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MONOPOLES ON ASYMPTOTICALLY EUCLIDEAN 3-MANIFOLDS

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ABSTRACT. We consider a generalization of Yang-Mills-Higgs theory on Euclidean $\mathbf{R}^3$ to connected sums of $\mathbf{R}^3$ with compact closed 3-manifolds.

In this note, we describe progress in Yang-Mills-Higgs theory on 3-dimensional Riemannian manifolds. In particular, we are interested in the set of minima of the Yang-Mills action

$$\mathfrak{A}(A,\Phi) = \int_M (|F_A(x)|^2 + |\nabla_A \Phi(x)|^2)\, d\mu(x), \tag{1}$$

where the "Higgs field" Φ is a section of a metric vector bundle E over M and A is a linear connection on E preserving the metric. For simplicity, we will restrict ourselves to the case where $E = \mathrm{ad}(P)$ is the adjoint bundle of an SU_2-principal bundle P over M.

The functional $\mathfrak{A}$ has been studied in great detail in the case where M is the Euclidean $\mathbf{R}^3$, see [**8**]. We recall here briefly the main results: The length of the Higgs field Φ of any finite action configuration $c = (A, \Phi)$ obtains in some sense an asymptotic value $m(c)$ at infinity. For each $m > 0$, the space of finite action configurations c with $m(c) = m$ decomposes into a family of components indexed by a "topological charge" k. On each of these components, the minima of the action functional (1) can be shown to be solutions of the Bogomolny equation

$$\mathfrak{b}_\pm(c) = \nabla_A \Phi \mp {}^*F_A = 0 \tag{2}$$

with the sign equal to the sign of k. Since reversing the sign of Φ changes the sign of k while leaving $\mathfrak{A}$ invariant, we can restrict ourselves to the case $k > 0$ and write $\mathfrak{b} = \mathfrak{b}_+$. The set of gauge equivalence classes of solutions of (2), also called monopoles, is a smooth manifold $\mathcal{M}_k$ of dimension $4k$. It can be described by means of algebraic geometry, see [**7** and **3**]. One usually considers the $(4k-1)$-dimensional submanifolds $\mathcal{M}_k^m$ of monopoles $[c]$ with fixed "mass" $m(c) = m$. In fact, one can without loss of generality set $m = 1$, since a scaling involving a dilation of $\mathbf{R}^3$ shows that $\mathcal{M}_k^m \cong \mathcal{M}_k^1$ for all $m > 0$.

In order to generalize these ideas, we replace $\mathbf{R}^3$ by an asymptotically flat manifold M, see [**9**]. This means that M is the connected sum of $\mathbf{R}^3$ with a compact manifold, equipped with a metric which at the end of M is a perturbation of the Euclidean metric in a certain sense. Since in this

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case, the methods of algebraic geometry do not apply any more, we pursue an entirely analytic approach. At first sight, an obstruction to applying standard methods of infinite-dimensional topology to this problem is the fact that due to the lack of decay conditions for finite action configurations (A, Φ) at the ends, the function $\mathfrak{b}$ is not defined on a Banach space. However, it turns out that this difficulty is entirely due to the action of the gauge group. As we prove in [**4**], the space $\mathcal{Q}_k$ of gauge equivalence classes of finite action configuration carries a natural Hilbert manifold structure, so that the gauge equivariant function $\mathfrak{b}$ yields a smooth Fredholm section of a smooth bundle over $\mathcal{Q}$. Its index can be determined to be $4k$.

Since $m(c)$ is gauge invariant, $\mathcal{Q}$ foliates into 1-codimensional submanifolds $\mathcal{Q}^m$. Our main result is that certain parts of the zero set $\mathcal{M}_k^m$ of $\mathfrak{b}$ on $\mathcal{Q}_k^m$ are regular, i.e., they are locally $(4k-1)$-dimensional smooth manifolds, and can be described in terms of the space of monopoles on Euclidean $\mathbf{R}^3$ and the topology of M. In particular, the homology of M enters at this point. For simplicity, we restrict ourselves here to the case $k = 1$ (see [**5**] for general charges). We can then define a real number $R([c])$ measuring the "distance" of the monopole from the center of M.

THEOREM. *If M is an asymptotically flat manifold with $H_1(M, Z) = 0$, then the set $\mathcal{M}^m = \mathcal{M}_1^m$ of self-dual gauge equivalence classes in $\mathcal{Q}^m = \mathcal{Q}_1^m$ is a smooth 3-dimensional manifold if either $m(c)$ or $R(c)$ are large enough. In particular, if $m(c)$ is large enough, we have a diffeomorphism*

$$z\colon \mathcal{M}^m \cong M \tag{3}$$

and for any mass $m > 0$ and r large enough a diffeomorphism

$$z\colon \{[c] \in \mathcal{M}^m \mid R(c) > r\} \cong \mathbf{R}^3 - B_r. \tag{4}$$

In both cases, z is defined by the unique zero of the Higgs field. Moreover, for m small enough, $\mathcal{M}^m$ contains a component diffeomorphic to $\mathbf{R}^3$ and a compact component K.

The diffeomorphism (4) is constructed by a grafting procedure similar to the one used in [**8**] to construct multimonopole solutions on $\mathbf{R}^3$. We define approximately self-dual configurations on M from self-dual configurations on $\mathbf{R}^3$ by means of cutoff functions. These are then deformed into self-dual configurations using a variant of the implicit function theorem. The same method is used in the small mass limit. The diffeomorphism (3) is obtained by a limit argument using a scale invariance of the self-duality equation.

Note that by transversality theory, a generic perturbation of the Fredholm section $\mathfrak{b}$ yields a cobordism of M to $\mathbf{R}^3 \cup K$, which is product outside a compact set. The compact manifolds K are related to flat SU_2-connections on M, whose relation to the topology of homology 3-spheres has been recently investigated by Casson [**1**].

One can prove similar statements about the monopole spaces for higher charges. For example, each $\mathcal{M}_k^m$ contains for m small enough a component diffeomorphic to $\mathcal{M}_k^m(\mathbf{R}^3)$. Moreover, if $H_1(M, \mathbf{Z})$ is nonzero but finite, we can state a result similar to the above theorem with diffeomorphisms replaced by coverings with covering group $H_1(M, \mathbf{Z})$. If it contains a free part, then the maps in (3) and (4) are not surjective in general; see [**5**]. If M is any asymptotically flat manifold with nonnegative Ricci curvature, then the Fredholm section $\mathfrak{b}$ is "manifestly" regular on each $\mathcal{M}^m$. This yields f.e. an alternative proof of the following result:

THEOREM (SEE [**10** AND **6**]). *Every asymptotically Euclidean* 3*-manifold with nonnegative Ricci curvature is diffeomorphic to* $\mathbf{R}^3$.

Finally, we should note that generalizations of Euclidean Yang-Mills-Higgs theory have been pursued in a different direction by A. Chakrabarti [**2**], replacing $\mathbf{R}^3$ by certain hyperbolic manifolds, and using essentially methods of algebraic geometry.

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