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The asymptotic solution of linear differential systems, applications of the Levinson theorem, by M. S. P. Eastham. Clarendon Press, Oxford, 1989, vii + 240 pp., $63.00. ISBN 0-19-853299-7

The earliest results on the asymptotic approximation to solutions of ordinary differential equations can be traced back to the latter part of the ninteenth century and the early part of the twentieth century in the works of Poincaré, Kneser, Horn, Bôcher, Dunkel, Birkhoff, and others. These early works had as their focus the asymptotic approximation of solutions by means of divergent series which are formal solutions of the differential equations. However, by taking any fixed finite number of terms of the formal divergent series, one gets an explicit function which is asymptotic to a solution of the differential equation as the independent variable approaches a point on the extended real line. In these two senses the word "asymptotic" has two slightly different meanings. During the last forty to fifty years a good deal of work on asymptotic solutions has concentrated on the second meaning. The results so obtained may not be as precise as the consideration of asymptotic series, but they have the enormous advantage that much wider classes of differential equations and systems can be dealt with. These wider classes of equations and asymptotic solutions are of importance for applications, which has been the main driving force behind these investigations.

In the book under review, the author deals only with the second aspect of asymptotic solution. In particular, the bulk of the book concentrates on two important applications of a remarkable asymptotic theorem of N. Levinson which the latter established in 1948 in an essentially definitive form. The Levinson theorem deals with $n \times n$ systems of ordinary differential equations of the form

$$Y'(x) = [A + V(x) + R(x)]Y(x), \qquad 0 < x < \infty,$$

where A is a matrix with n distinct characteristic roots, $V(x) \to 0$ as $x \to \infty$, and

$$\int_0^\infty |V'(x)|\,dx < \infty, \qquad \int_0^\infty |R(x)|\,dx < \infty.$$

Then, under suitable restrictions on the eigenvalues of $A + V(x)$, there are n linearly independent solutions, $y_1, \ldots, y_n$, of the differential system and an x_0, $0 \le x_0 < \infty$, so that

$$\lim_{x\to\infty} y_k(x) \exp\left[-\int_{x_0}^{x} \lambda_k(t)\,dt\right] = p_k,$$

where $\lambda_k(t)$ is an eigenvalue of $A+V(t)$ and p_k is an eigenvector of A. The author provides full details of this theorem in his first chapter.

One of the applications which motivated a good deal of the research on asymptotic solutions was the problem of deficiency index for a formally self-adjoint ordinary differential operator defined, say, on $(0, \infty)$. This is one of the applications explored in detail in Eastham's book. If L is a formally self-adjoint differential operator, then the deficiency index (n_+, n_-) of L is composed of the number of square integrable solutions, respectively, to the equations

$$Lu = \pm iu.$$

Clearly, the numbers n_+ and n_- can be determined if one has sufficiently accurate asymptotic approximations to solutions of these equations in a neighborhood of infinity. Beginning in the early 1950s a great deal of work in this direction was accomplished by a Russian school of mathematicians. A report of this work may be found in a book of Neumark published in Russian in 1954, translated into German in 1963, and into English in 1968. The proofs of the asymptotic results obtained at that time appeared to be of a somewhat ad hoc nature. However, with hindsight and a closer reading of the proofs, it can be seen that they are essentially reproofs of Levinson's theorem in special cases.

However, it took almost two decades after Levinson's paper that a younger generation of mathematicians, starting with the reviewer in 1966, realized that the asymptotic approach to the deficiency index problem was, after some transformations of the independent or dependent variables, simply a direct application of Levinson's theorem. And it was yet another eight years, in a series of papers beginning in 1974, that Harris and Lutz proposed the elegantly elementary techniques which made evident the significance and range of the applications of the theorem.

Since the mid 1960s work on both refinements and applications of Levinson's theorem have continued in a host of publications up to the present day. The refinements include the cases where the matrix A of the differential system has multiple characteristic roots and where A is in the Jordan canonical form, and where the differentiability of the matrix $V(x)$ and the integrability of the matrix $R(x)$ are relaxed. Some of these refinements are reported in detail in the work under review, while others are mentioned in notes to the first chapter.

For the applications, one usually starts with a first order system of ordinary differential equations, or converts a single differential equation to a first order system. The strategy is to apply transformations to the independent and/or dependent variables so as to bring the system to a form where Levinson's theorem may be applied. While the strategy is clear, the tactics of which transformations to apply are not always so clear and depend on the systems under consideration. This at times requires considerable ingenuity, and the author presents, in the remaining three chapters of the book, rather detailed analyses for transforming various classes of systems to the Levinson form. Because of the large literature on the subject it is not possible to list even the important contributions. However, a paper of Hinton (1968) may be singled out as an early work in which transformations of a differential system lead to a direct application of Levinson's theorem. Also, in the last decade, Eastham together with his students and collaborators may be singled out for providing a continuing impetus to research in this area.

Aside from the deficiency index problem, a second application explored in some detail in this work is the phenomenon of resonance and nonresonance. This is modeled by a differential system which in its simplest form is given by

$$Y'(x) = \{i\Lambda_0 + R(x)\}Y(x),$$

where Λ_0 is a real constant matrix and $R(x) = \xi(x)P(x)$, where $\xi(x)$ is a scalar factor such that $\xi(x) \to 0$ as $x \to \infty$ and $\xi(x)$ is not integrable, while $P(x)$ is nonconstant matrix which is either periodic or more generally a finite sum of periodic matrices with different periods. The new feature which appears in the theory, say for periodic $P(x)$, is the influence of the period on the size of the solutions $Y(x)$. It is possible for $Y(x)$ to possess an amplitude factor $\rho(x)$ such that

$$|Y(x)| = \rho(x)\{1 + o(1)\}, \qquad x \to \infty,$$

where either $\rho(x) \to \infty$ or $\rho(x) \to 0$ as $x \to \infty$. When such a factor $\rho(x)$ appears the system is said to be resonant, and otherwise nonresonant. The knowledge that resonance can occur goes back to Perron (1930) but the first systematic analysis of resonance and nonresonance appears to be due to Atkinson (1954).

Other applications of Levinson's theorem that the reviewer is familiar with, but not covered in this book, include spectral and scattering theory for ordinary and partial differential operators and to wave propagation in stratified fluids. The reviewer feels sure that there are other applications or possible applications to areas with which he is unfamiliar.

This book is written in the author's usual elegant style. The exposition is crisp, the explanations and proofs are clear. It can certainly be recommended for the bookshelf of anyone interested in its subject matter. Indeed, it can be recommended for anyone who enjoys reading well-written mathematics and learning a bit about a small, but important, corner of mathematics.

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Conformal geometry, Ravi S. Kulkarni and Ulrich Pinkall, eds. Aspects of Mathematics, Friedr. Vieweg & Sohn, Braunsweig and Wiesbaden, 1988, viii + 236 pp. ISBN 3-528-08982-2

The book under review is the proceedings of a seminar on conformal geometry held at the Max-Planck Institute in 1985–1986. This subfield of differential geometry is rather vast and multifaceted, and therefore it would be impossible for a single volume to deal with all aspects of this subject. The book contains several survey articles which deal with various aspects of conformal Riemannian structures, from the points of view of both topology/synthetic-geometry and local differential geometry. From both points of view flat conformal structures play a central role and all of the papers in this volume deal with at least some aspects of the theory of conformally flat manifolds. For this reason this review will concentrate on conformally flat manifolds and will