BOOK REVIEWS

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 16, Number 2, April 1987

0273-0979/87 $1.00 + $.25 per page

Treatise on the shift operator. Spectral function theory, by N. K. Nikol′skiĭ, with an appendix by S. V. Hruščev and V. V. Peller. Translated from the Russian by Jaak Peetre. Grundlehren der mathematischen Wissenschaften, vol. 237, Springer-Verlag, Berlin, Heidelberg, New York, and Tokyo, 1986, $64.50. ISBN 0-387-15021-8

The shift (or unilateral shift) operator, in one of its two common guises, is the operator on the complex Hilbert space l^2 that sends the sequence $(c_0, c_1, c_2, \ldots)$ to the sequence $(0, c_0, c_1, \ldots)$. It is a typically infinite-dimensional operator and as such is often useful to the instructor of an introductory operator theory course as a handy concrete illustration of various phenomena–for instance, it is left invertible but not right invertible; it has no eigenvalues, yet the eigenvalues of its adjoint fill the open unit disk; the powers of its adjoint tend to 0 in the strong operator topology, yet its powers do not.

In its other common guise, the shift operator is the operator of "multiplication by z" on the Hardy space H^2, the space of holomorphic functions in the unit disk whose power series coefficients at the origin are square summable. That guise is the appropriate one to visualize when one investigates the structure of the shift operator more deeply, a point strikingly brought home by A. Beurling in 1949, when he showed that the invariant subspace structure of the shift operator mimics the factorization theory in H^2 (the inner-outer factorization). Since Beurling's pioneering work, it has been recognized that this innocent-looking operator is connected with a surprisingly vast body of function theory.

The adjoint of the direct sum of countably many copies of the shift operator, the so-called backward shift operator of multiplicity $\aleph_0$, has a remarkable universality property discovered by G.-C. Rota: every operator on a separable Hilbert space whose spectrum is contained in the open unit disk is similar to a part of that operator (that is, similar to the restriction of that operator to one of its invariant subspaces). From this and related results one sees that, in principle, complete knowledge of the shift operator of multiplicity $\aleph_0$ would entail complete knowledge of all Hilbert space operators. In a less fanciful vein, one can hope on the basis of such results that by better understanding the shift operators of various multiplicities one will gain better insight into the structures of other operators and possibly of operators in general. The pursuit of that goal is a major ongoing program in operator theory.

The book under review, which is the outgrowth of a series of introductory lectures, is written on two levels. The initial portion of each chapter deals with the shift operator of unit multiplicity and delves quite deeply into the associated function theory, paying special attention to the spectral analysis of the parts of the backward shift operator. This material could serve as an introduction to shift-related operator theory and function theory for someone with a basic background in functional analysis and complex analysis. All but a few chapters contain supplementary sections where the earlier material is refined and extended; in particular, multiple shift operators are studied. The style here becomes that of an advanced monograph. Besides the eleven main chapters there are five appendices, themselves comprising about 45 percent of the text. One, of 100 pages, gives an introduction to the spectral theory of two kinds of operators closely related to the shift operator, Hankel and Toeplitz operators; another, of 56 pages and contributed by S. V. Hruščev and V. V. Peller, further develops the theory of Hankel operators, especially the connections of those operators with approximation problems and with stationary Gaussian sequences.

This is a book for the devotee, or the would-be devotee. If my experience is typical, those who love the subject will love the book.

DONALD SARASON

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 16, Number 2, April 1987

0273-0979/87 $1.00 + $.25 per page

An introduction to nonstandard real analysis, by A. E. Hurd and P. A. Loeb, Pure and Applied Mathematics, vol. 118, Academic Press, 1985, xii + 232 pp., $35.00. ISBN 0-12-362440-1

Nonstandard analysis is now widely applied in a number of different mathematical fields. A partial list of the applications includes functional analysis (Bernstein and Robinson [**15**], and the survey by Henson and Moore [**21**]), perturbation theory (Lutz and Goze [**38**]), mathematical physics (Arkeryd [**9, 10, 11, 12, 13**]), potential theory (Loeb [**37**]), mathematical economics (Brown and Robinson [**16, 17**], Anderson [**4, 6, 8**], the references in [**7**], and the forthcoming book by Rashid [**46**]), and probability theory (see the survey by Cutland [**18**]).

Standard mathematicians tend to test the worth of nonstandard analysis by asking whether it has led to new standard results in their fields. It is not clear that this is the right test: after all, most fields yield far more results of internal interest than applications to other fields. Nonetheless, it is a test which nonstandard analysis is beginning to meet.

In most of the above areas, nonstandard analysis has led to new standard theorems. A metatheorem guarantees that any standard theorem provable by nonstandard methods has a standard proof; this is important, since it tells us that any theorem with a nonstandard proof follows from the usual axioms of