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Ordinary differential equations, by V. I. Arnold, translated from the Russian by Richard A. Silverman, MIT Press, Cambridge, Massachusetts, 1978, x + 280 pp., $8.95.

The past twenty years have witnessed a revolution in the field of ordinary differential equations. It is not uncommon to attend a seminar on differential equations and not even hear the words differential equations, let alone see one written on the board. The "in phrase" these days is dynamical systems, and the language spoken is often the language of topology and differential geometry. *Ordinary differential equations* by the famed Soviet mathematician V. I. Arnold is a superb introduction to the modern theory of differential equations, and while reviewing this book, it is instructive to take a closer look at the profound changes that have occurred in this field. We begin with the fundamental concept of a dynamical system.

Consider a system that is evolving in time. Let x denote the initial state of the system, and $g^t x$ its state at time t, with $g^0 x = x$. The set M of all possible states is called the phase space of the system, and the individual states x are called phase points. Suppose, moreover, that the mappings g^t satisfy the group property

$$g^{t+s}x = g^t(g^s x) \tag{1}$$

and that g^t and $(g^t)^{-1}$ satisfy appropriate continuity conditions. The set of mappings g^t, together with the phase space M is called a dynamical system.

Dynamical systems occur very naturally in the study of ordinary differential equations. Let

$$\dot{x} = v(x) \tag{2}$$

be a differential equation defined on a domain M of n dimensional Euclidean space, and let

$$x(t) = g(t, x_0)$$

be the solution of (2) which at time $t = 0$ has the value x_0. Assume, moreover, that $g(t, x_0)$ is defined for all t. Then, the group of transformations

$$g^t: x \to g(t, x)$$

together with the phase space M is a dynamical system. If $v(x)$ is of class C^1, then the mappings g^t and $(g^t)^{-1}$ are also differentiable; i.e., they form a one-parameter group of diffeomorphisms.

We study differential equations from the abstract viewpoint of dynamical systems for many reasons. The best, and simplest reason, is that the description of the states of a process as points of a suitable phase space often turns out to be extraordinarily useful. Professor Arnold begins his text with the following example which shows that the simple act of just introducing a phase space often allows us to solve a difficult problem.

Problem. Two nonintersecting roads lead from City A to City B. Suppose it is known that two cars connected by a rope of length less than $2l$ manage to go from A to B along different roads without breaking the rope. Can two circular wagons of radius l, whose centers move along the roads in opposite directions, pass each other without colliding?

Solution. Consider the square

$$M = \{(x_1, x_2): 0 \leqslant x_1 \leqslant 1, 0 \leqslant x_2 \leqslant 1\}$$

(see Figure 1 below).

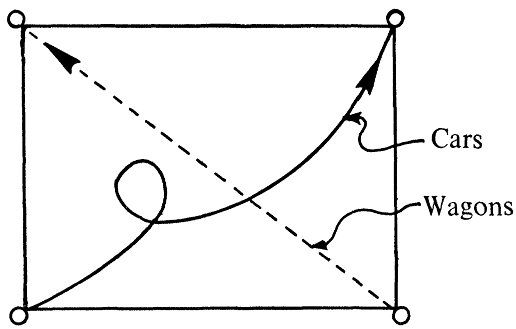

FIGURE 1

The position of two vehicles (one on the first road, the other on the second road) can be identified with a point of the square M: we simply let x_i denote the fraction of the distance from A to B along the ith road which lies between A and the vehicle on the given road. The square M is called the phase space and its points are phase points.

In this manner, every motion of the vehicles is represented by a motion of the phase point in the phase space. The motion of the cars from A to B is represented by a curve going from the lower left hand corner of the square to the upper right hand corner, while the motion of the wagons is represented by a curve going from the lower right hand corner to the upper left hand corner. But every pair of curves in the square joining different pairs of opposite corners must intersect. Thus, no matter how the wagons move, there comes a time when the pair of wagons occupy a position occupied at some time by the pair of cars. At this time, the distance between the centers of the wagons will be less than $2l$, and they will not manage to pass each other.

As another illustration of the usefulness of introducing a dynamical systems setting into a problem, we would like to briefly describe some recent and very fascinating work of Furstenberg, Katznelson and Weiss [6], [7], [8]. Two classical theorems of number theory are the following:

THEOREM (VAN DER WAERDEN). *Suppose that the integers Z are partitioned into finitely many classes $C_1, C_2, \ldots, C_r$; i.e.,*

$$Z = C_1 \cup C_2 \cup \cdots \cup C_r.$$

Then, at least one of the classes C_j possesses an arbitrarily long arithmetic progression.

THEOREM (SHUR). *For any partition*

$$N = C_1 \cup C_2 \cup \cdots \cup C_r$$

of the positive integers, we can find numbers u, v in one of the classes C_j such that $u + v$ also belongs to C_j.

Furstenberg et al. present dynamical systems proofs of these (and even more sophisticated) theorems. First, they consider the set of all sequences

$$x = (\ldots, a_{-2}, a_{-1}, a_0, a_1, a_2, \ldots)$$

of symbols a_j, where the a_j are chosen from a fixed set of r symbols, and then they introduce a metric into this space. Essentially, two elements

$$x = (\ldots, a_{-1}, a_0, a_1, \ldots) \text{ and } y = (\ldots, b_{-1}, b_0, b_1, \ldots)$$

are close to each other if $a_j = b_j, j = -k, \ldots, k$. This is the phase space M. Second, they introduce the shift mapping ϕ on M, defined by

$$(\varphi x)_k = a_{k-1}$$

and the iterates ϕ^n of ϕ. The phase space M together with the set of mappings ϕ^n is now a discrete (as opposed to continuous) dynamical system. Finally, they introduce the geometric concept of recurrence in a dynamical system.

DEFINITION. A point x is recurrent if there exists a sequence of times $t_{n_k} \to \infty$ such that

$$g^{t_{n_k}} x \to x.$$

Starting with a theorem of Birkhoff which states that a dynamical system with a compact phase space possesses at least one recurrent point, Furstenberg et al. present startling proofs of the above theorems.

Another, and more fundamental reason for introducing a dynamical systems setting into ordinary differential equations, is that partial differential equations and functional differential equations can also be put into a similar setting: the only difference is that the phase space is now an appropriate Banach space of functions. Thus we can try to generalize some of the theorems and techniques of O.D.E's to these more difficult problems. As an illustration, consider the P.D.E.

$$u_t = u_{xx} + \lambda f(u) \tag{3a}$$

with the boundary and initial conditions

$$u(0, t) = u(\pi, t) = 0, \qquad t \geqslant 0 \tag{3b}$$

$$u(x, 0) = \phi(x), \qquad 0 \leqslant x \leqslant \pi. \tag{3c}$$

Here f is a given function defined on the whole real line with $f(0) = 0$; ϕ is an arbitrarily specified function defined on $[0, \pi]$ with $\phi(0) = 0$, $\phi(\pi) = 0$, and λ is a nonnegative parameter.

The hypothesis $f(0) = 0$ implies that (3a) has a trivial solution $u_0 \equiv 0$, and it is well known that this solution is stable, when $\lambda = 0$. But what happens as we allow λ to increase from the value zero? Does the solution $u_0 \equiv 0$ lose its property of stability, and, if so, do there appear any new equilibrium solutions which inherit this property? We have the following theorem.

THEOREM (CHAFEE AND INFANTE [2]). *Assume that*

$$f'(0) > 0, \qquad \overline{\lim_{|u| \to \infty}}\ f(u)u^{-1} \leqslant 0, \qquad \operatorname{sgn} f''(u) = -\operatorname{sgn} u.$$

For each integer $n \geqslant 1$, let $\lambda_n = n^2 f'(0)$. As λ increases through the value λ_n, a pair of equilibrium solutions $u_n^{\pm}(\lambda)$ of (3a) bifurcate from the solution $u_0 \equiv 0$. For $\lambda \leqslant \lambda_1$, the solution $u_0 \equiv 0$ is asymptotically stable, but for $\lambda > \lambda_1$, this property is pre-empted by the pair $u_1^{\pm}(\lambda)$. The other solutions $u_n^{\pm}(\lambda)$, $n \geqslant 2$, are all unstable. In addition to this, every solution of (1) approaches one of the above equilibrium solutions.

A much weaker version of the above theorem was originally proven by Matkowsky [12], using the method of "two-time asymptotic series expansions". The method of Chafee and Infante [2] is a dynamical systems proof based on the method of constructing Lyapunov functionals for O.D.E.'s. Let

$$\dot{x} = v(x) \tag{4}$$

be a system of O.D.E.'s, and suppose that

$$V(x)\colon R^n \to R$$

is a Lyapunov functional with the property that

$$\dot{V} \leqslant 0$$

along the orbits of (4). The LaSalle Invariance Principle [10] states, under suitable conditions on the boundedness of orbits, that any orbit of (4) approaches the (invariant) set on which $\dot{V} = 0$.

Chafee and Infante generalize this approach to the above problem. They construct a suitable Lyapunov functional and show that $\dot{V} \leqslant 0$ along solutions of (3a). Indeed, the only difference between the O.D.E. and P.D.E. case, and the only difficulty, is that in the infinite dimensional phase-space of (3), bounded orbits do not necessarily belong to compact subsets. Indeed this is always the main difficulty, or stumbling block, in trying to apply Lyapunov theory to functional and partial differential equations. Chafee and Infante show how to overcome this difficulty for (3), and in an excellent survey paper, Hale [9] indicates several classes of problems which can be handled via this approach. It is also interesting to note that Chafee even generalizes the concepts of stable and unstable manifolds for some parabolic P.D.E.'s. In this manner he obtains some very strong instability results [3].

A second profound change that has occurred in the theory of differential equations is the introduction of many sophisticated topological concepts. One reason for this, of course, is that many of the problems raised during the past twenty-five years have been topological in nature. For example, instead of asking for the detailed orbit structure of (4), we now ask whether vector fields which are C^1 close to $v(x)$ have the same orbit structure. Specifically, we seek to find vector fields $v(x)$ with the property that the orbits of the system

$$\dot{x} = u(x)$$

can be mapped homeomorphically onto the orbits of (4) if $u(x)$ is C^1 close to $v(x)$. This question of "structural stability of vector fields" is clearly a topological one, and Smale and his school have made significant contributions during the past two decades. Indeed, during the middle and late 1960s global analysis was the "in" subject in mathematics. It is interesting to note that Poincaré's thesis was concerned with the problem of linearizing the system of equations

$$\dot{z} = Az + f(z). \tag{5}$$

Assuming that $f(z)$ was analytic, he attempted to find an analytic $u(\xi)$ such that the substitution

$$z = \xi + u(\xi)$$

transformed the equation (5) into the linear equation

$$\dot{\xi} = A\xi. \tag{6}$$

However, Poincaré found that even a formal series for $u(\xi)$ was impossible to obtain if one of the eigenvalues α_k of A was an integer linear combination of all the eigenvalues $\alpha_1, \ldots, \alpha_n$ of A. Thus, Poincaré was forced to impose the conditions

$$\alpha_k \neq j_1\alpha_1 + \cdots + j_n\alpha_n; \qquad k = 1, \ldots, n \tag{7}$$

for any set of positive integers $j_1, \ldots, j_n$ to obtain even formal equivalence. As we now know, Poincaré's difficulty arose because he required analytical equivalence of orbits. If we only require topological equivalence, then we have the following theorem of Hartman which was proven around 1960.

THEOREM [14]. *Suppose that* Re $\alpha_j \neq 0$ *for all eigenvalues* α_j *of* A. *Then, there exists a homeomorphism* $z = \xi + u(\xi)$ *of a neighborhood of* $\xi = 0$ *mapping the solutions of* (5) *into those of the linear system* (6).

A second, and more fundamental, reason for the proliferation of topological concepts, in this reviewer's opinion, is that many of the classical problems in differential equations have solutions which are really topological in nature. Here are two examples to illustrate this point.

1. One of the earliest, and most fundamental, problems of celestial mechanics was to continue solutions through collisions. More precisely, consider N

planets of mass $m_1, \ldots, m_N$ respectively, under the influence of their mutual gravitational attraction. If x_k, $k = 1, \ldots, N$ denote the N three-dimensional vectors describing the position of the centers of mass of the N planets, then

$$m_k \frac{d^2 x_k}{dt^2} = \frac{\partial U}{\partial x_k}$$

$$U = \sum_{1 \leq k < l \leq N} \frac{m_k m_l}{|x_k - x_l|}. \tag{8}$$

These equations are singular of course, when two planets collide; i.e., $x_k = x_l$ for some k and l. The problem of regularizing the singularity is to find solutions $x_k(t)$ which are valid for all time, even if collisions occur. This problem was solved by Sundman **[18]** in the special case that $N = 3$ and the total angular momentum is nonzero (this implies that there cannot be any triple collisions). Sundman's technique was to find an appropriate change of coordinates and a change of time scale along orbits which eliminated the singularity due to binary collisions. However, Sundman's results yield no information on the orbit structure of (8) near binary collisions. Further, these results offer no insight into how to regularize (if possible) the N body problem for $N > 3$.

Recently, Easton **[5]** presented a very general geometric theory of regularization which is applicable to the 3-body problem. It is called "regularization by surgery" and can be described very briefly as follows. Consider the system

$$\dot{x} = v(x) \tag{9}$$

and let the origin be a singular point of v. Suppose we can find an isolating block **[4]** which surrounds $x = 0$. This means, essentially, that we can find a surface S surrounding $x = 0$ with the property that at a point of tangency, an orbit of (4) points away from S (see Figure 2). We identify the endpoints of

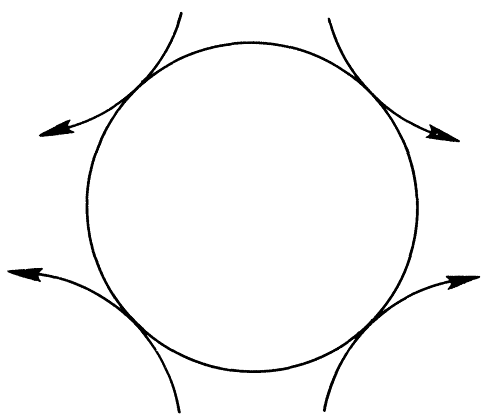

FIGURE 2

orbits which cross the block S, and then show that this identification has a unique extension to an identification which pairs the endpoints of orbits entering the singularity with the endpoints of orbits leaving the singularity. We then use this identification to close the gap left by surgery, thus obtaining the regularized phase space for the flow.

2. Another classical problem is the existence of periodic solutions of (9). Consider the linear system

$$\dot{x} = Ax \tag{10}$$

and suppose that the matrix A has purely imaginary eigenvalues $\pm\alpha_1, \pm\alpha_2, \ldots, \pm\alpha_n$. Then, the system (10) has n one parameter families of periodic solutions, with periods $2\pi i/\alpha_1, \ldots, 2\pi i/\alpha_n$ respectively. Consider now the perturbed equation

$$\dot{x} = Ax + f(x) \tag{11}$$

where f is analytic and begins with quadratic terms. A famous theorem of Liapunov **[11]** states the following:

THEOREM. *Let $\pm\alpha_1, \ldots, \pm\alpha_n$ be the purely imaginary eigenvalues of A, and assume that α_k/α_1 is not an integer, $k = 2, \ldots, n$. Moreover, assume that the system* (11) *possesses an integral with nondegenerate Hessian. Then, there exists a one parameter family of periodic solutions with period near $2\pi i/\alpha_1$. If $\alpha_k/\alpha_l \neq$ integer, $k \neq l$, then n such families exist.*

The next question, of course, is what happens if α_k/α_l is an integer. Liapunov's proof, which involves a series construction, appears to fall through in this case. About ten years ago Roels **[15]**, **[16]** showed that these series could be salvaged if the ratio α_k/α_l was at least three. Shortly thereafter, Schmidt and Sweet **[17]**, using a bifurcation theory developed by Hale **[1]**, presented a new proof of these results, and also obtained some results for the ratios 1 and 2. In 1973, Alan Weinstein **[19]**, **[20]** presented two extremely deep and brilliant topological theorems on the existence of periodic solutions of (9). He first specialized to Hamiltonian systems of equations of the form

$$\dot{z} = JH_z, z = \begin{bmatrix} z_1 \\ \vdots \\ z_{2n} \end{bmatrix}, J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}. \tag{12}$$

Very briefly, Weinstein studied C^1 perturbations of a Hamiltonian vector field on a manifold M which possessed a submanifold P consisting entirely of periodic orbits. Under small perturbations, only a finite number of periodic orbits can be expected to survive, in general. Weinstein reduced the determination of the minimal number of surviving periodic orbits to studying the intersection of two close Lagrange manifolds, and this problem in turn, is played back to estimating the minimal number of critical points on some orbit manifold. He then obtained the following theorem.

THEOREM. *If $H \in C^2$ near $z = 0$, and the Hessian matrix is positive definite, then for sufficiently small ε, any energy surface*

$$H(z) = H(0) + \varepsilon^2$$

contains at least n periodic orbits of (12) *with periods close to those of the linearized system.*

Now, it's true that Moser **[13]** presented an alternate proof of Weinstein's results which is more classical in nature. Nevertheless, Weinstein's theorems clearly motivated Moser's proof, and they stand as a monument to the topological nature of the flow.

As we mentioned at the beginning of this review, Professor Arnold's book is a splendid introduction to the modern theory of differential equations. All the fundamental concepts such as phase space, phase flows, vector fields, one parameter groups of diffeomorphisms, smooth manifolds and tangent bundles, which remain in the shadows in the traditional coordinate based approach, are presented here in a clear and rigorous fashion. Professor Arnold does much more than just pay lip service to these concepts, as many other books do: he does his darndest to make sure the reader understands these concepts. For example, after defining the tangent space TU_x to a domain U at the point x as the set of all velocity vectors of the curves leaving x, he remarks: "If the reader is accustomed to regard the velocity vector of a curve as lying in the same space as the curve itself, then the distinction between a tangent space to a linear space and the linear space itself may lead to certain psychological difficulties. In this case it is helpful to repeat the preceding considerations with U thought of as the surface of a sphere. Then TU_x is the ordinary tangent plane."

Another strong feature of this book is the many nontrivial examples and problems drawn from mechanics. One aspect of differential equations which hasn't changed is the fact that many theorems in dynamical systems owe their origin to similar theorems for Hamiltonian systems governing the mechanical interaction of a system of particles. Anyone who is serious about the study of differential equations should know the basic concepts of physics: someone who doesn't know any physics has no business undertaking a serious study of differential equations.

One possible objection of the "traditionalists" to this text is the absence of some of the elementary methods of solution. I fully agree with Professor Arnold that these topics are most conveniently studied in the guise of exercises. And to those who still object and claim that exactly soluble equations are important since they open up the possibility of solving neighboring equations by perturbation theory, Professor Arnold offers the following disclaimer: "However, it is dangerous to extend results obtained by studying an exactly solvable problem, to neighboring problems of a general form. In fact, an exactly integrable equation is often integrable precisely because its solutions are more simply behaved than those of neighboring nonintegrable problems."

Finally, this text is worth reading, even for an expert in the field, because of the many keen and penetrating insights presented by the author. Consider, for example, the problem of uniqueness for solutions of the equation $\dot{x} = v(x)$. If the vector field v is not differentiable (or Lipshitz) then an initial value problem may have several solutions. Now, numerical analysts are taught that a vector field v must be differentiable if we are to have any hope of containing the build-up of numerical round off error. But how many mathematics texts present a geometric reason for why uniqueness requires a differentiable vector field? Professor Arnold carefully shows that solutions of equations with differentiable vector fields cannot approach each other more rapidly than exponentially, thereby accounting for the uniqueness of solutions.

In short, Professor Arnold is one of the truly great stars of mathematics, and in this outstanding text, he shares his knowledge and understanding with us.

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Methods of modern mathematical physics, vol. III, *Scattering theory*, by Michael Reed and Barry Simon, Academic Press, New York, 1979, xv + 463 pp., $42.00.

Methods of modern mathematical physics, vol. IV, *Analysis of operators*, by