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THE VANISHING OF INTERSECTION MULTIPLICITIES OF PERFECT COMPLEXES

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Let R be a commutative Noetherian local ring, and let M and N be finitely generated R-modules of finite projective dimension such that $M \otimes_R N$ is a module of finite length. Under these hypotheses, $\mathrm{Tor}_i(M,N)$ is a module of finite length for all i and is zero for large i, and one can, following Serre [**7**], define the intersection multiplicity

$$\chi(M,N) = \sum_{i \geq 0} (-1)^i \text{ length}(\mathrm{Tor}_i(M,N)).$$

For any R-module Q, let $\dim Q$ denote the Krull dimension of Q. Serre proved that if R is a regular local ring and M and N are as above, then

$$\dim M + \dim N \leq \dim R.$$

In addition, if R is unramified, he showed that there is a relation between the dimensions of M and N and their intersection multiplicities:

$$\text{(1)} \qquad \text{If } \dim M + \dim N < \dim R, \quad \text{then } \chi(M,N) = 0.$$

$$\text{(2)} \qquad \text{If } \dim M + \dim N = \dim R, \quad \text{then } \chi(M,N) > 0.$$

Serre conjectured that these two statements hold for all regular local rings. In this paper we outline a proof that the first of these statements holds not only for arbitrary regular local rings, but for complete intersections and isolated singularities as well. This result has been proven independently in the case of complete intersections, using K-theoretic methods, by H. Gillet and C. Soulé [**5**].

Somewhat more generally, we prove a result on the vanishing of multiplicities for bounded complexes of finitely generated free modules; such complexes we call *perfect*. Let $Y = \operatorname{Spec} R$, and let E_* and F_* be perfect complexes. Let $X = \operatorname{Supp}(E_*) =$ the support of $E_* = \{P \in \operatorname{Spec} R | (E_*)_P \text{ is not exact}\}$; let $W = \operatorname{Supp}(F_*)$. Let p denote the closed point of Y. If G_* is a complex with $\operatorname{Supp}(G_*) = p$, let $\chi(G_*) = \sum_{i \in Z} (-1)^i \text{ length}(H_i(G_*))$.

THEOREM 1. *Let R be a local ring which is a homomorphic image of a regular local ring. Assume that R is either a complete intersection or an isolated singularity. Let E_* and F_* be perfect complexes with supports X and*

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W respectively. If

1. $X \cap W = p$, *and*

2. $\dim X + \dim W < \dim Y$,

then $\chi(E_* \otimes_R F_*) = 0$.

The result on the vanishing of $\chi(M, N)$ stated above follows by applying Theorem 1 to free resolutions of M and N.

The proof of Theorem 1 uses the theory of localized Chern characters as defined in Baum, Fulton and MacPherson [**1**] and described in Fulton [**4**, Chapter 18]. To obtain the result in the desired generality, we use the theory for schemes of finite type over a regular scheme as described in Fulton [**4**, Chapter 20].

It had been conjectured (see Szpiro [**8**]) that a system of invariants of this kind, which generalize the Euler characteristic in codimension zero and the MacRae invariant in codimension one, could be used to prove a vanishing theorem for $\chi(M, N)$ in the more general case where only M was assumed to have finite projective dimension. This was proven in this way by Peskine and Szpiro [**6**] in the graded case and by Foxby [**3**] when N has dimension one, providing more evidence that this should be possible. However, a recent example of Dutta, Hochster and McLaughlin [**2**] showed that the more general version is false; in their example the ring R is both a complete intersection (in fact, a hypersurface) and an isolated singularity. It thus appears that the hypothesis in Theorem 1, where both modules (or complexes) are assumed to have finite projective dimension, is the most reasonable.

Let R, $Y = \operatorname{Spec} R$, E_*, F_*, $X = \operatorname{Supp}(E_*)$, and $W = \operatorname{Supp}(F_*)$ be as above. Let $A_*Y = \bigoplus A_k Y$ be the group of cycles on Y modulo rational equivalence. The localized Chern character of E_* (and similarly for any perfect complex) is a sum,

$$\mathrm{ch}_X^Y(E_*) = \mathrm{ch}_0(E_*) + \mathrm{ch}_1(E_*) + \cdots$$

where, for each i and k, and for every subscheme Z of Y, $\mathrm{ch}_i(E_*)$ defines a map from $A_k Z$ to $A_{k-i}(Z \cap X)$. These operators satisfy the following properties (among others):

1 (MULTIPLICATIVITY). Given two perfect complexes E_* and F_*, we have

$$\mathrm{ch}_{X\cap W}^Y(E_* \otimes F_*) = \mathrm{ch}_X^Y(E_*)\mathrm{ch}_W^Y(F_*).$$

Note that $\operatorname{Supp}(E_* \otimes F_*) = X \cap W$. The multiplication on the right is defined for components $\mathrm{ch}_i(E_*)$ and $\mathrm{ch}_j(F_*)$ as the composition

$$A_k(Z) \xrightarrow{\mathrm{ch}_j(F_*)} A_{k-j}(Z \cap W) \xrightarrow{\mathrm{ch}_i(E_*)} A_{k-j-i}(Z \cap W \cap X).$$

2 (THE LOCAL RIEMANN-ROCH FORMULA). There is a class $\tau(Y)$ in $A_*(Y)$ such that if E_* is a perfect complex with support p, we have

$$\chi(E_*) = \mathrm{ch}_p^Y(E_*)(\tau(Y)).$$

If R is a complete intersection, then $\tau(Y) = [Y]$ and lies in $A_n Y$, where n is the dimension of Y.

These formulas can be found in Fulton [4, Chapter 18]. The main lemma in the proof of Theorem 1 is to show that the localized Chern characters also satisfy the following property:

3 (COMMUTATIVITY). For all i and j, we have

$$\mathrm{ch}_i(E_*)\mathrm{ch}_j(F_*) = \mathrm{ch}_j(F_*)\mathrm{ch}_i(E_*).$$

For the remainder of the paper, we let n denote the dimension of Y.

PROPOSITION 1. *If E_* and F_* satisfy hypotheses 1 and 2 of Theorem 1, then $\mathrm{ch}_n(E_* \otimes F_*)(\alpha) = 0$ for all α in $A_n Y$.*

PROOF. From the multiplicativity property, we have

$$\mathrm{ch}_n(E_* \otimes F_*)(\alpha) = \sum_{i+j=n} \mathrm{ch}_i(E_*)\mathrm{ch}_j(F_*)(\alpha).$$

Assume first that $j < n - \dim W$, so that $\dim W < n-j$. Then $\mathrm{ch}_j(F_*)(\alpha)$ is an element of $A_{n-j}W$, which is zero, so $\mathrm{ch}_i(E_*)\mathrm{ch}_j(F_*)(\alpha) = 0$. Using commutativity, we similarly deduce that if $i < n - \dim X$, then $\mathrm{ch}_i(E_*)\mathrm{ch}_j(F_*)(\alpha) = \mathrm{ch}_j(F_*)\mathrm{ch}_i(E_*)(\alpha) = 0$. Hypothesis 2 of Theorem 1 implies that one of these inequalities must hold for every i and j with $i + j = n$, so we conclude that $\mathrm{ch}_n(E_* \otimes F_*)(\alpha) = 0$.

Theorem 1 for complete intersections now follows from Proposition 1, the fact that $\tau(Y)$ is an element of $A_n Y$ in this case, and the local Riemann-Roch formula.

If Y is a regular scheme, then every closed integral subscheme V of dimension k has a resolution G_* by vector bundles over Y, and $\mathrm{ch}_{n-k}(G_*)([Y]) = [V]$ in $A_k Y$. Using this fact, the commutativity property, and an argument similar to that in Proposition 1, one can show that $\mathrm{ch}_i(E_*)(\alpha) = 0$ for all $\alpha \in A_k Y$ for all k when $i < n - \dim X$ and Y is regular.

PROPOSITION 2. *Let R be an isolated singularity (i.e., $\tilde{Y} = Y - p$ is a regular scheme). Then, under the hypotheses of Theorem 1, for any integer k and all $\alpha \in A_k Y$, we have $\mathrm{ch}_k(E_* \otimes F_*)(\alpha) = 0$.*

PROOF. As in the proof of Proposition 1, we can assume that $j < n - \dim W$. Let $\tilde{W} = W - p$. Since $\tilde{Y}$ is regular, the element $\mathrm{ch}_j(F_*)(\alpha)$ of $A_{k-j}W$ must become zero when restricted to $\tilde{W}$, so it is the image of an element of $A_{k-j}(p)$. Consider the commutative diagram

$$\begin{array}{ccc} A_{k-j}(p) & \longrightarrow & A_{k-j}W \\ {\scriptstyle \mathrm{ch}_i(E_*)} \downarrow & & \downarrow {\scriptstyle \mathrm{ch}_i(E_*)} \\ A_0(p) & \longrightarrow & A_0(p). \end{array}$$

Since $A_m(p) \neq 0$ only for $m = 0$, the only case when $\mathrm{ch}_i(E_*)\mathrm{ch}_j(F_*)(\alpha)$ might not be zero is when $k = j$ and $i = 0$. However, $\mathrm{ch}_0(E_*)$ is the operator which multiplies an element by $\sum_{i\in Z}(-1)^i \operatorname{rank}(E_i)$, and, since $\dim X < \dim Y$, this number is zero. Hence the product vanishes in this case also, and this concludes the proof.

Theorem 1 for isolated singularities now follows immediately from Proposition 2 and the local Riemann-Roch formula.

REFERENCES

1. P. Baum, W. Fulton and R. MacPherson, *Riemann-Roch for singular varieties*, Inst. Hautes Études Sci. Publ. Math. **45** (1975), 101–145.

2. S. P. Dutta, M. Hochster and J. E. McLaughlin, *Modules of finite projective dimension with negative intersection multiplicities*, Invent. Math. **79** (1985), 253–291.

3. H.-B. Foxby, *The MacRae invariant*, Commutative Algebra, Durham, 1981, London Math. Soc. Lecture Notes Ser. **72** (1982), 121–128.

4. W. Fulton, *Intersection theory*, Springer-Verlag, 1984.

5. H. Gillet and C. Soulé, *K-théorie et nullité des multiplicités d'intersection*, C. R. Acad. Sci. Paris, Sér. I, no. 3, t. **300** (1985), 71–74.

6. C. Peskine and L. Szpiro, *Syzygies et multiplicités*, C. R. Acad. Sci. Paris, Sér. A-B. **278** (1974), 1421–1424.

7. J.-P. Serre, *Algèbre locale. Multiplicités*, Lecture Notes in Math., vol. 11, Springer-Verlag, Berlin and New York, 1975.

8. L. Szpiro, *Sur la théorie des complexes parfaits*, Commutative Algebra, Durham, 1981, London Math. Soc. Lecture Note Ser. **72** (1982), 83–90.

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