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THE DUALITY OPERATION IN THE CHARACTER RING OF A FINITE CHEVALLEY GROUP

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It is possible (as in [4]) to define a duality operation $\zeta \longrightarrow \zeta^*$ in the ring of virtual characters of an arbitrary finite group with a split (B, N)-pair of characteristic p. Such a group arises as the fixed points under a Frobenius map of a connected reductive algebraic group, defined over a finite field [1]. This paper contains statements of several general properties of the duality map $\zeta \longrightarrow \zeta^*$ and two related operations (see §§2 and 4). The duality map $\zeta \longrightarrow \zeta^*$ generalizes the construction in [2] of the Steinberg character, and interacts well with the organization of the characters from the point of view of cuspidal characters (§6). It is hoped that there is also a useful interaction with the Deligne-Lusztig virtual characters $R_{\mathrm{T}}^{\mathrm{G}}\theta$. Partial results have been obtained in this direction (§5). Detailed proofs will appear elsewhere.

1. Let G be a finite group with split (B, N)-pair of characteristic p. Let (W, R) be the Coxeter system, and let $P_J = L_J V_J$ be the standard parabolic subgroup corresponding to $J \subseteq R$, with $V_J = O_p(P_J)$ (see [3] for definitions and notations). Let char(G) denote the ring of virtual characters of G, and Irr(G) the set of irreducible characters of G, all taken in the complex field. For $J \subseteq R$ and $\zeta \in$ char(G) define

$$\zeta_{(P_J/V_J)} = \Sigma(\zeta, \tilde{\lambda}^G)_G \lambda \tag{1.1}$$

where $\sim$ denotes extension to P_J via the projection $P_J \longrightarrow L_J \cong P_J/V_J$, and the sum is over all $\lambda \in$ Irr(L_J). Let $\zeta_{(P_J)} = \zeta_{(P_J/V_J)}{}^{\sim}$. The duality map is then defined by:

1.2 DEFINITION. $\zeta^* = \Sigma_{J\subseteq R}(-1)^{|J|}\zeta_{(P_J)}{}^G$, for all $\zeta \in$ char(G).

2. The truncation map $\zeta \longrightarrow \zeta_{(P_J/V_J)}$ and the map $\lambda \longrightarrow \tilde{\lambda}^G$ behave in much the same way as ordinary restriction and induction. The following basic properties follow directly from the structure theorems [3].

2.1 FROBENIUS RECIPROCITY. *Let* $\zeta \in$ char(G) *and* $\lambda \in$ char(L_J). *Then*

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$$(\zeta, \tilde{\lambda}^G)_G = (\zeta_{(P_J)}, \tilde{\lambda})_{P_J} = (\zeta_{P_J/V_J)}, \lambda)_{L_J}.$$

2.2 TRANSITIVITY. *If $K \subseteq J \subseteq R$, let Q_K be the standard parabolic subgroup $P_K \cap L_J$ of L_J and let $V_{J,K} = O_p(Q_K) = L_J \cap V_K$. Then if $\zeta \in$* char(G) *and $\zeta \in$* char(L_J), *we have*

$$(\zeta_{(P_J/V_J)})_{(Q_K/V_{J,K})} = \zeta_{(P_K/V_K)},$$

and

$$(\tilde{\lambda}^{L_J})^{\sim G} = \tilde{\lambda}^G.$$

2.3 INTERTWINING NUMBER THEOREM. *Let $\lambda_i \in$ char(L_{J_i}) for $i = 1,2$. Then*

$$(\tilde{\lambda}_1^G, \tilde{\lambda}_2^G)_G = \sum_{w \in W_{J_1,J_2}} (\lambda_{1(Q_{K_1}/V_{J_1})}, {}^w\lambda_{2(Q_{K_2}/V_{J_2,K_2})})_{L_{K_1}}$$

where W_{J_1,J_2} is the set of distinguished $W_{J_1} - W_{J_2}$ double coset representatives, $W_{K_1} = W_{J_1} \cap {}^wW_{J_2}$ and $W_{K_2} = W_{J_2} \cap {}^{w^{-1}}W_{J_1}$.

2.4 SUBGROUP THEOREM. *Let $\lambda \in$ char(L_{J_1}). Then*

$$(\tilde{\lambda}^G)_{(P_{J_2}/V_{J_2})} = \sum_{w \in W_{J_1,J_2}} {}^{w^{-1}}(\lambda_{(Q_{K_1}/V_{J_1,K_1})})^{\sim L_{J_2}}.$$

Here K_1 is as in 2.3 (note: ${}^{w^{-1}}L_{K_1} = L_{K_2}$).

3. The results of this section are of independent interest, and are due to Curtis ([4]). They are needed to apply the results of §2 to the duality operation.

3.1. LEMMA. *Let $w \in W$, ${}^wL_{J_2} = L_{J_1}$, ${}^w\lambda_2 = \lambda_1$, where $\lambda_i \in$ char(L_{J_i}). Then $\lambda_1^G = \lambda_2^G$.*

The idea of the proof is to show that the numbers $(\tilde{\lambda}_i^G, \tilde{\lambda}_j^G)_G$ are all the same for $i, j = 1, 2$. The proof in [3] (for the special case when λ_1, λ_2 are cuspidal) can be modified to work in the present situation.

The following is Lemma 2.5 of [4].

3.2. LEMMA. Let $a_{J_2,J_1,K} = |\{w \in W_{J_1,J_2} | W_K = W_{J_1} \cap {}^wW_{J_2}\}|$. *Then*

$$\sum_{J_2 \subseteq R} (-1)^{|J_2|} a_{J_2,J_1,K} = (-1)^{|K|}.$$

4. The first main result relates duality and the operations $\zeta \longrightarrow \zeta_{(P_J/V_J)}$ and $\lambda \longrightarrow \widetilde{\lambda}^G$. Part (1) is Theorem 1.3 of [4].

THEOREM. (1) $(\zeta^*)_{(P_J/V_J)} = (\zeta_{(P_J/V_J)})^*$ *for* $J \subseteq R$, $\zeta \in \mathrm{char}(G)$
(2) $(\widetilde{\lambda}^G)^* = (\lambda^*)^{\sim G}$ *for* $J \subseteq R$, $\lambda \in \mathrm{char}(L_J)$.

We provide a sketch of the proof of (2). Let $J_1 = J$. Using 2.4, 2.2, and then Lemma 3.1 (noting that $L_{K_1} = {}^wL_{K_2}$ by Proposition 2.6 of [3]) we have

$$(\widetilde{\lambda}^G)^* = \sum_{J_2 \subseteq R} (-1)^{|J_2|} \sum_{w \in W_{J_1,J_2}} \lambda_{(Q_{K_1}/V_{J_1,K_1})}{}^{\sim G}$$

The proof is then completed by applying Lemma 3.2 and 2.2.

4.2 THEOREM. *The map* $\zeta \longrightarrow \zeta^*$, *from* $\mathrm{char}(G) \longrightarrow \mathrm{char}(G)$ *is an isometry of order two. In particular,* $\zeta^{**} = \zeta$ *and* $\pm\, \zeta^* \in \mathrm{Irr}(G)$, *whenever* $\zeta \in \mathrm{Irr}(G)$.

In order to prove Theorem 4.2, one first proves that $(\zeta_1, \zeta_2^*)_G = (\zeta_1^*, \zeta_2)_G$. It then suffices to prove $\zeta^{**} = \zeta$. The key is to apply Theorem 4.1 part (1) to the expression for ζ^{**}. We have

$$\zeta^{**} = \sum_{J \subseteq R} (-1)^{|J|} \zeta_{(P_J/V_J)}{}^{*\sim G}$$

$$= \sum_{J \subseteq R} (-1)^{|J|} \sum_{K \subseteq J} (-1)^{|K|} \zeta_{(P_K)}{}^G$$

using 2.2. To finish the proof, note that $\Sigma\,(-1)^{|J|}$ summed over all J such that $K \subseteq J \subseteq R$ is zero unless $K = R$.

5. It is clear that $\zeta^* = (-1)^{|R|}\zeta$ for any cuspidal $\zeta \in \mathrm{Irr}(G)$. Thus by applying Theorem 4.1 part (2) we have:

5.1 COROLLARY. *Let* $\lambda \in \mathrm{Irr}(L_\lambda)$ *be cuspidal. Then* $(\widetilde{\lambda}^G)^* = (-1)^{|J|}\widetilde{\lambda}^G$.

Thus duality permutes (up to sign) the components of $\widetilde{\lambda}^G$. We can thus determine the "sign" of ζ^* as follows: $(-1)^{|J|}\zeta^*$ is in $\mathrm{Irr}(G)$ if $\zeta \in \mathrm{Irr}(G)$ is a component of $\widetilde{\lambda}^G$, $\lambda \in \mathrm{Irr}(L_J)$ cuspidal. In particular, $\zeta \longrightarrow \zeta^*$ permutes the principal series characters, i.e. the components of $\widetilde{\lambda}^G$, $\lambda \in \mathrm{Irr}(L_\varnothing)$. A more explicit result is known for the components $\zeta_{\varphi,q}$ of $1_{B(q)}^{G(q)}$ in a system of groups $\{G(q)\}$ of type (W, R). Specifically, $\zeta^*_{\varphi,q} = \zeta_{\epsilon\varphi,q}$ where ϵ is the sign character of W ([4]).

Finally, consider the case $G = \mathbf{G}^F$ where $\mathbf{G}$ is a reductive algebraic group and $F : \mathbf{G} \longrightarrow \mathbf{G}$ is a Frobenius map over F_q. Let $R_{\mathbf{T}}^{\mathbf{G}}\theta$ denote the Deligne-Lusztig generalized character of G ($\mathbf{T}$ an F-stable maximal torus of $\mathbf{G}$, θ a linear character of $\mathbf{T}^F$). It is natural to ask whether

$$(5.2) \qquad (R_{\mathbf{T}}^{\mathbf{G}}\theta)^* = \pm R_{\mathbf{T}}^{\mathbf{G}}\theta$$

holds. The following suggests the answer is yes.

$$(5.3) \qquad (R_{\mathbf{T}}^{\mathbf{G}}\theta)^*(s) = \pm R_{\mathbf{T}}^{\mathbf{G}}\theta(s)$$

for semisimple elements s of G. The $\pm$ sign in 5.3 does not depend on the particular element s of G. The proof of 5.3 uses several results of [5]. (Note added in proof: The conjecture 5.2 has been proved by G. Lusztig.)

5.4 Example. Let $G = \mathbf{G}^F$ as above, with (relative) Coxeter system (W, R). Let V be the set of unipotent elements of G and let ϵ_V be the characteristic function of V. A recent result of Springer (Theorem 1 of [6])[1] shows

$$\epsilon_V = q^d \sum_{J \subseteq R} (-1)^{|J|} |P_J|^{-1} 1_{V_J}^G$$

where $d = \dim(\mathbf{G}/\mathbf{B})$, $\mathbf{B}$ a Borel subgroup of $\mathbf{G}$. Applying Theorems 4.1 and 4.2 we have:

5.5 Theorem. (1) *$\epsilon_V^* = (q^d/|G|)\rho_G$ where ρ_G is the regular character of G.*

(2) *For $\zeta \in \mathrm{Irr}(G)$,*

$$\frac{1}{\zeta(1)} \sum_{v \in V} \zeta(v) = q^d(\zeta^*(1)/\zeta(1)).$$

(3) *For $\zeta \in \mathrm{Irr}(G)$, $|\zeta^*(1)|_{p'} = \zeta(1)_{p'}$ where p is the characteristic of F_q and $n_{p'}$ is the p' part of n.*

(4) *For $\zeta \in \mathrm{Irr}(G)$, $1/\zeta(1)\ \Sigma_{v \in V}\zeta(v)$ is, up to sign, a power of p.*

Part (4) of Theorem 5.5 confirms a special case of a conjecture of Macdonald (see [6]), namely the case when $q = p$ is prime.

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[1] The author is indebted to T. A. Springer for communicating both his results in [6] and the suggestion of G. Lusztig of combining them with duality.

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