a text for a seminar or a reading text for graduate students with some background in analytic number theory.

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Geometrical methods in theoretical physics, by K. Bleuler and M. Werner, Kluwer Academic Publ., NATO ASI Series, 1988, 471 pp., ISBN 90-277-2820

INTRODUCTION

Like a newspaper's headline, a Conference Proceeding can capture a moment in the history of science and thought. As if reading old newsprint, you will enter that moment as a time traveler. But, you travel in time without the context or ties to the newspaper's moment, though the context gives meaning to the newspaper's words—turns two dimensional pictures to three dimensional lives.

Geometrical methods in theoretical physics presents us with proceedings from a conference in Como, the country was Italy, the time was late summer, two years ago. Read it as newsprint; it reports of events in mathematical physics, excitement that dates from those summers ago. But, for most of you readers, the context is missing, so I will provide a taste in the space that's allotted, for with no context, these proceedings can be dusty and dry.

Geometrical physics can be traced back to Einstein and ideas that were born seven decades ago. He taught us to think of space-time together, one geometrical object—not abstract points. Space-time is a manifold, locally Euclidean, but curved in the large and in the four directions of space and time. This curvature we humans interpret as gravity—that most mundane of forces that holds us to Earth. Einstein taught that not only do we feel the curving, but the curvature feels us too. Einstein's Equation equates a part of that curving to the energy of the matter that sits in the space. Mathematically put:

$$\text{Ricci Curvature} \;-\; 1/2 \cdot \text{ Scalar curvature} \;=\; \text{Stress Energy.}$$

Tests have been done which confirm Einstein's equations.

While Einstein pursued his theory of space-time, physics experienced an unprecedented upheaval, the birth of the quantum mechanical view. According to the laws of quantum mechanics, an isolated physical system (an atom, for example, or the whole universe for another) is (loosely) described in the following way: A complex vector space V must first be proposed, and V must have an inner product: $\langle\ ,\ \rangle$. This inner product must be hermitian and positive too. (Hermitian requires that $\langle c \cdot \psi, \phi\rangle = c^* \cdot \langle \psi, \phi\rangle$ and $\langle \psi, c \cdot \phi\rangle = c \cdot \langle \psi, \phi\rangle$ for all complex numbers c and vectors ψ and ϕ in V. Positivity requires that $\langle \phi, \phi\rangle$ is positive unless $\phi = 0$.)

The possible states of the (isolated) system should each be assigned a complex line in V. And, each performable measurement of the system should be assigned a selfadjoint operator on V. (An operator O on V is selfadjoint if $\langle\psi, O\phi\rangle = \langle O\psi, \phi\rangle$ for all vectors ψ and ϕ in V.) Once all of these assignments are made, quantum mechanics makes the following additional postulate: The evolution of the states as a function of the parameter that we call time is to be effected by the action on V of a 1-parameter (time) group of unitary operators. Futhermore, this group has a generator which is to be a multiple of that specific selfadjoint operator which corresponds to the total energy of the system.

Quantum mechanics has been tested by experiment again and again, now for almost 70 years. All tests have been passed with flying colors. Quantum mechanics is as good a theory as one could hope to find.

Make your peace with these last two historical points; for the greatest conceptual problem in physics which yet stands unsolved is: Formulate a version of quantum mechanics which is consistent with Einstein's theory of gravity. Or, modify Einstein's gravity to be consistent with quantum mechanics.

The unification of gravity with quantum mechanics defies, to date, even a *formally* consistent solution. By comparison, the unification of quantum mechanics with Einstein's special relativity has had a formal solution (quantum field theory) now for forty years. Indeed, the formulation is easy: Unification with special relativity requires that our unitary representation on V of the linear group of time translations should be naturally embedded as a subgroup in a unitary representation on the space V of the Poincaré group (the semidirect product of the Lorentz group $SO(3, 1)$ and the Euclidean group of translations R^4).

This unification of quantum mechanics with special relativity is, in practice, remarkably difficult to achieve for systems with interparticle interactions. In fact, for realistic physical systems, quantum field theories exist only in a formal sense: There is but an algorithm for calculating the probabilities of the various outcomes of experiments and observations. The algorithm is reasonably successful though it does not (as yet) have any rigorous mathematical justification. All physicists of repute believe the calculations from these formal quantum field theories. (Much difficult mathematical work (see, e.g. [G-J]) justifies an optimistic outlook for eventual construction of the mathematical foundations.)

With the preceding broad remarks understood, let me turn to the specific issues which faced the mathematical physicists at Como. At Como, most of the papers discuss aspects and spin-offs of two recent, popular (and not unrelated) approaches to the aforementioned reconciliation problem. These being String theory and

Supersymmetry. I will describe both, but let me first end any suspense: Neither has yet successfully unified gravity with quantum mechanics, not even in the formal manner that constitutes a "proof" for my physicist friends.

Supersymmetry

In supersymmetric quantum mechanics (as I have learned from the writings of E. Witten, [W] for one), the vector space V is posited to decompose as a direct sum of $V_0 \oplus V_1$, called the "even" and "odd" elements of V. Furthermore, the selfadjoint operator which represents the total energy of the system in question must be the square of a selfadjoint operator which changes even to odd and vice versa. Call this new operator $\mathbf{Q}$, then with respect to the aforementioned direct sum decomposition of V,

$$\mathbf{Q} = \begin{pmatrix} 0 & q \\ q^* & 0 \end{pmatrix}.$$

Marrying supersymmetry with special relativity introduces an exceptionally rich structure. One is led fairly rapidly to interesting mathematical investigations of extensions of the lie algebra of the Poincaré group. This is a consequence of the fact that the total energy of a system is not an invariant notion; two observers, both moving at constant, but different velocities will disagree about the total energy.

A simple experiment with a thrown baseball and a plate glass window will convince you that a ball in motion has more energy than a ball at rest. So, the fact that the energy of a system is a function of the observer's velocity is easy to establish (though the correct functional dependence is not as accessible to the arm chair physicist.)

The fact is that energy in special relativity is one component in a vector representation of the Lorentz group, $SO(3, 1)$. Thus, our operator on V which represents the total energy must be one of the four components of a 4-vector of operators on V, where the other three components generate an action on V of the Euclidean group of spatial translations. These other three operators are called the momentum operators.

With this last fact understood, we see that a relativistic theory which asserts a square root operator for the energy operator must assert the existence of a square root operator for each of the three momentum operators. And, each of these new square root operators must change even to odd and vice-versa. Call our original energy operator $\mathbf{P}_0$ and each of the three momentum operators

$\mathbf{P}_{1,2,3}$. Then each has a square root $\mathbf{Q}_{0,1,2,3}$. The invariant way of saying this:

$$\mathbf{Q}_k\mathbf{Q}_j + \mathbf{Q}_j\mathbf{Q}_k = \mathbf{P}_j \cdot \delta_{jk}.$$

Here, it is explicit that the representation on V of the Lie algebra of the Poincaré group has been augmented by operators whose *anticommutators* (rather than commutators) has been prescribed. Such an augmentation is called a "super Lie algebra," and a quantum field theory with such a super Lie algebra represented on its vector space V of states is called a "supersymmetric" quantum field theory.

The dream among the supersymmetry afficianados is that a supersymmetric quantum field theory might be found which contains Einstein's gravity as a natural classical limit. There is some mathematical evidence that certain pathologies which arise when quantizing gravity are incompatible with the existence of an underlying supersymmetry. So there is some evidence that a supersymmetric theory should be easier to quantize than Einstein's.

I will refer you for further details and references to §V of *Differential geometric methods in theoretical physics*. But first, a comment of a personal nature: I will begin with the apology that I am a complete novice with supersymmetry and the related fields. With this last remark understood, I can say that I found, with two exceptions, the contributions to §V to require significant effort by the reader. That is, the writing is terse and full of specialized jargon. But, in any event, the collection of references seems to be timely; and such a collection is one of the prime benefits to owning such a book as this.

My two exceptions are: First, the article by U. Bruzzo titled *Supermanifolds, supermanifold cohomology and super vector bundles* which has an imposing name, but gave me a nice taste of what a supermanifold is. My second exception is the article by Manfred Scheunert which is an introduction (sans proofs) to the representation theory of super Lie algebras.

STRING THEORY

String theory and its kin, the conformal field theories, have had a remarkable impact on mathematics in the past five years. Notice that I said mathematics. Their impact on actual physics has been minimal, except for the nonminimal effect of turning a whole generation of young theoretical physicists into mathematicians. I will explain.

The basic novelty in string theory is the replacement of the point as the idealized particle with an extended object, the string. To describe a point, one needs only state its position as a function of time. But, the string requires a position (a center of mass) as a function of time, and also the specification of the vibrational modes of the string about this center of mass.

These facts enter physics in the following way: There are on the order of twenty or so elementary particles in nature—all other particles (and forces) can be described as composites of these basic elementary ones. Each particle has a mass plus some other basic properties which must be a priori specified as the values of the various parameters which enter in the standard theories of elementary particles. So, there are on the order of twenty or thirty free parameters which must be specified by the theoretical physicist before calculations can be done. Providentially, there are more than thirty independent experiments which can be done, for if you have only twenty-nine experiments, and thirty free parameters, then you don't have a theory.

A point describes each of the twenty or so elementary particles—in quantum field theory, one uses quantized points.

A "better" theory of elementary particles would have equal (or better) predictive power and *fewer* free parameters to be specified. With this goal in mind, imagine a quantum theory where there is only one basic object, a string. The one string is to replace the twenty or so elementary points. Furthermore, hypothesize that each vibrational mode of this elementary string is interpreted by humans as a *different* elementary particle. For example, *A* above middle *C* is an electron, and *A* below middle *C* is a photon...etc. If such a theory of propagating strings can be constructed, then it would only have one parameter, a sort of string tension which sets the basic vibrational frequency. With a consistent mathematical formulation, and a consistent physical interpretation, such a theory would be a marvelous improvement over the "standard model."

And, just maybe, maybe, one of the octaves of the string can be given a consistent physical interpretation as the carrier of the *gravitational* force. Wouldn't it be wonderful? Gravity and all of the other forces and particles in nature would have a unified, accurate description. Physics as we know it would be finished, the book closed and only Chemists would remain!

I am being facetious here. I have slightly exaggerated the euphoric remarks that string people spoke as their models were seen, after some hand waving, to come close to predicting elementary particle behavior.

But, hand waving is not, ultimately, credible physics, and "close" only counts in horse shoes. To date, string theory has made no flat out prediction which is testable by any experiment in the forseeable future, even a future with the Super Conducting Super Collider—this being a proposed particle accellerator with a forty mile circumference. (Your tax dollars at work.) As an aside, this collider might be able to confirm some predictions of certain supersymmetric theories.

So much for real physics from string theories. However, the mathematical outwash from string theories and conformal field theories has been truly prodigious with applications from algebraic geometry through algebraic topology to quantum groups and knot theory.

After reading the writings of Graeme Segal (a nice abridged version appears in *Differential geometric methods in theoretical physics*), I have enough courage to give a brief outline of how all of this comes about. To begin, it is rather crucial to consider *parameterized* strings (a string with a coordinate on it). Deal with parameterized strings first and then regain strings by forgetting the extra knowledge of the parameter. This strategy exploits the basic fact that the set of all parameterized strings is simpler to describe than the set of all unparameterized strings. Indeed, the set of all parameterized strings is the space of all maps from the circle into space (if the strings are closed); this is the configuration space for the classical theory of moving, parameterized strings. When space is a Euclidean space, this configuration space is itself a vector space.

The task of forgetting the parameterization is simplified if one remembers the following facts: First, two different parameterizations of the same string differ by a diffeomorphism of the circle. Second, the set of all diffeomorphisms of the circle forms a group, in that two can be composed to form a third, and each has an inverse. This group is called $\operatorname{Diff} S^1$. It has a Lie algebra which is the lie algebra of vector fields on the circle (the Virasoro algebra). Finally, the group $\operatorname{Diff} S^1$ acts on the space of parameterized strings, and the space of *orbits* is just the space of unparameterized strings.

Pick a vector subspace of the vector space of all complex valued functions on the space of parameterized strings. This vector space, or some generalization of it will be the V for the quantum theory. But, make sure that V is sufficiently large so that the group $\operatorname{Diff} S^1$ acts on it; if $\operatorname{Diff} S^1$ does not act on V, there will be no hope of separating $\operatorname{Diff} S^1$ invariant products at the end. Indeed, because parameterizations are ultimately forgotten, all constructions must preserve the action of $\operatorname{Diff} S^1$—how else can $\operatorname{Diff} S^1$ invariant data be sorted from the surrounding chaff? (To accomplish this preservation, one is lead to study the representation theory for the group $\mathrm{Diff}\, S^1$; both V and the space of operators on V are, by definition such representations.)

Our vector space V , above, is the vector space for the quantized theory of a single string. According to Segal, the vector space for a quantized theory of interacting strings should be a many-fold tensor product of V .

To understand the operators in string theory, first digress to picture a string propagating in time: It sweeps out a two dimensional surface. This surface has the a priori topology of a cylinder. Next, imagine the propagation of many strings at once (a physical necessity because the world contains more than one particle). This propagation sweeps out a set of disjoint cylinders.

Imagine these many strings interacting via collisions (a physical necessity because real particles do interact). There is a natural way for two closed strings to interact; they can collide and merge into one. Conversely, one string can pinch off into a figure eight, and then split into two closed strings. This kind of interacting propagation of many strings sweeps out a surface with (possibly) many connected components, where each component can have the topology of a many holed torus with cylindrical ends. One end for each incoming string, and one for each outgoing string.

This is a crucial point: There is a natural interaction between strings which causes their propagation to sweep out surfaces with complicated topologies.

The point above is encapsuled by the quantum field theory by postulating that a quantum theory of many interacting strings should contain, for each topological surface (with parameterized boundary components), an operator on the big tensor product, $\otimes V$. For each such surface, the associated operator corresponds to a propagation which sweeps out the topological surface in question. In addition, this assignment of surface to operator should be $\mathrm{Diff}\, S^1$ equivariant—so the $\mathrm{Diff}\, S^1$ invariant structure can be identified. One final demand, the assignment should be natural with respect to the action of sewing surfaces together along identified boundary components. This corresponds to the fact that the propagation from time A to time B and then to time C should be the same as the propagation from time A to time C .

Conformal field theories

The preceeding paragraph summarizes most of Graeme Segal's axioms for string theory. It is not easy to find nontrivial examples of these axioms, and few are known. The hard part is to build a natural assignment from topological surfaces to operators. Somewhat easier (but still not generally complete) is the task of building an assignment from a Riemann surface (a smooth surface with a metric on its tangent bundle) to an operator. Midway in difficulty is to build an assignment from Riemann surface to an operator which depends only on the *conformal* class of the metric. Such field theories are called *conformal*.

The space of conformal structure on a topological surface is the same as the moduli space of complex structures on the surface. As explained in Segal's article, these moduli spaces, taken together, form a sort of algebra under the operation of sewing together identified boundary components. Then, a conformal field theory is naught but a representation of this algebra by operators in a vector space. Here, of course, one is already deep into classical algebraic geometry, for the study of natural structures on the moduli spaces of algebraic curves is at the heart of a great deal of 20th-century mathematics. And this conformal field theory certainly looks natural.

Profound relationships between conformal field theories, statistical mechanics, quantum groups and knot theory have recently been discovered. The relationships between field theories and knots is currently being explored by Ed Witten [W′], while a parallel development is being carried out by Soviet mathematicians (Turaev and Reshitikin are prominent [R-T]) who are uncovering marvelous relationships between quantum groups and knots. These relationships are also being studied by Japanese researchers (Sato, Jimbo and their colleagues), Vaughn Jones here in the United States, and M. F. Atiyah and N. Hitchin at Oxford.

The biggest breakthroughs in the subject (by Witten and the Soviets) occurred since summer, after the closing of the conference at Como, though they are presaged by J. Frolich's contribution in *Differential geometric methods in theoretical physics*, and likewise M. F. Atiyah's. Frolich discusses a mechanism whereby representations of Artin's braid group can be obtained from conformal field theories, while Atiyah speculates about a possible relationship between Vaughn Jones' knot invariants and the work of Donaldson and Floer on gauge theories. Also, H. J. de Vega contributes to *Differential geometric methods in theoretical physics* with an article which describes the relationships between quantum groups and statistical mechanics. On the subject of quantum groups I profess a granitic ignorance and will refer the reader directly to de Vega's article, or to the recent book by Fadeev and Takhtajan [F-T]. The relationship between conformal field theories and statistical mechanics is discussed in the contribution of M. Karowski, but those who are interested might look also at a recent Bourbaki contribution by K. Gawedski [G].

Conclusion

As conference proceedings go, *Differential geometric methods in theoretical physics* is probably worth having; if nothing else, for the timely set of references. For myself, I found these proceedings giving an interesting snapshot of mathematical physics in the mid 1980s.

For those who buy the book, I recommend the following strategy: Start with George Mackey's contribution, "Weyl's program and modern physics," a thought provoking piece which talks about quantum mechanics, both relativistic and not. Mackey's article gives a nice historical perspective to the conference. Then, read Graeme Segal's contribution, *The definition of conformal field theory*; his formulation is already influential. Next, read the article by Y. Ne'eman and D. Sijacki; *Towards a renormalizable theory of quantum gravity* to see where quantum gravity is. Then, read the article by U. Bruzzo and that by M. Scheunert to learn about supersymmetry. Look at J. Frolich's article, because you will be hearing a lot about knots and quantum field theory. Scan de Vega's article because you will also be hearing a lot about quantum groups. Finally, read Atiyah's introduction, *The impact of physics on geometry*, as anything by Atiyah is thought provoking.

There are contributions of which I have said nothing. These are what look like research articles, for example, there is one by V. Kac, R. V. Moody and M. Wakimoto; one by F. Hirzebruch, one by B. Kostant, one by S. Sternberg, one by A. Lichenrowicz, and other contributions by other authors. Certainly a distinguished cast of characters.

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Asymptotic behavior of dissipative systems, by Jack K. Hale. Mathematical Surveys and Monographs, vol. 25, American Mathematical Society, Providence, R.I., 1988, ix + 198 pp., $54.00. ISBN 0-8218-1527-x

The dynamical systems encountered in physical or biological sciences can be grouped roughly into two classes: the conservative ones (including the Hamiltonian systems) and those exhibiting some type of dissipation. These dynamical systems are often generated by partial differential equations and thus the underlying state space is infinite dimensional.

I. It is natural to expect that the flow defined by a dissipative system shall be simpler than the one of a conservative system. It is perhaps even possible to isolate an interesting class of systems for which one can adapt several ideas coming from the ordinary differential equations (O.D.E's) to the analysis of the flow. If this can be done, then one must overcome the difficulties that arise due to the nonlocal compactness of the state space. This will require some type of "smoothing" property of the dynamical system. There are also problems that can arise at infinity due to the unboundedness of the space. This problem is avoided by imposing specific dissipative conditions. To make the discussion more meaningful and to motivate the class of systems considered in the book under review, it is instructive to recall the situation for the ordinary differential equations.

In his study of the forced van der Pol equation, Levinson [13] introduced the concept "point dissipative." To keep the technicality at a minimum, let us discuss at first discrete dynamical systems; that is, those defined by a map $T : \mathbb{R}^n \to \mathbb{R}^n$. The