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Certain Identities on Derivatives of Radial Homogeneous and Logarithmic Functions

Kei Morii *
Department of Mathematics
Kyoto University
Kyoto, 606-8502, Japan

Tokushi Sato †
Mathematical Institute
Tohoku University
Sendai, 980-8578, Japan

Yoshihiro Sawano ‡
Department of Mathematics
Kyoto University
Kyoto, 606-8502, Japan

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Abstract

Let k be a natural number and s be real. In the 1-dimensional case, the k-th order derivatives of the functions $|x|^s$ and $\log|x|$ are multiples of $|x|^{s-k}$ and $|x|^{-k}$, respectively. In the present paper, we generalize this fact to higher dimensions by introducing a suitable norm of the derivatives, and give the exact values of the multiples.

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1 Introduction

In the present paper, we show two identities for derivatives of radial homogeneous functions and a radial logarithmic function. A logarithm $\log r$ always stands for the natural logarithm

*E-mail address: k-morii@math.kyoto-u.ac.jp
†E-mail address: tokushi@math.tohoku.ac.jp
‡E-mail address: yosihiro@math.kyoto-u.ac.jp

$\log_e r$. Let $k \in \mathbb{N} = \{1, 2, \dots\}$ and $s \in \mathbb{R}$. In the 1-dimensional case, we readily have that the functions $(d/dx)^k[|x|^s]$, $(d/dx)^k[\log|x|]$ are homogeneous of degree $s-k$, $-k$, respectively. Precisely we have

$$|x|^{k-s}\left|\left(\frac{d}{dx}\right)^k[|x|^s]\right| = |(s)_k|\,,\ |x|^k\left|\left(\frac{d}{dx}\right)^k[\log|x|]\right| = (k-1)!\ \text{ for } x \in \mathbb{R}\setminus\{0\}. \qquad (1.1)$$

Here we use the Pochhammer symbol for the falling factorial (lower factorial);

$$(\nu)_k = \begin{cases} \prod_{j=0}^{k-1}(\nu - j) & \text{for } \nu \in \mathbb{R}, k \in \mathbb{N}, \\ 1 & \text{for } \nu \in \mathbb{R}, k = 0. \end{cases}$$

We denote the space dimension by $N \in \mathbb{N}$. Let ∇^k be a partial differential operator on $\mathbb{R}^N$ which contains only k-th order derivatives. Then the functions $\nabla^k[|x|^s]$, $\nabla^k[\log|x|]$ for $x \in \mathbb{R}^N \setminus \{0\}$ are also homogeneous of degree $s-k$, $-k$, respectively. However, it is not trivial whether the functions

$$|x|^{k-s}|\nabla^k[|x|^s]|,\ |x|^k|\nabla^k[\log|x|]| \qquad (1.2)$$

are constants or not. It deeply depends on the definition of the norm $|\nabla^k u(x)|$ of the vector $\nabla^k u(x)$ for a smooth function u defined on a domain in $\mathbb{R}^N$. See Remark 1.5 below for a counterexample.

In the present paper, we shall define an appropriate norm of the vector $\nabla^k u(x)$ to solve this problem affirmatively, and specify the constants in (1.2).

In what follows, we specify the dimension N as a sub- or super-script and denote by $|\cdot|_N$ the Euclidean norm on $\mathbb{R}^N$;

$$|x|_N = (x_1^2 + x_2^2 + \cdots + x_N^2)^{1/2} \ \text{ for } x = (x_1, x_2, \dots, x_N) \in \mathbb{R}^N.$$

Let us write $I_N = \{1, 2, \dots, N\}$ for short. For a k-tuple of indices $i = (i_1, i_2, \dots, i_k) \in I_N^k$, define the k-th order partial differential operator D_i as

$$D_i = D_{i_1} D_{i_2} \cdots D_{i_k} = \frac{\partial}{\partial x_{i_1}} \frac{\partial}{\partial x_{i_2}} \cdots \frac{\partial}{\partial x_{i_k}}.$$

For a smooth real-valued function u on a domain Ω in $\mathbb{R}^N$, define the vector

$$\nabla_N^k u(x) = (D_i u(x))_{i \in I_N^k} \ \text{ for } x \in \Omega$$

and its norm as

$$\begin{aligned} |\nabla_N^k u(x)|_{N^k} &= \left(\sum_{i \in I_N^k} (D_i u(x))^2\right)^{1/2} \\ &= \left(\sum_{i_1=1}^N \sum_{i_2=1}^N \cdots \sum_{i_k=1}^N \left(\frac{\partial}{\partial x_{i_1}} \frac{\partial}{\partial x_{i_2}} \cdots \frac{\partial}{\partial x_{i_k}} u(x)\right)^2\right)^{1/2} \quad \text{for } x \in \Omega; \end{aligned}$$

we make the agreement $\nabla_N^0 u(x) = u(x)$ and then $|\nabla_N^0 u(x)|_1 = |u(x)|$. When $k=1$, $\nabla_N^1 u(x)$ coincides with the gradient vector of $u(x)$, and $|\nabla_N^1 u(x)|_N$ is its Euclidean norm. When $k=2$, $\nabla_N^2 u(x)$ can be identified with the Hessian matrix of $u(x)$, and $|\nabla_N^2 u(x)|_{N^2}$ is its Frobenius norm. Then we have the following results. Let $\mathbb{Z}_+ = \{0\} \cup \mathbb{N} = \{0,1,2,\ldots\}$.

Theorem 1.1. *Let $N \in \mathbb{N}$.*

(i) *For any $k \in \mathbb{Z}_+$ and $s \in \mathbb{R}$, there exists a constant $\gamma_N^{s,k} \geq 0$ such that*

$$(|x|_N^{k-s}|\nabla_N^k[|x|_N^s]|_{N^k})^2 = \gamma_N^{s,k} \quad \text{for } x \in \mathbb{R}^N \setminus \{0\}.$$

(ii) *For any $k \in \mathbb{N}$, there exists a constant $\ell_N^k > 0$ such that*

$$(|x|_N^k|\nabla_N^k[\log|x|_N]|_{N^k})^2 = \ell_N^k \quad \text{for } x \in \mathbb{R}^N \setminus \{0\}.$$

It follows from (1.1) that for any $k \in \mathbb{N}$ and $s \in \mathbb{R}$,

$$\gamma_1^{s,k} = ((s)_k)^2,\ \ell_1^k = ((k-1)!)^2. \tag{1.3}$$

We can determine explicitly the constants $\gamma_N^{s,k}$ and ℓ_N^k given in Theorem 1.1 for a general dimension N as follows. Before we go into the detail, we provide some notation. Let

$$\lfloor \nu \rfloor = \max\{k \in \mathbb{Z};\, k \leq \nu\},\ \lceil \nu \rceil = \min\{k \in \mathbb{Z};\, k \geq \nu\} \quad \text{for } \nu \in \mathbb{R}.$$

Define the binomial coefficient

$$\binom{\nu}{k} = \frac{(\nu)_k}{k!} \quad \text{for } \nu \in \mathbb{R},\, k \in \mathbb{Z}_+.$$

The following theorem provides the explicit values of the constants $\gamma_N^{s,k}$ and ℓ_N^k.

Theorem 1.2. *Let $N \in \mathbb{N}$.*

(i) *For any $k \in \mathbb{Z}_+$ and $s \in \mathbb{R}$, it holds*

$$\gamma_N^{s,k} = k! \sum_{l=0}^{\lfloor k/2 \rfloor} (k-2l)!\, l! \left(\frac{N-3}{2}+l\right)_l \left(\sum_{n=\lceil k/2 \rceil}^{k-l} 2^{2n-k+l} \binom{s/2}{n} \binom{n}{k-n} \binom{k-n}{l}\right)^2.$$

(ii) *For any $k \in \mathbb{N}$, it holds*

$$\ell_N^k = k! \sum_{l=0}^{\lfloor k/2 \rfloor} (k-2l)!\, l! \left(\frac{N-3}{2}+l\right)_l \left(\sum_{n=\lceil k/2 \rceil}^{k-l} 2^{2n-k+l} \frac{(-1)^n}{2n} \binom{n}{k-n} \binom{k-n}{l}\right)^2.$$

We also obtain the following result as a special case of Theorem 1.2.

Theorem 1.3. (i) *For any $N \in \mathbb{N}$ and $k \in \mathbb{Z}_+$, it holds*

$$\gamma_N^{-(N-2),k} = 2^k (N+k-3)_k \left(\frac{N}{2}+k-2\right)_k.$$

(ii) *For any $k \in \mathbb{N}$, it holds*

$$\ell_2^k = 2^{k-1}((k-1)!)^2.$$

Remark 1.4. For small k, we have calculated the concrete values of $\gamma_N^{s,k}$ and ℓ_N^k;

$$\begin{aligned}
&\gamma_N^{s,1} = s^2,\ \gamma_N^{s,2} = s^2(s^2-2s+N),\ \gamma_N^{s,3} = s^2(s-2)^2(s^2-2s+3N-2),\\
&\gamma_N^{s,4} = s^2(s-2)^2(s^4-8s^3+(16+6N)s^2+(12-36N)s+3N^2+54N-48),\\
&\ell_N^1 = 1,\ \ell_N^2 = N,\ \ell_N^3 = 4(3N-2),\ \ell_N^4 = 12(N^2+18N-16),\\
&\ell_N^5 = 192(5N^2+30N-32),\ \ell_N^6 = 960(N^3+78N^2+224N-288),\\
&\ell_N^7 = 34560(7N^3+196N^2+308N-496),\\
&\ell_N^8 = 241920(N^4+204N^3+3052N^2+2736N-5888).
\end{aligned}$$

As we mentioned before, it is essential to define the norm $|\nabla^k u(x)|$ appropriately.

Remark 1.5. One may also adopt some other plausible definition instead of $|\nabla_N^k u(x)|_{N^k}$ defined before. For instance, let us define

$$\begin{aligned}
|\tilde{\nabla}_N^k u(x)|_{\binom{N+k-1}{k}} &= \left(\sum_{\alpha_1+\alpha_2+\cdots+\alpha_N=k} (D_1^{\alpha_1} D_2^{\alpha_2} \cdots D_N^{\alpha_N} u(x))^2 \right)^{1/2}\\
&= \left(\sum_{1 \le i_1 \le i_2 \le \cdots \le i_k \le N} (D_{i_1} D_{i_2} \cdots D_{i_k} u(x))^2 \right)^{1/2},
\end{aligned}$$

which gives a norm of $\nabla^k u(x)$. Putting $k=2$, we see that both the functions

$$\begin{aligned}
(|x|_N^{2-s} |\tilde{\nabla}_N^2[|x|_N^s]|_{N(N+1)/2})^2 &= s^2 \left(N+2s-4+(s-2)^2 \sum_{1 \le i_1 \le i_2 \le N} \frac{x_{i_1}^2 x_{i_2}^2}{|x|_N^4} \right),\\
(|x|_N^2 |\tilde{\nabla}_N^2[\log|x|_N]|_{N(N+1)/2})^2 &= N-4+4 \sum_{1 \le i_1 \le i_2 \le N} \frac{x_{i_1}^2 x_{i_2}^2}{|x|_N^4}
\end{aligned}$$

are not constants on $\mathbb{R}^N \setminus \{0\}$ unless $N=1$ or $s \in \{0,2\}$. To illustrate how they are different clearly, note that

$$\begin{aligned}
|\nabla_N^k u(x)|_{N^k} &= \left(\sum_{\alpha_1+\alpha_2+\cdots+\alpha_N=k} \frac{k!}{\alpha_1!\,\alpha_2! \cdots \alpha_N!} (D_1^{\alpha_1} D_2^{\alpha_2} \cdots D_N^{\alpha_N} u(x))^2 \right)^{1/2}\\
&= \left(\sum_{1 \le i_1 \le i_2 \le \cdots \le i_k \le N} \frac{k!}{\prod_{l=1}^N \sharp\{n;\, i_n = l\}!} (D_{i_1} D_{i_2} \cdots D_{i_k} u(x))^2 \right)^{1/2},
\end{aligned}$$

where $\sharp S$ denotes the cardinality of a finite set S.

The present work is originated in our desire to investigate Brézis-Gallouët-Wainger type inequalities. The authors together with Wadade [6], [7] and [8] investigated the sharp constants of such inequalities in the first order critical Sobolev space $W_0^{1,N}(\Omega)$ on a bounded domain Ω in $\mathbb{R}^N$ with $N \in \mathbb{N} \setminus \{1\}$. In their forthcoming paper [5], they shall give a lower bound in terms of ℓ_N^k for the sharp constants of such inequalities in the k-th order critical Sobolev space $W_0^{k,N/k}(\Omega)$ by calculating the exact values of homogeneous Sobolev norms of the radial logarithmic function on annuli. To explain more concretely, we can give a sufficient condition for $\lambda_1 > 0$ and $\lambda_2 \in \mathbb{R}$ that the inequality

$$\|u\|_{L^\infty(\Omega)}^{N/(N-k)} \leq \lambda_1 \log(1+\|u\|_{A^{s,N/(s-\alpha),q}(\Omega)}) + \lambda_2 \log(1+\log(1+\|u\|_{A^{s,N/(s-\alpha),q}(\Omega)})) + C$$
$$\text{for } u \in W_0^{k,N/k}(\Omega) \cap A^{s,N/(s-\alpha),q}(\Omega) \ \text{ with } \|\nabla^k u\|_{L^{N/k}(\Omega)} = 1$$

fails for any constant C independent of u, where $k \in \{1,2,\dots,N-1\}$, $0 < \alpha \leq s < \infty$, $0 < q < \infty$ and we denote by $A^{s,p,q}$ either the Besov space $B^{s,p,q}$ or the Triebel-Lizorkin space $F^{s,p,q}$. The results in [2] and [3] obtained by Brézis, Gallouët and Wainger imply that this inequality holds for sufficiently large λ_1 and arbitrary λ_2 with a suitable constant C provided that $A^{s,N/(s-\alpha),q}(\Omega)$ is replaced by the Sobolev space (or the potential space) $H^{s,N/(s-\alpha)}(\Omega)$. Since $H^{s,N/(s-\alpha)}(\Omega) = F^{s,N/(s-\alpha),2}(\Omega)$, the same assertion holds in the cases $A^{s,N/(s-\alpha),q}(\Omega) = F^{s,N/(s-\alpha),q}(\Omega)$ with $0 < q \leq 2$ and $A^{s,N/(s-\alpha),q}(\Omega) = B^{s,N/(s-\alpha),q}(\Omega)$ with $0 < q \leq \min\{N/(s-\alpha), 2\}$ by virtue of the embedding theorems of Besov and Triebel-Lizorkin spaces.

We now describe how we organized the present paper; Sections 2, 3 and 4 are devoted to proving Theorems 1.1, 1.2 and 1.3, respectively.

2 Proof of Theorem 1.1

In this section, we shall prove Theorem 1.1. The following two propositions are easy to prove.

Proposition 2.1. *Let $s \in \mathbb{R}$ and $u \in C(\mathbb{R}^N \setminus \{0\})$ be homogeneous of degree s, that is,*

$$u(\lambda x) = \lambda^s u(x) \ \textit{ for } x \in \mathbb{R}^N \setminus \{0\}, \lambda > 0.$$

(i) *If $v \in C(\mathbb{R}^N \setminus \{0\})$ is homogeneous of degree s as well, then so is $u + v$.*

(ii) *For $\nu \in \mathbb{R}$, $|u|^\nu$ is homogeneous of degree $s\nu$.*

(iii) *If $u \in C^1(\mathbb{R}^N \setminus \{0\})$ and $i \in I_N$, then $D_i u$ is homogeneous of degree $s-1$.*

For a square matrix A of order N, let us define

$$A[x] = {}^{\mathrm{t}}(A\,{}^{\mathrm{t}}x) = x\,{}^{\mathrm{t}}A \ \text{ for } x = (x_1, x_2, \dots, x_N) \in \mathbb{R}^N.$$

Proposition 2.2. *Let $s \in \mathbb{R}$ and $u \in C^1(\mathbb{R}^N \setminus \{0\})$ be homogeneous of degree s and radially symmetric, that is,*

$$u(A[x]) = u(x) \ \textit{ for } x \in \mathbb{R}^N \setminus \{0\}, A \in O(N),$$

where $O(N)$ denotes the orthogonal group of order N. Then there exists a constant $c \in \mathbb{R}$ such that

$$u(x) = c|x|_N^s \ \text{for } x \in \mathbb{R}^N \setminus \{0\}.$$

To prove Theorem 1.1, we need to use the Fourier transform on $\mathbb{R}^N$. Let $\mathcal{S}(\mathbb{R}^N)$ denote the Schwartz class on $\mathbb{R}^N$. Define the Fourier transform $\mathcal{F}_N$ and its inverse $\mathcal{F}_N^{-1}$ on $\mathbb{R}^N$ by

$$\mathcal{F}_N u(\xi) = \frac{1}{(2\pi)^{N/2}} \int_{\mathbb{R}^N} e^{-\sqrt{-1}\,(x,\xi)_N} u(x) dx \ \text{ for } \xi \in \mathbb{R}^N,$$
$$\mathcal{F}_N^{-1} u(x) = \frac{1}{(2\pi)^{N/2}} \int_{\mathbb{R}^N} e^{\sqrt{-1}\,(x,\xi)_N} u(\xi) d\xi \ \text{ for } x \in \mathbb{R}^N, u \in \mathcal{S}(\mathbb{R}^N),$$

respectively, where $\sqrt{-1}$ denotes the imaginary unit and

$$(x,\xi)_N = \sum_{i=1}^N x_i \xi_i \ \text{ for } x = (x_1, x_2, \ldots, x_N), \xi = (\xi_1, \xi_2, \ldots, \xi_N) \in \mathbb{R}^N.$$

The crux of Theorem 1.1 is the following observation by using the Fourier transform.

Lemma 2.3. *If $u \in \mathcal{S}(\mathbb{R}^N)$ is real-valued and radially symmetric, then so is $|\nabla_N^k u|_{N^k}^2$ for $k \in \mathbb{Z}_+$.*

Proof. This is trivial if $k = 0$; we may assume $k \in \mathbb{N}$ below. Let $i = (i_1, i_2, \ldots, i_k) \in I_N^k$. By the Fourier inversion formula and [4, Proposition 2.2.11 (10)], we have two expressions of $D_i u(x)$;

$$\begin{aligned} D_i u(x) &= D_{i_1} D_{i_2} \cdots D_{i_k} [\mathcal{F}_N^{-1}[\mathcal{F}_N u]](x) \\ &= \frac{(\sqrt{-1})^k}{(2\pi)^{N/2}} \int_{\mathbb{R}^N} e^{\sqrt{-1}\,(x,\xi)_N} \xi_{i_1} \xi_{i_2} \cdots \xi_{i_k} \mathcal{F}_N u(\xi) d\xi \ \text{ for } x \in \mathbb{R}^N \end{aligned}$$

and

$$\begin{aligned} D_i u(x) &= D_{i_1} D_{i_2} \cdots D_{i_k} [\mathcal{F}_N[\mathcal{F}_N^{-1} u]](x) \\ &= \frac{(-\sqrt{-1})^k}{(2\pi)^{N/2}} \int_{\mathbb{R}^N} e^{-\sqrt{-1}\,(x,\eta)_N} \eta_{i_1} \eta_{i_2} \cdots \eta_{i_k} \mathcal{F}_N^{-1} u(\eta) d\eta \ \text{ for } x \in \mathbb{R}^N. \end{aligned}$$

Thus we deduce

$$\begin{aligned} &(D_i u(x))^2 \\ &= \frac{1}{(2\pi)^N} \iint_{\mathbb{R}^N \times \mathbb{R}^N} e^{\sqrt{-1}\,(x,\xi-\eta)_N} \xi_{i_1} \xi_{i_2} \cdots \xi_{i_k} \eta_{i_1} \eta_{i_2} \cdots \eta_{i_k} \mathcal{F}_N u(\xi) \mathcal{F}_N^{-1} u(\eta) d\xi d\eta \ \text{ for } x \in \mathbb{R}^N. \end{aligned}$$

Hence we obtain

$$\begin{aligned} &|\nabla_N^k u(x)|_{N^k}^2 \\ &= \sum_{i \in I_N^k} (D_i u(x))^2 \\ &= \frac{1}{(2\pi)^N} \sum_{i \in I_N^k} \iint_{\mathbb{R}^N \times \mathbb{R}^N} e^{\sqrt{-1}\,(x,\xi-\eta)_N} \xi_{i_1} \xi_{i_2} \cdots \xi_{i_k} \eta_{i_1} \eta_{i_2} \cdots \eta_{i_k} \mathcal{F}_N u(\xi) \mathcal{F}_N^{-1} u(\eta) d\xi d\eta \\ &= \frac{1}{(2\pi)^N} \iint_{\mathbb{R}^N \times \mathbb{R}^N} e^{\sqrt{-1}\,(x,\xi-\eta)_N} (\xi,\eta)_N^k \mathcal{F}_N u(\xi) \mathcal{F}_N^{-1} u(\eta) d\xi d\eta \ \text{ for } x \in \mathbb{R}^N. \end{aligned}$$

For $A \in O(N)$, we have

$$(A[x], y)_N = (x, {}^{\mathrm{t}}A[y])_N,\ (A[x], A[y])_N = (x, y)_N \quad \text{for } x, y \in \mathbb{R}^N.$$

Since Fourier transform and its inverse of a radially symmetric function are also radially symmetric (see e.g. [4, Proposition 2.2.11 (13)]), we see that

$$[\mathcal{F}_N u](A[\xi]) = \mathcal{F}_N u(\xi),\ [\mathcal{F}_N^{-1} u](A[\xi]) = \mathcal{F}_N^{-1} u(\xi) \quad \text{for } \xi \in \mathbb{R}^N.$$

Changing variables $(\xi, \eta) = (A[\tilde{\xi}], A[\tilde{\eta}])$, we have

$$\begin{aligned}
|\nabla_N^k u(A[x])|_{N^k}^2 &= \frac{1}{(2\pi)^N} \iint_{\mathbb{R}^N \times \mathbb{R}^N} e^{\sqrt{-1}\,(A[x], \xi - \eta)_N} (\xi, \eta)_N^k \mathcal{F}_N u(\xi) \mathcal{F}_N^{-1} u(\eta) d\xi\, d\eta \\
&= \frac{1}{(2\pi)^N} \iint_{\mathbb{R}^N \times \mathbb{R}^N} e^{\sqrt{-1}\,(x, {}^{\mathrm{t}}A[\xi - \eta])_N} (\xi, \eta)_N^k \mathcal{F}_N u(\xi) \mathcal{F}_N^{-1} u(\eta) d\xi\, d\eta \\
&= \frac{1}{(2\pi)^N} \iint_{\mathbb{R}^N \times \mathbb{R}^N} e^{\sqrt{-1}\,(x, \tilde{\xi} - \tilde{\eta})_N} (A[\tilde{\xi}], A[\tilde{\eta}])_N^k [\mathcal{F}_N u](A[\tilde{\xi}]) [\mathcal{F}_N^{-1} u](A[\tilde{\eta}]) d\tilde{\xi}\, d\tilde{\eta} \\
&= \frac{1}{(2\pi)^N} \iint_{\mathbb{R}^N \times \mathbb{R}^N} e^{\sqrt{-1}\,(x, \tilde{\xi} - \tilde{\eta})_N} (\tilde{\xi}, \tilde{\eta})_N^k \mathcal{F}_N u(\tilde{\xi}) \mathcal{F}_N^{-1} u(\tilde{\eta}) d\tilde{\xi}\, d\tilde{\eta} \\
&= |\nabla_N^k u(x)|_{N^k}^2,
\end{aligned}$$

which shows that $|\nabla_N^k u|_{N^k}^2$ is radially symmetric. □

We conclude the proof of Theorem 1.1. Let

$$B_r^N = \{x \in \mathbb{R}^N;\ |x|_N < r\} \quad \text{for } r > 0.$$

Proof of Theorem 1.1. For $j \in \mathbb{N}$, choose $\psi_j \in C_{\mathrm{c}}^\infty((0, \infty))$ satisfying

$$\psi_j(r) = \begin{cases} 1 & \text{for } \dfrac{1}{j} < r < j, \\ 0 & \text{for } 0 < r < \dfrac{1}{2j} \ \text{ or } r > 2j. \end{cases}$$

Then the functions $\psi_j(|x|_N)|x|_N^s$, $\psi_j(|x|_N)\log|x|_N$ belong to $\mathcal{S}(\mathbb{R}^N)$ and are real-valued, radially symmetric. Also, they satisfy

$$\psi_j(|x|_N)|x|_N^s = |x|_N^s,\ \psi_j(|x|_N)\log|x|_N = \log|x|_N \quad \text{for } x \in B_j^N \setminus \overline{B_{1/j}^N}.$$

Since Lemma 2.3 yields that $|\nabla_N^k[\psi_j(|x|_N)|x|_N^s]|_{N^k}^2$ and $|\nabla_N^k[\psi_j(|x|_N)\log|x|_N]|_{N^k}^2$ are radially symmetric, we deduce that so are $|\nabla_N^k[|x|_N^s]|_{N^k}^2$ and $|\nabla_N^k[\log|x|_N]|_{N^k}^2$ on $B_j^N \setminus \overline{B_{1/j}^N}$, and then on $\mathbb{R}^N \setminus \{0\}$ because $j \in \mathbb{N}$ is arbitrary.

(i) It follows from Proposition 2.1 that for $i \in I_N^k$, the functions $|x|_N^s$, $D_i[|x|_N^s]$ and $(D_i[|x|_N^s])^2$ are homogeneous of degree s, $s-k$ and $2(s-k)$, respectively. Hence $|\nabla_N^k[|x|_N^s]|_{N^k}^2$ is also homogeneous of degree $2(s-k)$. Then the desired conclusion immediately follows from Proposition 2.2.

(ii) Since

$$D_i[\log|x|_N] = \frac{x_i}{|x|_N^2} \quad \text{for } x \in \mathbb{R}^N \setminus \{0\},\ i \in I_N,$$

we deduce that this function is homogeneous of degree -1. The rest of the proof is quite similar to (i). □

3 Proof of Theorem 1.2

In this section, we prove Theorem 1.2. We decompose it into the following three lemmas.

Lemma 3.1. *Theorem 1.2 holds true for* $N = 1$. *Namely:*

(i) *For any* $k \in \mathbb{Z}_+$ *and* $s \in \mathbb{R}$, *it holds*

$$\gamma_1^{s,k} = \left(k! \sum_{n=\lceil k/2 \rceil}^{k} 2^{2n-k} \binom{s/2}{n} \binom{n}{k-n} \right)^2.$$

(ii) *For any* $k \in \mathbb{N}$, *it holds*

$$\ell_1^k = \left(k! \sum_{n=\lceil k/2 \rceil}^{k} 2^{2n-k} \frac{(-1)^n}{2n} \binom{n}{k-n} \right)^2.$$

Lemma 3.2. *Let* $N \in \mathbb{N} \setminus \{1\}$.

(i) *For* $k \in \mathbb{Z}_+$ *and* $s \in \mathbb{R}$, *it holds*

$$\gamma_N^{s,k} = k! \sum_{l=0}^{\lfloor k/2 \rfloor} \frac{(k-2l)!}{(2l)!} \left(\sum_{n=\lceil k/2 \rceil}^{k-l} 2^{2n-k} \binom{s/2}{n} \binom{n}{k-n} \binom{k-n}{l} \right)^2 \gamma_{N-1}^{2l,2l}.$$

In particular, for $m \in \mathbb{Z}_+$, *it holds*

$$\gamma_N^{2m,2m} = (2m)! \sum_{l=0}^{m} \frac{(2(m-l))!}{(2l)!} \binom{m}{l}^2 \gamma_{N-1}^{2l,2l}.$$

(ii) *For* $k \in \mathbb{N}$, *it holds*

$$\ell_N^k = k! \sum_{l=0}^{\lfloor k/2 \rfloor} \frac{(k-2l)!}{(2l)!} \left(\sum_{n=\lceil k/2 \rceil}^{k-l} 2^{2n-k} \frac{(-1)^n}{2n} \binom{n}{k-n} \binom{k-n}{l} \right)^2 \gamma_{N-1}^{2l,2l}.$$

Lemma 3.3. *For* $N \in \mathbb{N}$ *and* $m \in \mathbb{Z}_+$, *it holds*

$$\gamma_N^{2m,2m} = 2^{2m} m!\,(2m)! \left(\frac{N}{2} + m - 1 \right)_m. \tag{3.1}$$

Combining these three lemmas yields Theorem 1.2. We now concentrate on proving them. We need some propositions. For $m \in \mathbb{Z}_+$, define

$$\phi^m(t) = (t^2 + 2t)^m \quad \text{for } t \in \mathbb{R}.$$

Proposition 3.4. *Let* $m, k \in \mathbb{Z}_+$.

(i) *It holds*

$$[\phi^m]^{(k)}(0) = \chi_{[m,2m]}(k) 2^{2m-k} k! \binom{m}{k-m},$$

where χ_S denotes the characteristic function of a set S.

(ii) *It holds*

$$|\nabla_N^k [|\cdot|_N^{2m}](0)|_{N^k}^2 = \delta_{k,2m} \gamma_N^{2m,2m} = \delta_{k,2m} \gamma_N^{k,k}.$$

Proof. (i) Expand ϕ^m by means of the binomial theorem;

$$\phi^m(t) = \sum_{j=0}^{m} 2^{m-j} \binom{m}{j} t^{m+j} \quad \text{for } t \in \mathbb{R}.$$

Let $\nu_+ = \max\{\nu, 0\}$. For $k \in \mathbb{Z}_+$, we have

$$\begin{aligned} [\phi^m]^{(k)}(t) &= \sum_{j=(k-m)_+}^{m} 2^{m-j} \binom{m}{j} (m+j)_k t^{m+j-k} \\ &= \sum_{l=(m-k)_+}^{2m-k} 2^{2m-k-l} \binom{m}{k-m+l} (k+l)_k t^l \quad \text{for } t \in \mathbb{R}, \end{aligned}$$

which implies the assertion.

(ii) If $k > 2m$ and $i \in I_N^k$, then $D_i[|x|_N^{2m}] = 0$ for $x \in \mathbb{R}^N$, which implies

$$|\nabla_N^k [|x|_N^{2m}]|_{N^k}^2 = 0 \quad \text{for } x \in \mathbb{R}^N.$$

Meanwhile, if $k \le 2m$, then Theorem 1.1 (i) shows that

$$|\nabla_N^k [|x|_N^{2m}]|_{N^k}^2 = \gamma_N^{2m,k} |x|_N^{2(2m-k)} \quad \text{for } x \in \mathbb{R}^N \setminus \{0\}.$$

Hence a passage to the limit as $x \to 0$ yields the assertion. □

In what follows, we use the notation

$$x' = (x_1, x_2, \ldots, x_{N-1}) \in \mathbb{R}^{N-1} \quad \text{for } x = (x_1, x_2, \ldots, x_{N-1}, x_N) \in \mathbb{R}^N$$

when $N \in \mathbb{N} \setminus \{1\}$. Let Ω be a domain in $\mathbb{R}^N$, and for $u \in C^k(\Omega)$, we write

$$|\nabla_{N-1}^k u(x)|_{(N-1)^k}^2 = \sum_{i' \in I_{N-1}^k} (D_{i'} u(x))^2 \quad \text{for } x \in \Omega.$$

Proposition 3.5. *Let $N \in \mathbb{N} \setminus \{1\}$, $k \in \mathbb{Z}_+$ and Ω be a domain in $\mathbb{R}^N$. Then for $u \in C^k(\Omega)$, we have*

$$|\nabla_N^k u(x)|_{N^k}^2 = \sum_{j=0}^{k} \binom{k}{j} |\nabla_{N-1}^j [D_N^{k-j} u](x)|_{(N-1)^j}^2 \quad \textit{for } x \in \Omega.$$

Proof. The conclusion is trivial if $k=0$; we may assume $k \in \mathbb{N}$ below. Define

$$j_{N-1}[i] = \sharp\{n \in \{1,2,\ldots,k\};\, i_n \in I_{N-1}\} \quad \text{for } i=(i_1,i_2,\ldots,i_k) \in I_N^k$$

and

$$I_{N-1}^{j;k} = \{i \in I_N^k;\, j_{N-1}[i]=j\} \quad \text{for } j \in \{0,1,\ldots,k\}.$$

For $i=(i_1,i_2,\ldots,i_k) \in I_{N-1}^{j;k}$, let

$$(n'_1[i], n'_2[i], \ldots, n'_j[i])$$

be all the n's listed in ascending order such that $i_n \in I_{N-1}$, and let

$$(\tilde{n}_1[i], \tilde{n}_2[i], \ldots, \tilde{n}_{k-j}[i])$$

be all the n's listed in ascending order such that $i_n = N$. If we define

$$i'_{N-1}[i] = (i_{n'_1[i]}, i_{n'_2[i]}, \ldots, i_{n'_j[i]}),\ \tilde{i}_N[i] = (i_{\tilde{n}_1[i]}, i_{\tilde{n}_2[i]}, \ldots, i_{\tilde{n}_{k-j}[i]}),$$

then

$$i'_{N-1}[i] \in I_{N-1}^j,\ \tilde{i}_N[i] = (N,N,\ldots,N) \in \{N\}^{k-j} \quad \text{for } i \in I_{N-1}^{j;k}$$

and

$$D_i u(x) = D_{i'_{N-1}[i]} D_{\tilde{i}_N[i]} u(x) = D_{i'_{N-1}[i]}[D_N^{k-j}u](x) \quad \text{for } x \in \Omega,\ i \in I_{N-1}^{j;k}.$$

We next define

$$\Sigma_k^{k-j} = \{\sigma = (\sigma_1,\sigma_2,\ldots,\sigma_{k-j}) \in \{1,2,\ldots,k\}^{k-j};\, \sigma_1 < \sigma_2 < \cdots < \sigma_{k-j}\}$$
$$\text{for } j \in \{0,1,\ldots,k-1\}$$

and

$$I_{N-1}^{j;k}(\sigma) = \{i \in I_{N-1}^{j;k};\, (\tilde{n}_1[i], \tilde{n}_2[i], \ldots, \tilde{n}_{k-j}[i]) = \sigma\} \quad \text{for } \sigma \in \Sigma_k^{k-j}.$$

Since the mapping $I_{N-1}^{j;k}(\sigma) \ni i \mapsto i'_{N-1}[i] \in I_{N-1}^j$ is bijective for any $\sigma \in \Sigma_k^{k-j}$, we have

$$\begin{aligned}
\sum_{i \in I_{N-1}^{j;k}(\sigma)} (D_i u(x))^2 &= \sum_{i \in I_{N-1}^{j;k}(\sigma)} (D_{i'_{N-1}[i]}[D_N^{k-j}u](x))^2 \\
&= \sum_{i' \in I_{N-1}^j} (D_{i'}[D_N^{k-j}u](x))^2 \\
&= |\nabla_{N-1}^j [D_N^{k-j}u](x)|_{(N-1)^j}^2 \quad \text{for } x \in \Omega,\ \sigma \in \Sigma_k^{k-j}.
\end{aligned}$$

Since

$$\sharp\Sigma_k^{k-j} = \binom{k}{k-j} = \binom{k}{j} \quad \text{for } j \in \{0,1,\ldots,k-1\},$$

we deduce

$$\begin{aligned}
|\nabla_N^k u(x)|_{N^k}^2 &= \sum_{i \in I_N^k \setminus I_{N-1}^k} (D_i u(x))^2 + \sum_{i' \in I_{N-1}^k} (D_{i'} u(x))^2 \\
&= \sum_{j=0}^{k-1} \sum_{\sigma \in \Sigma_k^{k-j}} \sum_{i \in I_{N-1}^{j;k}(\sigma)} (D_i u(x))^2 + |\nabla_{N-1}^k u(x)|_{(N-1)^k}^2 \\
&= \sum_{j=0}^{k} \binom{k}{j} |\nabla_{N-1}^j [D_N^{k-j}u](x)|_{(N-1)^j}^2 \quad \text{for } x \in \Omega.
\end{aligned}$$

This completes the proof. □

Define $e_N = (0,0,\dots,0,1) \in \mathbb{R}^N$ and

$$\rho_N(x) = |x+e_N|_N^2 - 1 \quad \text{for } x \in \mathbb{R}^N,$$

which becomes

$$\rho_N(x) = \begin{cases} \phi^1(x) & \text{for } x \in \mathbb{R} \quad \text{if } N = 1, \\ |x'|_{N-1}^2 + \phi^1(x_N) & \text{for } x \in \mathbb{R}^N \quad \text{if } N \in \mathbb{N} \setminus \{1\}. \end{cases}$$

Note that

$$|\rho_N(x)| = |\,|x|_N^2 + 2x_N| \leq |x|_N^2 + 2|x_N| \leq |x|_N^2 + 2|x|_N < \varepsilon \quad \text{for } x \in B^N_{(1+\varepsilon)^{1/2}-1}$$

for all $\varepsilon > 0$.

Proposition 3.6. *Let* $\varepsilon > 0$ *and*

$$f(t) = \sum_{n=0}^{\infty} a_n t^n \quad \textit{for } -\varepsilon < t < \varepsilon$$

be analytic, where $\{a_n\}_{n=0}^{\infty} \subset \mathbb{R}$.

(i) *Let* $N = 1$ *and* $k \in \mathbb{Z}_+$. *Then it holds*

$$[f(\rho_1)]^{(k)}(0) = k! \sum_{n=\lceil k/2 \rceil}^{k} 2^{2n-k} a_n \binom{n}{k-n}.$$

(ii) *Let* $N \in \mathbb{N} \setminus \{1\}$ *and* $k \in \mathbb{Z}_+$. *Then it holds*

$$|\nabla_N^k [f(\rho_N)](0)|_{N^k}^2 = k! \sum_{l=0}^{\lfloor k/2 \rfloor} \frac{(k-2l)!}{(2l)!} \left(\sum_{n=\lceil k/2 \rceil}^{k-l} 2^{2n-k} a_n \binom{n}{k-n} \binom{k-n}{l} \right)^2 \gamma_{N-1}^{2l,2l}.$$

Proof. (i) It follows from the definition of ρ_1 and ϕ^n that

$$[f(\rho_1)]^{(k)}(x) = [f(\phi^1)]^{(k)}(x) = \sum_{n=0}^{\infty} a_n [\phi^n]^{(k)}(x)$$
$$\text{for } -((1+\varepsilon)^{1/2} - 1) < x < (1+\varepsilon)^{1/2} - 1.$$

If we invoke Proposition 3.4 (i), then we have

$$[f(\rho_1)]^{(k)}(0) = \sum_{n=0}^{\infty} a_n [\phi^n]^{(k)}(0) = k! \sum_{n=\lceil k/2 \rceil}^{k} 2^{2n-k} a_n \binom{n}{k-n}.$$

Thus, (i) is established.

(ii) Using the binomial expansion, we have

$$\begin{aligned}
f(\rho_N(x)) &= \sum_{n=0}^{\infty} a_n(\rho_N(x))^n \\
&= \sum_{n=0}^{\infty} a_n(|x'|_{N-1}^2 + \phi^1(x_N))^n \\
&= \sum_{n=0}^{\infty} a_n \sum_{m=0}^{n} \binom{n}{m} \phi^{n-m}(x_N)|x'|_{N-1}^{2m} \quad \text{for } x \in B^N_{(1+\varepsilon)^{1/2}-1}.
\end{aligned}$$

Proposition 3.5 gives

$$\begin{aligned}
&|\nabla_N^k[f(\rho_N)](x)|_{N^k}^2 \\
&= \sum_{j=0}^{k} \binom{k}{j} |\nabla_{N-1}^j[D_N^{k-j}[f(\rho_N)]](x)|_{(N-1)^j}^2 \\
&= \sum_{j=0}^{k} \binom{k}{j} \left| \sum_{n=0}^{\infty} a_n \sum_{m=0}^{n} \binom{n}{m} [\phi^{n-m}]^{(k-j)}(x_N)\nabla_{N-1}^j[|x'|_{N-1}^{2m}] \right|_{(N-1)^j}^2 \\
&= \sum_{l=0}^{\lfloor (k-1)/2 \rfloor} \binom{k}{2l+1} \left| \sum_{n=0}^{\infty} a_n \sum_{m=0}^{n} \binom{n}{m} [\phi^{n-m}]^{(k-2l-1)}(x_N)\nabla_{N-1}^{2l+1}[|x'|_{N-1}^{2m}] \right|_{(N-1)^{2l+1}}^2 \\
&\quad + \sum_{l=0}^{\lfloor k/2 \rfloor} \binom{k}{2l} \left| \sum_{n=0}^{\infty} a_n \sum_{m=0}^{n} \binom{n}{m} [\phi^{n-m}]^{(k-2l)}(x_N)\nabla_{N-1}^{2l}[|x'|_{N-1}^{2m}] \right|_{(N-1)^{2l}}^2 \quad \text{for } x \in B^N_{(1+\varepsilon)^{1/2}-1}.
\end{aligned}$$

Here, we decomposed the summation with respect to j into two parts consisting odd j's and even j's. Note that Proposition 3.4 (ii) gives $\nabla_{N-1}^j[|\cdot|_{N-1}^{2m}](0) = 0$ unless $j = 2m$. It follows from Proposition 3.4 (i) that

$$\binom{n}{l}[\phi^{n-l}]^{(k-2l)}(0) = \chi_{[k/2,k-l]}(n)2^{2n-k}(k-2l)!\binom{n}{k-n}\binom{k-n}{l}.$$

Using these equalities, we have

$$\begin{aligned}
&|\nabla_N^k[f(\rho_N)](0)|_{N^k}^2 \\
&= \sum_{l=0}^{\lfloor (k-1)/2 \rfloor} \binom{k}{2l+1} \left| \sum_{n=0}^{\infty} a_n \sum_{m=0}^{n} \binom{n}{m} [\phi^{n-m}]^{(k-2l-1)}(0)\nabla_{N-1}^{2l+1}[|\cdot|_{N-1}^{2m}](0) \right|_{(N-1)^{2l+1}}^2 \\
&\quad + \sum_{l=0}^{\lfloor k/2 \rfloor} \binom{k}{2l} \left| \sum_{n=0}^{\infty} a_n \sum_{m=0}^{n} \binom{n}{m} [\phi^{n-m}]^{(k-2l)}(0)\nabla_{N-1}^{2l}[|\cdot|_{N-1}^{2m}](0) \right|_{(N-1)^{2l}}^2 \\
&= \sum_{l=0}^{\lfloor k/2 \rfloor} \binom{k}{2l} \left| \sum_{n=l}^{\infty} a_n \binom{n}{l} [\phi^{n-l}]^{(k-2l)}(0)\nabla_{N-1}^{2l}[|\cdot|_{N-1}^{2l}](0) \right|_{(N-1)^{2l}}^2 \\
&= k! \sum_{l=0}^{\lfloor k/2 \rfloor} \frac{(k-2l)!}{(2l)!} \left(\sum_{n=\lceil k/2 \rceil}^{k-l} 2^{2n-k} a_n \binom{n}{k-n}\binom{k-n}{l} \right)^2 \gamma_{N-1}^{2l,2l}.
\end{aligned}$$

$\square$

For $s \in \mathbb{R}$, define

$$f_s(t) = (1+t)^{s/2},\ f_*(t) = \frac{1}{2}\log(1+t) \ \text{ for } -1 < t < 1.$$

Then the Taylor expansion formula (see e.g. [1, p. 361]) immediately yields

$$f_s(t) = \sum_{n=0}^{\infty} \binom{s/2}{n} t^n,\ f_*(t) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n} t^n \ \text{ for } -1 < t < 1.$$

We now prove Lemmas 3.1 and 3.2 by applying Proposition 3.6. First we prove Lemma 3.2.

Proof of Lemma 3.2. Since $|e_N|_N = 1$ and

$$|x+e_N|_N^s = f_s(\rho_N(x)),\ \log|x+e_N|_N = f_*(\rho_N(x)) \ \text{ for } x \in B^N_{2^{1/2}-1},$$

we deduce

$$\begin{aligned} \gamma_N^{s,k} &= |\nabla_N^k[|\cdot|_N^s](e_N)|^2_{N^k} = |\nabla_N^k[|\cdot+e_N|_N^s](0)|^2_{N^k} = |\nabla_N^k[f_s(\rho_N)](0)|^2_{N^k}, \\ \ell_N^k &= |\nabla_N^k[\log|\cdot|_N](e_N)|^2_{N^k} = |\nabla_N^k[\log|\cdot+e_N|_N](0)|^2_{N^k} = |\nabla_N^k[f_*(\rho_N)](0)|^2_{N^k}. \end{aligned}$$

Applying Proposition 3.6 (ii), we obtain both the assertions (i) and (ii). $\square$

Next we prove Lemma 3.1.

Proof of Lemma 3.1. We argue as in the proof of Lemma 3.2 with applying Proposition 3.6 (i) instead of Proposition 3.6 (ii) to obtain the assertion. $\square$

We need the following proposition to prove Lemma 3.3.

Proposition 3.7. *For $\nu \in \mathbb{R}$ and $m \in \mathbb{Z}_+$, it holds*

$$\sum_{l=0}^{m} \frac{(2l)!}{2^{2l} l!} (\nu+m-l)_{m-l} \binom{m}{l} = \left(\nu+m+\frac{1}{2}\right)_m. \tag{3.2}$$

Proof. We use an induction on m. When $m=0$, (3.2) trivially holds. Fix $m \in \mathbb{N}$ and assume that (3.2) holds for $m-1$, that is,

$$\sum_{l=0}^{m-1} \frac{(2l)!}{2^{2l} l!} (\nu+m-l-1)_{m-l-1} \binom{m-1}{l} = \left(\nu+m-\frac{1}{2}\right)_{m-1}. \tag{3.3}$$

We use the following identities

$$\begin{aligned} \binom{m}{l} &= \binom{m-1}{l} + \binom{m-1}{l-1} \ \text{ for } l \in \mathbb{N}, \\ (\nu+m-l)_{m-l} &= (\nu+m-l)(\nu+m-l-1)_{m-l-1} \ \text{ for } l \in \mathbb{Z}_+. \end{aligned}$$

Then we have

$$\begin{aligned}\sum_{l=0}^{m}\frac{(2l)!}{2^{2l}l!}(\nu+m-l)_{m-l}\binom{m}{l} &= (\nu+m)_m+\sum_{l=1}^{m-1}\frac{(2l)!}{2^{2l}l!}(\nu+m-l)_{m-l}\binom{m-1}{l}\\ &\quad+\sum_{l=1}^{m-1}\frac{(2l)!}{2^{2l}l!}(\nu+m-l)_{m-l}\binom{m-1}{l-1}+\frac{(2m)!}{2^{2m}m!}\\ &=\sum_{l=0}^{m-1}\frac{(2l)!}{2^{2l}l!}(\nu+m-l)(\nu+m-l-1)_{m-l-1}\binom{m-1}{l}\\ &\quad+\sum_{l=0}^{m-1}\frac{(2(l+1))!}{2^{2(l+1)}(l+1)!}(\nu+m-l-1)_{m-l-1}\binom{m-1}{l}\\ &=\left(\nu+m+\frac{1}{2}\right)\sum_{l=0}^{m-1}\frac{(2l)!}{2^{2l}l!}(\nu+m-l-1)_{m-l-1}\binom{m-1}{l}.\end{aligned}$$

Applying (3.3), we have

$$\sum_{l=0}^{m}\frac{(2l)!}{2^{2l}l!}(\nu+m-l)_{m-l}\binom{m}{l}=\left(\nu+m+\frac{1}{2}\right)_m,$$

which shows that (3.2) holds also for m. The calculation above works also for $m=1$; as usual, we regard any empty sum as 0. Thus (3.2) is proved. □

We now prove Lemma 3.3.

Proof of Lemma 3.3. We use an induction on N. First, (1.3) gives

$$\gamma_1^{2m,2m}=((2m)!)^2 \quad \text{for } m\in\mathbb{Z}_+.$$

Meanwhile we have

$$2^{2m}m!\,(2m)!\left(m-\frac{1}{2}\right)_m=2^m m!\,(2m)!\prod_{j=1}^{m}(2j-1)=((2m)!)^2 \quad \text{for } m\in\mathbb{Z}_+.$$

The equality above is valid also for $m=0$; as usual, we regard any empty product as 1. Thus (3.1) holds for $N=1$. Fix $N\in\mathbb{N}\setminus\{1\}$ and assume that (3.1) holds for $N-1$, that is,

$$\gamma_{N-1}^{2m,2m}=2^{2m}m!\,(2m)!\left(\frac{N-3}{2}+m\right)_m \quad \text{for } m\in\mathbb{Z}_+. \tag{3.4}$$

It follows from Lemma 3.2 that

$$\gamma_N^{2m,2m}=(2m)!\sum_{l=0}^{m}\frac{(2(m-l))!}{(2l)!}\binom{m}{l}^2\gamma_{N-1}^{2l,2l}=(2m)!\sum_{l=0}^{m}\frac{(2l)!}{(2(m-l))!}\binom{m}{l}^2\gamma_{N-1}^{2(m-l),2(m-l)}.$$

Applying (3.4) and Proposition 3.7, we have

$$\begin{aligned}\gamma_N^{2m,2m}&=(2m)!\sum_{l=0}^{m}(2l)!\binom{m}{l}^2 2^{2(m-l)}(m-l)!\left(\frac{N-3}{2}+m-l\right)_{m-l}\\ &=2^{2m}m!\,(2m)!\sum_{l=0}^{m}\frac{(2l)!}{2^{2l}l!}\left(\frac{N-3}{2}+m-l\right)_{m-l}\binom{m}{l}\\ &=2^{2m}m!\,(2m)!\left(\frac{N}{2}+m-1\right)_m \quad \text{for } m\in\mathbb{Z}_+,\end{aligned}$$

which shows that (3.1) holds also for N. Thus (3.1) is proved. □

Thus we have proved Theorem 1.2.

4 Proof of Theorem 1.3

We can easily prove Theorem 1.3 by applying Theorem 1.1.

Proof of Theorem 1.3. Let $k \in \mathbb{N}$. For $u \in C^{k+1}(\mathbb{R}^N \setminus \{0\})$, a direct calculation shows

$$\Delta_N[|\nabla_N^{k-1}u|^2_{N^{k-1}}] = 2|\nabla_N^k u|^2_{N^k} + 2(\nabla_N^{k-1}u, \nabla_N^{k-1}[\Delta_N u])_{N^{k-1}} \quad \text{on } \mathbb{R}^N \setminus \{0\}, \tag{4.1}$$

where $\Delta_N = D_1^2 + D_2^2 + \cdots + D_N^2$ is the usual Laplacian on $\mathbb{R}^N$. We see that for $\nu \in \mathbb{R}$,

$$\Delta_N[|x|_N^\nu] = \nu(\nu + N - 2)|x|_N^{\nu-2} \quad \text{for } x \in \mathbb{R}^N \setminus \{0\}. \tag{4.2}$$

(i) It follows from (4.1) and (4.2) that

$$2\left|\nabla_N^k\left[\frac{1}{|x|_N^{N-2}}\right]\right|^2_{N^k} = \Delta_N\left[\left|\nabla_N^{k-1}\left[\frac{1}{|x|_N^{N-2}}\right]\right|^2_{N^{k-1}}\right] \quad \text{for } x \in \mathbb{R}^N \setminus \{0\}.$$

By virtue of Theorem 1.1 and (4.2), we deduce

$$\begin{aligned}
&2\gamma_N^{-(N-2),k}\frac{1}{|x|_N^{2(N+k-2)}}\\
&= \gamma_N^{-(N-2),k-1}\Delta_N\left[\frac{1}{|x|_N^{2(N+k-3)}}\right]\\
&= 2(N+k-3)(N+2k-4)\gamma_N^{-(N-2),k-1}\frac{1}{|x|_N^{2(N+k-2)}} \quad \text{for } x \in \mathbb{R}^N \setminus \{0\},
\end{aligned}$$

which implies

$$\gamma_N^{-(N-2),k} = 2(N+k-3)\left(\frac{N}{2}+k-2\right)\gamma_N^{-(N-2),k-1}.$$

The desired conclusion now follows inductively since $\gamma_N^{-(N-2),0} = 1$.

(ii) Note that

$$\Delta_2[\log|x|_2] = 0 \quad \text{for } x \in \mathbb{R}^2 \setminus \{0\}.$$

We argue as in (i) to deduce

$$\ell_2^k = 2(k-1)^2\ell_2^{k-1}.$$

The desired conclusion now follows inductively since $\ell_2^1 = 1$. □

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