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BOOK REVIEWS

Quasiconformal mappings and Riemann surfaces, by Samuil L. Krušhkal′, edited by Irwin Kra, John Wiley and Sons, New York, 1979, xii + 319 pp., $24.95.

Under review is a book on quasiconformal mappings and their role in the study of Riemann surfaces and Kleinian groups. Since quasiconformal mappings are beginning to find applications outside of complex analysis (see, for example, [5]), it may be worthwhile to begin by listing some basic facts about them.

Suppose that D and D' are domains in euclidean n-space R^n, $n \geqslant 2$, and suppose that f is a sense preserving homeomorphism of D onto D'. For each x in D set

$$H(x, f) = \limsup_{r \to 0} \frac{\max_{|y-x|=r} |f(y) - f(x)|}{\min_{|y-x|=r} |f(y) - f(x)|}.$$

If $H(x, f)$ is bounded in D, then f is said to be *quasiconformal* and

$$K(f) = \operatorname*{ess\,sup}_{x \in D} H(x, f)$$

is called the *maximal dilatation* of f. Obviously $1 \leqslant K(f) < \infty$. When $n = 2$, $K(f) = 1$ if and only if f, regarded as a complex valued function of a complex variable, is analytic in D [13]. When $n \geqslant 3$, $K(f) = 1$ if and only if there exists a Möbius transformation, i.e. the composition of an even number of reflections in $(n-1)$-spheres and planes, which coincides with f in D, [7] and [15]. Thus f is *conformal* if and only if it is quasiconformal with $K(f) = 1$.

Compositions and inverses of quasiconformal mappings are again quasiconformal and

$$K(f \circ g) \leqslant K(f)K(g), \qquad K(f^{-1}) = K(f).$$

Thus the quasiconformality and maximal dilatation of a given mapping f remain unchanged after composition with conformal mappings, and one can extend these concepts to homeomorphisms between two *conformally flat n-dimensional manifolds* S and S', i.e. n-dimensional manifolds for which the transition mappings are conformal. Of central importance is the case $n = 2$ when S and S' are Riemann surfaces.

Plane quasiconformal mappings have been studied for over fifty years. They occur first in the papers of Grötzsch, who posed and solved several important and suggestive extremal problems in [9]. They appear next in Ahlfors' fundamental paper on covering surfaces [1]; here one first sees that the geometric aspects of value distribution theory hold for functions which are locally uniformly quasiconformal rather than just locally conformal. Forty years later Drasin appealed to this fact to obtain his complete solution for the inverse problem of the Nevanlinna theory [6]. Finally Lavrentieff studied quasiconformal mappings while seeking a geometric interpretation of the partial differential equations governing the flow of a compressible fluid, and he established in [11] a version of the generalized Riemann mapping theorem. This result says that when D is simply connected, for almost all points x in D one can specify the eccentricity and inclination of the infinitesimal ellipses onto which infinitesimal circles about x are mapped by f. For the most general version of this important result see, for example, [3].

Quasiconformal mappings slowly gained recognition as an interesting branch of mathematics and as a useful and flexible tool in treating a range of problems in complex analysis. However it was Teichmüller who pointed out the fundamental role played by extremal mappings in the study of Riemann surfaces. Suppose that $\mathfrak{F}$ is a family of quasiconformal mappings f between two domains D and D' in R^n or between two conformally flat n-dimensional manifolds S and S'. A mapping f_0 in $\mathfrak{F}$ is said to be an *extremal* if

$$K(f_0) = \inf\{K(f): f \in \mathfrak{F}\}.$$

If S and S' are compact Riemann surfaces of genus $g > 1$ and if $\mathfrak{F}$ consists of all quasiconformal f: $S \to S'$ which are homotopic to a fixed mapping of S onto S', then Teichmüller argued in [16] that $\mathfrak{F}$ contains a unique extremal f_0 which is intimately connected with the holomorphic quadratic differentials on S and S'. He showed also that by fixing S and varying S', the extremals f_0 allow one to parametrize all Riemann surfaces of genus g by means of a metric space T_g which is homeomorphic to R^{6g-6}. The metric on T_g is given by

$$d(S_1', S_2') = \log K(f_2 \circ f_1^{-1}),$$

where f_1 and f_2 are the extremals corresponding to S_1' and S_2'. Teichmüller's proofs for these results were not complete, and the significance of his work was recognized only after more rigorous arguments had been published by Ahlfors [2] and Bers [4].

The study of higher dimensional quasiconformal mappings was initiated by Loewner, who introduced a conformally invariant n-dimensional capacity in [12], and the theory was subsequently developed by a number of mathematicians. Unfortunately virtually nothing is yet known about the existence or nature of extremals in the family $\mathfrak{F}$ of quasiconformal mappings between conformally flat n-dimensional manifolds S and S' when $n \geqslant 3$. The situation is complicated by the following rigidity phenomenon discovered by Mostow [14]. If $S = B/\Gamma$ and $S' = B/\Gamma'$, where Γ and Γ' are discrete groups of Möbius transformations acting on B, and if S and S' have finite non-euclidean volume, then either $\mathfrak{F}$ is empty or there exists f_0 in $\mathfrak{F}$ with $K(f_0) = 1$. Thus in contrast to the situation when $n = 2$, S and S' are quasiconformally equivalent if and only if they are conformally equivalent.

Teichmüller's theorem yields complete information about extremals in the family $\mathfrak{F}$ of quasiconformal mappings between two finitely connected domains in R^2, while the fragmentary results in [8] and [10] represent almost all that is known about the corresponding problem in R^n when $n \geqslant 3$. The study of extremal n-dimensional quasiconformal mappings is a difficult and challenging area which deserves further attention.

The book under consideration is devoted primarily to the development of variational methods to study a range of extremal problems for two-dimensional quasiconformal mappings. The author begins with background information in Chapter 1 and gives a variational proof for Teichmüller's theorem in Chapter 2. In Chapters 3 and 4 he treats a number of other extremal problems for quasiconformal mappings between open Riemann surfaces or between plane domains. The problem of moduli for Riemann surfaces is discussed in Chapter 5, and the author concludes with results on Kleinian groups in Chapters 6 and 7. The bibliography is an impressive collection of 226 references.

The author has presented a wealth of information on an important area of complex analysis where current interest is high. Moreover he has packed this material into seven chapters totaling a mere 302 pages. Unfortunately he has achieved this concise presentation only by sacrificing precision or completeness in the statements or proofs of certain results. For example, the author's argument for Theorem 5 of Chapter 7–a proof for Bers' conjecture that finitely generated Kleinian groups are conditionally stable–contains assertions which require further verification, and the conjecture appears still to be an open problem. Thus although the reader will gain a global view of the subject, he will find it necessary to consult the extensive bibliography or the clarifying footnotes kindly provided by the translation editor of the English edition. Since there are few other general presentations of this active field, the author's book constitutes a timely contribution to current mathematical literature.

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Optimization and approximation, by W. Krabs, translated by Philip M. Anselone and Ronald B. Guenther, John Wiley and Sons, Chichester-New York-Brisbane-Toronto, 1979, xii + 220 pp., $28.95.

0. Introduction. Although approximation theory is a discipline with very wide spectrum overlapping with areas ranging from classical function theory to operator theory, the very basic problems are those that deal with best approximation. More precisely, questions of existence, uniqueness, and characterization of the best approximants from a certain subspace or a certain submanifold, as well as the questions of determination and estimation of the orders of best approximation are of utmost importance. Very frequently, with the possible exception of the problem of uniqueness, the techniques and theory of optimization turn out to be quite useful. This book is devoted to the connection between best approximation and optimization; it contains numerous examples showing how approximation problems of different types arising in physical as well as technical applications can be treated in a unified way within the general framework of optimization theory. The main body of the book consists of three chapters: Chapter I treats the general linear theory of optimization including the standard existence and duality theorems, while Chapters II and III deal with the convex problems and the general nonlinear problems respectively. Each of these chapters begins with a series of examples followed by the development of the theoretical principles. The advantage of this presentation is that each chapter can be read almost independently. An appendix on the background of functional analysis is also included. We will divide our discussion into three parts, each of which deals with one chapter of the book.

I. Linear problems. The general topic treated throughout this book is that of continuous approximation problems. These lead to semi-infinite (completely infinite) optimization problems in which finitely (infinitely) many free parameters can occur along with infinitely many side conditions. Such infinite optimization problems arise when one seeks to approximate a continuous real-valued function f on a compact set S by linear combinations of continuous functions on S as well as possible in the sense of the maximum norm. As the author mentions, such a situation occurs when one seeks simplified representations of functions for the purpose of evaluation on a computer. This situation also occurs in the approximate solution of boundary and initial value problems for ordinary and partial differential equations.

There are many such examples scattered throughout the book. A typical problem, as given by Krabs, concerns mathematically treating the problem of