BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 1, Number 6, November 1979
© 1979 American Mathematical Society
0002-9904/79/0000-0524/$01.75

Gesammelte Schriften, by Gustav Herglotz, edited by Hans Schwerdtfeger, Vandenhoeck & Ruprecht, Gottingen and Zurich, 1979, xl + 652 pp.

Herglotz published only moderately, and not "spectacularly". Although his work is quite meaningful throughout, not much of it has become widely known. But he contributed to the fertilization of several areas of mathematics, and he was the kind of mathematician whom faculty colleagues, even in a place like Göttingen, prize having around. (A regard for Herglotz was implanted in me by C. Carathéodory during my Munich years 1927–1932.)

And so Hans Schwerdtfeger, a student of his in Montreal, Canada, decided to bring Herglotz before the public by this very handsomely printed collection of all his papers. It even includes a survey article of 42 pages in the German Encyclopedia on the determination of orbits of planets and comets–Herglotz started out as a "double-major" in mathematics and (planetary) astronomy–and it also includes summarizing reports by six specialists on his achievements in relativity, geometry, differential equations and potential theory, applied mathematics, number theory, and complex analysis.

Although I am no Herglotzologist, I have a few things to say that are not properly emphasized or even not to be found in the volume (under review).

Herglotz shows off his "hardest" analysis in papers on linear partial differential equations of hyperbolic type (we will omit those of elliptic type) with constant coefficients. In the notation of [**1**, Chapter 2], let $Q(\eta_1, \ldots, \eta_n, \lambda)$ be a homogeneous polynomial of even degree m, with $m \geqslant n + 1$, such that for $\eta \neq 0$ all zeros of $Q(\eta, \lambda)$ are real and distinct (strict hyperbolicity).

Consider now the Cauchy problem

$$Q\left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}, \frac{\partial}{\partial t}\right)u(x, t) = 0$$

with the initial conditions for $t = 0$

$$\frac{\partial^k u}{\partial t^k} = 0, \quad k = 0, \ldots, m-2, \quad \text{and} \quad = f(x_1, \ldots, x_k) \quad \text{for } k = m-1.$$

Subject to assumptions, the solution is of the form

$$u(x, t) = \int f(y)K(x - y, t)\, dy$$

where the kernel $K(x, t)$ is such that $K(x - y, t)$ is the solution of our problem for $f(x)$ being the Dirac function $\delta(x - y)$. Now, emphasis ought to be put on the fact that $K(x, t)$ can be expressed by a certain specific integral taken over the socalled "normal surface" $Q(\eta, 1) = 0$, which we will not undertake to write out, and that extensions of the original result(s) of Herglotz are linked to extensions of this integral; thus, Gelfand and Shilov [**2**] speak of a Herglotz-Petrovskii formula, that fits a case less restrictive than Herglotz' own.

Next, the neatest result in the volume is the following. If $f(z)$ is holomorphic in the disk $|z| < 1$ then its real part is positive, $Rf(z) > 0$, if and only if there is a (unique) Stieltjes integral representation

$$f(z) = \oint \frac{e^{i\alpha} + z}{e^{i\alpha} - z}\, d_\mu(\alpha) + ic, \tag{1}$$

with nonnegative measure, $d_\mu \geqslant 0$. The summarizer reports on various subsequent extensions and generalizations. But he does not (want to) observe that contingent to the proof of (1), Herglotz stated the first version, a relatively easy one, of the structure theorem for positive-definite functions. It is a fact though that the role of the structure theorem in stochastic theory and functional analysis was established before the anticipation of Herglotz (in 1911) became known. Thus, A. Ya. Khintchine (1934), and A. Weil (1940) and R. Godement (1948) quote only my work [3] of 1932; and I myself became aware of the early Herglotz contribution only around 1950.

To this context of work of Herglotz it is worth noting that around 1911 substantive operations involving Stieltjes integrals were still avant-garde activities on the rim of analysis, as can be seen from the circumspect manner in which Hilbert himself derived his spectral theorem.

In turning over the pages of the volume I was incredulous not to come across a delightful classroom contribution of Herglotz to a classical 19th century finding. At issue is the formula

$$\pi \cot \pi z = \frac{1}{z} + \sum_{-\infty}^{\infty}{}' \left(\frac{1}{z+n} - \frac{1}{n} \right) = \frac{1}{z} + \sum_{1}^{\infty} \frac{2z}{z^2 - n^2}.$$

It follows from Mittag-Leffler that the difference

$$g(z) = \pi \cot \pi z - \frac{1}{z} - \sum_{-\infty}^{\infty}{}' \left(\frac{1}{z+n} - \frac{1}{n} \right)$$

is holomorphic and $g(0) = 0$. We have now to show that $g(z) = c$, and the traditional ungainly procedure has been to show that $g(z)$ is bounded, so that the theorem of Liouville applies. It was only slightly less ungainly to apply Liouville's theorem to

$$g'(z) = -\frac{\pi^2}{(\sin \pi z)^2} + \sum_{-\infty}^{\infty} \frac{1}{(z+n)^2}.$$

Herglotz' innovation was to find the functional relation

$$4g'(z) = g'\left(\frac{z}{2}\right) + g'\left(\frac{z+1}{2}\right). \tag{2}$$

If M is the maximum of $|g'(z)|$ in $|z| \leqslant 2$ it follows from (2) that

$$4M \leqslant M + M.$$

Thus $M = 0$, $g'(z) \equiv 0$. Finished.

In print this device of Herglotz has surfaced only in 1950 in the posthumous book [4], but in mimeographed form it appeared already in my lecture notes [5] of 1936.

Herglotz had a great skill in computing definite integrals. This does not manifest itself sufficiently in the volume, but C. Carathéodory handed down to me a telling anecdote about it. An aggressive young contempory of

Herglotz–let's call him R. C.–ran up against a complicated definite integral in 5 dimensions. With great effort he reduced it to a much simpler integral in 3 dimensions, and this he passed on to Herglotz for his assistance in computing. After a while Herglotz came back with the comment that if by a nonobvious substitution one transforms the integral into a 5-dimensional one–which was R. C.'s original expression–then the computation is trivial. And he proceeded to show him how trivial.

On the personal side, Herglotz had great charm and was a perfect gentleman. The volume cites testimonials for that, and I can add the following corroboration. From what I can remember, I never had epistolary or personal encounters with Herglotz except for meeting him once, in May 1932, when I was in Göttingen for a lecture. (It was on Greek mathematics, and I own a clipping from a leading Berlin newspaper reporting on it.) My return train to Munich departed at 2 a.m. By academic seniority Herglotz towered over me, but he came to the station to see me off. And he stayed with me until the train started moving.

REFERENCES

1. Fritz John, *Plane waves and spherical means applied to partial differential equations*, Interscience Publishers, New York, 1955.

2. J. M. Gelfand and G. E. Shilov, *Generalized functions*, vol. I, Academic Press, New York, 1964.

3. S. Bochner, *Vorlesugen über Fouriersche Integrale*, Leipzig, 1932.

4. C. Carathéodory, *Theory of functions of a complex variable*, volume 1, Chelsea Publishing Co., New York, 1954.

5. S. Bochner, *Lectures on several complex variables*, Vol. I, Princeton, N. J., 1936.

SALOMON BOCHNER

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 1, Number 6, November 1979
© 1979 American Mathematical Society
0002-9904/79/0000-0524/$01.75

Choice sequences, A chapter of intuitionistic mathematics, by A. S. Troelstra, Clarendon Press, Oxford, 1977, ix + 170 pp., $10.95.

The book under review is based on lecture notes of a course on choice sequences given by the author at Oxford in 1975. Choice sequences are a paradigm case of specifically intuitionistic notions, that is notions which cannot be classically understood, and one of the principal virtues of the book is that it demonstrates the possibility of coherent reasoning about such notions.

The simplest sort of "freely" chosen sequence is a lawless sequence (of natural numbers). Such sequences are necessarily incomplete, only finite initial segments having been constructed at any time. At future times one is completely free in the choice of additional elements. If Γ is any well-defined operation on such sequences η whose values are completed objects x (so we have a proof that $\forall\eta\exists!x\Gamma(\eta) = x$) then Γ is continuous in the product topology; for a proof that $\Gamma(\eta) = x$ can depend only on a finite initial segment of η. Since the axiom of choice is, intuitionistically, logically valid, we have for well-defined relations R the principle of $\forall\alpha\exists x$-continuity: