u^2,u_{2x}^2,u_{6x},\widetilde{u}^3u,\widetilde{u}^2u_{2x},\widetilde{u}u_{4x}\},$
  $R=P=\{u^4,u^2u_{2x},\widetilde{u}u^3,\widetilde{u}^2u^2,u_{2x}^2,\widetilde{u}^3u\}.$

  From R, a candidate density can be written as

  $$\rho_2=c_1u^4+c_2u^2u_{2x}+c_3\widetilde{u}u^3+c_4\widetilde{u}^2u^2+c_5u_{2x}^2+c_6\widetilde{u}^3u.$$

  Condition (3.8) is satisfied if and only if $c_1=c_2=c_3=c_5=0$ and for any constants c_4 and c_6. Setting $c_4=c_6=1$ gives $\rho_2=\widetilde{u}^2u^2+\widetilde{u}^3u$. Let $E_2=-D_t\rho_2|_{u_t=F(t,x,u^{(5)},\widetilde{u})}$. Applying the one dimensional homotopy operator to the differential function E_2, we obtain the flux

  $$\begin{aligned} J_2 &= 2\widetilde{u}^3(uu_{2x}-\frac{1}{2}u_x^2)+\widetilde{u}^3\gamma u_{4x}+\frac{1}{2}\widetilde{u}^2\alpha u^4\\ &+ 2\widetilde{u}^2\gamma(uu_{4x}-u_xu_{3x}+\frac{1}{2}u_{2x}^2)\frac{1}{3}\widetilde{u}^3\alpha u^3+\widetilde{u}^4u_{2x}. \end{aligned}$$

3.2 Construction of time-space dependent conservation laws

We start this section with the following statement.

Proposition 3.3. *If*

$$\rho=\sum_{\nu=1}^{m}\rho_\nu \quad \text{such that} \quad \sum_{i=1}^{n}D_{x^i}\rho_\nu=F_\nu \tag{3.17}$$

is a conserved density for equation (3.2) whose associated flux is $J=(J^1,\dots,J^n)$, *then*

$$\widetilde{\rho}=t\rho+\frac{1}{n}\left(\sum_{j=1}^{n}x^j\right)\sum_{\nu=1}^{m}u^\nu \tag{3.18}$$

is also a conserved density of equation (3.2) with the associated flux $\widetilde{J}=(\widetilde{J}^1,\dots,\widetilde{J}^n)$, *where*

$$\widetilde{J}^i=tJ^i-\frac{1}{n}\left(\sum_{j=1}^{n}x^j\right)\rho. \tag{3.19}$$

Proof.

$$D_t\widetilde{\rho}=tD_t\rho+\rho+\frac{1}{n}\left(\sum_{j=1}^{n}x^j\right)\sum_{\nu=1}^{m}(u^\nu)_t,\ \ D_{x^i}\widetilde{J}^i=tD_{x^i}J^i-\frac{1}{n}\rho-\frac{1}{n}\left(\sum_{j=1}^{n}x^j\right)\sum_{\nu=1}^{m}D_{x^i}\rho_\nu.$$

Therefore,

$$D_t\widetilde{\rho}+\mathrm{Div}\widetilde{J}=t\,[D_t\rho+\mathrm{Div}J]+\frac{1}{n}\left(\sum_{j=1}^{n}x^j\right)\sum_{\nu=1}^{m}\left[(u^\nu)_t-\sum_{i=1}^{n}D_{x^i}\rho_\nu\right]=0$$

whenever $u_t=F(t,x,u^{(s)},\widetilde{u}^{(s)})$ ■ □

We consider now the systems of the form (3.4). Let $d_{i\nu}$ be the derivation order of the function F_ν with respect to x^i and let d_ν be the order of the function F_ν. The condition (3.17) shows that the order of the function ρ_ν must be $d_\nu - 1$. Thus, setting

$$\begin{aligned}
J_\nu &= \{j \in \{1,\dots,m\},\ D^{\eta}_{x^i} u^j \text{ is in } F_\nu \text{ for some } \eta \in \mathbb{N}\},\\
L_\nu &= \{l \in \{1,\dots,q\},\ \widetilde{u}^l \text{ is in } F_\nu\},\\
R_{i\nu} &= (d_{i\nu}-1)w(D_{x^i}) + \sum_{k\neq i} d_{k\nu} w(D_{x^k}) + \sum_{j\in J_\nu} w(u^j) + \sum_{l\in L_\nu} w(\widetilde{u}^l),\\
R_\nu &= \max\{R_{i\nu},\ i = 1,\dots,n\}, \quad R = \max\{R_\nu,\ \nu = 1,\dots,m\}
\end{aligned}$$

affords the following proposition.

Proposition 3.4. *Let ρ be a conserved density of equation (3.4) satisfying the condition (3.17). Then,*

$$\operatorname{rank}\rho = R, \quad \operatorname{rank}\rho_\nu = R_\nu, \quad \frac{\delta}{\delta u^j}\left(D_t \rho|_{u_t = F(t,x,u^{(s)},\widetilde{u}^{(s)})}\right) = 0. \tag{3.20}$$

Therefore, a basic algorithm for the construction of time-space dependent conservation laws for the equation (3.4) can be established as follows:

Step 1. *Find a scaling symmetry group for the PDEs system (3.4).* See details of computation in the previous subsection.

Step 2. *Construct a rank R conserved density ρ satisfying conditions (3.17) and (3.20)* as follows: First, for each $\nu \in \{1,\dots,m\}$, form a set $Q_{0\nu}$ such that

$$Q_{0\nu} = \{M_{\nu\lambda}, \lambda \in \Lambda \subset \mathbb{N},\ M_{\nu\lambda} = \prod_{j\in J_\nu} (u^j)^{\beta_{\nu\lambda j}},\ \beta_{\nu\lambda j} \in \mathbb{N}, \operatorname{rank} M_\nu \le R_\nu\}.$$

Second, for each $M_{\nu\lambda}$, $\nu \in \{1,\dots,m\}$ and $\lambda \in \Lambda$, form a set $Q_{0\nu\lambda}$ such that

$$\begin{aligned}
Q_{0\nu\lambda} = \{M_{0\nu\lambda},\ M_{0\nu\lambda} = &\left(\prod_{l\in L_\nu} (\widetilde{u}^l)^{\widetilde{\beta}_{\nu\lambda l}}\right)\left(\prod_{i=1}^{n} (D_{x^i})^{\eta_{\nu\lambda i}}\right) M_{\nu\lambda},\\
&\operatorname{rank} M_{0\nu\lambda} \le R_\nu,\ \sum_{i=1}^{n} \eta_{\nu\lambda i} \le d_\nu - 1,\ \widetilde{\beta}_{\nu\lambda l}, \eta_{\nu\lambda i} \in \mathbb{N}\}.
\end{aligned}$$

Remark 3.5. Note that:

(i) $Q_{0\nu\lambda} \neq \emptyset$ since $M_{\nu\lambda} \in Q_{0\nu\lambda}$.

(ii) All monomials that define $M_{0\nu\lambda}$ are separately taken in $Q_{0\nu\lambda}$ without their coefficients.

Third, form a set Q_ν such that

$$Q_\nu = \bigcup_{\lambda\in\Lambda} Q_{0\nu\lambda} = \{M_{0\nu\lambda},\ \operatorname{rank} M_{0\nu\lambda} \le R_\nu,\ \lambda = 1,\dots,q_\nu\} \tag{3.21}$$

and define ρ_ν and a rank R candidate density ρ as

$$\rho_\nu = \sum_{\lambda=1}^{q_\nu} c_{\nu\lambda} M_{0\nu\lambda}, \quad \rho = \sum_{\nu=1}^{m} \rho_\nu. \tag{3.22}$$

Fourth, explicitly determine the values of the undetermined coefficients $c_{\nu\lambda}$ in such a way that both conditions (3.17) and (3.20) are satisfied.

Step 3. *Compute a flux J associated with ρ by applying the homotopy operator to the differential function $-D_t\,\rho|_{u_t=F}$.*

Step 4. *Determine the time-space dependent conservation laws of equation (3.4) using the relations (3.18) and (3.19).*

Example 3.6. Consider the $(1+1)$-dimensional fifth order KdV equation (3.10) with the parameter $p=1$, namely [4]

$$u_t = -\gamma u_{5x} - \beta u_{3x} - \alpha u u_x \equiv F(t,x,u^{(5)}). \tag{3.23}$$

In Example 3.2, we have shown that equation (3.23) does not have a scaling symmetry group, but the substitution $\beta \leftrightarrow \widetilde{u}(t,x)$, where $\widetilde{u}(t,x)$ is a new dependent variable, leads to the following equation

$$u_t = -\gamma u_{5x} - \widetilde{u} u_{3x} - \alpha u u_x \equiv F(t,x,u^{(5)},\widetilde{u}) \tag{3.24}$$

in which the weights of different variables and total derivatives are

$$w(D_t) = 5w(D_x),\ w(u) = 4w(D_x),\ w(\widetilde{u}) = 2w(D_x).$$

Seeking $w(D_x)=1$ gives: $w(x)=-1, \quad w(D_t)=5, \quad w(t)=-5, \quad w(u)=4, \quad w(\widetilde{u})=2$ from which we deduce the corresponding one parameter dilatation group as

$$(t,x,u,\widetilde{u}) \longmapsto \left(\lambda^{-5}t, \lambda^{-1}x, \lambda^4 u, \lambda^2 \widetilde{u}\right)$$

and $d_1=5,\ d_{11}=5,\ R_{11}=10,\ R_1=10,\ R=10.$

- **Computation of a conserved density ρ of rank $R=10$ and its associated flux**
  We have:

$$\begin{aligned}
&Q_{01} = \{u,u^2\},\ Q_{012} = \{u^2, u_x u, u_{2x}u, u_x^2, \widetilde{u}u^2\},\\
&Q_{011} = \{u, u_x, u_{2x}, u_{3x}, u_{4x}, \widetilde{u}u, \widetilde{u}^2 u, \widetilde{u}, \widetilde{u}u_x, \widetilde{u}^2 u_x, \widetilde{u}u_{2x}, \widetilde{u}^2 u_{2x}, \widetilde{u}u_{3x}, \widetilde{u}u_{4x}\},\\
&Q_1 = Q_{011} \cup Q_{012}.
\end{aligned}$$

Determining a candidate density as a linear combination of all elements in the set Q_1 and looking for the undetermined coefficients in such a way that both conditions (3.17) and (3.20) are satisfied, we obtain $\rho = -\gamma u_{4x} - \beta u_{2x} - \frac{\alpha}{2}u^2$.
Applying the one dimensional homotopy operator to the differential function $E = -D_t\rho|_{u_t=F(t,x,u^{(5)},\widetilde{u})}$ yields the flux

$$\begin{aligned}
J &= -\frac{1}{3}\alpha^2 u^3 - 2u_{6x}\beta\gamma - 2u\beta\alpha u_{2x} - u_{8x}\gamma^2 - 2u\gamma\alpha u_{4x} - \frac{1}{2}\beta\alpha u_x^2\\
&- 3u_x\gamma\alpha u_{3x} - \frac{7}{2}u_{2x}^2\gamma\alpha - u_{4x}\beta^2.
\end{aligned}$$

- **Determination of a time-space dependent conservation laws of equation (3.24)**
By using relations (3.18) and (3.19), we obtain the conserved density

$$\widetilde{\rho} = t\left(-\gamma u_{4x} - \beta u_{2x} - \frac{\alpha}{2}u^2\right) + xu$$

with the associated flux

$$\begin{aligned}\widetilde{J} &= t\left(-\frac{1}{3}\alpha^2 u^3 - 2u_{6x}\beta\gamma - 2u\beta\alpha u_{2x} - u_{8x}\gamma^2 - 2u\gamma\alpha u_{4x} - \frac{1}{2}\beta\alpha u_x^2\right.\\ &- \left. 3u_x\gamma\alpha u_{3x} - \frac{7}{2}u_{2x}^2\gamma\alpha - u_{4x}\beta^2\right) - x\left(-\gamma u_{4x} - \beta u_{2x} - \frac{\alpha}{2}u^2\right).\end{aligned}$$

Example 3.7. Consider the modified $(2+1)$-dimensional Zakharov-Kuznetsov (ZK) equation [8]

$$u_t = -\alpha(uu_x + uu_y) - \beta\left[(u_{2x}+u_{2y})_x + (u_{2x}+u_{2y})_y\right] \equiv F(t,x,y,u^{(2)}). \tag{3.25}$$

The uniformity condition applied to the function F leads to the weight system

$$\begin{aligned} w(D_t)+w(u) &= 2w(u)+w(D_x)\\ &= 2w(u)+w(D_y)\\ &= w(u)+3w(D_x)\\ &= w(u)+3w(D_y)\\ &= w(u)+w(D_x)+2w(D_y)\\ &= w(u)+2w(D_x)+w(D_y)\end{aligned} \tag{3.26}$$

whose solution can be written as

$$w(D_t) = 3w(D_x), \quad w(D_y) = w(D_x), \quad w(u) = 2w(D_x).$$

Setting $w(D_x)=1$ provides $w(x)=-1,\ w(D_t)=3,\ w(t)=-3,\ w(D_y)=1,\ w(y)=-1,\ w(u)=2$ from which we deduce the one parameter scaling symmetry group of (3.25) as:

$$(t,x,y,u) \mapsto (\lambda^{-3}t, \lambda^{-1}x, \lambda^{-1}y, \lambda^2 u).$$

We get: $d_1=3,\ d_{11}=3,\ d_{21}=3,\ R_{11}=7,\ R_{21}=7,\ R_1=7,\ R=7.$

- **Computation of a conserved density ρ of rank $R=7$ and its associated flux**
We have:

$Q_{01}=\{u,u^2,u^3\},\ Q_{011}=\{u,u_x,u_y,u_{2x},u_{2y},u_{xy}\},\ Q_{013}=\{u^3\},$
$Q_{012}=\{u^2,uu_x,u_x^2,u_y^2,uu_y,uu_{2x},u_xu_y,uu_{2y},uu_{xy}\},\ Q_1=Q_{011}\cup Q_{012}\cup Q_{013}.$

Constructing a candidate density as a linear combination of all elements in the set Q_1 and looking for the undetermined coefficients in such a way that both conditions (3.17) and (3.20) are satisfied, we find $\rho = -\frac{1}{2}\alpha u^2 - \beta u_{2x} - \beta u_{2y}$.

Applying the two dimensional homotopy operator to the differential function $E = -D_t\rho|_{u_t=F(t,x,y,u^{(2)})}$ we obtain the flux $J = (J^1, J^2)$, where

$$\begin{aligned}
J^1 &= -\frac{1}{3}u^3\alpha^2 - \frac{1}{2}u_x^2\beta\alpha - \frac{4}{3}u\beta\alpha u_{xy} - \frac{4}{5}u_{x3y}\beta^2 - \frac{2}{3}u\beta\alpha u_{2y} - \frac{6}{5}u_{2x2y}\beta^2 \\
&- 2u\beta\alpha u_{2x} - \frac{4}{5}u_{3xy}\beta^2 - \frac{1}{3}u_y\beta\alpha u_x - u_{4x}\beta^2 - \frac{1}{6}u_y^2\beta\alpha - \frac{1}{5}u_{4y}\beta^2, \\
J^2 &= -\frac{1}{3}u^3\alpha^2 - \frac{1}{2}u_y^2\beta\alpha - \frac{2}{3}u\beta\alpha u_{2x} - \frac{4}{5}u_{3xy}\beta^2 - \frac{4}{3}u\beta\alpha u_{xy} - \frac{6}{5}u_{2x2y}\beta^2 \\
&- 2u\beta\alpha u_{2y} - \frac{4}{5}u_{x3y}\beta^2 - \frac{1}{6}u_x^2\beta\alpha - u_{4y}\beta^2 - \frac{1}{3}u_y\beta\alpha u_x - \frac{1}{5}u_{4x}\beta^2.
\end{aligned}$$

- **Determination of a time-space dependent conservation laws of equation (3.25)** By using relations (3.18) and (3.19), we compute the conserved density

$$\widetilde{\rho} = t\left(-\frac{1}{2}\alpha u^2 - \beta u_{2x} - \beta u_{2y}\right) + \frac{1}{2}(x+y)u$$

with the associated flux $\widetilde{J} = (\widetilde{J}^1, \widetilde{J}^2)$, where

$$\begin{aligned}
\widetilde{J}^1 &= t\left(-\frac{1}{3}u^3\alpha^2 - \frac{1}{2}u_x^2\beta\alpha - \frac{4}{3}u\beta\alpha u_{xy} - \frac{4}{5}u_{x3y}\beta^2 - \frac{2}{3}u\beta\alpha u_{2y} - \frac{6}{5}u_{2x2y}\beta^2\right. \\
&- \left. 2u\beta\alpha u_{2x} - \frac{4}{5}u_{3xy}\beta^2 - \frac{1}{3}u_y\beta\alpha u_x - u_{4x}\beta^2 - \frac{1}{6}u_y^2\beta\alpha - \frac{1}{5}u_{4y}\beta^2\right) \\
&- \frac{1}{2}(x+y)\left(-\frac{1}{2}\alpha u^2 - \beta u_{2x} - \beta u_{2y}\right), \\
\widetilde{J}^2 &= t\left(-\frac{1}{3}u^3\alpha^2 - \frac{1}{2}u_y^2\beta\alpha - \frac{2}{3}u\beta\alpha u_{2x} - \frac{4}{5}u_{3xy}\beta^2 - \frac{4}{3}u\beta\alpha u_{xy} - \frac{6}{5}u_{2x2y}\beta^2\right. \\
&- \left. 2u\beta\alpha u_{2y} - \frac{4}{5}u_{x3y}\beta^2 - \frac{1}{6}u_x^2\beta\alpha - u_{4y}\beta^2 - \frac{1}{3}u_y\beta\alpha u_x - \frac{1}{5}u_{4x}\beta^2\right) \\
&- \frac{1}{2}(x+y)\left(-\frac{1}{2}\alpha u^2 - \beta u_{2x} - \beta u_{2y}\right).
\end{aligned}$$

Example 3.8. Consider the modified $(3+1)$-dimensional Zakharov-Kuznetsov (ZK) equation [8]

$$\begin{aligned}
u_t &= -\alpha(uu_x + uu_y + uu_z) - \beta\left[(u_{2x} + u_{2y} + u_{2z})_x + (u_{2x} + u_{2y} + u_{2z})_y\right. \\
&+ \left.(u_{2x} + u_{2y} + u_{2z})_z\right] \equiv F(t,x,y,z,u^{(2)}).
\end{aligned} \tag{3.27}$$

The uniformity condition applying to the function F leads to the weight system

$$\begin{aligned}
&w(D_t) + w(u) = 2w(u) + w(D_x), && w(D_t) + w(u) = w(u) + w(D_x) + 2w(D_y) \\
&w(D_t) + w(u) = 2w(u) + w(D_z), && w(D_t) + w(u) = w(u) + w(D_y) + 2w(D_x) \\
&w(D_t) + w(u) = w(u) + 3w(D_y), && w(D_t) + w(u) = w(u) + w(D_z) + 2w(D_x) \\
&w(D_t) + w(u) = 2w(u) + w(D_y), && w(D_t) + w(u) = w(u) + w(D_x) + 2w(D_z) \\
&w(D_t) + w(u) = w(u) + 3w(D_x), && w(D_t) + w(u) = w(u) + w(D_y) + 2w(D_z) \\
&w(D_t) + w(u) = w(u) + 3w(D_z), && w(D_t) + w(u) = w(u) + w(D_z) + 2w(D_y)
\end{aligned}$$

whose solution is

$$w(D_t) = 3w(D_x), \quad w(D_y) = w(D_x), \quad w(D_z) = w(D_x), \quad w(u) = 2w(D_x).$$

Setting $w(D_x) = 1$ gives $w(D_t) = 3$, $w(D_y) = w(D_z) = 1$, $w(u) = 2$, $w(x) = w(y) = w(z) = -1$, $w(t) = -3$ from which we deduce the associated one parameter scaling symmetry group as:

$$(t,x,y,z,u) \mapsto (\lambda^{-3}t, \lambda^{-1}x, \lambda^{-1}y, \lambda^{-1}z, \lambda^2 u).$$

We obtain: $d_1 = 3$, $d_{11} = d_{21} = d_{31} = 3$, $R_{11} = R_{21} = R_{31} = 10$, $R_1 = 10$, $R = 10$.

- **Computation of a conserved density ρ of rank $R = 10$ and its associated flux**
  We have:

$$\begin{aligned}
Q_{01} &= \{u, u^2, u^3, u^4, u^5\}, \quad Q_{015} = \{u^5\},\\
Q_{011} &= \{u, u_x, u_y, u_{2x}, u_{2y}, u_{xy}, u_z, u_{2z}, u_{xz}, u_{yz}\},\\
Q_{012} &= \{u^2, uu_x, uu_y, uu_z, u_x^2, u_y^2, u_z^2, uu_{2x}, uu_{2y}, uu_{2z},\\
&\quad u_xu_y, uu_{xy}, uu_{xz}, uu_{yz}, u_xu_z, u_yu_z\},\\
Q_{013} &= \{u^3, u^2u_x, u^2u_y, u^2u_z, uu_x^2, uu_y^2, uu_z^2, u^2u_{2x}, u^2u_{2y}, u^2u_{2z},\\
&\quad uu_xu_y, u^2u_{xy}, u^2u_{xz}, u^2u_{yz}, uu_xu_z, uu_yu_z\},\\
Q_{014} &= \{u^4, u^3u_x, u^3u_y, u^3u_z, u^2u_x^2, u^2u_y^2, u^2u_z^2, u^3u_{2x}, u^3u_{2y}, u^3u_{2z},\\
&\quad u^2u_xu_y, u^3u_{xy}, u^3u_{xz}, u^3u_{yz}, u^2u_xu_z, u^2u_yu_z\},\\
Q_1 &= Q_{011} \cup Q_{012} \cup Q_{013} \cup Q_{014} \cup Q_{015}.
\end{aligned}$$

  Forming a candidate density as a linear combination of all elements in the set Q_1 and looking for the undetermined coefficients in such a way that both conditions (3.17) and (3.20) are satisfied, we obtain

$$\rho = -\frac{1}{2}\alpha u^2 - \beta u_{2x} - \beta u_{2y} - \beta u_{2z}.$$

  Applying the three dimensional homotopy operator to the differential function $E = -D_t\rho|_{u_t=F(t,x,y,z,u^{(2)})}$, with the help of the computer algebraic system Maple, yields the flux $J=(J^1,J^2,J^3)$, where

$$\begin{aligned}
J^1 &= -\frac{4}{5}u_{x2yz}\beta^2-\frac{4}{3}u\beta\alpha u_{xy}-\frac{2}{5}u_{2y2z}\beta^2-\frac{1}{3}u_y\beta\alpha u_x-u_{4x}\beta^2-\frac{4}{3}u\beta\alpha u_{xz}\\
&- \frac{6}{5}u_{2x2z}\beta^2-2u\beta\alpha u_{2x}-\frac{1}{5}u_{4z}\beta^2-\frac{2}{3}u\beta\alpha u_{2y}-\frac{4}{5}u_{x3z}\beta^2-\frac{2}{3}u\beta\alpha u_{2z}\\
&- \frac{1}{3}\alpha^2u^3-\frac{6}{5}u_{2x2y}\beta^2-\frac{4}{5}u_{3xz}\beta^2-\frac{4}{5}u_{xy2z}\beta^2-\frac{1}{5}u_{4y}\beta^2-\frac{4}{5}u_{x3y}\beta^2\\
&- \frac{4}{5}u_{3xy}\beta^2-\frac{1}{3}u_z\beta\alpha u_x-\frac{1}{6}u_z^2\beta\alpha-\frac{1}{2}u_x^2\beta\alpha-\frac{1}{6}u_y^2\beta\alpha,\\
J^2 &= -\frac{1}{2}u_y^2\beta\alpha-\frac{4}{5}u_{3yz}\beta^2-\frac{1}{6}u_z^2\beta\alpha-\frac{4}{5}u_{y3z}\beta^2-\frac{1}{6}u_x^2\beta\alpha-\frac{4}{5}u_{2xyz}\beta^2\\
&- \frac{4}{3}u\beta\alpha u_{xy}-\frac{4}{5}u_{xy2z}\beta^2-\frac{1}{5}u_{4z}\beta^2-\frac{4}{5}u_{x3y}\beta^2-u_{4y}\beta^2-\frac{2}{5}u_{2x2z}\beta^2\\
&- \frac{6}{5}u_{2x2y}\beta^2-\frac{1}{3}u_y\beta\alpha u_x-\frac{1}{5}u_{4x}\beta^2-2u\beta\alpha u_{2y}-\frac{6}{5}u_{2y2z}\beta^2-\frac{1}{3}u_z\beta\alpha u_y\\
&- \frac{1}{3}\alpha^2u^3-\frac{4}{3}u\beta\alpha u_{yz}-\frac{2}{3}u\beta\alpha u_{2x}-\frac{2}{3}u\beta\alpha u_{2z}-\frac{4}{5}u_{3xy}\beta^2,\\
J^3 &= -\frac{1}{5}u_{4y}\beta^2-\frac{4}{5}u_{y3z}\beta^2-\frac{2}{5}u_{2x2y}\beta^2-\frac{4}{5}u_{3yz}\beta^2-\frac{6}{5}u_{2x2z}\beta^2-\frac{6}{5}u_{2y2z}\beta^2\\
&- \frac{4}{5}u_{3xz}\beta^2-\frac{1}{5}u_{4x}\beta^2-\frac{4}{5}u_{x3z}\beta^2-\frac{1}{2}u_z^2\beta\alpha-\frac{4}{5}u_{x2yz}\beta^2-\frac{1}{6}u_y^2\beta\alpha\\
&- \frac{4}{5}u_{2xyz}\beta^2-\frac{1}{6}u_x^2\beta\alpha-\frac{4}{3}u\beta\alpha u_{yz}-\frac{1}{3}\alpha^2u^3-\frac{1}{3}u_z\beta\alpha u_x-\frac{1}{3}u_z\beta\alpha u_y\\
&- u_{4z}\beta^2-\frac{2}{3}u\beta\alpha u_{2y}-2u\beta\alpha u_{2z}-\frac{4}{3}u\beta\alpha u_{xz}-\frac{2}{3}u\beta\alpha u_{2x}.
\end{aligned}$$

- **Determination of a time-space dependent conservation laws of equation (3.25)**
  By using relations (3.18) and (3.19), we compute the conserved density

$$\widetilde{\rho}=t\left(-\frac{1}{2}\alpha u^2-\beta u_{2x}-\beta u_{2y}-\beta u_{2z}\right)+\frac{1}{3}(x+y+z)u$$

with the associated flux $\widetilde{J}=(\widetilde{J^1},\widetilde{J^2},\widetilde{J^3})$, where

$$\begin{aligned}
\widetilde{J^1} &= t\left(-\frac{4}{5}u_{x2yz}\beta^2-\frac{4}{3}u\beta\alpha u_{xy}-\frac{2}{5}u_{2y2z}\beta^2-\frac{1}{3}u_y\beta\alpha u_x-u_{4x}\beta^2-\frac{4}{3}u\beta\alpha u_{xz}\right.\\
&- \frac{6}{5}u_{2x2z}\beta^2-2u\beta\alpha u_{2x}-\frac{1}{5}u_{4z}\beta^2-\frac{2}{3}u\beta\alpha u_{2y}-\frac{4}{5}u_{x3z}\beta^2-\frac{2}{3}u\beta\alpha u_{2z}\\
&- \frac{1}{3}\alpha^2u^3-\frac{6}{5}u_{2x2y}\beta^2-\frac{4}{5}u_{3xz}\beta^2-\frac{4}{5}u_{xy2z}\beta^2-\frac{1}{5}u_{4y}\beta^2-\frac{4}{5}u_{x3y}\beta^2\\
&- \left.\frac{4}{5}u_{3xy}\beta^2-\frac{1}{3}u_z\beta\alpha u_x-\frac{1}{6}u_z^2\beta\alpha-\frac{1}{2}u_x^2\beta\alpha-\frac{1}{6}u_y^2\beta\alpha\right)\\
&- \frac{1}{3}(x+y+z)\left(-\frac{1}{2}\alpha u^2-\beta u_{2x}-\beta u_{2y}-\beta u_{2z}\right),
\end{aligned} \tag{3.28}$$

$$
\begin{aligned}
\widetilde{J}^2 &= t\left(-\frac{1}{2}u_y^2\beta\alpha - \frac{4}{5}u_{3yz}\beta^2 - \frac{1}{6}u_z^2\beta\alpha - \frac{4}{5}u_{y3z}\beta^2 - \frac{1}{6}u_x^2\beta\alpha - \frac{4}{5}u_{2xyz}\beta^2\right. \\
&- \frac{4}{3}u\beta\alpha u_{xy} - \frac{4}{5}u_{xy2z}\beta^2 - \frac{1}{5}u_{4z}\beta^2 - \frac{4}{5}u_{x3y}\beta^2 - u_{4y}\beta^2 - \frac{2}{5}u_{2x2z}\beta^2 \\
&- \frac{6}{5}u_{2x2y}\beta^2 - \frac{1}{3}u_y\beta\alpha u_x - \frac{1}{5}u_{4x}\beta^2 - 2u\beta\alpha u_{2y} - \frac{6}{5}u_{2y2z}\beta^2 - \frac{1}{3}u_z\beta\alpha u_y \\
&- \left.\frac{1}{3}\alpha^2u^3 - \frac{4}{3}u\beta\alpha u_{yz} - \frac{2}{3}u\beta\alpha u_{2x} - \frac{2}{3}u\beta\alpha u_{2z} - \frac{4}{5}u_{3xy}\beta^2\right) \\
&- \frac{1}{3}(x+y+z)\left(-\frac{1}{2}\alpha u^2 - \beta u_{2x} - \beta u_{2y} - \beta u_{2z}\right), \\
\widetilde{J}^3 &= t\left(-\frac{1}{5}u_{4y}\beta^2 - \frac{4}{5}u_{y3z}\beta^2 - \frac{2}{5}u_{2x2y}\beta^2 - \frac{4}{5}u_{3yz}\beta^2 - \frac{6}{5}u_{2x2z}\beta^2 - \frac{6}{5}u_{2y2z}\beta^2\right. \\
&- \frac{4}{5}u_{3xz}\beta^2 - \frac{1}{5}u_{4x}\beta^2 - \frac{4}{5}u_{x3z}\beta^2 - \frac{1}{2}u_z^2\beta\alpha - \frac{4}{5}u_{x2yz}\beta^2 - \frac{1}{6}u_y^2\beta\alpha \\
&- \frac{4}{5}u_{2xyz}\beta^2 - \frac{1}{6}u_x^2\beta\alpha - \frac{4}{3}u\beta\alpha u_{yz} - \frac{1}{3}\alpha^2u^3 - \frac{1}{3}u_z\beta\alpha u_x - \frac{1}{3}u_z\beta\alpha u_y \\
&- \left.u_{4z}\beta^2 - \frac{2}{3}u\beta\alpha u_{2y} - 2u\beta\alpha u_{2z} - \frac{4}{3}u\beta\alpha u_{xz} - \frac{2}{3}u\beta\alpha u_{2x}\right) \\
&- \frac{1}{3}(x+y+z)\left(-\frac{1}{2}\alpha u^2 - \beta u_{2x} - \beta u_{2y} - \beta u_{2z}\right).
\end{aligned}
$$

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