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Sobolev spaces of infinite order and differential equations, by Julij A. Dubinskij. B. G. Teubner, Leipzig, and D. Reidel, Dordrecht, Boston, Lancaster, and Tokyo, 1986, 161 pp., $48.00. ISBN 90-277-2147-5

The early fifties, marked by the publication of the book *Applications of functional analysis in mathematical physics* by Sergej L. Sobolev in 1950 (see [**1**]), can be considered the beginnings of a systematical study of function spaces, which soon obtained the name of *Sobolev spaces*. Of course, the foundations of this research had been laid by Sobolev as early as in the thirties by his three papers that appeared in the years 1935–1938. In 1939 Sobolev became the youngest member of the Soviet Academy of Sciences at the age of 31. His approach during this period, which involved among other things the foundations of the theory of distributions, can be seen in his book [**2**] published in 1974. It is not my intention to incite discussions on the question of priority: as is quite frequent in mathematics, there were other mathematicians reflecting on similar problems (J. W. Calkin, Ch. B. Morrey, Jr.), and even the Dirichlet integral can be considered a basis for the theory of Sobolev spaces.

It was discovered that the Sobolev spaces form a very useful tool for introducing modern methods of solution of partial differential equations. Their stormy development was not restricted to the country of their origin (some generalizations and new views of these spaces can be found, e.g., in the books [**3, 4**] by S. M. Nikol′skij and his colleagues): monographs devoted to the theory of Sobolev spaces appeared also outside the Soviet Union. At first, the elements of their theory appeared in books on partial differential equations—see, e.g., S. Agmon [**5**] and above all J. Nečas [**6**]. It was the last monograph, published in 1966, which contained perhaps the most systematical and most complete (at the time) description of the theory of Sobolev spaces. Unfortunately, it did not become a *universal* reference book; this aim was later realized by R. A. Adams' book [**7**], from 1975. The last monograph in the series devoted to Sobolev spaces is (at the moment) V. G. Maz′ja's book [**8**], from 1985.

The Sobolev spaces have been modified and generalized in various ways. A survey of the situation as it was in 1977 is given in the book [**9**] by A. Kufner, O. John and S. Fučík; let us mention here at least the Besov spaces, the Triebel-Lizorkin spaces (among numerous books by H. Triebel and his school let us cite [**10**, **11**, **12**, and **13**]), spaces of Bessel potentials, weighted spaces (see A. Kufner [**14**]), and the Sobolev-Orlicz spaces (see [**7**]).

The author of the book under review is also one of those who have generalized the Sobolev spaces. While the "usual" Sobolev spaces $W^{k,p}(G)$ are defined as linear sets of functions $u = u(x)$ defined on an open set $G \subset \mathbf{R}^n$, whose (distributional) derivatives $D^\alpha u$ of order α, $|\alpha| \leq k$, belong to the Lebesgue space $L^p(G)$, the intense study of Sobolev spaces of *infinite* order was started by Ju. A. Dubinskij in the seventies: Assuming $a_\alpha \geq 0$, $p_\alpha \geq 1$, and $r_\alpha \geq 1$, he denoted these spaces by $W^\infty\{a_\alpha, p_\alpha\}$ and defined them as sets of functions u for which the number

$$\rho(u) = \sum_{|\alpha|=0}^{\infty} a_\alpha \|D^\alpha u\|_{r_\alpha}^{p_\alpha} \tag{1}$$

is finite, provided $\|\cdot\|_r$ denotes the norm in the Lebesgue space $L^r(G)$. As is more or less the rule with function spaces, the study of these "infinite-order" spaces was motivated by boundary value problems for differential equations, here of course of infinite order of the form

$$L(u) \equiv \sum_{|\alpha|=0}^{\infty} (-1)^{|\alpha|} D^\alpha A_\alpha(x; D^\gamma u) = h(x), \qquad x \in G, \tag{2}$$

for which the spaces $W^\infty\{a_\alpha, p_\alpha\}$ represent the "energy" spaces.

Dubinskij's book is very well organized and—in spite of its relatively small size (160 pp.)—contains a very large amount of material. Let us at least briefly describe the contents of the book.

The main problem arising in the study of the spaces $W^\infty\{a_\alpha, p_\alpha\}$ is that of their nontriviality, that is, of the existence of a nontrivial function $u(x)$ such that $\rho(u) < \infty$. This problem is solved in Chapter I. Necessary and sufficient conditions of nontriviality are given for the cases when G is a bounded domain in $\mathbf{R}^n$, for the space

$$\overset{\circ}{W}{}^\infty\{a_\alpha, p_\alpha\} = \{u \in C_0^\infty(G);\ \rho(u) < \infty\};$$

when G is the whole space $\mathbf{R}^n$, or the n-dimensional torus T^n, for

$$W^\infty\{a_\alpha, p_\alpha\} = \{u \in C^\infty(G);\ \rho(u) < \infty\};$$

or when G is the strip $]0,a[\times\mathbf{R}^\nu$, for the space

$$\overset{\circ}{W}{}^\infty\{a_\alpha,p\}=\left\{u=u(t,x)\in C^\infty(G);\ \rho(u)=\sum_{n+|\alpha|=0}^{\infty}a_{n\alpha}\|D_t^nD_x^\alpha u\|_p^p<\infty,\right.$$

$$\left.D_t^m u(0,x)=D_t^m u(a,x)=0,\ m=0,1,\dots\right\}.$$

These conditions are expressed in terms of existence of a vector q such that $\sum_{|\alpha|=0}^\infty a_\alpha q^{\alpha p_\alpha}<\infty$ (for $G=\mathbf{R}^n$, $G=T^n$) or, as the case may be, by Hadamard's quasianalyticity connected with the solutions M_N of the equations $\sum_{|\alpha|=N}a_\alpha M_N^{p_\alpha}=1$, $N=0,1,\dots$ (for G a bounded domain); for G a strip, the conditions of nontriviality are a little more complicated.

The theory of boundary value problems of infinite order is dealt with in Chapter II (elliptic problems, namely the Dirichlet problem with homogeneous boundary data), Chapter III (inhomogeneous Dirichlet problem), and Chapter VI (nonstationary—parabolic and hyperbolic—equations of infinite order). The main tool is the theory of the boundary value problems of order $2m$ and the limiting process for $m\to\infty$. Of course, certain growth and coercivity conditions concerning the "coefficients" $A_\alpha(x;\xi)$ of equation (2) are required, while the monotonicity condition is only one of the possible conditions which guarantee the *uniqueness* of the solution of the infinite-order boundary value problem.

The study of these boundary value problems requires a more detailed knowledge of properties of the spaces considered. Therefore, in Chapter III, some criteria for the existence of a trace of a function from $W^\infty\{a_\alpha,p_\alpha\}$ are derived. Since the universal necessary and sufficient condition is rather complicated, a simpler sufficient algebraic condition is given for the case of a strip. Chapter IV deals with the spaces $W^\infty\{a_\alpha,p_\alpha\}$ as limits of sequences of Banach spaces (of finite order) and the properties developed there are used in Chapter V for deriving imbedding theorems of the type $W^\infty\{a_\alpha,p\}\subset W^\infty\{b_\alpha,q\}$. Since the conditions for the imbedding and compact imbedding are again rather complicated (being expressed in terms of the asymptotic behaviour of the norms of the imbedding operators between the corresponding finite-order spaces, which approximate the spaces W^∞), some sufficient algebraic criteria are given for the case $G=\mathbf{R}^n$, $p=q=2$, and $G=\mathbf{R}^1$, $p=q>1$ arbitrary.

Ju. A. Dubinskij built in the USSR a school of equations and spaces of infinite orders; in the present book he attempted a compilation of the present state. It is rather curious that he was "overtaken" by his student Chan Dyk Van who in 1983 published (in Russian) the book [**15**] collecting the results concerning the Sobolev-Orlicz spaces (and their application to the solution of the boundary value problems). On the other hand, we should point out that a survey of the results concerning the properties and applications of Sobolev spaces of infinite order was given by Ju. A. Dubinskij in the proceedings [**16**], which appeared in 1979.

REMARK. Dubinskij's book appeared simultaneously in two publishing houses: the reviewed edition, published by Reidel, is based on that appearing (as Volume 87) in the series *Teubner-Texte zur Mathematik*, published by

Teubner Publishing House in Leipzig. The two versions actually differ only by their cover. The reviewer would like to use this opportunity to turn the reader's attention to this series, which apparently is only rarely available in North American mathematical libraries. Indeed, it was the books cited below under references [**10**, **11**, **14**, **16**] which appeared as Volumes 7, 15, 31 and 19 of the series *Teubner-Texte zur Mathematik.*

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