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ON REAL ALGEBRAIC MODELS OF SMOOTH MANIFOLDS

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An affine nonsingular real algebraic variety X diffeomorphic to a smooth manifold M is said to be an *algebraic model* of M. The remarkable theorem of Nash-Tognoli asserts that each compact smooth manifold M has an algebraic model [**16** or **6**, Theorem 14.1.10]. In fact, there exists an infinite family $\{X_i\}_{i\in\mathbf{N}}$ of irreducible algebraic models of M such that X_i and X_j are birationally nonisomorphic for $i \neq j$ [**10**] (cf. also [**7**] for a proof in a special case). In view of these results, it seems natural and interesting to investigate algebro-geometric properties of various algebraic models of a given smooth manifold. This paper addresses a few questions of this type. For notions and results of real algebraic geometry we refer the reader to the book [**6**].

Given a compact affine nonsingular real algebraic variety X, denote by $H_k^{\mathrm{alg}}(X, \mathbf{Z}/2)$ the subgroup of $H_k(X, \mathbf{Z}/2)$ of the homology classes represented by (Zariski closed) k-dimensional algebraic subvarieties of X [**6**, Chapter 11 or **11**]. Let $H^k_{\mathrm{alg}}(X, \mathbf{Z}/2)$ be the image of $H^{\mathrm{alg}}_{d-k}(X, \mathbf{Z}/2)$, $d = \dim X$, under the Poincaré duality isomorphism $H_{d-k}(X, \mathbf{Z}/2) \to H^k(X, \mathbf{Z}/2)$. Although the groups $H^k_{\mathrm{alg}}(X, \mathbf{Z}/2)$ are one of the most important invariants of X (a sample of applications can be found in [**1, 2, 3, 6, 8, 9**]), our knowledge of their behavior is still rather limited. Here we consider the following.

PROBLEM. Let M be a compact smooth manifold and let G be a subgroup of $H^k(M, \mathbf{Z}/2)$. When are there an algebraic model X of M and a diffeomorphism $\varphi: X \to M$ such that the induced isomorphism

$$\varphi^*: H^k(M, \mathbf{Z}/2) \to H^k(X, \mathbf{Z}/2)$$

maps G onto $H^k_{\mathrm{alg}}(X, \mathbf{Z}/2)$?

This problem has attracted the attention of several mathematicians (cf. [**3, 4, 5, 6, 12, 14, 15**]), however, the results are far from complete. We have a solution for $k = 1$, M connected, and $\dim M \geq 3$.

THEOREM 1. *Let M be a compact connected smooth manifold with* $\dim M \geq 3$ *and let G be a subgroup of $H^1(M, \mathbf{Z}/2)$. Then the following conditions are equivalent*:

(i) *There exists an algebraic model X of M and a diffeomorphism $\varphi: X \to M$ such that $\varphi^*(G) = H^1_{\mathrm{alg}}(X, \mathbf{Z}/2)$.*

(ii) *The first Stiefel-Whitney class $w_1(M)$ of M is in G.*

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In particular, if M is orientable, then (i) *is always satisfied.*

For smooth surfaces and $k = 1$ we have only a partial, but quite satisfactory, solution. First let us define the following invariants of a compact nonsingular real algebraic surface X:

$$\beta(X) = \dim_{\mathbf{Z}/2} H^1_{\text{alg}}(X, \mathbf{Z}/2),$$
$$\delta(X) = \dim_{\mathbf{Z}/2}\{v \in H^1_{\text{alg}}(X, \mathbf{Z}/2) | v \cup v = 0\}.$$

If X is connected, orientable (resp. nonorientable of odd topological genus), then $\beta(X) = \delta(X)$ (resp. $\beta(X) = \delta(X)+1$; indeed, $w_1(X)$ is in $H^1_{\text{alg}}(X, \mathbf{Z}/2)$ [**6**, Theorem 12.4.8] and $w_1(X) \cup w_1(X) \neq 0$). For X connected, nonorientable of even topological genus, one has either $\beta(X) = \delta(X)$ or $\beta(X) = \delta(X) + 1$ and, in accordance with Theorem 2 below, all topologically possible cases can be realized algebraically.

THEOREM 2. *Let M be a compact connected smooth surface of genus g.*
(i) *If M is orientable* (*resp. nonorientable of odd genus*) *and β is an integer satisfying $0 \leq \beta \leq 2g$* (*resp.* $1 \leq \beta \leq g$), *then there exists an algebraic model X_β of M with $\beta(X_\beta) = \beta$.*
(ii) *If M is nonorientable of even genus and β and δ are integers satisfying either $\beta = \delta$, $1 \leq \beta \leq g - 1$, or $\beta = \delta + 1$, $2 \leq \beta \leq g$, then there exists an algebraic model $X_{\beta,\delta}$ of M such that $\beta(X_{\beta,\delta}) = \beta$ and $\delta(X_{\beta,\delta}) = \delta$.*

Our interest in the invariants $\beta(X)$ and $\delta(X)$ is explained by the fact that they determine the projective module group $K_0(\mathcal{R}(X))$ of the ring $\mathcal{R}(X)$ of regular functions from X to $\mathbf{R}$.

THEOREM 3. (i) *Let X be a compact connected affine nonsingular real algebraic surface. Then*

$$K_0(\mathcal{R}(X)) \cong \mathbf{Z} \oplus (\mathbf{Z}/4)^{\beta(X)-\delta(X)} \oplus (\mathbf{Z}/2)^{\beta(X)+1-2(\beta(X)-\delta(X))}.$$

(ii) *As X runs through all algebraic models of a compact, connected surface M of genus g, then the groups $K_0(\mathcal{R}(X))$ take, up to isomorphism, precisely $q(M)$ values, where*

$$q(M) = \begin{cases} 2g+1 & \text{if } M \text{ is orientable,} \\ g & \text{if } M \text{ is nonorientable and } g \text{ is odd,} \\ 2g-2 & \text{if } M \text{ is nonorientable and } g \text{ is even.} \end{cases}$$

Condition (i) is proved in [**9**], while (ii) follows immediately from (i) and Theorem 2.

Here is another application. Given a compact affine nonsingular real algebraic variety X, let $C^\infty(X, S^1)$ denote the topological group of C^∞ mappings from X to the unit circle $S^1 = \{z \in \mathbf{C} | |z| = 1\}$ (the group structure on $C^\infty(X, S^1)$ is induced from that on S^1 and the topology is the C^∞ one). Let $\overline{\mathcal{R}(X, S^1)}$ be the closure in $C^\infty(X, S^1)$ of the subgroup $\mathcal{R}(X, S^1)$ of regular mappings from X to S^1. Below we are concerned with the quotient group

$$\Gamma(X) = C^\infty(X, S^1)/\overline{\mathcal{R}(X, S^1)}.$$

THEOREM 4. *Let M be a compact connected smooth manifold with* $\dim M$
≥ 2. *Let*

$$r_M: H^1(M, \mathbf{Z}) \otimes \mathbf{Z}/2 \to H^1(M, \mathbf{Z}/2)$$

be the canonical monomorphism and let

$$\alpha(M) = \begin{cases} \operatorname{rank} H^1(M, \mathbf{Z}) - 1 & \text{if } M \text{ is nonorientable,} \\ & \text{and } w_1(M) \in \operatorname{Image} r_M, \\ \operatorname{rank} H^1(M, \mathbf{Z}) & \text{otherwise.} \end{cases}$$

(i) *For each algebraic model X of M, one has $\Gamma(X) \cong (\mathbf{Z}/2)^s$ for some s with $0 \leq s \leq \alpha(M)$.*

(ii) *For each integer s satisfying $0 \leq s \leq \alpha(M)$, there exists an algebraic model X_s of M with $\Gamma(X_s) \cong (\mathbf{Z}/2)^s$.*

SKETCH OF PROOF. Let X be a compact affine nonsingular real algebraic variety. By [**8**, Theorem 1.4], a mapping f in $C^\infty(X, S^1)$ is in $\overline{\mathcal{R}(X, S^1)}$ if and only if $f^*(H^1(S^1, \mathbf{Z}/2))$ is contained in $H^1_{\text{alg}}(X, \mathbf{Z}/2)$. It follows that $\Gamma(X)$ is isomorphic to the quotient group A/B, where $A = \operatorname{Image} r_X$ and $B = A \cap H^1_{\text{alg}}(X, \mathbf{Z}/2)$. This implies (i) and, applying Theorems 1 and 2, also (ii).

Proofs of Theorems 1 and 2 are quite involved. Here we can only sketch the proof of Theorem 1.

The implication (i) $\Rightarrow$ (ii) is well known (cf. [**6**, Theorem 12.4.8]). Suppose then that (ii) holds. One easily finds a C^∞ embedding $i: M \to P$ of M into the product $P = \mathbf{R}P^{k_1} \times \cdots \times \mathbf{R}P^{k_r}$ of real projective spaces with $i^*(H^1(P, \mathbf{Z}/2)) = G$. It requires some care to construct a compact smooth surface S in M such that $j^*(v) \neq 0$ for all v in $H = H^1(M, \mathbf{Z}/2) \backslash G$, where $j: S \to M$ is the inclusion mapping, and S bounds a compact smooth submanifold W of M with the property that the normal vector bundles of $i(W)$ in $i(M)$ and P are trivial. Using the triviality of the normal vector bundle of $i(W)$ in P, one shows the existence of a C^∞ diffeomorphism $h: P \to P$, arbitrarily close in the C^∞ topology to the identity mapping, such that $Y = h(i(S))$ is a nonsingular algebraic subvariety of P (cf. [**12**, §2]). Moreover, and this is the hard part of the construction, h can be chosen in such a way that $H^1_{\text{alg}}(Y, \mathbf{Z}/2) = 0$. To achieve this, one uses, in particular, appropriate real algebraic versions of the theorem of Gherardelli [**13**, Theorem 6.5] and the theorem of Noether-Lefschetz-Moishezon [**13**, Theorem 7.5] concerning the Picard group of complex projective varieties.

Since $H^*_{\text{alg}}(P, \mathbf{Z}/2) = H^*(P, \mathbf{Z}/2)$ and Y has trivial normal vector bundle in $N = h(i(M))$, it follows from [**1**, Proposition 2.8] that there exist a positive integer q and a C^∞ embedding $e: N \times \{0\} \to P \times \mathbf{R}^q$, arbitrarily close in the C^∞ topology to the inclusion mapping $N \times \{0\} \to P \times \mathbf{R}^q$, such that $X = e(N \times \{0\})$ is a nonsingular algebraic subvariety of $P \times \mathbf{R}^q$ containing $Y \times \{0\}$. Clearly, $\varphi: X \to M$, defined by the condition $e(h(i(\varphi(x))), 0) = x$ for x in X, is a C^∞ diffeomorphism and $H^1_{\text{alg}}(X, \mathbf{Z}/2)$ contains $\varphi^*(G)$. Since $H^1_{\text{alg}}(Y \times \{0\}, \mathbf{Z}/2) = 0$ and $f^*(\varphi^*(v)) \neq 0$ for all

v in H, where $f: Y \times \{0\} \to X$ is the inclusion mapping, it follows that $\varphi^*(G) = H^1_{\text{alg}}(X, \mathbf{Z}/2)$, i.e., (i) is satisfied.

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