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GLOBAL WEAK SOLUTIONS TO THE 1D COMPRESSIBLE NAVIER-STOKES EQUATIONS WITH RADIATION

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Abstract

We consider an initial-boundary value problem for the equations of one-dimensional motions of a compressible viscous heat-conducting gas coupled with radiation through a radiative transfer equation. Assuming suitable hypotheses on the transport coefficients, we prove that the problem admits a unique weak solution.

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1 Introduction

The purpose of radiation hydrodynamics is to include the effects of radiation into the hydrodynamical framework. When the equilibrium holds between the matter and the radiation, a simple way to do that is to include local radiative terms into the state functions and the transport coefficients. On the other hand radiation is described by its quanta, the photons,

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which are massless particles traveling at the speed c of light, characterized by their frequency $\nu \in \mathbb{R}_+$, their energy $E = h\nu$ (where h is the Planck's constant), their momentum $\vec{p} = \frac{\hbar\nu}{c}\,\vec{\Omega}$, where $\vec{\Omega} \in S^2$ is a vector of the 2-unit sphere. Statistical mechanics allows us to describe macroscopically an assembly of massless photons of energy E and momentum $\vec{p}$ by using a distribution function: the radiative intensity $I(x,t,\vec{\Omega},\nu)$. Using this intensity, one can derive global quantities by integrating with respect to the angular and frequency variables: the spectral radiative energy density $E_R(x,t)$ per unit volume is then $E_R(x,t) := \frac{1}{c}\int\int I(x,t,\vec{\Omega},\nu)\, d\Omega\, d\nu$, and the spectral radiative flux $\vec{F}_R(x,t) = \int\int \vec{\Omega}\, I(x,t,\vec{\Omega},\nu)\, d\Omega\, d\nu$.

In the absence of radiation the hydrodynamical system is derived from the fundamental conservation laws (mass, momentum and energy) by using the Boltzmann's equation satisfied by the $f_m(x,\vec{v},t)$ and Chapman-Enskog's expansion [12]. One gets then formally the compressible Navier-Stokes system for matter. When radiation is taken into account at a macroscopic level, supplementary source terms appear, coupling matter variables to radiative intensity I, which is supposed to satisfy a transport equation: the so called radiative transfert equation, an integro-differential equation early discussed by Chandrasekhar in [5].

Supposing that the matter is at local thermodynamical equilibrium (LTE) and in the non-relativistic framework (the velocity of matter is less than the velocity of light: $\vec{u}^2 < c^2$), the coupled system satisfied by the density ρ, the velocity $\vec{u}$, the temperature θ and the radiative intensity I in $\mathbb{R}^3$ reads [26] [28] [4]

$$
\begin{cases}
\partial_t \rho + \mathrm{div}_x(\rho\vec{u}) = 0, \\
\partial_t(\rho\vec{u}) + \mathrm{div}_x(\rho\vec{u}\otimes\vec{u}) + \nabla_x p = \mathrm{div}_x \overset{\Rightarrow}{S} - \vec{S}_F, \\
\partial_t(\rho E) + \mathrm{div}_x((\rho E + p)\vec{u}) + \mathrm{div}_x(\vec{q} - \overset{\Rightarrow}{S}\,\vec{u}) = -S_E, \\
\dfrac{1}{c}\dfrac{\partial}{\partial t} I + \vec{\Omega}\cdot\nabla_x I = S,
\end{cases}
\tag{1.1}
$$

where $E = \frac{1}{2}\,\vec{v}^2 + e$ is the total energy withe e the internal energy, $\overset{\Rightarrow}{S}$ the viscous stress tensor (we treat the matter as a newtonian fluid with viscous contribution), p the pressure, $\vec{q}$ the thermal heat flux and $\vec{S}_F$ and S_E are the radiative force and energy source terms, described below.

Some remarks are in order concerning the relativistic characteristics of the transfer equation (we refer to the paper by Buet and Desprès [4] for a detailed analysis).

In fact this transport equation is written, as the hydrodynamical part, in Eulerian variables (also called laboratory frame in physics books [26]). However scattering and absorption-emission processes are best described in Lagrangian frame (also called comoving frame), and we need to use transformation formulas from special relativity to pass from a frame to the other. We will affect an index 0 to a given quantity in the comoving frame and no index for the same quantity in the laboratory frame.

Now we can describe the various coupling terms in the right-hand sides of (1.1) (see[25]).

In the radiative source splitted in two parts $S = S_{a,e} + S_s$ the first contribution

$$
S_{a,e}(x,t,\vec{\Omega},\nu) = -\frac{\nu_0}{\nu}\,\sigma_a I\left(x,t,\vec{\Omega},\nu\right) + \left(\frac{\nu_0}{\nu}\right)^2\,\sigma_a B\left(x,t,\vec{\Omega},\nu\right),
$$

is the absorption-emission contribution, where variables appear in both frames

$$\nu_0 = \gamma_L \left(1 - \frac{\vec{\Omega}\cdot\vec{u}}{c}\right)\nu \ \text{ and } \ \vec{\Omega}_0 = \frac{\nu}{\nu_0}\left[\vec{\Omega} - \frac{\gamma_L \vec{u}}{c}\left(1 - \frac{\vec{\Omega}\cdot\vec{u}}{c}\left(\frac{\gamma_L}{1+\gamma_L}\right)\right)\right],$$

with the Lorentz factor $\gamma_L = \left(1 - \frac{\vec{u}^2}{c^2}\right)^{-1/2}$, and the second one

$$S_s(x,t,\vec{\Omega},\nu) = -\frac{\nu_0}{\nu}\,\sigma_s I\left(x,t,\vec{\Omega},\nu\right) + \frac{\sigma_s}{\pi}\int \frac{\nu_0}{\nu'}\, I\left(x,t,\vec{\Omega}',\nu'\right)\, d\vec{\Omega}',$$

is the scattering contribution with $\nu' = \frac{1-\frac{\vec{\Omega}\cdot\vec{u}}{c}}{1-\frac{\vec{\Omega}'\cdot\vec{u}}{c}}\,\nu$.

In the $(\vec{\Omega},\nu)$-space we will use the invariance of the measure $\nu\, d\nu\, d\vec{\Omega}$ i.e.

$$\nu\, d\nu\, d\vec{\Omega} = \nu_0\, d\nu_0\, d\vec{\Omega}_0, \tag{1.2}$$

and the following formulas for quantities in the laboratory frame

$$I = \left(\frac{\nu_0}{\nu}\right)^3 I_0,\ \ B = \left(\frac{\nu_0}{\nu}\right)^3 B_0,\ \ S_{a,e} = \left(\frac{\nu_0}{\nu}\right)^2 (S_{a,e})_0,\ \ S_s = \left(\frac{\nu_0}{\nu}\right)^2 (S_s)_0. \tag{1.3}$$

The radiative energy is

$$S_E(r,t) := \int_{\mathbb{R}_+}\int_{S^2} S(x,t,\vec{\Omega},\nu)\, d\vec{\Omega}\, d\nu,$$

The radiative flux is

$$\vec{S}_F(x,t) := \frac{1}{c}\int_{\mathbb{R}_+}\int_{S^2} \vec{\Omega}\, S(x,t,\vec{\Omega},\nu)\, d\vec{\Omega}\, d\nu.$$

In the radiative transfer equation (the last equation (1.1)) the functions σ_a and σ_s appearing in the radiative source S describe in a phenomenological way the absorption-emission and scattering properties (frequency and angular transitions) of the interaction photon-matter and the function B describes the thermodynamical equilibrium distribution.

Let us note that the foundations of the previous system have been described by Pomraning [28] and Mihalas and Weibel-Mihalas [26] in the full framework of special relativity (oversimplified in the previous considerations). The coupled system (1.1) has been recently investigated (in the inviscid case) by Lowrie, Morel, Hittinger [25], Buet, Després [4] with a special attention to asymptotic regimes, and by Dubroca-Feugeas [8], Lin [23] and Lin-Coulombel-Goudon [24] for numerical aspects. Concerning the existence of solutions, a proof of local-in-time existence and blow-up of solutions (in the inviscid case) has been proposed by Zhong and Jiang [31] (see also the recent papers by Jiang and Wang [18] [19] for a 1D related "Euler -Boltzmann" model). Moreover, a simplified version of the system has been investigated by Golse and Perthame [13].

At present time, the multidimensional situation described by the system (1.1) is far from been understood even at the formal level (however see [10] for a simplified treatment of radiation) and we restrict our study to the monodimensional geometry. The fluid part of system (1.1) reads

$$\begin{cases} \rho_\tau + (\rho v)_y = 0, \\ (\rho v)_\tau + (\rho v^2)_y + p_y = (\mu v_y)_y - (S_F)_R, \\ \left[\rho\left(e + \frac{1}{2}\, v^2\right)\right]_\tau + \left[\rho v\left(e + \frac{1}{2}\, v^2\right) + pv - \kappa\theta_y - \mu v v_y\right]_y = -(S_E)_R, \end{cases} \tag{1.4}$$

and the transfer equation is

$$\frac{1}{c} I_\tau + \omega I_y = S, \tag{1.5}$$

in the slab $(0,L)\times\mathbb{R}_+$ with $L>0$, where the density ρ, the velocity v (with $|v|<c$), the temperature θ depend on the coordinates (y,τ). The radiative intensity $I = I(y,\tau,\nu,\omega)$, depends also on two extra variables: the radiation frequency $\nu\in\mathbb{R}_+$ and the angular variable $\omega\in S^1 := [-1,1]$ (let us stress that here S^1 is *not* the unit circle). The state functions are the pressure p, the internal energy e, the heat conductivity κ and the viscosity coefficient μ .

The absorption-emission term is

$$S_{a,e}(y,\tau,\nu,\omega) = \frac{\nu_0}{\nu}\,\sigma_a(\nu_0,\rho,\theta)\left[\left(\frac{\nu}{\nu_0}\right)^3 B(\nu_0,\theta) - I(y,\tau,\nu,\omega)\right], \tag{1.6}$$

and the scattering term is

$$S_s(y,\tau,\nu,\omega) = \frac{\nu_0}{\nu}\,\sigma_s(\nu_0,\rho,\theta)\,\left[\left(\frac{\nu}{\nu_0}\right)^3 \tilde{I}(y,\tau,\nu_0,\theta) - I(y,\tau,\nu,\omega)\right], \tag{1.7}$$

where $\tilde{I}(y,\tau,\nu) := \frac{1}{2}\int_{-1}^{1}\frac{\nu}{\nu'}\, I(y,\tau,\nu',\omega)\, d\omega$.

In the following we assume that the fluid moves at a low speed so we consider a one order model, neglecting terms of order greater than 1. Then we take $\gamma_L = 0$ for the Lorentz factor and in the previous formulas

$$\nu_0 = \left(1 - \frac{\omega v}{c}\right)\nu,\ \ \nu' = \left(1 - \frac{(\omega-\omega')v}{c}\right)\,\nu,\ \text{ and }\ \omega_0 = \omega - (1-\omega^2)\frac{v}{c}. \tag{1.8}$$

The function $B(\nu_0,\theta)$ depending on the temperature and the frequency, describing the equilibrium state.

Typically, taking

$$B(\nu_0,\theta) = 2h\nu_0^3 c^{-2}\,\left(e^{\frac{h\nu_0}{k_B}} - 1\right)^{-1}, \tag{1.9}$$

where k_B is the Boltzmann's constant and h is the Planck's constant, corresponds to the Planck's equilibrium distribution of photons in a cavity at temperature θ (black body).

The coefficients σ_a and σ_s are positive but their evaluation is a difficult problem of quantum mechanics and their general form is not known (an expression of σ_a used for stars of moderate stars is given by the Kramers formula $\sigma_a(\nu_0,\theta) = \frac{C(\theta)}{\nu_0^3}\,\left(1 - e^{-\frac{h\nu_0}{k_B\theta}}\right)$, where C is a positive function).

In the following we will also assume a density dependence and growth hypotheses of σ_a and σ_s in function of temperature and density.

Defining the radiative energy

$$E_R := \frac{1}{c}\int_{-1}^{1}\int_{0}^{\infty} I(y,\tau,\nu,\omega)\, d\nu\, d\omega, \tag{1.10}$$

the radiative flux

$$F_R := \int_{-1}^{1}\int_{0}^{\infty} \omega I(y,\tau,\nu,\omega)\, d\nu\, d\omega, \tag{1.11}$$

and the radiative pressure

$$P_R := \frac{1}{c}\int_{-1}^{1}\int_{0}^{\infty} \omega^2 I(y,\tau,\nu,\omega)\, d\nu\, d\omega, \tag{1.12}$$

one defines in turn the radiative energy source

$$(S_E)_R := \int_{-1}^{1}\int_{0}^{\infty} S(y,\tau,\nu,\omega)\, d\nu\, d\omega, \tag{1.13}$$

and the radiative force

$$(S_F)_R := \frac{1}{c}\int_{-1}^{1}\int_{0}^{\infty} \omega S(y,\tau,\nu,\omega)\, d\nu\, d\omega. \tag{1.14}$$

Taking benefit of the one-dimensional geometry, it is now convenient to switch now to Lagrange (mass) coordinates relative to matter flow: $(y,\tau) \to (x,t)$ with the transformation rules [3]: $\partial_y \to \rho\partial_x$ and $\partial_\tau + v\partial_y \to \partial_t$.

The fluid part of the previous system reads now

$$\begin{cases} \eta_t = v_x, \\ v_t = \sigma_x - \eta(S_F)_R, \\ \left(e + \dfrac{1}{2}\, v^2\right)_t = (\sigma v - q)_x - \eta(S_E)_R, \end{cases} \tag{1.15}$$

and the transfer equation is

$$I_t + \eta^{-1}(c\omega_0 - v)I_x = cS_0, \tag{1.16}$$

in the transformed domain $Q := \Omega \times \mathbf{R}^+$ with $\Omega := (0,M)$ (M is the total mass of matter in the slab $(0,L)$), where the specific volume η (with $\eta := \frac{1}{\rho}$), the velocity v (with $|v| < c$), the temperature θ and the radiative intensity I depend now on the Lagrangian mass coordinates (x,t). We also denote by $\sigma := -p + \mu\, \frac{v_x}{\eta}$ the stress and by $q := -\kappa\, \frac{\theta_x}{\eta}$ the heat flux.

We denote by $O = (0,\infty) \times S^1$ the (ν,ω)-space, with $O_+ = (0,\infty) \times (0,1)$ and $O_- = (0,\infty) \times (-1,0)$, and by O_0 the corresponding (ν_0,ω_0)-space for the Lagrangian (comoving) variables.

Taking into account transformations formulas (1.3), the source term in the right-hand side of (1.16) takes the simple form

$$(S(x,t,\nu_0,\omega_0))_0 = \sigma_a(\nu_0;\eta,\theta)\,[B(\nu_0;\theta) - I(x,t;\nu_0,\omega_0)]$$
$$+\sigma_s(\nu_0;\eta,\theta)\,\left[\frac{1}{2}\int_{-1}^{1} I(x,t,\nu_0,\omega')\,d\omega' - I(x,t,\nu,\omega)\right]. \tag{1.17}$$

In the last term we used the following formula adapted from Bruet and Després (see Appendix A in [4])

$$\frac{\nu_0}{\nu}\,\sigma_s(\nu_0,\rho,\theta)\,\left[\left(\frac{\nu}{\nu_0}\right)^3 \frac{1}{2}\int_{-1}^{1} \frac{\nu}{\nu'}\, I(x,t,\nu',\omega')\,d\omega' - I(x,t,\nu,\omega)\right]$$
$$= \sigma_s(\nu_0,\rho,\theta)\,\left[\frac{1}{2}\int_{-1}^{1} I_0(x,t,\nu_0,\omega')\,d\omega' - I_0(x,t,\nu_0,\omega_0)\right].$$

Moreover as σ_s is frequency-independent, using formulas (1.2) and (1.3), the following relation holds, characterizing the isotropy of the scattering in the comoving frame

$$\int_O \frac{\nu_0}{\nu}\, S_s(\nu,\omega)\,d\omega\,d\nu = \int_{O_0} (S_s)_0(\nu_0,\omega_0)\,d\omega_0\,d\nu_0 = 0. \tag{1.18}$$

From the last equation (1.15) and the definitions (1.10)-(1.14), one derives the equations

$$(\eta I)_t + ((c\omega - v)I)_x = c\eta S_0. \tag{1.19}$$

and after integrating in frequency and angular variables

$$\begin{cases} (\eta E_R)_t + (F_R - vE_R)_x = \eta\,(S_E)_R\,, \\ \\ (\eta F_R)_t + (P_R - vF_R)_x = \eta\,(S_F)_R\,. \end{cases} \tag{1.20}$$

From now on, we make the nonrelativistic approximation $O_0 = O$ and, unless we specify the contrary, we suppress the index 0 in the comoving quantities.

We consider Dirichlet-Neumann boundary conditions for the fluid unknowns

$$\begin{cases} v|_{x=0} = v|_{x=M} = 0, \\ \\ q|_{x=0} = q|_{x=M} = 0, \end{cases} \tag{1.21}$$

and transparent boundary conditions for the radiative intensity (see [7])

$$\begin{cases} I|_{x=0} = 0 \quad \text{for } \omega \in (0,1) \\ \\ I|_{x=M} = 0 \quad \text{for } \omega \in (-1,0), \end{cases} \tag{1.22}$$

for $t > 0$, and initial conditions

$$\eta|_{t=0} = \eta^0(x), \quad v|_{t=0} = v^0(x), \quad \theta|_{t=0} = \theta^0(x), \ \text{ on } \Omega. \tag{1.23}$$

and

$$I|_{t=0} = I^0(x, \nu_0, \omega_0) \ \text{ on } \ \Omega \times O. \tag{1.24}$$

Pressure and energy of the matter are related by the thermodynamical relation

$$e_\eta(\eta,\theta) = -p(\eta,\theta) + \theta p_\theta(\eta,\theta). \tag{1.25}$$

Finally we assume that state functions e, p and κ (resp. σ_a and σ_s) are C^2 (resp C^0) functions of their arguments for $0 < \eta < \infty$ and $0 \leq \theta < \infty$, and, for any $\underline{\eta} \geq 0$ we suppose the following growth conditions for $\eta \geq \underline{\eta}$ and $\theta \geq 0$

$$\begin{cases} e(\eta,0) \geq 0, \quad c_1(1+\theta^r) \leq e_\theta(\eta,\theta) \leq C_1(\underline{\eta})(1+\theta^r), \\ -c_2\eta^{-2}(1+\theta^{1+r}) \leq p_\eta(\eta,\theta) \leq -C_2\eta^{-2}(1+\theta^{1+r}), \\ |p_\theta(\eta,\theta)| \leq C_3(\underline{\eta})\eta^{-1}(1+\theta^r), \\ \eta p(\eta,\theta) \leq C_4(1+\theta^{1+r}), \\ c_5(\underline{\eta})(1+\theta^{1+r}) \leq p(\eta,\theta) \leq C_5(\underline{\eta})(1+\theta^{1+r}), \\ c_6(1+\theta^q) \leq \kappa(\eta,\theta) \leq C_6(\underline{\eta})(1+\theta^q), \\ |\kappa_\eta(\eta,\theta)| + |\kappa_{\eta\eta}(\eta,\theta)| \leq C_7(\underline{\eta})(1+\theta^q), \\ \eta\sigma_a(\nu;\eta,\theta)B^m(\nu,\theta) \leq C_8\theta^\alpha f(\nu) \quad \text{for } m = 1,2, \\ \eta\sigma_a(\nu,\eta,\theta) \leq C_9 g(\nu), \\ \left(\left|(\sigma_a)_\eta\right| + |(\sigma_a)_\theta|\right)(1 + B + |B_\theta| + |B_\nu|) \leq C_{10}h(\nu), \\ \eta\sigma_s(\nu;\eta,\theta) \leq C_{11}k(\nu), \\ \left(\left|(\sigma_s)_\eta\right| + |(\sigma_s)_\theta|\right)(1 + B + |B_\theta|) \leq C_{12}\ell(\nu), \end{cases} \tag{1.26}$$

where the numbers $c_j, C_j,\ j = 1,\dots,12$ are positive constants and the functions f,g,h,k,ℓ are such that

$$f,g,h \ \in L^1(\mathbb{R}_+) \cap L^\infty(\mathbb{R}_+),$$

and

$$k,\ell \ \in L^{1+\gamma}(\mathbb{R}_+) \cap L^\infty(\mathbb{R}_+),$$

for an arbitrary small $\gamma > 0$.

The importance of relative growth of the exponents $r \geq 0$ and $q \geq 0$ has been the subject of a number of works in the context of real gas flows. A common choice, relying on physics motivations and covering a number of situations (see [30]) is $r \in [0,1],\ q \geq r+1$, used for example by Dafermos and Hsiao [6], Kawohl [20] or Jiang [16].

Recently, the following "quasi-optimal" choice was proposed by Qin [29]

$$\begin{cases} q > 1/3 \quad \text{if } 0 \le r \le 1/3, \\ q > (2r+1)/5 \quad \text{if } 1/3 \le r \le 4/7, \\ q > (5r+1)/9 \quad \text{if } 4/7 \le r \le 1, \\ q > (9r+1)/15 \quad \text{if } 1 \le r \le 13/3, \\ q > (11r+3)/19 \quad \text{if } 13/3 < r. \end{cases} \tag{1.27}$$

For simplicity, we choose the previous choice $r \in [0,1]$, $q \ge r+1$, but one can check that our results also hold for the general choice (1.27). However, for technical reasons made clear below, we must adapt the method of existence proof given by Dafermos-Hsiao, Kawohl, Jiang. In fact the original strategy followed by these authors does not apply verbatim in our case, due to the coupling to the transfert equation, and we use the clever idea of Qin to consider the uniform bound $\|\theta\|_{L^\infty(Q_T)}$ for the temperature as the fundamental quantity.

Finally we also suppose that

$$0 \le \alpha < r.$$

Concerning the viscosity, we suppose that it does not depend on temperature and that

$$0 < \mu_0 \le \mu(\eta) \le \mu_1, \tag{1.28}$$

for some positive constants μ_0 and μ_1.

We study weak solutions for the above problem with properties

$$\begin{cases} \eta \in L^\infty(Q_T), \ \ \eta_t \in L^\infty([0,T], L^2(\Omega)), \\ v \in L^\infty([0,T], L^4(\Omega)), \ \ v_t \in L^\infty([0,T], L^2(\Omega)), \ \ v_x \in L^\infty([0,T], L^2(\Omega)), \\ \sigma_x \in L^\infty([0,T], L^2(\Omega)), \\ \theta \in L^\infty([0,T], L^q(\Omega)), \ \ \theta_x \in L^\infty([0,T], L^2(\Omega)), \\ I \in L^\infty([0,T], L^1(\Omega \times O)), \end{cases} \tag{1.29}$$

where $Q_T := \Omega \times (0,T)$, and we assume the following conditions on the data:

$$\begin{cases} \eta^0 > 0 \text{ on } \Omega, \ \eta^0 \in L^1(\Omega), \\ v^0 \in L^2(\Omega), \ \ v_x^0 \in L^2(\Omega), \\ \theta^0 \in L^2(\Omega), \ \ \inf_\Omega \theta^0 \ge 0, \\ I^0 \in L^1(\Omega \times O). \end{cases} \tag{1.30}$$

Then our definition of a weak solution for the previous problem is the following

Definition 1.1. *We call (η, v, θ, I) a weak solution of (1.15) if it satisfies*

$$\eta(x,t) = \eta^0(x) + \int_0^t v_x \, ds, \tag{1.31}$$

for a.e. $x \in \Omega$ and any $t > 0$, and if, for any test function $\phi \in L^2([0,T], H^1(\Omega))$ with $\phi_t \in L^1([0,T], L^2(\Omega))$ such that $\phi(\cdot, T) = 0$, one has

$$\begin{aligned}\int_Q \left[\phi_t v + \phi_x p - \frac{\mu \phi_x}{\eta} v_x\right] dx\, dt \\ = \int_\Omega \phi(0,x)\, v^0(x)\, dx,\end{aligned} \tag{1.32}$$

$$\begin{aligned}\int_Q \left[\phi_t \left(e + \frac{1}{2} v^2\right) + \phi_x (\sigma v - q) + \phi \eta (S_E)_R\right] dx\, dt \\ = \int_\Omega \phi(0,x) \left(e^0(x) + \frac{1}{2} v^0(x)^2\right) dx,\end{aligned} \tag{1.33}$$

and if, for any test function $\psi \in L^2([0,T], H^1(\Omega \times O))$ with $\phi_t \in L^1([0,T], L^2(\Omega \times O))$ such that $\phi(\cdot, T, \cdot, \cdot) = 0$, one has

$$\begin{aligned}\int_{Q \times O} [\phi_t \eta I + \phi_x (v - c\omega) I + \phi \eta S] \, d\nu\, d\omega\, dx\, dt \\ = \int_{\Omega \times O} \phi(0,x)\, \eta^0(x) I^0(x)\, d\nu\, d\omega\, dx.\end{aligned} \tag{1.34}$$

In the following we use the following notation for the integrated radiative intensity

$$I(x,t) := \int_O I(x,t;\omega,\nu)\, d\omega\, d\nu.$$

Then our main result is the following

Theorem 1.2. *Suppose that the initial data satisfy (1.30) and that T is an arbitrary positive number.*

Then the problem (1.15)(1.21)(1.22)(1.23)(1.24) possesses a global weak solution satisfying (1.29) together with properties (1.31), (1.32) and (1.33).

Moreover one has the uniqueness result

Theorem 1.3. *Suppose that the initial data satisfy (1.30) and that T is an arbitrary positive number.*

Then the problem (1.15)(1.21)(1.22)(1.23)(1.24) possesses a global unique weak solution satisfying (1.29) together with properties (1.31), (1.32) and (1.33).

For that purpose, we first prove a classical existence result in the Hölder category. We denote by $C^\alpha(\Omega)$ the Banach space of real-valued functions on Ω which are uniformly Hölder continuous with exponent $\alpha \in \Omega$, and $C^{\alpha,\alpha/2}(Q_T)$ the Banach space of real-valued functions on $Q_T := \Omega \times (0,T)$ which are uniformly Hölder continuous with exponent α in x and $\alpha/2$ in t. The norms of $C^\alpha(\Omega)$ (resp. $C^{\alpha,\alpha/2}(Q_T)$) will be denoted by $\|\cdot\|_\alpha$ (resp. $|||\cdot|||_\alpha$).

Theorem 1.4. *Suppose that the initial data satisfy*

$$\left(\eta^0, \eta^0_x, v^0, v^0_x, v^0_{xx}, \theta^0, \theta^0_x, \theta^0_{xx}, I^0, I^0_x\right) \in \left(C^\alpha(\Omega)\right)^{10},$$

for some $\alpha \in \Omega$. *Suppose also that* $\eta^0(x) > 0$, $\theta^0(x) > 0$ *and* $I^0(x) > 0$ *on* Ω, *and that the compatibility conditions*

$$\theta^0_x(0) = \theta^0_x(M) = 0, \quad v^0(0) = v^0(M) = 0,$$

and

$$I^0(0, \omega, \nu) = 0 \text{ in } O_+, \; I^0(M, \omega, \nu) = 0 \text{ in } O_-,$$

hold. Then, for any $T > 0$, *there exists a unique solution* (η, v, θ, I) *with* $\eta > 0$, $\theta > 0$ *and* $I > 0$ *to the initial-boundary value problem (1.15)(1.21)(1.22)(1.23) (1.24) on* $Q = \Omega \times [0,T] \times O_+$ *such that*

$$(\eta, \eta_x, \eta_t, \eta_{xt}, v, v_x, v_t, v_{xx}, \theta, \theta_x, \theta_t, \theta_{xx}, I, I_x) \in \left(C^\alpha(Q_T)\right)^{14},$$

and

$$(\eta_{tt}, v_{xt}, \theta_{xt}) \in \left(L^2(Q_T)\right)^3.$$

Then Theorem 1.2 will be obtained from Theorem 1.4 through a regularization process.

Let us recall that the investigation of 1D viscous flows for compressible media goes back to the pioneer work of Antonsev-Kazhikov-Monakov [3] (see also [14] [15] and [29] for more recent presentations).

The strategy we use to prove these results consists in an adaptation to the radiative case of the ideas of Dafermos-Hsiao [6], Kawohl [20], Jiang [16] and Qin [29]: in Section 2 we give necessary a-priori estimates sufficient to get existence and uniqueness of a solution (Section 3). Finally we give in Section 4 a simple negative result on the absence of stationary solution for the previous problem.

2 A priori estimates

Let us suppose that the solution is classical in the following sense

$$\left\{ \begin{array}{c} \eta \in C^1(Q_T), \;\; \rho > 0, \\ \\ v, \; \theta \in C^1([0,T], C^0(\Omega)) \cap C^0([0,T], C^2(\Omega)), \; \theta > 0, \\ \\ I \in C^1(Q_T, C^0(O)), \end{array} \right. \tag{2.1}$$

In the following T be an arbitrary positive number and we denote by C or $C(T)$ various positive constants depending on T and the physical data of the problem and by K various positive constants depending also on the physical data of the problem but *not* on T.

We first prove the following regularity result

Theorem 2.1. *Assuming that the initial-boundary value problem (1.15)(1.21)(1.23) has a classical solution described by Theorem 1.4, the solution* $(\eta, v, v_x, \theta, \theta_x, I)$ *is bounded in the anisotropic Hölder space* $C^{1/3,1/6}(Q_T)$ *such that*

$$|||\eta|||_{1/3} + |||v|||_{1/3} + |||v_x|||_{1/3} + |||\theta|||_{1/3} + |||\theta_x|||_{1/3} \leq C,$$

and

$$|||I|||_{1/3} \leq C.$$

Moreover

$$0 < \underline{\eta} \leq \eta \leq \overline{\eta}, \quad 0 < \underline{\theta} \leq \theta \leq \overline{\theta},$$

where $\underline{\eta}, \overline{\eta}, \underline{\theta}, \overline{\theta}$ *are positive constants depending on* T*, the physical data of the problem and the initial data.*

We first quote a simple mass-energy estimate

Lemma 2.2. *Under the following condition on the data*

$$\left\|v^0\right\|_{L^2(\Omega)} + \left\|\eta^0\right\|_{L^1(\Omega)} + \left\|\theta^0\right\|_{L^1 \cap L^{r+1}(\Omega)} + \left\|I^0\right\|_{L^1(\Omega \times \mathbb{R}_+ \times S^1)} \leq N, \tag{2.2}$$

there exist a positive constant $K = K(N)$ *such that*

1. *the mass conservation*
$$\int_\Omega \eta \, dx = \int_\Omega \eta^0 \, dx, \tag{2.3}$$

2. *the energy equality*
$$\begin{aligned} &\int_\Omega \left[e + \frac{1}{2} v^2 + \eta E_R \right] \, dx + \int_0^t \int_0^\infty \int_0^1 \omega I(M, s, \nu, \omega) \, d\nu \, d\omega \, ds \\ &\quad - \int_0^t \int_0^\infty \int_{-1}^0 \omega I(0, s, \nu, \omega) \, d\nu \, d\omega \, ds \\ &\quad = \int_\Omega \left[e^0 + \frac{1}{2} (v^0)^2 + \eta^0 E_R^0 \right] \, dx, \end{aligned} \tag{2.4}$$
where $E_R^0(x) = \frac{1}{c} \int_0^\infty \int_{S^1} I^0(x, \nu, \omega) \, d\nu \, d\omega$*,*

3. *the estimate*
$$\|\eta\|_{L^\infty(0,T;L^1(\Omega))} + \|v\|_{L^\infty(0,T;L^2(\Omega))} + \|\theta\|_{L^\infty(0,T;L^\delta(\Omega))} \leq K, \tag{2.5}$$
for any $1 \leq \delta \leq r+1$*,*

4. *the condition*
$$\theta(x,t) > 0 \quad \text{for any } (x,t) \in Q_T, \tag{2.6}$$

hold.

Proof. 1. Integrating the first equation (1.15) and using boundary conditions give (2.3).

2. Integrating on frequencies and angular momentum the first equation (1.20) and plugging the result in the third equation (1.15), we get

$$\left(e+\frac{1}{2}\,v^2+\eta E_R\right)_t = (\sigma v - q - F_R + vE_R)_x. \tag{2.7}$$

Integrating on Ω and using boundary conditions gives (2.4).

3. Estimate (2.5) follows directly from (1.26), (2.3) and (2.4).

4. Using (1.26), the positivity of $\theta(x,t)$ follows from that of $\theta^0(x)$ after the maximum principle applied to the third equation (1.15) $\square$ $\square$

Lemma 2.3. *Radiative intensity satisfies the following bounds*

$$\max_{[0,T]} \int_\Omega \int_O \eta I(x,t;\nu,\omega)\; d\omega\, d\nu\, dx \le C(T), \tag{2.8}$$

$$\int_{Q_T} \int_O \eta\sigma_a(\eta,\theta;\nu,\omega) I(x,t;\nu,\omega)\; d\omega\, d\nu\, dx\, dt \le C(T), \tag{2.9}$$

$$\max_{[0,T]} \int_\Omega \int_O \eta I^2(x,t;\nu,\omega)\; d\omega\, d\nu\, dx \le C(T), \tag{2.10}$$

$$\int_{Q_T} \int_O \eta\sigma_a(\eta,\theta;\nu,\omega) I^2(x,t;\nu,\omega)\; d\omega\, d\nu\, dx\, dt \le C(T), \tag{2.11}$$

$$\int_{Q_T} \int_O \eta\sigma_s(\eta,\theta;\nu) \left(\tilde{I}(x,t;\nu) - I(x,t;\nu,\omega)\right)^2\; d\omega\, d\nu\, dx\, dt \le C(T), \tag{2.12}$$

$$\left|\int_{Q_T} \eta\,(S_E)_R\; dx\, dt\right| \le K, \quad \left|\int_{Q_T} \eta\,(S_F)_R\; dx\, dt\right| \le C(T). \tag{2.13}$$

Proof. 1. Integrating equation (1.16) on $\Omega \times O$ and using boundary conditions, we get

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega \int_O \eta I\; d\omega\, d\nu\, dx + \int_O (c\omega - v(M,t))\, I(M,t;\nu,\omega)\; d\omega\, d\nu \\
&- \int_O (c\omega - v(0,t))\, I(0,t;\nu,\omega)\; d\omega\, d\nu + \int_\Omega \int_O \eta\sigma_a(\nu_0;\eta,\theta) I\; d\omega\, d\nu\, dx \\
&= \int_\Omega \int_O \eta\sigma_a(\nu;\eta,\theta) B(\nu;v,\theta)\; d\omega\, d\nu\, dx \\
&+ \int_\Omega \int_O \eta\sigma_s(\nu;\eta,\theta)\; \left[\tilde{I}(x,t,\nu) - I(x,t,\nu,\omega_0)\right]\; d\omega\, d\nu\, dx,
\end{aligned} \tag{2.14}$$

As σ_s does not depend on ω, the last integral in the right-hand side is zero and, using boundary conditions, the boundary terms in the left-hand side are non negative, so integrating on time, we find

$$\begin{aligned}
&\int_\Omega \int_O \eta I(x,t;\nu,\omega)\; d\omega\, d\nu\, dx - \int_\Omega \int_O \eta^0 I^0(x;\nu,\omega)\; d\omega\, d\nu\, dx \\
&+ c\int_0^T \int_{O_+} \omega I(M,t;\nu,\omega)\; d\omega\, d\nu\, dt - c\int_0^T \int_{O_-} \omega I(0,t;\nu,\omega)\; d\omega\, d\nu\, dt
\end{aligned}$$

$$+c\int_{Q_T}\int_O \eta\sigma_a I \, d\omega\, d\nu\, dx\, dt = \int_{Q_T}\int_O \eta\sigma_a B \, d\omega\, d\nu\, dx\, dt.$$

Observing that, after (1.26), the integral in the right-hand side is bounded by $C_8\int_{Q_T}\theta^\alpha\int_O f\, d\omega\, d\nu\, dx\, dt$, we get then (2.8) and (2.9) by using (2.4).

2. Multiplying (1.16) by I, integrating on $\Omega\times O$ and using boundary conditions, we get

$$\frac{1}{2}\frac{d}{dt}\int_\Omega\int_O \eta I^2 \, d\omega\, d\nu\, dx + \frac{1}{2}\int_O (c\omega - v(M,t))\, I^2(M,t;\nu,\omega)\, d\omega\, d\nu$$
$$-\frac{1}{2}\int_O (c\omega - v(0,t))\, I^2(0,t;\nu,\omega)\, d\omega\, d\nu + \int_\Omega\int_O \eta\sigma_a I^2 \, d\omega\, d\nu\, dx$$
$$+\int_\Omega\int_O \eta\sigma_s \left(\tilde{I}-I\right)^2 \, d\omega\, d\nu\, dx = \int_\Omega\int_O \eta\sigma_a BI \, d\omega\, d\nu\, dx.$$

Integrating on time, using (1.26) and estimating the right-hand side by Cauchy-Schwarz, we have

$$\frac{1}{2}\int_\Omega\int_O \eta I^2 \, d\omega\, d\nu\, dx - \frac{1}{2}\int_\Omega\int_O \eta (I^0)^2 \, d\omega\, d\nu\, dx$$
$$+\frac{c}{2}\int_0^T\int_{O_+} \omega I^2(M,t;\nu,\omega)\, d\omega\, d\nu\, dt - \frac{c}{2}\int_0^T\int_{O_-} \omega I^2(0,t;\nu,\omega)\, d\omega\, d\nu\, dt$$
$$+\int_{Q_T}\int_O \eta\sigma_a I^2 \, dx\, d\omega\, d\nu\, dx\, dt + \int_{Q_T}\int_O \eta\sigma_s\left(\tilde{I}-I\right)^2 \, d\omega\, d\nu\, dx\, dt$$
$$\leq \frac{1}{2}\int_{Q_T}\int_O \eta\sigma_a I^2 \, d\omega\, d\nu\, dx\, dt + \frac{1}{2}\int_{Q_T}\int_O \eta\sigma_a B^2 \, d\omega\, d\nu\, dx\, dt.$$

As, using (1.26), the last integral in the right-hand side is bounded by

$$\leq C_8\int_{Q_T}\theta^\alpha\int_O f\, d\omega\, d\nu\, dx\, dt,$$

we get (2.10), (2.11) and (2.12).

3. Using the definitions of the radiative sources, we get the inequalities

$$\left|\int_{Q_T}\eta\, (S_E)_R \, dx\, dt\right| \leq \int_{Q_T}\int_O \eta\sigma_a B \, dx\, d\omega\, d\nu\, dx\, dt, \tag{2.15}$$

and

$$\left|\int_{Q_T}\eta\, (S_F)_R \, dx\, dt\right| \leq \int_{Q_T}\int_O \eta\sigma_a B \, dx\, d\omega\, d\nu\, dx\, dt$$
$$+\int_{Q_T}\int_O \eta\sigma_s |\tilde{I}-I| \, dx\, d\omega\, d\nu\, dx\, dt. \tag{2.16}$$

Using (1.26), the right-hand side of (2.15) is bounded, and using (1.26), Cauchy-Schwarz inequality and (2.12) we bound the right-hand side of (2.16) which proves (2.13) □ □

Lemma 2.4. *Under conditions (2.2) on the data, the following entropy inequality holds*

$$\int_{Q_t}\left(\frac{\kappa(\eta,\theta)}{\eta\theta^2}\,\theta_x^2 + \frac{\mu(\eta)}{\eta\theta}\, v_x^2\right) dx\, ds \leq K, \tag{2.17}$$

Proof. Total entropy $s = s_m + s_R$ is the sum of the entropy of matter s_m and entropy of radiation s_R.

From the second principle of thermodynamics

$$\theta(s_m)_t = e_t + p\eta_t.$$

Using (1.15), (1.8) and (1.18), one finds

$$(s_m)_t = -\left(\frac{\kappa\theta_x}{\eta\theta}\right)_x + \frac{\mu v_x^2}{\eta\theta} + \frac{\kappa\theta_x^2}{\eta\theta^2} + \frac{\eta}{\theta}\int_O \left(\frac{\omega v}{c} - 1\right) S_{a,e}\, d\omega\, d\nu. \tag{2.18}$$

From statistical mechanics mechanics [27], we know that the entropy per mode of a boson gas is $k_B[(n+1)\log(n+1) - n\log n]$, where n is the occupation number related to I by

$$n = n(I) := \frac{c^2}{2h}\,\frac{I}{\nu^3}.$$

Multiplying by the number of modes, we find the entropy per mass unit

$$s_R = \eta\int_0^\infty \int_{S^1} \frac{2k_B\nu^2}{c^3}\,[(n+1)\log(n+1) - n\log n]\, d\nu\, d\omega.$$

Using the last equation (1.15), observing that for any regular function $n \to \chi(n)$ one has the identity

$$(\eta\chi)_t + [(c\omega - v)\chi]_x = \frac{c^3}{2h\nu^3}\,\chi'\eta S,$$

and choosing $\chi(n) = (n+1)\log(n+1) - n\log n$, we get after a direct computation

$$(s_R)_t + \left[\int_0^\infty \int_{S^1} \frac{2k_B\nu^2}{c^3}(c\omega - v)\,[(n+1)\log(n+1) - n\log n]\, d\nu\, d\omega\right]_x$$
$$= \eta\int_0^\infty \int_{S^1} \frac{k_B}{h\nu}\,\log\frac{n+1}{n}\, S\, d\nu\, d\omega =: Q_R. \tag{2.19}$$

Decomposing

$$\eta\int_0^\infty \int_{S^1} \frac{k_B}{h\nu}\,\log\frac{n+1}{n}\, S\, d\nu\, d\omega$$
$$= \eta\int_0^\infty \int_{S^1} \frac{k_B}{h\nu}\,\log\frac{n+1}{n}\,\sigma_a(B - I)\, d\nu\, d\omega$$
$$+\eta\int_0^\infty \int_{S^1} \frac{k_B}{h\nu}\,\log\frac{n+1}{n}\,\sigma_s(\tilde{I} - I)\, d\nu\, d\omega,$$

and checking the identity

$$\log\frac{n(B)+1}{n(B)} = \left(1 - \frac{\omega v}{c}\right)\frac{h\nu}{k_B\theta},$$

the right-hand side of (2.19) reads

$$Q_R = \eta\int_0^\infty \int_{S^1} \frac{k_B}{h\nu}\,\left[\log\frac{n(I)+1}{n(I)} - \log\frac{n(B)+1}{n(B)}\right]\,\sigma_a(B - I)\, d\nu\, d\omega$$

$$+\frac{\eta}{\theta}\,(S_E)_R - \frac{\eta}{\theta}\,v\,(S_F)_R$$

$$+\eta\int_0^\infty\int_{S^1}\frac{k_B}{h\nu}\left[\log\frac{n(I)+1}{n(I)} - \log\frac{n(\tilde I)+1}{n(\tilde I)}\right]\,\sigma_s(\tilde I - I)\,d\nu\,d\omega.$$

As $u \to \log\frac{u+1}{u}$ is decreasing for $u > 0$, the first and last terms are positive, and we get finally

$$(s_R)_t + \left[\int_0^\infty\int_{S^1}\frac{2k_B\nu^2}{c^3}(c\omega - v)\,[(n+1)\log(n+1) - n\log n]\,d\nu\,d\omega\right]_x$$

$$= \eta\int_0^\infty\int_{S^1}\frac{k_B}{h\nu}\left[\log\frac{n(I)+1}{n(I)} - \log\frac{n(\tilde I)+1}{n(\tilde I)}\right]\,\sigma_s(\tilde I - I)\,d\nu\,d\omega$$

$$+\eta\int_0^\infty\int_{S^1}\frac{k_B}{h\nu}\left[\log\frac{n(I)+1}{n(I)} - \log\frac{n(B)+1}{n(B)}\right]\,\sigma_a(B - I)\,d\nu\,d\omega$$

$$+\frac{\eta}{\theta}\int_O\left(1 - \frac{\omega v}{c}\right)S_{a,e}\,d\omega\,d\nu. \tag{2.20}$$

Using the technique of [16] and defining the free energy $\psi := e - \theta s_m$ of the fluid, with $\psi_\theta = -s_m$ and $\psi_\eta = -p$, let us introduce the auxiliary function

$$\mathcal{E}(\eta,\theta) := \psi(\eta,\theta) - \psi(1,\theta_0) - (\eta - 1)\psi_\eta(1,\theta_0) - (\theta - \theta_0)\psi_\theta(\eta,\theta) - \theta_0 s_R.$$

A direct computation gives

$$\left(\mathcal{E} + \frac{1}{2}\,v^2 + \eta E_R\right)_t = [\sigma v + p(1,\theta_0)v - q - F_R + vE_R]_x - \theta_0 s_t.$$

Plugging (2.18) and (2.20) in the right-hand side, we get finally

$$\left(\mathcal{E} + \frac{1}{2}\,v^2 + \eta E_R\right)_t + \theta_0\left(\frac{\mu v_x^2}{\eta\theta} + \frac{\kappa\theta_x^2}{\eta\theta^2}\right)$$

$$+\theta_0\eta\int_0^\infty\int_{S^1}\frac{k_B}{h\nu}\left[\log\frac{n(I)+1}{n(I)} - \log\frac{n(\tilde I)+1}{n(\tilde I)}\right]\,\sigma_s(I - \tilde I)\,d\nu\,d\omega$$

$$+\theta_0\eta\int_0^\infty\int_{S^1}\frac{k_B}{h\nu}\left[\log\frac{n(I)+1}{n(I)} - \log\frac{n(B)+1}{n(B)}\right]\,\sigma_a(I - B)\,d\nu\,d\omega$$

$$= [\sigma v + p(1,\theta_0)v - q - F_R + vE_R$$

$$+\theta_0\int_0^\infty\int_{S^1}\frac{2k_B\nu^2}{c^3}(c\omega - v)\,[(n+1)\log(n+1) - n\log n]\,d\nu\,d\omega\Big]_x.$$

Integrating on Q_t and using (2.4) and (1.21) the contribution of the first three boundary term is zero. Moreover using (1.22) to compute the contribution of the radiative terms boundary terms we have the final equality

$$\int_\Omega\left(\mathcal{E} + \frac{1}{2}\,v^2 + \eta E_R\right)\,dx + \theta_0\int_{Q_t}\left(\frac{\mu v_x^2}{\eta\theta} + \frac{\kappa\theta_x^2}{\eta\theta^2}\right)\,dx\,ds$$

$$-\theta_0 \int_{Q_t} \eta \int_0^\infty \int_{S^1} \frac{k_B}{h\nu} \left[\log \frac{n(I)+1}{n(I)} - \log \frac{n(\tilde{I})+1}{n(\tilde{I})} \right] \sigma_s (I - \tilde{I})\, d\nu\, d\omega\, dx\, ds$$

$$-\theta_0 \int_{Q_t} \eta \int_0^\infty \int_{S^1} \frac{k_B}{h\nu} \left[\log \frac{n(I)+1}{n(I)} - \log \frac{n(B)+1}{n(B)} \right] \sigma_a (I - B)\, d\nu\, d\omega\, dx\, ds$$

$$+ \int_0^t \int_0^\infty \int_0^1 \omega I(M,s;\omega,\nu)\, d\nu\, d\omega\, ds$$

$$- \int_0^t \int_0^\infty \int_{-1}^0 \omega I(0,s;\omega,\nu)\, d\nu\, d\omega\, ds$$

$$+\theta_0 \int_0^t \int_0^\infty \int_0^1 \frac{2k_B\nu^2}{c^2}\, \omega\, [(n+1)\log(n+1) - n\log n](M,s;\omega,\nu)\, d\nu\, d\omega\, ds$$

$$-\theta_0 \int_0^t \int_0^\infty \int_{-1}^0 \frac{2k_B\nu^2}{c^2}\, \omega\, [(n+1)\log(n+1) - n\log n](0,s;\omega,\nu)\, d\nu\, d\omega\, ds$$

$$= \int_\Omega \left(\mathcal{E}^0 + \frac{1}{2}\, v^{0^2} + \eta^0 E_R^0 \right) dx. \tag{2.21}$$

Now we argue in the same way as [16] noting that, by using Taylor formula, for any $\eta > 0$

$$\mathcal{E}(\eta,\theta) - \psi(\eta,\theta) + \psi(\eta,\theta_0) + (\theta - \theta_0)\psi_\theta(\eta,\theta) - \theta_0 s_R$$

$$= \psi(\eta,\theta_0) - \psi(1,\theta_0) - (\eta - 1)\psi_\eta(1,\theta_0) \geq 0,$$

and that

$$\psi(\eta,\theta) - \psi(\eta,\theta_0) - (\theta - \theta_0)\psi_\theta(\eta,\theta) = -(\theta - \theta_0)^2 \int_0^1 (1-\alpha)\psi_{\theta\theta}(\eta, \theta + \alpha(\theta_0 - \theta))\, d\alpha.$$

Using $\psi_{\theta\theta} = -\theta^{-1} e_\theta$ and estimates (1.26), we find

$$\psi(\eta,\theta) - \psi(\eta,\theta_0) - (\theta - \theta_0)\psi_\theta(\eta,\theta) \geq \frac{1}{4}\, K\left(\theta + \theta^{1+r}\right) - K.$$

Now one checks by elementary computations that $\eta E_R - \theta_0 s_R \geq K$, so we deduce that

$$\mathcal{E}(\eta,\theta) + \eta E_R \geq \frac{1}{4}\, K\left(\theta + \theta^{1+r}\right) - K,$$

and by plugging this into (2.21) we conclude, after (2.4), that (2.17) holds □ □

Lemma 2.5. *Under the previous condition on the data (2.2), there exists positive constants* $\underline{\eta}$ *and* $\overline{\eta}$ *depending on* T *and the data, such that*

$$\underline{\eta} \leq \eta(x,t) \leq \overline{\eta} \quad \textit{for } (t,x) \in Q_T. \tag{2.22}$$

Proof. 1. Introducing the strictly increasing function $s \to \mathcal{M}(s) := \int_1^s \frac{\mu(\xi)}{\xi}\, d\xi$, one observes that $\mathcal{M}$ maps $(0, \inf_\Omega \eta^0]$ onto $(-\infty, 0)$. So the lower bound in (2.22) follows after

$$M(\eta(x,t)) \geq -C \quad \text{on } Q_T. \tag{2.23}$$

Using (1.20), the second equation (1.15) rewrites

$$(v+A)_t = (\sigma - B)_x,$$

where $A(x,t) := \eta F_R$ and $B(x,t) := P_R - vF_R$.

If $\phi(x,t) := \int_0^t \sigma\, ds + \int_0^x v^0\, dy + \int_0^x A^0\, dy$, then ϕ satisfies the equations

$$\phi_x = v + A,$$

and

$$\phi_t = \frac{\mu(\eta)}{\eta}\, v_x - p - B.$$

Multiplying this last equation by η we find that

$$(\eta\phi)_t = (v\phi)_x + \mu(\eta) v_x - v^2 - p\eta - \eta P_R.$$

Integrating on Q_t, we find

$$\int_\Omega \phi\eta\, dx = \int_\Omega \big(\mu(\eta)v_x - p\eta - v^2 - \eta P_R\big)\, dx\, ds + \int_\Omega \phi^0\eta^0\, dx. \tag{2.24}$$

Using (2.3) and a standard argument of [3], there exists a point $X(t) \in \Omega$ such that $\phi(X(t),t) = \frac{1}{R} \int_\Omega \phi\eta\, dx$ with $R := \int_\Omega \eta\, dx$. Then after the definition of ϕ and (2.24), we find

$$\int_0^t \sigma(X(t),t)\, ds = \int_\Omega \big(\mu(\eta)v_x - p\eta - v^2 - \eta P_R\big)\, dx\, ds + \int_\Omega \eta^0(x) \int_0^x v^0(y) dy\, dx - \int_0^{X(t)} v^0(y)\, dy. \tag{2.25}$$

Now integrating first on $[0,t]$, then on $[X(t),x]$ the second equation (1.15) rewritten as

$$\mathcal{M}_{xt} = v_t + p_x + \eta\,(S_F)_R, \tag{2.26}$$

we find

$$\mathcal{M}(\eta(x,t)) = \mathcal{M}(\eta^0(x)) + \int_0^t p\, ds + \int_{X(t)}^x \big(v(y,t) - v^0(y)\big)\, dy + \int_0^t \sigma(X(t),t)\, ds + \int_{X(t)}^x \int_0^t \eta\,(S_F)_R\, dt\, dx, \tag{2.27}$$

and using (2.25), we get

$$\begin{aligned}\mathcal{M}(\eta(x,t)) = \mathcal{M}(\eta^0(x)) + \int_0^t p\, ds + \int_{X(t)}^x \big(v(y,t) - v^0(y)\big)\, dy \\ + \int_\Omega \big(\mu(\eta)v_x - p\eta - v^2 - \eta P_R\big)\, dx \\ + \int_\Omega \eta^0(x) \int_0^x v^0(y) dy\, dx - \int_0^{X(t)} v^0(y)\, dy + \int_{X(t)}^x \int_0^t \eta\,(S_F)_R\, dt\, dx.\end{aligned} \tag{2.28}$$

Let us estimates all of the contributions in the right-hand side.

Using Cauchy-Schwarz inequality, we know after a standard argument gives (see [14]) that $\theta^{1/2}(x,t) \le C + C\left(\int_\Omega \frac{(1+\theta^q)\theta_x^2}{\eta\theta^2}\,dx\right)^{1/2}$, which implies

$$\int_0^t \max_\Omega \theta(\cdot,t)\,ds \le C\left(1+\int_{Q_t} \frac{(1+\theta^q)\theta_x^2}{\eta\theta^2}\,dx\,ds\right) \le C.$$

Then, using (2.3) and (2.17) we get

$$\left|\int_{Q_t} \mu v_x\,dx\,ds\right| \le \frac{1}{2}\int_{Q_t} \frac{\mu v_x^2}{\eta\theta}\,dx\,ds + \frac{1}{2}\int_{Q_t} \eta\theta\,dx\,ds$$

$$\le C + C\int_0^t \max_\Omega \theta(\cdot,t)\int_\Omega \eta\,dx\,ds \le C.$$

We have also, using (1.26)

$$\int_{Q_t} \eta p\,dx\,ds \le C.$$

Using Lemma 2.3, one get finally that

$$\int_\Omega \eta P_R\,dx = \int_\Omega\int_O \eta\omega^2 I\,d\omega\,d\nu\,dx \le \int_\Omega\int_O \eta I\,d\omega\,d\nu\,dx \le C,$$

and

$$\left|\int_{X(t)}^x\int_0^t \eta\,(S_F)_R\,dt\,dx\right| \le \int_{Q_t}\int_O \eta|\,(S_F)_R\,|\,dt\,dx$$

$$\le \int_{Q_t}\int_O \eta\left(\sigma_a B + \sigma_a I + \sigma_s|\tilde{I} - I|\right)\,d\omega\,d\nu\,dx\,ds \le C.$$

We conclude, by plugging all of these estimates into (2.28) and using (2.4), that (2.23) is valid, which implies the lower bound in (2.22).

2. It is clearly sufficient to get an upper bound for $\mathcal{M}(\eta)$. Revisiting the previous proof, we find

$$\mathcal{M}(\eta) \le C + C\int_0^t p\,ds \le C + C\int_0^t \max_\Omega \theta^{1+r}\,ds.$$

But

$$D := \int_0^t \max_\Omega \theta^{1+r}\,ds \le C + C\int_{Q_t} \theta^r|\theta_x|\,dx\,ds$$

$$\le C + \frac{1}{2\|\eta\|_{L^1(\Omega)}}\int_{Q_t} \eta\theta^{1+r}\,dx\,ds + C\int_{Q_t} \frac{\theta^{1+r}\theta_x^2}{\eta\theta^2}\,dx\,ds\,dx\,ds$$

$$\le C + \frac{1}{2}\int_0^t \max_\Omega \theta^{1+r}\,ds.$$

Then we see that

$$\int_0^t \max_\Omega \theta^{1+r}\,ds \le C, \tag{2.29}$$

which implies that $D \le C$ then $\mathcal{M}(\eta) \le C$ and ends the proof $\square$ $\square$

Lemma 2.6.
$$C - CV(t) \le \theta^{2\lambda}(x,t) \le C + CV(t), \tag{2.30}$$
where $V(t) := \int_\Omega \frac{1+\theta^q}{\theta^2}\,\theta_x^2\,dx$*, for any* $\lambda \le \frac{q+r+1}{2}$.

Proof. Just use the inequality $\theta^\lambda(x,t) \le C + C\int_\Omega \theta^{\lambda-1}|\theta_x|\,dx$ together with (2.17) and Lemma 2.5 □ □

Lemma 2.7. *For any* $t \in (0,T)$
$$\int_{Q_T} (S_E)_R^2\,dx\,dt \le C, \quad \int_{Q_T} (S_F)_R^2\,dx\,dt \le C. \tag{2.31}$$

Proof. After the definition of $(S_E)_R$ and using (2.22), we get
$$\int_{Q_t} (S_E)_R^2\,dx\,dt \le \int_{Q_t}\left(\int_O \eta\sigma_a B\,d\omega\,d\nu\right)^2 dx\,ds$$
$$+\int_{Q_t}\left(\int_O \eta\sigma_a I\,d\omega\,d\nu\right)^2 dx\,ds + \int_{Q_t}\left(\int_O \eta\sigma_s(\tilde{I}-I)\,d\omega\,d\nu\right)^2 dx\,ds$$
$$=: J_1 + J_2 + J_3.$$
As $J_3 = 0$, we estimate J_1 and J_2.

After (1.26) and (2.29), we have first
$$J_1 \le C\int_0^t \max_\Omega \theta^{2\alpha}\,ds \le C\int_0^t \max_\Omega \theta^{1+r}\,ds \le C.$$
Then applying Cauchy-Schwarz inequality and using (1.26) and Lemma 2.3, we get
$$J_2 \le \frac{1}{2}\int_{Q_T}\int_0^\infty \eta\sigma_a\,d\omega\,d\nu\,dx\,ds + \frac{1}{2}\int_{Q_T}\int_0^\infty \eta\sigma_a I^2\,d\omega\,d\nu\,dx\,ds \le C,$$
which implies the first bound (2.31).

In the same stroke one get the second bound (2.31) □ □

Lemma 2.8.
$$\int_{Q_T} v_x^2\,dx\,dt \le C. \tag{2.32}$$

Proof. Multiplying the second equation (1.15) by v and integrating by parts on Q_t for any $t \in [0,T]$, we get
$$\int_\Omega v^2\,dx + \int_{Q_t} \frac{\mu}{\eta}\,v_x^2\,dx\,ds \le C + \int_{Q_t} p|v_x|\,dx\,ds + \overline{\eta}\int_{Q_t} |v(S_E)_R|\,dx\,ds. \tag{2.33}$$

To get (2.32), it is sufficient to observe that in the right-hand side, the last term is bounded after Lemma 2.7 and that we can estimate the first integral by
$$\int_{Q_t} p|v_x|\,dx\,ds \le \frac{1}{2}\int_{Q_t} \frac{\mu}{\eta}\,v_x^2\,dx\,ds + C\int_{Q_t} p^2\,dx\,ds$$
$$\le \frac{1}{2}\int_{Q_t} \frac{\mu}{\eta}\,v_x^2\,dx\,ds + C\int_0^t \max_\Omega \theta^{1+r}\,ds,$$
where the last term is bounded after Lemma 2.6. Plugging this into (2.33), we get (2.32) □
□

After Qin [29], we introduce the auxiliary function

$$\mathcal{T}(t) := 1 + \max_{Q_t} \theta(x,s). \tag{2.34}$$

Lemma 2.9. *The following estimates hold for any $t \in [0,T]$*

1.

$$\int_0^t \max_\Omega v^2\, ds \le C, \tag{2.35}$$

2.

$$\int_{Q_t} \left(1+\theta^{q+1+r}\right)\, v^2\, dx\, ds \le C, \tag{2.36}$$

3.

$$\max_{[0,t]} \int_\Omega \eta_x^2\, dx + \int_{Q_t} \left(1+\theta^{1+r}\right)\, \eta_x^2\, dx\, ds \le C. \tag{2.37}$$

4.

$$\int_{Q_t} \left(1+\theta^{q+r+1}\right)\, \eta_x^2\, dx\, ds \le C. \tag{2.38}$$

Proof. 1. As $v(x,t) \le C \int_\Omega \frac{\mu}{\eta}\, v_x^2\, dx$, (2.35) follows from (2.4).

2. One has

$$\int_{Q_t} \left(1+\theta^{q+1+r}\right)\, v^2\, dx\, ds \le \int_0^t \max_\Omega \left(1+\theta^{q+1+r}\right) \int_\Omega v^2\, dx\, dt$$

$$\le \int_0^t \max_\Omega \left(1+\theta^{q+1+r}\right)\, ds \le C,$$

after Lemma 2.6.

3. From (2.26)

$$\frac{1}{2}\frac{d}{dt}\int_\Omega (\mathcal{M}_x - v)^2\, dx = \int_\Omega (\mathcal{M}_x - v)\, p_\eta \eta_x\, dx + \int_\Omega (\mathcal{M}_x - v)\, p_\theta \theta_x\, dx$$

$$+ \int_\Omega (\mathcal{M}_x - v)\, \eta (S_F)_R\, dx.$$

The first integral in the right-hand side reads

$$\int_\Omega (\mathcal{M}_x - v)\, p_\eta \eta_x\, dx = \int_\Omega (\mathcal{M}_x - v)^2 \frac{\eta p_\eta}{\mu}\, dx + \int_\Omega v\, (\mathcal{M}_x - v)\, \frac{\eta p_\eta}{\mu}\, dx.$$

As, after (1.26), the first integral in the right-hand side is bounded by

$$-C_2 \int_\Omega (\mathcal{M}_x - v)^2 \left(1+\theta^{r+1}\right)\, \frac{1}{\eta\mu}\, dx,$$

we have

$$\frac{1}{2}\frac{d}{dt}\int_\Omega (\mathcal{M}_x - v)^2\, dx + \int_\Omega \left(1+\theta^{r+1}\right) (\mathcal{M}_x - v)^2\, dx$$

$$\leq C\int_\Omega (\mathcal{M}_x - v)\left(1+\theta^{r+1}\right)\theta_x\,dx + C\int_\Omega \left|\mathcal{M}_x - v\right|(1+\theta^r)\,|v|\,dx$$

$$+C\int_\Omega \left|\mathcal{M}_x - v\right|\eta|(S_F)_R|\,dx.$$

Integrating in t and using Lemma 2.7 we get

$$\int_\Omega (\mathcal{M}_x - v)^2\,dx + \int_{Q_t}\left(1+\theta^{r+1}\right)(\mathcal{M}_x - v)^2\,dx\,ds$$

$$\leq C + C\int_{Q_t}\left(1+\theta^{r-1}\right)\theta_x^2\,dx\,ds + C\int_{Q_t}\left(1+\theta^{r+1}\right)v^2\,dx\,ds.$$

As $r-1\leq q-2$ and $\max_\Omega \theta^{r+1}\leq V(t)$, the right-hand side is bounded, which ends the proof of (2.37).

4. we have

$$\int_{Q_t}\left(1+\theta^{q+r+1}\right)\,\eta_x^2\,dx\,ds \leq \int_0^t \max_\Omega \int_\Omega \left(1+\theta^{q+1+r}\right)\,\max_{[0,t]}\int_\Omega \eta_x^2\,dx\,ds$$

$$\leq C\int_0^t \max_\Omega \int_\Omega \left(1+\theta^{q+1+r}\right)\,ds \leq C,$$

where we used (2.37) and Lemma 2.6 □ □

Lemma 2.10. *The following estimates hold for any $t\in[0,T]$*

1.

$$\int_\Omega v_x^2\,dx + \int_{Q_t} v_{xx}^2\,dx\,ds \leq C\mathcal{T}^{\beta_1}, \tag{2.39}$$

where $\beta_1 = \max\{2r-q+2, 0\}$.

2.

$$\max_{[0,t]}\int_\Omega v_x^2(x,s)\,dx + \int_{Q_t} v_t^2\,dx\,ds \leq C\mathcal{T}^{\beta_2}, \tag{2.40}$$

where $\beta_2 = \frac{3\beta_1}{4}$.

3.

$$\int_0^t \max_\Omega v_x^2\,ds \leq C\mathcal{T}^{\beta_3}, \tag{2.41}$$

where $\beta_3 = \beta_1$.

Proof. 1. Multiplying the second equation (1.15) by v_{xx} and integrating by parts on Ω, we get

$$\frac{1}{2}\frac{d}{dt}\int_\Omega v_x^2\,dx = \int_\Omega p_x v_{xx}\,dx - \int_\Omega \left[\left(\frac{\mu}{\eta}\right)'\eta_x v_x + \frac{\mu}{\eta}\,v_{xx}\right]v_{xx}\,dx\,ds$$

$$-\int_\Omega \eta(S_F)_R v_{xx}\,dx.$$

Integrating now in t and using Cauchy-Schwarz inequality

$$\int_\Omega v_x^2\,dx + \frac{\mu_0}{\overline{\eta}}\int_{Q_t} v_{xx}^2\,dx\,ds$$

$$\leq C+C\int_{Q_t}\left(1+\theta^{2r+2}\right)\eta_x^2\,dx\,ds+C\int_{Q_t}\eta_x^2 v_x^2\,dx\,ds+C\int_{Q_t}\left(1+\theta^{2r}\right)\theta_x^2\,dx\,ds,$$

$$\leq C+C\int_{Q_t}\left(1+\theta^{r+q+1}\right)\eta_x^2\,dx\,ds+C\int_{Q_t}\max_{\Omega} v_x^2\int_{\Omega}\eta_x^2 dx\,ds$$

$$+C\int_{Q_t}\left(1+\theta^{2r}\right)\theta_x^2\,dx\,ds,$$

$$\leq C+C\int_0^t V(s)\,\max_{Q_t}\theta^{2r+2-q}\,ds,$$

after Lemma 2.9, which implies (2.39).

2. After the elementary inequality $|v_x|\leq C\int_\Omega |v_{xx}|\,dx$, we have

$$\int_0^t \max_{\Omega} v_x^2\,ds\leq C\int_{Q_t} v_{xx}^2\,dx\,ds,$$

which implies (2.41) after (2.39).

3. Multiplying the second equation (1.15) by v_t and integrating by parts on Ω, we get

$$\int_\Omega v_t^2\,dx=\int_\Omega p_x v_t\,dx-\int_\Omega\left[\left(\frac{\mu}{\eta}\right)'\eta_x v_x+\frac{\mu}{\eta}\,v_{xx}\right]v_t\,dx\,ds-\int_\Omega \eta(S_F)_R v_t\,dx.$$

Integrating in t

$$\int_{Q_t} v_t^2\,dx\,ds+\int_\Omega v_x^2\,dx\leq C+C\int_{Q_t} p_x^2\,dx\,ds+C\int_{Q_t}|v_x|^3\,dx\,ds$$

$$\leq C+C\max_{Q_t}\left(\int_{Q_t} v_x^2\,dx\,ds\right)^{3/4}\left(\int_{Q_t} v_{xx}^2\,dx\,ds\right)^{1/4}$$

$$\leq C+C\max_{[0,t]}\left(\int_\Omega v_x^2\,dx\right)^{1/2}\left(\int_{Q_t} v_{xx}^2\,dx\,ds\right)^{1/4},$$

which implies (2.40) □ □

Lemma 2.11. *The following estimates hold for any $t\in[0,T]$*

1.

$$\int_{Q_t}\left(1+\theta^{q+r+1}\right)v_x^2\,dx\,ds\leq C\mathcal{T}^{\beta_4}, \tag{2.42}$$

where $\beta_4=\beta_1$.

2.

$$\int_{Q_t}\left(1+\theta^{q+r+2}\right)v_x^2\,dx\,ds\leq C\mathcal{T}^{\beta_5}, \tag{2.43}$$

where $\beta_5=\min\{1+\beta_4,q+r+2\}$.

3.

$$\int_{Q_t}\left(1+\theta^{q+1}\right)|v_x|^3\,dx\,ds\leq C\mathcal{T}^{\beta_6}, \tag{2.44}$$

where $\beta_6=\min\{\frac{3\beta_4}{2},\frac{3\beta_4}{4}+1\}$.

4.

$$\int_{Q_t} \left(1+\theta^{q-r}\right) v_x^4 \, dx\, ds \le C\mathcal{T}^{\beta_7}, \tag{2.45}$$

where $\beta_7 = \min\{q_2 + 2\beta_1, q - r + \frac{3\beta_4}{2}\}$.

Proof. 1. After (2.39)

$$\int_{Q_t} \left(1+\theta^{q+r+1}\right) v_x^2 \, dx\, ds \le C + \int_0^t V(s) \int_\Omega v_x^2 \, dx\, ds \le C + C\mathcal{T}^{\beta_1}.$$

2. We observe first that

$$\int_{Q_t} \left(1+\theta^{q+r+2}\right) v_x^2 \, dx\, ds \le C\mathcal{T}^{\beta_4+1}.$$

But we have also

$$\int_{Q_t} \left(1+\theta^{q+r+2}\right) v_x^2 \, dx\, ds \le C\mathcal{T}^{q+r-2} \int_{Q_t} v_x^2 \, dx\, ds \le C\mathcal{T}^{q+r-2}.$$

These estimates give (2.43).

3. We have

$$\begin{aligned}
\int_0^t \max_\Omega \left(1+\theta^{q+1}\right) |v_x|^3 \, ds &\le \int_0^t \max_\Omega \left(1+\theta^{\frac{3}{2}(q+r+1)}\right) |v_x|^3 \, ds \\
&\le C\int_{Q_t} |v_x|^3 \, dx\, ds + C\int_{Q_t} V^{3/4}(s) |v_x|^3 \, dx\, ds \\
&\le C \max_{[0,t]} \int_\Omega v_x^2 \, dx \left(\int_{Q_t} v_x^2 \, dx\, ds\right)^{3/4} \left(\int_{Q_t} v_{xx}^2 \, dx\, ds\right)^{1/4} \\
&\quad + C\left(\max_{[0,t]} \int_\Omega v_x^2 \, dx\right)^{5/4} \left(\int_{Q_t} v_{xx}^2 \, dx\, ds\right)^{1/4} \le C\mathcal{T}^{\frac{3\beta_4}{2}}.
\end{aligned}$$

But we have also

$$\begin{aligned}
\int_0^t \max_\Omega \left(1+\theta^{q+1}\right) |v_x|^3 \, ds &\le C\mathcal{T}^{q+1} \int_0^t \left(\int_\Omega v_x^2 \, dx\right)^{5/4} \left(\int_{Q_t} v_{xx}^2 \, dx\, ds\right)^{1/2} ds \\
&\le C\mathcal{T}^{1+\frac{3\beta_4}{4}}.
\end{aligned}$$

These estimates give (2.44).

4. If we note $q_2 := \max\{\frac{q-3r-1}{2}, 0\}$, we observe that

$$\begin{aligned}
\int_{Q_t} \left(1+\theta^{q-r}\right) v_x^4 \, dx\, ds &\le C\mathcal{T}^{q_2} \int_{Q_t} \left(1+\theta^{\frac{q+r+1}{2}}\right) v_x^4 \, dx\, ds. \\
&\le C\mathcal{T}^{q_2} \int_0^t \left\{ \left(\int_\Omega v_x^2 \, dx\right)^{3/2} \left(\int_{Q_t} v_{xx}^2 \, dx\, ds\right)^{1/2} ds \right.
\end{aligned}$$

$$+V^{1/2}(s)\left(\int_\Omega v_x^2\,dx\right)^{3/2}\left(\int_{Q_t} v_{xx}^2\,dx\,ds\right)^{1/2}\Bigg\}\,ds$$

$$\leq C\mathcal{T}^{q_2}\left\{\max_{[0,t]}\int_\Omega v_x^2\,dx\times\left(\int_{Q_t} v_x^2\,dx\,ds\right)^{1/2}\left(\int_{Q_t} v_{xx}^2\,dx\,ds\right)^{1/2}\,ds\right.$$

$$\left.+\max_{[0,t]}\int_\Omega |v_x|^3\,dx\times\left(\int_0^t V(s)\,ds\right)^{1/2}\left(\int_{Q_t} v_{xx}^2\,dx\,ds\right)^{1/2}\right\}$$

$$\leq C\mathcal{T}^{q_2+2\beta_1}.$$

But we have also

$$\int_{Q_t}\left(1+\theta^{q-r}\right)v_x^4\,dx\,ds\leq C\mathcal{T}^{q-r}\int_{Q_t} v_x^4\,dx\,ds$$

$$\leq C\mathcal{T}^{q-r}\max_{[0,t]}\int_\Omega v_x^2\,dx\times\left(\int_{Q_t} v_x^2\,dx\,ds\right)^{1/2}\left(\int_{Q_t} v_{xx}^2\,dx\,ds\right)^{1/2}\,ds$$

$$\leq C\mathcal{T}^{q-r+\frac{3}{2}\beta_1}.$$

These estimates give (2.45) □ □

Lemma 2.12. *The following estimate holds*

$$\int_\Omega\left(\theta+\theta^{r+1}\right)^2\,dx+\int_{Q_T}\left(1+\theta^{q+r}\right)\theta_x^2\,dx\,dt\leq C\mathcal{T}^{\frac{3}{2}(r+1)}. \tag{2.46}$$

Proof. Multiplying the second equation (1.15) by $e+\frac{1}{2}\,v^2$ and integrating by parts on Q_t, we get

$$\frac{1}{2}\int_\Omega\left(e+\frac{1}{2}v^2\right)^2\,dx-\frac{1}{2}\int_\Omega\left(e^0+\frac{1}{2}v^{0^2}\right)^2\,dx+\int_{Q_t}\left(\frac{\mu}{\eta}\,v^2v_x^2+\frac{\kappa}{\eta}\,e_\theta\theta_x^2\right)\,dx\,ds$$

$$=\int_{Q_t} pve_\eta\eta_x\,dx\,ds+\int_{Q_t} pve_\theta\theta_x\,dx\,ds+\int_{Q_t} pv^2v_x\,dx\,ds-\int_{Q_t}\frac{\mu}{\eta}\,ve_\eta v_x\eta_x\,dx\,ds$$

$$-\int_{Q_t}\frac{\mu}{\eta}\,ve_\theta v_x\theta_x\,dx\,ds-\int_{Q_t}\frac{\kappa}{\eta}\,e_\eta\eta_x\theta_x\,dx\,ds-\int_{Q_t}\frac{\kappa}{\eta}\,vv_x\theta_x\,dx\,ds$$

$$-\int_{Q_t}\eta(s_E)_R\left(e+\frac{1}{2}\,v^2\right)\,dx\,ds=:\sum_{k=1}^{8}J_k.$$

First note that the left-hand side has the lower bound

$$\int_\Omega\left(e+\frac{1}{2}v^2\right)^2\,dx+\int_{Q_t}\frac{\mu}{\eta}\,v^2v_x^2\,dx\,ds+\frac{c_1c_6}{\overline{\eta}}\int_{Q_t}(1+\theta)^{r+q}\theta_x^2\,dx\,ds-C.$$

Let us estimate the right-hand side. As we follows the lines of [29], we only detail the first estimate and sketch the subsequent computations.

After Lemmas 2.2 to 2.11, we have the inequalities

$$\max_{[0,t]} v^2\leq C\mathcal{T}^{\frac{r+1}{2}},\ \int_\Omega v^2v_x^2\,dx\leq C\mathcal{T}^{\frac{3(r+1)}{2}}\ \text{ and }\int_{Q_t} v^2v_x^2\,dx\,ds\leq C\mathcal{T}^{\frac{r+1}{2}}.$$

Let ε be an arbitrary positive number.

$$|J_1| \le C\int_{Q_t}(1+\theta)^{2r+2}|v\eta_x|\,dx\,ds$$

$$\le C\int_{Q_t}(1+\theta)^{r+q+1}\eta_x^2\,dx\,ds + C\int_{Q_t}(1+\theta)^{r+q+1}v^2\,dx\,ds \le C.$$

$$|J_2| \le C\int_{Q_t}(1+\theta)^{2r+1}|v\theta_x|\,dx\,ds \le C + \varepsilon\int_{Q_t}(1+\theta)^{2r+1}|v\theta_x|\,dx\,ds.$$

$$|J_3| \le C\int_{Q_t}(1+\theta)^{r+1}v^2|v_x|\,dx\,ds \le C\mathcal{T}^{\frac{r+1}{4}}.$$

$$|J_4| \le C\int_{Q_t}(1+\theta)^{q+r+1}v^2|v_x|\,dx\,ds \le C\mathcal{T} + \varepsilon\int_{Q_t}(1+\theta)^{r+q}\theta_x^2\,dx\,ds.$$

$$|J_5| \le C\int_{Q_t}(1+\theta)^{q}|vv_x\theta_x|\,dx\,ds \le C\mathcal{T}^{\frac{3(r+1)}{2}} + \varepsilon\int_{Q_t}(1+\theta)^{r+q}\theta_x^2\,dx\,ds.$$

$$|J_6| \le C\int_{Q_t}(1+\theta)^{r}|vv_x\theta_x|\,dx\,ds \le C\mathcal{T}^{\frac{r+1}{2}} + \varepsilon\int_{Q_t}(1+\theta)^{r+q}\theta_x^2\,dx\,ds.$$

$$|J_7| \le C\int_{Q_t}(1+\theta)^{r}|vv_x\eta_x|\,dx\,ds \le C\mathcal{T}^{\frac{r+1}{2}} + \varepsilon\int_{Q_t}(1+\theta)^{r+q}\theta_x^2\,dx\,ds.$$

$$|J_8| \le \int_{Q_t}\eta|(s_E)_R|\left(e+\frac{1}{2}\,v^2\right)\,dx\,ds \le C + \int_{Q_t}\left(e+\frac{1}{2}v^2\right)^2\,dx\,ds.$$

Gathering all of these estimates, choosing $\varepsilon < \frac{c_1c_6}{5\eta}$ and using Gronwall's Lemma, we get (2.46) □ □

Lemma 2.13. *Let us introduce the two quantities*

$$Y(t) := \int_{\Omega}\left(1+\theta^{2q}\right)\theta_x^2\,dx, \quad X(t) := \int_{Q_t}\left(1+\theta^{q+r}\right)\theta_t^2\,dx\,ds.$$

The following estimate holds

$$X(t)+Y(t) \le C\mathcal{T}^{\beta_8}, \tag{2.47}$$

where $\beta_8 = \frac{6q-3r+3}{2}$.

Proof. From (1.15), the equation for the internal energy reads

$$e_\theta\theta_t + \theta p_\theta v_x - \frac{\mu}{\eta}\,v_x^2 = \left(\frac{\kappa\theta_x}{\eta}\right)_x - \eta\left(S_E\right)_R.$$

Defining the auxiliary function $K(\eta,\theta) := \int_0^\theta \frac{\kappa(\eta,u)}{u}\,du$, multiplying the previous equation by K_t and integrating by parts, we get

$$\int_{Q_T}\left(e_\theta\theta_t + \theta p_\theta v_x - \frac{\mu}{\eta}\,v_x^2\right)K_t\,dx\,ds + \int_{Q_T}\left(\frac{\kappa\theta_x}{\eta}\right)K_{tx}\,dx\,ds$$
$$-\int_{Q_T}\eta\left(S_E\right)_R K_t\,dx\,ds = 0. \tag{2.48}$$

Observing that $K_t = K_\eta v_x + \frac{\kappa}{\eta}\,\theta_t$, $K_{xt} = \left(\frac{\kappa\theta_x}{\eta}\right)_t + K_{\eta\eta} v_x \eta_x + \left(\frac{\kappa}{\eta}\right)_\eta \eta_x\theta_t$ and that after (1.26) $|K_\eta| + |K_{\eta\eta}| \le C(1+\theta^{q+1})$, we can estimate all the contributions in (2.48).

After (1.26) we have the lower bound

$$\int_{Q_T} \frac{\kappa e_\theta}{\eta}\,\theta_t^2\,dx\,ds \ge \frac{c_6 c_1}{\overline{\eta}}\,X(t),$$

Using (1.26) and Lemma 2.5

$$\left|\int_{Q_T} e_\theta\theta_t K_\eta v_x\,dx\,ds\right| \le C\int_{Q_T}(1+\theta)^{q+r+1}|\theta_t v_x|\,dx\,ds$$
$$\le \frac{c_6 c_1}{8\overline{\eta}}\,X + C\mathcal{T}^{r+2}.$$

In the same stroke

$$\left|\int_{Q_T}\left(\theta p_\theta v_x - \frac{\mu}{\eta}\,v_x^2\right)\frac{\kappa\theta_t}{\eta}\,dx\,ds\right| \le C\int_{Q_T}(1+\theta)^{q+r+1}|\theta_t v_x|\,dx\,ds$$
$$+C\int_{Q_T}(1+\theta)^q|\theta_t|v_x^2\,dx\,ds$$
$$\le \frac{c_6 c_1}{8\overline{\eta}}\,X + C\mathcal{T}^{r+2} + \int_{Q_T}(1+\theta)^{q-r}v_x^4\,dx\,ds.$$

The last term can be estimated as follows

$$\int_{Q_T}(1+\theta)^{q-r}v_x^4\,dx\,ds \le C\mathcal{T}^{q_3}\left(\int_{Q_T}v_x^4\,dx\,ds + \int_0^t V^{1/2}(s)\int_\Omega v_x^4\,dx\,ds\right)$$
$$\le C\mathcal{T}^{q_3}\left[\left(\int_{Q_T}v_x^2\,dx\,ds\right)^{1/2}\left(\int_{Q_T}v_{xx}^2\,dx\,ds\right)^{1/2}\max_{[0,t]}\int_\Omega v_x^2\,dx\right.$$
$$\left.+\max_{[0,t]}\left(\int_\Omega v_x^2\,dx\right)^{3/2}\left(\int_0^t V(s)\,ds\right)^{1/2}\left(\int_\Omega v_{xx}^2\,dx\right)^{1/2}\right] \le C\mathcal{T}^{q_4},$$

with $q_3 = \max\{\frac{q-3r-1}{2}, 0\}$ and $\beta_9 = q_3 + 2r + 2$. So we find

$$\left|\int_{Q_T}\left(\theta p_\theta v_x - \frac{\mu}{\eta}\,v_x^2\right)\frac{\kappa\eta_t}{\eta}\,dx\,ds\right| \le \frac{c_6 c_1}{8\overline{\eta}}\,X + C\mathcal{T}^{r+2} + C\mathcal{T}^{\beta_9}.$$

$$\left|\int_{Q_T}\left(\theta p_\theta v_x - \frac{\mu}{\eta}\,v_x^2\right)K_\eta v_x\,dx\,ds\right| \le C\int_{Q_T}(1+\theta)^{q+r+2}v_x^2\,dx\,ds$$
$$+C\int_{Q_T}(1+\theta)^{q+1}|v_x|^3\,dx\,ds \le C\mathcal{T}^{\beta_{10}},$$

with $\beta_{10} = \max\{r+2, q_4 + \frac{3(r+1)}{2}\}$ and $q_4 = \max\{0, \frac{q-3r+1}{4}\}$. Using (1.26) and Lemmas 2.10 and 2.12 we have

$$\left|\int_{Q_T}\frac{\kappa\theta_x}{\eta}\left(\frac{\kappa\theta_x}{\eta}\right)_s\,dx\,ds\right| \ge CY(t) - C,$$

$$\left|\int_{Q_T} \frac{\kappa\theta_x}{\eta} K_\eta v_{xx}\, dx\, ds\right| \le C\left(\int_{Q_T} (1+\theta)^{q+r}\theta_x^2\, dx\, ds\right)^{1/2}$$

$$\left(\int_{Q_T} (1+\theta)^{3q-r+2}|v_{xx}^2\, dx\, ds\right)^{1/2} \le C\mathcal{T}^{\beta_8},$$

and

$$\left|\int_{Q_T} \frac{\kappa\theta_x}{\eta} K_{\eta\eta} v_x \eta_x\, dx\, ds\right| \le C\mathcal{T}^{\beta_{11}},$$

with $\beta_{11} = \frac{3q+r+4}{2}$. Using (1.26) and Lemma 2.10 we have also

$$\left|\int_{Q_T} \frac{\kappa\theta_x}{\eta} \left(\frac{\kappa}{\eta}\right)_\eta \eta_x\theta_t\, dx\, ds\right| \le \frac{c_6 c_1}{8\overline{\eta}} X + C\mathcal{T}^{\beta_{12}},$$

with $\beta_{12} = \frac{6q-3r+3}{2}$.

Let us estimate the last term in (2.48).

$$\left|\int_{Q_T} \eta\,(S_E)_R K_t\, dx\, ds\right| \le \int_{Q_T}\left(\int_0^\infty\int_{S^1} \eta\sigma_a B\, d\nu\, d\omega\right)|K_t|\, dx\, ds$$

$$+\int_{Q_T}\left(\int_0^\infty\int_{S^1} \eta\sigma_a I\, d\nu\, d\omega\right)|K_t|\, dx\, ds$$

$$+\int_{Q_T}\left(\int_0^\infty\int_{S^1} \eta\sigma_s|\tilde{I}-I|\, d\nu\, d\omega\right)|K_t|\, dx\, ds =: P+Q+R.$$

After (1.26) and Lemma 2.3

$$P \le C\int_{Q_T} |K_t|\,(1+\theta^\alpha)\, dx\, ds$$

$$\le C\int_{Q_T} \left(1+\theta^{q+\alpha+1}\right)|v_x|\, dx\, ds + C\int_{Q_T}\left(1+\theta^{q+\alpha}\right)|\theta_t|\, dx\, ds =: A+B.$$

Using Cauchy-Schwarz inequality and Lemma 2.6 we have

$$A \le C\mathcal{T}^{r+2} + C\int_{Q_T}\left(1+\theta^{q+2\alpha-r}\right)\, dx\, ds \le C + C\mathcal{T}^{r+2},$$

and

$$B \le \frac{c_6 c_1}{8\overline{\eta}} X + C\int_{Q_T}\left(1+\theta^{q+r}\right)\, dx\, ds \le \frac{c_6 c_1}{8\overline{\eta}} X + C.$$

Using (1.26), Lemma 2.3 and Cauchy-Schwarz inequality we have

$$R \le C\int_{Q_T}\int_0^\infty\int_{S^1} \eta\sigma_s\left[|(\tilde{I}-I)K_\eta v_x + (\tilde{I}-I)K_\theta\theta_t|\right]\, d\nu\, d\omega\, dx\, ds$$

$$\le C\int_{Q_T}\int_0^\infty\int_{S^1} \eta\sigma_s|\tilde{I}-I|^2\, d\nu\, d\omega\, dx\, ds + C\int_{Q_T}\left(1+\theta^{2q+2}\right)v_x^2\, dx\, ds$$

$$+C\int_{Q_T}(1+\theta^q)\,|\tilde{I}-I|\,|\theta_t|\, dx\, ds$$

$$\leq C + \int_{Q_T} \left(1+\theta^{2q+2}\right) v_x^2 \, dx \, ds + \frac{c_6 c_1}{8\overline{\eta}} X + \int_{Q_T} \int_0^\infty \int_{S^1} \left(1+\theta^{q-r}\right) (\tilde{I}-I)^2 d\nu \, d\omega \, dx \, ds$$

$$\leq C + C\mathcal{T}^{q+2} + \frac{c_6 c_1}{8\overline{\eta}} X + C\mathcal{T}^{q-r}.$$

Using the same technique, we get also

$$Q \leq C + C\mathcal{T}^{q+2} + \frac{c_6 c_1}{8\overline{\eta}} X + C\mathcal{T}^{q-r}.$$

Combining all the previous inequalities, we get (2.47) which ends the proof □ □

Lemma 2.14. *All the quantities*

$$\max_{Q_T} \theta, \;\; X(t), \;\; Y(t), \tag{2.49}$$

for $t \in [0,T]$ are bounded.

Proof. As $\theta^{2q+2} \leq C \int_\Omega \theta^{2q+2} \, dx + (2q+2) \int_\Omega \theta^{2q+1} |\theta_x| \, dx$, we have

$$\mathcal{T}^{2q+2} \leq C\mathcal{T}^{2q+1-r} + C\left(\int_\Omega \theta^{2q+2} \, dx\right)^{1/2} \left(\int_\Omega \theta^{2q} \theta_x^2 \, dx\right)^{1/2} \leq C + C\mathcal{T}^{\beta_{13}},$$

where $\beta_{13} = \max\left\{2q+1-r, \frac{2q+1-r+\beta_8}{2}\right\}$. As one checks that $\beta_{13} < 2q+2$, (2.49) follows after Lemma 2.13 □ □

Corollary 2.15. *All the quantities*

$$\int_{Q_T} v_x^2 \, dx \, dt, \;\; \int_{Q_T} \theta_x^2 \, dx \, dt, \;\; \int_{Q_T} v_t^2 \, dx \, dt, \;\; \int_{Q_T} \theta_t^2 \, dx \, dt, \tag{2.50}$$

are bounded.

Proof. The first bound follows after Lemmas 2.11 and 2.14 the second bound after Lemmas 2.12 and 2.14, the third bound after Lemmas 2.10 and 2.14 and the last one after Lemmas 2.13 and 2.14 □ □

Lemma 2.16.

$$\max_{[0,T]} \int_\Omega \int_0^\infty \int_{S^1} I_t^2 \, d\omega \, d\nu \, dx \leq C, \tag{2.51}$$

$$\max_{[0,T]} \int_\Omega \int_0^\infty \int_{S^1} I_x^2 \, d\omega \, d\nu \, dx \leq C. \tag{2.52}$$

Proof. Going back to Eulerian coordinates, it is sufficient to prove that $I_\tau \in L^2(\mathbf{O}\times[0,T]\times O)$ with $\mathbf{O} := (0,L)$, and $I_y \in L^2(\mathbf{O}\times[0,T]\times O)$, where $I(y,\tau;\nu,\omega)$ solves the problem

$$\begin{cases} \dfrac{\partial}{\partial\tau} I(y,\tau;\nu,\omega) + c\omega \dfrac{\partial}{\partial y} I(y,\tau;\nu,\omega) \\ = c\sigma_a(\nu,\omega,\eta,\theta)\,[B(\nu,\theta) - I(y,\tau;\nu,\omega)] \\ +c\sigma_s(\nu,\eta,\theta)\left[\tilde{I}(y,\tau;\nu) - I(y,\tau;\nu,\omega)\right] =: S(I;y,\tau;\nu,\omega) \quad \text{on } \mathbf{O}\times[0,T]\times O, \\ I(0;\nu,\omega) = 0 \quad \text{for } \omega \in (0,1), \\ I(L;\nu,\omega) = 0 \quad \text{for } \omega \in (-1,0), \\ I(y,0;\nu,\omega) = I^0(y;\nu,\omega) \quad \text{on } \mathbf{O}\times O. \end{cases} \tag{2.53}$$

We can use a bootstrap method. Derivating the equation with respect to τ and putting $J := I_\tau$, one checks that J solves the problem

$$\begin{cases} J_\tau + c\omega J_y = S_\tau \quad \text{on } \mathbf{O}\times[0,T]\times O, \\ J(0;\nu,\omega) = 0 \quad \text{for } \omega \in (0,1), \\ J(L;\nu,\omega) = 0 \quad \text{for } \omega \in (-1,0), \\ J(y,0;\nu,\omega) = J^0(y,0;\nu,\omega) \\ = -\omega I^0_y(y;\nu,\omega) + S(I^0(y;\nu,\omega)) \quad \text{on } \mathbf{O}\times O, \end{cases} \tag{2.54}$$

with the right-hand side

$$S_\tau = S(J;y,\tau;\nu,\omega) + \Phi(I;y,\tau;\nu,\omega),$$

where $\Phi \in L^2(\mathbf{O}\times[0,T]\times O)$, after the Eulerian counterparts of Lemmas 2.2-2.16. Note that we have used the equation to derive the initial condition.

Now we proceed as in Lemma 2.3. Multiplying equation (2.54) by J integrating by parts on $[0,\tau]\times\mathbf{O}\times O$ and using Cauchy-Schwarz, we get

$$\begin{aligned} &\frac{1}{2}\int_{\mathbf{O}}\int_O J^2\,d\omega\,d\nu\,dy - \frac{1}{2}\int_{\mathbf{O}}\int_O (J^0)^2\,d\omega\,d\nu\,dy \\ &+\frac{c}{2}\int_0^\tau\int_{O_+}\omega J^2(L,\tau;\nu,\omega)\,d\omega\,d\nu\,ds - \frac{c}{2}\int_0^\tau\int_{O_-}\omega J^2(0,\tau;\nu,\omega)\,d\omega\,d\nu\,ds \\ &+\frac{1}{2}\int_0^\tau\int_{\mathbf{O}}\int_O \eta\sigma_a J^2\,d\omega\,d\nu\,dy\,ds + \int_0^\tau\int_{\mathbf{O}}\int_O \eta\sigma_s\left(\tilde{J}-J\right)^2\,d\omega\,d\nu\,dy\,ds \\ &\leq \frac{1}{2}\int_0^\tau\int_{\mathbf{O}}\int_O \eta\sigma_a\,B^2\,d\omega\,d\nu\,dy\,ds + \frac{1}{2}\int_0^\tau\int_{\mathbf{O}}\int_O \Phi^2\,d\omega\,d\nu\,dy\,ds. \end{aligned}$$

After (1.26) the right-hand side is bounded, so this last inequality clearly implies (2.51).

In the same stroke, derivating the equation with respect to y and putting $K := I_y$, one checks that K solves the problem

$$\left\{\begin{array}{c} K_\tau + c\omega K_y = S_\tau \quad \text{on } \mathbf{O}\times[0,T]\times O, \\ K(0;\nu,\omega) = \dfrac{\sigma_a B}{\omega}(0;\nu,\omega) \quad \text{for } \omega\in(0,1), \\ K(L;\nu,\omega) = \dfrac{\sigma_a B}{\omega}(L;\nu,\omega) \quad \text{for } \omega\in(-1,0), \\ K(y,0;\nu,\omega) = K^0(y,0;\nu,\omega) = I^0_y(y;\nu,\omega) \quad \text{on } \mathbf{O}\times O, \end{array}\right. \tag{2.55}$$

with the right-hand side

$$S_y = S(K;y,\tau;\nu,\omega) + \Psi(K;y,\tau;\nu,\omega),$$

where $\Psi\in L^2(\mathbf{O}\times[0,T]\times O)$, after the Eulerian counterparts of Lemmas 2.2-2.16. Observe that after (1.26), the boundary conditions are meaningfull.

As previously, multiplying equation (2.55) by K and integrating by parts on $[0,\tau]\times\mathbf{O}\times O$, we get

$$\begin{aligned} &\frac{1}{2}\int_{\mathbf{O}}\int_O K^2\,d\omega\,d\nu\,dy - \frac{1}{2}\int_{\mathbf{O}}\int_O (K^0)^2\,d\omega\,d\nu\,dy \\ &+c\int_0^\tau\int_{O_+}\omega K^2(L,\tau;\nu,\omega)\,d\omega\,d\nu\,ds - c\int_0^\tau\int_{O_-}\omega K^2(0,\tau;\nu,\omega)\,d\omega\,d\nu\,ds \\ &+\frac{1}{2}\int_0^\tau\int_{\mathbf{O}}\int_O \sigma_a K^2\,d\omega\,d\nu\,dy\,ds + \int_0^\tau\int_{\mathbf{O}}\int_O \eta\sigma_s\left(\tilde{K}-K\right)^2\,d\omega\,d\nu\,dy\,ds \\ &\le \frac{1}{2}\int_0^\tau\int_{\mathbf{O}}\int_O \eta\sigma_a\,B^2\,d\omega\,d\nu\,dy\,ds + \frac{1}{2}\int_0^\tau\int_{\mathbf{O}}\int_O \Phi^2\,d\omega\,d\nu\,dy\,ds. \end{aligned}$$

This inequality clearly implies (2.52) □ □

Lemma 2.17.

$$\max_{[0,T]}\int_\Omega v_t^2\,dx + \int_{Q_T} v_{xt}^2\,dx\,dt \le C, \tag{2.56}$$

$$\max_{[0,T]}\int_\Omega v_{xx}^2\,dx \le C, \tag{2.57}$$

$$\max_{[0,T]}\int_\Omega \eta_x^2\,dx \le C. \tag{2.58}$$

Proof. 1. Formally derivating the second equation (1.15) with respect to t, multiplying by v_t, integrating by parts and using (1.26), we find

$$\frac{1}{2}\int_\Omega v_t^2\,dx + \frac{\mu_0}{2\overline{\eta}}\int_{Q_T} v_{xt}^2\,dx\,dt$$

$$\leq C+C\int_{Q_T}\left[p_t^2+\left(\frac{\mu}{\eta}\right)_\eta^2 v_x^4\right]\,dx\,dt+\int_{Q_T}|v_t\,[(S_F)_R]_t|\,dx\,dt.$$

$$\leq C+C\int_{Q_T}\left(1+\theta^{2r+2}\right)v_x^2\,dx\,dt+C\int_{Q_T}\int_{Q_T}\left(1+\theta^{2r+2}\right)v_x^2\,dx\,dt\,dx\,dt$$

$$+C\int_{Q_T}v_x^4\,dx\,dt+\varepsilon\max_{[0,T]}\int_\Omega v_t^2\,dx\,dt+C\int_{Q_T}((S_F)_R)_t^2\,dx\,dt.$$

As $|((S_F)_R)_t|\leq C+C\,[(1+\theta^\alpha)|\theta_t|+|v_x|+|I_t|]$, we get

$$\frac{1}{2}\int_\Omega v_t^2\,dx+\frac{\mu_0}{2\overline{\eta}}\int_{Q_T}v_{xt}^2\,dx\,dt$$

$$\leq C+C\int_{Q_T}\left[v_x^2+v_x^4+\theta_t^2+I_t^2\right]\,dx\,dt+\varepsilon\max_{[0,T]}\int_\Omega v_t^2\,dx\,dt$$

$$\leq C+\varepsilon\max_{[0,T]}\int_\Omega v_t^2\,dx\,dt\leq C,$$

for ε small enough, which proves (2.56).

2. From the second equation (1.15)

$$v_{xx}=\frac{\eta}{\mu}\left[v_t+p_x-\left(\frac{\mu}{\eta}\right)_\eta\eta_x v_x+\eta(S_F)_R\right],$$

then we get

$$\int_\Omega v_{xx}^2\,dx\leq C+C\int_\Omega\left[v_t^2+\eta_x^2+\theta_x^2+\eta_x^2 v_x^2\right]\,dx,$$

which implies (2.57), after (2.56)

3. Using the first equation (1.15), one gets

$$|\eta_x|\leq C+C\left(\int_0^T v_{xx}^2\,dx\right)^{1/2},$$

so

$$\int_\Omega\eta_x^2\,dx\leq C+C\int_\Omega v_{xx}^2\,dx\,dt\leq C,$$

after (2.57) □ □

Lemma 2.18. *Under the previous condition on the data there exists positive constant* $\overline{\theta}$ *and* $\underline{\theta}$ *depending on* T *and* N *such that*

$$0<\underline{\theta}\leq\theta(x,t)\leq\overline{\theta}\ \textit{for}\ (t,x)\in Q_T. \tag{2.59}$$

Proof. Applying the maximum principle to the parabolic equation

$$e_\theta\theta_t+\theta p_\theta v_x-\frac{\mu}{\eta}\,v_x^2=\left(\frac{\kappa\theta_x}{\eta}\right)_x-\eta\,(S_E)_R,$$

observing that the terms $\theta p_\theta v_x$ and $\eta\,(S_E)_R$ are bounded and using Lemma 2.14, we get (2.59) □ □

Lemma 2.19. *All the quantities*

$$\max_{Q_T}|v_x|,\ \max_{[0,T]}\int_\Omega v_x^2\,dx,\ \int_{Q_T} v_x^4\,dx,\ \max_{[0,T]}\int_\Omega v_t^2\,dx,\ \int_{Q_T} v_{xt}^2\,dx\,dt,$$

are bounded.

Proof. The first quantity is bounded after Lemma 2.17, the second one is bounded after Lemma 2.10, the third is bounded after Lemma 2.11 and the boudedness of the two last quantities follows after Lemma 2.17 □ □

Lemma 2.20. *The following estimate holds*

$$\int_{Q_T}\theta_x^4\,dx\,dt \le C. \tag{2.60}$$

Proof. 1. After the inequality

$$\int_{Q_T}\theta_x^4\,dx\,dt \le \int_0^t \max_\Omega \theta_x^2 \int_\Omega \theta_x^2\,dx\,ds,$$

and Lemma 2.14, we have

$$\int_{Q_T}\theta_x^4\,dx\,dt \le C\int_0^t \max_\Omega \theta_x^2\,ds \tag{2.61}$$

so, in order to prove (2.60), it is sufficient to bound the right-hand side.

First multiplying the equation of the internal energy

$$e_\theta\theta_t + \theta p_\theta v_x - \frac{\mu}{\eta}\,v_x^2 = \left(\frac{\kappa\theta_x}{\eta}\right)_x - \eta\,(S_E)_R\,,$$

by $\frac{\eta}{\kappa}\,\theta_t$ and integrating on Q_t, we get

$$\begin{aligned}\int_{Q_t}\frac{\eta e_\theta}{\kappa}\,\theta_t^2\,dx\,dt + \int_{Q_t}\frac{\eta\theta p_\theta}{\kappa}\,\theta_t v_x\,dx\,dt - \int_{Q_t}\frac{\mu}{\kappa}\,\theta_t\,v_x^2\,dx\,dt \\ = \int_{Q_t}\frac{\eta}{\kappa}\,\theta_t\left(\frac{\kappa\theta_x}{\eta}\right)_x dx\,dt - \int_{Q_t}\frac{\eta^2}{\kappa}\,\theta_t\,(S_E)_R\,dx\,dt.\end{aligned} \tag{2.62}$$

The first term in the right-hand side of (2.62) rewrites

$$\begin{aligned}\int_{Q_t}\frac{\eta}{\kappa}\,\theta_t\left(\frac{\kappa\theta_x}{\eta}\right)_x dx\,dt = \int_{Q_t}\frac{\kappa_\theta}{\kappa}\,\theta_t\theta_x^2\,dx\,dt + \int_{Q_t}\left(\frac{\kappa_\eta}{\kappa} - \frac{1}{\kappa}\right)\theta_t\theta_x\eta_x\,dx\,dt \\ + \int_{Q_t}\theta_t\theta_{xx}\,dx\,dt.\end{aligned}$$

Integrating by parts, plugging into (2.62) and using Lemmas (2.5) and (2.18), we get

$$\int_{Q_t}\frac{\eta e_\theta}{\kappa}\,\theta_t^2\,dx\,dt + \frac{1}{2}\int_\Omega \theta_x^2\,dx\,dt$$

$$\leq C + C\int_{Q_t} \left\{ |\theta_t v_x| + |\theta_t| v_x^2 + |\theta_t| \theta_x^2 + |\theta_t \theta_x \eta_x| + |\theta_t (S_E)_R| \right\} \, dx\, dt.$$

So, for any $\varepsilon > 0$

$$\int_{Q_t} \frac{\eta e_\theta}{\kappa}\, \theta_t^2 \, dx\, dt + \frac{1}{2}\int_\Omega \theta_x^2 \, dx\, dt$$

$$\leq C + \varepsilon \int_{Q_t} \theta_t^2 \, dx\, dt + C\int_0^t \max_\Omega \theta_x^2 \, ds + \int_{Q_t} \theta_x^2 \eta_x^2 \, dx\, dt$$

$$\leq C + \varepsilon \int_{Q_t} \theta_t^2 \, dx\, dt + C\int_0^t \max_\Omega \theta_x^2 \, ds + \int_0^t \max_\Omega \theta_x^2 \int_\Omega \eta_x^2 \, dx\, ds.$$

Finally

$$\int_{Q_t} \theta_t^2 \, dx\, dt + \int_\Omega \theta_x^2 \, dx\, dt \leq C + C\int_0^t \max_\Omega \theta_x^2 \, ds. \tag{2.63}$$

2. Multiplying the previous equation of the internal energy by $\frac{(x-M)\kappa}{\eta}\, \theta_x$ and integrating on Ω, we get

$$\int_\Omega \frac{(x-M)\kappa e_\theta}{\eta}\, \theta_t \theta_x \, dx + \int_\Omega \frac{(x-M)\kappa}{\eta}\, \theta p_\theta \theta_x v_x \, dx - \int_\Omega \frac{(x-M)\mu\kappa}{\eta^2} \theta_x \, v_x^2 \, dx$$

$$= \int_\Omega (x-M)\, \frac{\kappa\theta_x}{\eta} \left(\frac{\kappa\theta_x}{\eta}\right)_x dx - \int_\Omega (x-M)\kappa\, \theta_x (S_E)_R \, dx.$$

Then integrating in t and using boundary conditions, we have the estimate

$$\frac{1}{2}\int_0^t \left(\frac{\kappa\theta_x}{\eta}\right)^2 (0,s)\, ds \leq C + C\int_{Q_t} \left\{ \theta_x^2 + \theta_t^2 + v_x^2 + v_x^4 \right\} \, dx\, dt.$$

So we end with

$$\int_0^t \left(\frac{\kappa\theta_x}{\eta}\right)^2 (0,s)\, ds \leq C. \tag{2.64}$$

3. Multiplying now the same equation of the internal energy by $\frac{\kappa}{\eta}\, \theta_x$ and integrating on $[0,x]$, we get

$$\int_0^x \frac{\kappa e_\theta}{\eta}\, \theta_t \theta_y \, dy + \int_0^x \frac{\kappa}{\eta}\, \theta p_\theta \theta_y v_y \, dy - \int_0^x \frac{\mu\kappa}{\eta^2} \theta_y \, v_y^2 \, dy$$

$$= \int_0^x \frac{\kappa\theta_y}{\eta} \left(\frac{\kappa\theta_y}{\eta}\right)_y dy - \int_0^x \kappa\, \theta_y (S_E)_R \, dy.$$

Then integrating in t and using boundary conditions, we have the estimate

$$\frac{1}{2}\int_0^t \left(\frac{\kappa\theta_x}{\eta}\right)^2 (x,s)\, ds \leq \frac{1}{2}\int_0^t \left(\frac{\kappa\theta_x}{\eta}\right)^2 (0,s)\, ds$$

$$+ C + C\int_{Q_t} \left\{ \theta_x^2 + \theta_t^2 + v_x^2 + v_x^4 \right\} \, dx\, dt.$$

So after (2.64) we end with

$$\int_0^t \left(\frac{\kappa\theta_x}{\eta}\right)^2 (x,s)\, ds \leq C, \tag{2.65}$$

which gives (2.60). □

Lemma 2.21. *All the quantities*

$$\max_{[0,T]} \int_\Omega \theta_x^2 \, dx, \ \ \max_{[0,T]} \int_\Omega \theta_{xx}^2 \, dx, \ \ \int_{Q_T} \theta_{xt}^2 \, dx \, dt, \tag{2.66}$$

are bounded.

Proof. 1. By derivating formally the internal energy equation with respect to t, multiplying by $e_\theta \theta_t$ and using integration by parts on Q_T, we get

$$\frac{1}{2}\int_\Omega (e_\theta\theta_t)^2(x,t)\,dx - \frac{1}{2}\int_\Omega (e_\theta\theta_t)^2(x,0)\,dx + \int_{Q_T} p_\theta v_x e_\theta \theta_t^2 \,dx\,dt$$

$$+\int_{Q_T} \theta p_{\theta\theta} v_x e_\theta \theta_t^2\,dx\,dt + \int_{Q_T} \theta p_{\theta\eta} v_x^2 e_\theta\theta_t\,dx\,dt + \int_{Q_T}\theta p_\theta v_{xt} e_\theta\theta_t\,dx\,dt$$

$$-\int_{Q_T}\left[\left(\frac{\mu(\eta)}{\eta}\right)_\eta v_x^3 + 2\frac{\mu(\eta)}{\eta}\,v_x v_{xt}\right] e_\theta\theta_t\,dx\,dt$$

$$= -\int_{Q_T}\frac{\kappa}{\eta} e_\theta \theta_{tx}^2\,dx\,dt - \int_{Q_T}\left[\left(\frac{\kappa}{\eta}\right)_\eta v_x\theta_x + \frac{\kappa_\theta}{\eta}\,\theta_t\theta_x\right](e_\theta\theta_t)_x\,dx\,dt$$

$$-\int_{Q_T}\theta_x\left(e_{\theta\eta}\eta_x + e_{\theta\theta}\theta_x\right)\,dx\,dt - \int_{Q_T}\eta\left[(S_E)_R\right]_t\, e_\theta\theta_t\,dx\,dt - \int_{Q_T} v_x (S_E)_R\, e_\theta\theta_t\,dx\,dt.$$

After [6] (see the proof of Lemma 3.6), we get

$$\frac{1}{2}\int_\Omega (e_\theta\theta_t)^2(x,t)\,dx + \int_{Q_T}\frac{\kappa}{\eta}e_\theta\theta_{tx}^2\,dx\,dt$$

$$\le C - \int_{Q_T}\left[(S_E)_R\right]_t\, e_\theta\theta_t\,dx\,dt - \int_{Q_T} v_x (S_E)_R\, e_\theta\theta_t\,dx\,dt. \tag{2.67}$$

As the two integrals in the right-hand side are bounded after Lemmas 2.16 and 2.20, we obtain the first two estimates (2.66).

2. From the internal energy equation

$$\frac{\kappa}{\eta}\,\theta_{xx} = \left(\frac{\kappa - \eta\kappa_\eta}{\eta^2}\right)\eta_x\theta_x - \frac{\kappa_\theta}{\eta}\,\theta_x^2 + e_\theta\theta_t + \theta p_\theta v_x - \frac{\mu}{\eta}\,v_x^2 + \eta\,(S_E)_R\,,$$

then

$$|\theta_{xx}| \le C\left(|\eta_x\theta_x| + \theta_x^2 + |\theta_t| + |v_x| + v_x^2 + |\,(S_E)_R\,|\right),$$

where all of the terms in the right-hand side are in $L^2(\Omega)$ after Lemmas 2.21, 2.20, 2.19 and 2.7, which proves the last bound (2.66) □ □

Proof of Theorem 2.1

1. As $\max_{Q_T}|v_x|$ is bounded we have

$$\left|\eta(x,t) - \eta(x,t')\right| \le |t-t'|^{1/2}\left(\int_0^T v_x^2\,dt\right)^{1/2} \le C|t-t'|^{1/2}.$$

We have also

$$\left|\eta(x,t)-\eta(x',t)\right| \le C|x-x'|^{1/2}\left(1+\int_\Omega \eta_x^2\,dx\right) \le C|x-x'|^{1/2},$$

so we find that $\eta \in C^{1/2,1/4}(Q_T)$.

2. After Lemma 2.21 we have

$$\left|\theta(x,t)-\theta(x,t')\right| \le |t-t'|^{1/2}\left(\int_0^T \theta_t^2\,dt\right)^{1/2}$$

$$\le C|t-t'|^{1/2}\left(\int_0^T\int_\Omega 2|\theta_t\theta_{xt}|\,dx\,dt\right)^{1/2} \le C|t-t'|^{1/2}.$$

We see also that

$$\left|\theta(x,t)-\theta(x',t)\right| \le C|x-x'|^{1/2}\left(T\cdot\max_{[0,T]}\int_\Omega \theta_t^2\,dx+\int_0^T\int_\Omega \theta_{xt}^2\,dx\right)$$

$$\le C|x-x'|^{1/2},$$

so we find that $\theta \in C^{1/2,1/4}(Q_T)$. We have also

$$\left|\theta_x(x,t)-\theta_x(x',t)\right| \le |x-x'|^{1/2}\left(\int_\Omega \theta_{xx}^2\,dt\right)^{1/2} \le |x-x'|^{1/2},$$

we conclude, by using an interpolation argument of [22], that $\theta_x \in C^{1/3,1/6}(Q_T)$.

3. The same arguments holding verbatim for v and v_x, we have that $v, v_x \in C^{1/3,1/6}(Q_T)$.

4. Let us note $I(x,t) := \int_0^\infty \int_{S^1} I(x,t;\omega,\nu)\,d\omega\,d\nu$.

As $\max_{[0,T]}\|I_t\|_{L^2(\Omega)} \le C$, after Lemma 2.16, it follows that

$$|I(x,t)-I(x',t)| \le \int_{x'}^x |I_y|\,dy \le C|x-x'|^{1/2}.$$

As $\max_{[0,T]}\|I_x\|_{L^2(\Omega)} \le C$, also after Lemma 2.16, it also follows that

$$|I(x,t)-I(x,t')| \le \int_{t'}^t |I_s|\,ds \le C|t-t'|^{1/2}.$$

Then we conclude in particular that $I \in C^{1/3,1/6}(Q_T)$, which ends the proof $\square$.

3 Existence and uniqueness of solutions

In this section we prove existence of classical solution by means of the classical Leray-Schauder fixed point theorem in the same spirit as in [20, 6], then using a limiting process as in [11] we will get the existence of a weak solution.

Let us recall the Leray- Schauder fixed point theorem

Theorem 3.1. *Let $\mathcal{B}$ be a Banach space and suppose that $P : [0,1]\times\mathcal{B} \to \mathcal{B}$ has the following properties:*

- *i) For any fixed $\lambda \in [0,1]$ the map $P(\lambda,.) : \mathcal{B} \to \mathcal{B}$ is completely continuous.*
- *ii) For every bounded subset $\mathcal{S} \subset \mathcal{B}$ the family of maps $P(.,\chi) : [0,1] \to \mathcal{B}$, $\chi \in \mathcal{S}$ is uniformly equicontinuous.*
- *iii) There is a bounded subset $\mathcal{S}$ of $\mathcal{B}$ such that any fixed point in $\mathcal{B}$ of $P(\lambda,.), \lambda \in [0,1]$ is contained in $\mathcal{S}$.*
- *iv) $P(0,.)$ has precisely one fixed point in $\mathcal{B}$.*

Then, $P(1,.)$ has at least one fixed point in $\mathcal{B}$.

In our case $\mathcal{B}$ will the Banach space of functions (η, v, θ, I) on Q_T such that $\eta, v, v_x, \theta, \theta_x, I \in C^{1/3,1/6}(Q_T)$, with the norm

$$|||(\eta, v, \theta)|||_{\mathcal{B}} := |||\eta|||_{1/3} + |||v|||_{1/3} + |||v_x|||_{1/3} + |||\theta|||_{1/3} + |||\theta_x|||_{1/3} + |||I|||_{1/3}.$$

For $\lambda \in [0,1]$ we define $P(\lambda,.)$ as the map which carries $\{\tilde{\eta}, \tilde{v}, \tilde{\theta}, \tilde{I}\} \in \mathcal{B}$ into $\{\eta, v, \theta, I\} \in \mathcal{B}$ by solving the system

$$\begin{cases} \eta_t = v_x, \\ v_t - \dfrac{\mu}{\tilde{\eta}}\, v_{xx} = -\dfrac{\mu}{\tilde{\eta}^2}\, \tilde{\eta}_x \tilde{v}_x - \tilde{p}_\eta(\tilde{\eta},\tilde{\theta})\eta_x - \tilde{p}_\theta(\tilde{\eta},\tilde{\theta})\theta_x - \lambda\tilde{\eta}\left(\tilde{S}_F\right)_R, \\ \tilde{e}_\theta(\tilde{u},\tilde{\theta})\theta_t - \dfrac{\tilde{\kappa}(\tilde{\eta},\tilde{\theta})}{\tilde{\eta}}\, \theta_{xx} = \left(\dfrac{\tilde{\kappa}(\tilde{\eta},\tilde{\theta})}{\tilde{\eta}}\right)_\eta \tilde{\theta}_x \eta_x + \dfrac{\tilde{\kappa}_\theta(\tilde{\eta},\tilde{\theta})}{\tilde{\eta}}\, \tilde{\theta}_x^2 \\ \qquad + \dfrac{\mu}{\tilde{\eta}}\, \tilde{v}_x^2 - \tilde{\theta}\tilde{p}_\theta(\tilde{\eta},\tilde{\theta})\tilde{v}_x - \lambda\tilde{\eta}\left(\tilde{S}_E\right)_R, \\ I_t + \tilde{\eta}^{-1}(c\omega - \tilde{v})I_x = \tilde{\sigma}_a(\tilde{B} - I) + \tilde{\sigma}_s\left(\displaystyle\int_{S^1} I\, d\omega - I\right), \end{cases} \tag{3.1}$$

together with the boundary conditions

$$\begin{cases} v|_{x=0,M} = 0, \\ \theta_x|_{x=0} = 0,\ \ \theta|_{x=M} = 0, \end{cases} \tag{3.2}$$

and

$$\begin{cases} I|_{x=0} = 0 \ \text{ for } \omega > 0, \\ I|_{x=M} = 0 \ \text{ for } \omega < 0, \end{cases} \tag{3.3}$$

for $t > 0$, and initial conditions

$$\begin{cases} \eta(x,0) = (1-\lambda) + \lambda\eta^0(x), \\ v(x,0) = \lambda v^0(x), \\ \theta(x,0) = (1-\lambda) + \lambda\theta^0(x), \\ I(x,0;\omega,\nu) = I^0(x;\omega,\nu). \end{cases} \tag{3.4}$$

We can consider the second and the third equations of (64) as parabolic type and apply the classical Schauder-Friedmann estimates

$$\|v\|_{1/3}+\|v_x\|_{1/3}\le c\{\|\eta\|_{1/3}+\|\tilde{v}\|_{1/3}+\|\tilde{\theta}_x\|_{1/3}+\|\tilde{I}\|_{1/3}\}$$

$$\|\theta\|_{1/3}+\|\theta_x\|_{1/3}\le c\{\|\tilde{\theta}_x\|_{1/3}+\|\tilde{v}_x\|_{1/3}+\|\tilde{I}\|_{1/3}\}.$$

After properties of strong solutions of transport equation (see [7]), we have also

$$\|I\|_{1/3}\le c\{\|\tilde{\eta}\|_{1/3}+\|\tilde{\theta}\|_{1/3}+\|\tilde{v}\|_{1/3}\}.$$

Moreover from the first equation (3.1), we get

$$\|\eta\|_{1/3}\le\|v_x\|_{1/3}.$$

It implies that $P(\lambda,.):\mathcal{B}\to\mathcal{B}$ is well defined and continuous.

Using a priori estimates from Section 2 it follows that for any $\{\tilde{\eta},\tilde{v},\tilde{\theta},\tilde{I}\}$ from any fixed bounded subset the family $P(.,\{\tilde{\eta},\tilde{v},\tilde{\theta},\tilde{I}\}):[0,1]\to\mathcal{B}$ of mappings is uniformly equicontinuous.

Now, in order to verify (iii), we observe that any fixed point of P will initially satisfy original problem, therefore η and θ cannot escape from $[\underline{\eta},\overline{\eta}]$, $[\underline{\theta},\overline{\theta}]$ up to time T. This fact follows clearly from Theorem 2.1.

In order to check (iv) we see by inspection that the unique fixed point of $P(0,.)$ is given by

$$\{\eta(x,t)=1,\ v(x,t)=0,\ \theta(x,t)=1,\ \phi(x,t)\},$$

where $\phi(x,t)$ is the unique solution (see [7]) of the linear equation

$$\phi_t+c\omega\phi_x=S^0(\phi),$$

where the source is

$$S^0(\phi)=\sigma_a(\nu,\omega;1,1)\,[B(\nu,\omega;0,1)-\phi]+\sigma_s(\nu;1,1)\,\left[\tilde{\phi}-\phi\right],$$

satisfying absorbing conditions (3.3) and initial condition $\phi(x,0;\omega,\nu)=I^0(x;\omega,\nu)$.

Note that the simpler (and "physical") choice $\phi^0(x;\omega,\nu))=B(\nu,\omega;0,1)$ is not allowed: as the compatibility boundary conditions are not satisfied, the corresponding solution ϕ is discontinuous and then lies out of $\mathcal{B}$.

All the previous facts allow us to apply Theorem 3.1, which imply the existence of classical solutions of (1.15)-(1.23) in $\Omega\times(0,T)$.

This ends the proof of Theorem 1.4.

Let us now consider the existence of a weak solution. From previous results it follows

- $v_k\to v$ in $L^p(0,T;C^0(\Omega))$ strongly and in $L^p(0,T;H^1(\Omega))$ weakly for $1<p<\infty$,
- $v_k\to v$ a.e. in $\Omega\times[0,T]$ and in $L^\infty(0,T);L^4(\Omega))$ weakly *,
- $(v_k)_t\to v_t$ in $L^2(0,T,L^2(\Omega))$ weakly,

- $\theta_k \to \theta$ in $L^2(0,T,C^0(\Omega))$ strongly and in $L^2(0,T,H^1(\Omega))$ weakly,
- $\theta_k \to \theta$ a.e. in $\Omega \times [0,T]$ and in $L^\infty(0,T;L^2(\Omega))$ weakly,
- $\sigma_k \to A_1$ in $L^2(0,T;H^1(\Omega))$ weakly,

It implies that

$$\eta_k \to \eta \text{ a.e. in } \Omega \times [0,T] \text{ and } L^s(\Omega \times [0,T]) \text{ strongly for all } s \in [1,\infty[.$$

All this implies that

- $\frac{\kappa_k(\theta_k)_x}{\eta_k} \to A_2$ weakly in $L^2(0,T,H^1(\Omega))$,
- $\frac{\mu(\eta_k)}{\eta_k}\,(v_k)_x \to A_3$ in $L^\infty(0,T,L^2(\Omega))$ weakly *,
- $\eta_k\{(S_E)_R\}_k \to A_4$ in $L^2(0,T;H^1(\Omega))$ weakly.

Then applying similar technique as in [11] it follows that

- $A_1 = \sigma$ in $L^2(0,T);H^1(\Omega)$,
- $A_2 = \frac{\kappa\theta_x}{\eta}$ in $L^2(0,T,L^2(\Omega))$,
- $A_3 = \frac{\mu(\eta)}{\eta}\,v_x$ in $L^2(0,T,H^1(\Omega))$,
- $A_4 = \eta(S_E)_R$ in $L^2(0,T,H^1(\Omega))$, which ends the proof of the existence of a weak solution.

Finally we prove uniqueness of the solution.

Let $\eta_i, v_i, \theta_i, I_i$, $i = 1,2$ be two solutions of (1.15), and let us consider the differences: $E = \eta_1 - \eta_2$, $T = \theta_1 - \theta_2$, $V = v_1 - v_2$, $J = I_1 - I_2$ and $\mathcal{J} = I_1 - I_2$.

From the first equation (1.15) written for η_1, v_1 and η_2, v_2 subtracting, multiplying by a test function χ, integrating by part and putting $\chi = E$ we obtain

$$\frac{1}{2}\frac{d}{dt}\int_\Omega E^2 dx = \int_\Omega E V_x\, dx \le \|E\|_2 \|V_x\|_2.$$

Using Cauchy Schwarz inequality for $\varepsilon > 0$

$$\frac{d}{dt}\int_\Omega E^2 dx \le \varepsilon \|V_x\|_2^2 + C_\varepsilon \big\|E\|_2^2. \tag{3.5}$$

Rewriting the second equation (1.15) for v_2 and v_1, subtracting, multiplying by a test function ϕ, integrating by part and putting $\phi = V$ we obtain the following

$$\frac{1}{2}\frac{d}{dt}\int_\Omega V^2 dx + \int_\Omega \frac{\mu_2}{\eta_2}\, V_x^2\, dx = -\sum_{i=1}^4 \mathcal{A}_i,$$

with

$$|\mathcal{A}_1| = \left|\int_\Omega (p_2 - p_1)\, V_x\, dx\right|$$

$$\leq C\|V_x\|_2\big(\|T\|_2+\|E\|_2\big) \leq \varepsilon\|V_x\|_2^2+C_\varepsilon\big(\|T\|_2^2+\|E\|_2^2\big),$$

where we used Cauchy Schwarz inequality for $\varepsilon > 0$.

In the same stroke

$$|\mathcal{A}_2| = \left|\int_\Omega \frac{1}{\eta_1}\,(\mu_1-\mu_2)v_{2x}V_x\,dx\right| \leq C\|V_x\|_2\|E\|_2 \leq \varepsilon\|V_x\|_2^2+C_\varepsilon\|E\|_2^2,$$

$$|\mathcal{A}_3| = \left|\int_\Omega \frac{\eta_2-\eta_1}{\eta_2\eta_1}\,\mu_2 v_{2x}V_x\,dx\right| \leq C\|E\|_2\|V_x\|_2 \leq \varepsilon\|V_x\|_2^2+C_\varepsilon\|E\|_2^2,$$

and

$$|\mathcal{A}_4| = \left|\int_\Omega\int_0^\infty\int_{S^1} (\eta_1[(S_E)_R]_1-\eta_2[(S_E)_R]_2)\,V\,d\omega\,d\nu\,dx\right|$$
$$\leq C\left(\|E\|_2^2+\|V\|_2^2+\|T\|_2^2+\|\mathcal{I}\|_2^2\right).$$

So we get finally, taking ε small enough

$$\frac{d}{dt}\int_\Omega V^2dx+\int_\Omega V_x^2\,dx \leq C\big(\|T\|_2^2+\|E\|_2^2+\|\mathcal{I}\|_2^2\big). \tag{3.6}$$

Now, dividing the energy equation by e_θ, we have

$$\theta_t = -\frac{\theta p_\theta}{e_\theta}\,w_x+\frac{q_x}{e_\theta}+\frac{\mu}{\eta e_\theta}\,v_x^2-\frac{\eta}{e_\theta}\,(S_E)_R.$$

Subtracting this equation written for η_1, v_1, θ_1 from the same for η_2, v_2, θ_2, multiplying by a test function ψ, integrating by part and putting $\psi = T$ we obtain

$$\frac{1}{2}\frac{d}{dt}\int_\Omega T^2dx = -\int_\Omega\left[\frac{\theta_1 p_\theta(\eta_1,\theta_1)}{e_\theta(\eta_1,\theta_1)}\,v_{1x}-\frac{\theta_2 p_\theta(\eta_2,\theta_2)}{e_\theta(\eta_2,\theta_2)}\,v_{2x}\right]T\,dx$$
$$+\int_\Omega\left[\frac{\kappa(\eta_1,\theta_1)}{\eta_1 e_\theta(\eta_1,\theta_1)}-\frac{\kappa(\eta_2,\theta_2)}{\eta_2 e_\theta(\eta_2,\theta_2)}\right]T\,dx$$
$$+\int_\Omega\left[\frac{\mu(\eta_1)v_{1x}^2}{\eta_1 e_\theta(\eta_1,\theta_1)}-\frac{\mu(\eta_2)v_{2x}^2}{\eta_2 e_\theta(\eta_2,\theta_2)}\right]T\,dx$$
$$\int_\Omega\{\eta_1[(S_E)_R]_1-\eta_2[(S_E)_R]_2\}\,T\,dx := -\sum_{i=1}^4\mathcal{B}_i.$$

Bounding the $\mathcal{B}_i$, using as previously Cauchy Schwarz inequality for $\varepsilon > 0$, we get

$$|\mathcal{B}_1| \leq \varepsilon\left(\|V_x\|_2^2+\|T_x\|_2^2\right)+C_\varepsilon\left(\|E\|_2^2+\|T\|_2^2\right),$$

$$|\mathcal{B}_2| \leq -\int_\Omega \frac{\kappa(\eta_2,\theta_2)}{\eta_2 e_\theta(\eta_2,\theta_2)}\,T_x^2\,dx+\varepsilon\int_\Omega T_x^2\,dx+C_\varepsilon\left(\|E\|_2^2+\|T\|_2^2\right),$$

$$|\mathcal{B}_3| \leq \varepsilon\|V_x\|_2^2+C_\varepsilon\|E\|_2^2,$$

and

$$|\mathcal{B}_4| \leq \left|\int_\Omega\int_0^\infty\int_{S^1} (\eta_1[(S_E)_R]_1-\eta_2[(S_E)_R]_2)\,T\,d\omega\,d\nu\,dx\right|$$

$$\leq C\left(\|E\|_2^2 + \|V\|_2^2 + \|T\|_2^2 + \|\mathcal{J}\|_2^2\right).$$

We obtain finally

$$\frac{d}{dt}\int_\Omega T^2 dx + \int_\Omega T_x^2\, dx$$
$$\leq \varepsilon \int_\Omega V_x^2\, dx + C\left(\|E\|_2^2 + \|T\|_2^2 + \|V\|_2^2 + \|T\|_2^2 + \|\mathcal{J}\|_2^2\right). \tag{3.7}$$

Finally writing the last equation (1.15) for I_2 and I_1, subtracting, multiplying by a test function ζ, integrating by part on $\Omega \times \mathbb{R}_+ \times S^1$ and putting $\zeta = J$ we obtain

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \mathcal{J}^2\, dx + \frac{c}{2\eta_1(M,t)}\int_0^\infty \int_{S^1} \omega J^2(M,t;\omega,\nu)\, d\omega\, d\nu$$
$$-\frac{c}{2\eta_1(0,t)}\int_0^\infty \int_{S^1} \omega J^2(0,t;\omega,\nu)\, d\omega\, d\nu$$
$$\leq C\left(\|E\|_2^2 + \|V\|_2^2 + \|T\|_2^2 + \|\mathcal{J}\|_2^2\right). \tag{3.8}$$

Then adding inequalities (3.5), (3.6), (3.7) and (3.8) and choosing ε small enough, we get

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \left(E^2 + V^2 + T^2 + \mathcal{J}^2\right) dx \leq C\left(\|E\|_2^2 + \|V\|_2^2 + \|T\|_2^2 + \|\mathcal{J}\|_2^2\right),$$

which clearly implies uniqueness.

4 A remark about the large time behaviour

We have the following negative result

Proposition 4.1. *The system (1.15) (1.21)(1.22)(1.23) admits no stationary solution.*

Proof. Any stationary solution of (1.15) satisfies

$$\begin{cases} v_x = 0, \\ p_x = -\eta (S_F)_R, \\ q_x = -\eta (S_E)_R, \\ \omega I_x = \eta S, \end{cases} \tag{4.1}$$

From the definitions (1.10)-(1.14), one also get the equations

$$\begin{cases} (F_R)_x = \eta\, (S_E)_R\,, \\ (P_R)_x = \eta\, (S_F)_R\,. \end{cases} \tag{4.2}$$

As before, we fix the boundary condition for the temperature

$$q|_{x=0} = q|_{x=M} = 0, \tag{4.3}$$

and the previous transparent boundary conditions for the radiative intensity

$$\begin{cases} I|_{x=0} = 0 \quad \text{for } \omega \in (0,1) \\ I|_{x=M} = 0 \quad \text{for } \omega \in (-1,0), \end{cases} \tag{4.4}$$

for $t > 0$. Integrating the previous system leads to

$$\begin{cases} v = 0, \\ p + P_R = C_1, \\ q + F_R = C_2, \\ \omega I_x = \eta S, \end{cases} \tag{4.5}$$

where $C_{1,2}$ are two constants. Using boundary conditions in the third equation (4.5), we get

$$C_2 = F_R(0) = F_R(M),$$

which implies

$$\int_0^\infty \int_{-1}^0 \omega\, I|_{x=0}\, d\omega\, d\nu = \int_0^\infty \int_0^1 \omega\, I|_{x=M}\, d\omega\, d\nu.$$

As $I \geq 0$, this implies that $I|_{x=0} = 0$ and $I|_{x=M} = 0$, for any $\omega \in S^1$.

From the last equation (4.5)

$$\omega I_x = \eta \sigma_a (B - I) + \sigma_s (\tilde{I} - I),$$

we get

$$\left(I(x;\omega,\nu) \exp\left\{ \int_0^x \frac{\eta(\sigma_a + \sigma_s)}{\omega}\, dy \right\} \right)_x$$
$$= \exp\left\{ \int_0^x \frac{\eta(\sigma_a + \sigma_s)}{\omega}\, dy \right\} \frac{\eta(\sigma_a B + \sigma_s \tilde{I})}{\omega},$$

so for $\omega \geq 0$, the function $x \to \phi(x,\omega,\nu) := I(x;\omega,\nu) \exp\left\{ \int_0^x \frac{\eta(\sigma_a+\sigma_s)}{\omega}\, dy \right\}$ is not decreasing. As $\phi(M;\omega,\nu) = 0$, we conclude that $I(x;\omega,\nu) = 0$ for $\omega \geq 0$.

Using the same argument, we have also

$$\left(I(x;\omega,\nu) \exp\left\{ \int_x^M \frac{\eta(\sigma_a + \sigma_s)}{\omega}\, dy \right\} \right)_x$$
$$= -\exp\left\{ \int_x^M \frac{\eta(\sigma_a + \sigma_s)}{\omega}\, dy \right\} \frac{\eta(\sigma_a B + \sigma_s \tilde{I})}{\omega},$$

so for $\omega \leq 0$, the function $x \to \psi(x,\omega,\nu) := I(x;\omega,\nu) \exp\left\{ \int_x^M \frac{\eta(\sigma_a+\sigma_s)}{\omega}\, dy \right\}$ is not increasing. As $\phi(0;\omega,\nu) = 0$, we conclude that $I(x;\omega,\nu) = 0$ for $\omega \leq 0$.

We get finally that I is identically zero on $\Omega \times \mathbb{R}_+ \times S^1$, which cannot be a solution of the last equation (4.5), as $B > 0$ $\square$ $\square$

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