topology in the first place. A large number of exercises add considerably to the usefulness of the book as a text.

The book appears to be well designed for research purposes. One could read the first two chapters and then proceed to the chapter of particular interest. So one could learn the "state of the art" in hyperspace theory with respect to the topics previously mentioned. He would also acquire a statement of the research problems of current interest in this area. This makes the book "a must" for researchers in the field.

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JACK SEGAL

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Skew field constructions, by P. M. Cohn, London Mathematical Society Lecture Note Series No. 27, Cambridge Univ. Press, Cambridge, London, New York, Melbourne, 1977, xii + 253 pp., $12.95.

A skew field, or division ring, is an associative ring with 1 in which every nonzero element is invertible. Why do we study such objects?

A natural answer is that they generalize fields, which are so important in commutative ring theory. But this analogy is not as strong as it seems. Skew fields generalize fields as characterized by the property that every nonzero element is invertible. But commutative fields have a number of other important characterizations. They are, for instance, the *simple* commutative rings. A consequence is that every nonzero commutative ring R has homomorphisms into fields, and the class of such homomorphisms (up to an obvious equivalence) gives a basic framework for studying R: It is (with a few extra trimmings) the "prime spectrum" Spec R of algebraic geometry.

It is similarly true that every nonzero associative ring has maps into simple associative rings, but these form a wider class than the skew fields. Two examples are the ring of $n \times n$ matrices over a field, and the algebra of operators on a polynomial ring $k[x]$ (k a field of characteristic 0) generated by p = multiplication by x and $q = d/dx$. (This is the *Weyl algebra*, which has the presentation $k\langle p, q | qp - pq = 1\rangle$.)

Commutative fields are also characterizable as commutative rings which have faithful *simple modules*. Associative rings with this property are called *primitive*, and form a larger class than the simple rings. The fact that any finitely generated module has simple homomorphic images again gives us a large class of maps of any ring R into primitive rings, and in particular yields the important technical result that if we write $J(R)$ for the intersection of the kernels of such maps (the Jacobson radical of R) then any finitely generated right R-module M such that $M = MJ(R)$ must be zero. This is Nakayama's Lemma; in the commutative theory it adds to the importance of fields; but in the noncommutative theory it lends the same importance to primitive rings.

Commutative fields are also those commutative rings, not decomposable as direct products, whose module theory has the neat property that any element of a free module generates a submodule which is a direct summand. In general, rings with this property are the direct-product-indecomposable *von Neumann regular rings*, and they form a thoroughly unruly menagerie.

Finally, commutative fields are easy to construct. Whenever we have commutative ring without zero-divisors, we can form its field of fractions. A noncommutative ring R without zero-divisors, on the other hand, may not be embeddable in any skew field, or may be embeddable in skew fields in many essentially different ways; and the construction of such skew fields from R when they exist may not be easy at all.

Again, some examples. For any integers $n > m > 1$, one may construct by generators and relations a ring R with a universal example of an $n \times m$ matrix X and an $m \times n$ matrix Y such that

$$XY = I_n, \qquad YX = I_m. \tag{1}$$

One can show that the resulting ring is nonzero, and in fact has no zero-divisors. Now the existence of matrices satisfying (1) means that as right R-modules,

$$R^m \cong R^n. \tag{2}$$

If there existed *any* homomorphism f from R to a skew field D, then the (entrywise) images of X and Y under f would be matrices over D satisfying (1), hence we would have an isomorphism of right D-vector-spaces $D^m \cong D^n$. But vector-spaces, whether over commutative fields or skew fields, have unique dimension, so no such homomorphism f can exist.

Rather than giving the details of two essentially different ways a ring can be embedded in skew fields, let me give an analogous example for the embedding of semigroups in groups. Let n be a positive integer, and consider the two affine maps of the real line into itself,

$$x_n(t) = t/n, \qquad y_n(t) = (t+1)/n.$$

The semigroup $S(n)$ generated by x_n and y_n is free on this pair of generators. This can be seen by noting that if W is any nontrivial composition of x_n's and y_n's, the *last* factor in this product can be determined by checking whether the image of the unit interval, $W([0, 1])$, lies in $[0, 1/n]$ or $[1/n, 2/n]$; hence by induction the full expression for W in terms of x_n and y_n may be recovered. Hence the semigroups $S(n)$ are isomorphic for all values of n. But the groups $G(n)$ which they generate are not: for each n one has the group relation $x_n^{-1}y_n = (y_n x_n^{-1})^n$, which is not satisfied by x_m, y_m for any $m \neq n$. (From this example one can in fact get an example of a ring with nonisomorphic skew-field embeddings, by a trick using formal power series over ordered groups.)

Finally, as an example where an analog of a field of fractions exists, but was not at all easy to find, we mention the problem which was outstanding for many years, of whether there existed a reasonable noncommutative analog of the field of rational functions in n variables. This should clearly be some sort of "skew field of fractions" of the free associative algebra in n indeterminates. Various ways were found over the years of embedding free associative algebras in division algebras D, but it was not until 1966 that S. A. Amitsur showed that there existed one such D having a universal property that justified thinking of it as the noncommutative "rational function" field. A large part of the difficulty of constructing such a skew field of fractions arises from the fact that there is no uniform simple form like

$$(\text{numerator})(\text{denominator})^{-1}$$

to which a rational expression in noncommuting elements can be reduced. An expression like

$$\left(x^{-1} + y^{-1} + (xy - yx)^{-1} - 1\right)^{-1} + (x - y)^{-1} \tag{3}$$

can be transformed in various ways, but not really put into any simple or canonical form.

From the above discussion it should be clear that skew fields can never play the enormously powerful role in noncommutative ring theory that fields do in the commutative theory. What, now, is to be said in their favor?

First, they are still the simplest possible rings to look at from the point of view of basic linear algebra. As far as the results that allow us to construct and extend bases, linear maps, etc. are concerned, a right (or left) vector space over a skew field behaves exactly like a vector space over a field.

Second, though we saw that a simple module M over a ring R does not generally look like a 1-dimensional vector-space over a skew-field factor-ring of R, as would be true in the commutative case, nevertheless the study of such modules does lead to skew fields. The endomorphism ring $\mathrm{End}_R(M)$ is a skew field D, and the factor-ring of R by the kernel I of its action on M is a dense subring (in an appropriate topology) of the ring of all endomorphisms of M as a (possibly infinite-dimensional) D-vector-space. In the important case of a ring R with polynomial identity–the theory of rings with polynomial identity being a sort of border-country between commutative and noncommutative ring theory–R/I will in fact be a matrix ring $M_n(D)$.

Finally, though the problem of embedding rings in "natural" ways into skew fields is certainly not as trivial as in the commutative theory, it is not as hopeless as it once seemed. To make things more comparable, let us make the commutative problem a little harder! Instead of taking a commutative integral domain and asking how it can be embedded in a field, let us be given an arbitrary commutative ring R, and ask how to describe all fields generated by *homomorphic images* of R.

The answer is still clear from standard commutative ring theory. Such fields will be precisely the fields of fractions of the factor-rings $R/\mathfrak{p}$ as $\mathfrak{p}$ ranges over the prime ideals of R. In other words, the structure of a field k generated by a homomorphic image of R is determined if we know what set $\mathfrak{p}$ of elements of R go to zero in k. In particular, if we write $\bar{r}$ for the image in k of $r \in R$, then every element of k can be written as a quotient $\bar{a}^{-1}\bar{b}$, such a quotient will define an element of k if and only if a $\notin \mathfrak{p}$, and we can write down a rule for when two such expressions represent the same element of k.

The major breakthrough in the study of the noncommutative situation was P. M. Cohn's discovery in 1970 that a skew field D generated by a homomorphic image of a ring R, though not generally determined by the set of *elements* of R which go to *zero* in D (we have noted that D may not be unique when this set is $\{0\}$) *is* determined, up to natural isomorphism, by the set of *square matrices* (of all sizes) over R which become *singular* over D! The key idea, which was borrowed from work on rational noncommuting formal power series by M. Schützenberger, M. Nivat *et al.*, who in turn borrowed it from the theory of differential equations, is that of transforming a single complicated equation into a system of *linear* equations in a larger number of variables. For instance one finds that a complicated expression like (3) can be described as the first component of the solution u of an equation $Au = b$, where A is a certain square matrix and b a certain column vector over R; and (3) will in fact be defined in the skew field D if and only if the image of A in D is nonsingular.

From this point on I will adopt Cohn's convention of dropping the adjective "skew", so that "fields" are now not necessarily commutative. An *R-field* will mean a field D given with a homomorphism $R \to D$. If D is an R-field, then the set of square matrices over R whose images over D are singular is called the *singular kernel* of D. Cohn has found fairly simple conditions on a set $\mathfrak{P}$ of square matrices over a ring R which are necessary and sufficient for $\mathfrak{P}$ to be the singular kernel of an R-field D. If a set $\mathfrak{P}$ satisfies these conditions he calls $\mathfrak{P}$ a *prime matrix ideal* of R, and has given a construction for the R-field D having $\mathfrak{P}$ as its singular kernel.

If D and E are two R-fields, then a *specialization* (over R) φ: $D \to E$ is defined as a homomorphism from a *local* subring $D_\varphi \subseteq D$ into E, having the maximal ideal of D_φ as its kernel; and such that D_φ contains the image of R in D and the map respects the R-field structures. (For a commutative example, let $R = \mathbf{Z}[x, y]$, let D be its field of fractions $\mathbf{Q}(x, y)$, and let F be any commutative field, made an R-ring by sending x and y to any elements ξ, $\eta \in F$. If we let $\mathbf{Q}(x, y)_\varphi$ be the set of elements that can be written $p(x, y)/q(x, y)$ $(p, q \in R)$ such that $q(\xi, \eta) \neq 0$ in F, then $p(x,y)/q(x,y) \mapsto p(\xi, \eta)/q(\xi, \eta)$ defines a specialization φ: $\mathbf{Q}(x, y) \to F$.) One now finds that given two R-fields D and E, a specialization $D \mapsto E$ exists *if and only if* the singular kernel of D is contained in the singular kernel of E. (This condition can also be stated "Every rational relation satisfied in D by images of elements of R is also satisfied by the images of these elements in E," but I won't give details here on how to make this statement precise.) It follows that if R has a *smallest* prime matrix ideal, $\mathfrak{P}$, then the associated R-field U will have the property that for any R-field D, there exists a specialization $U \to D$. (This specialization becomes unique when one throws on an auxiliary condition to exclude certain sorts of "irrelevant" extensions of specializations.) U is then called a *universal* R-field.

For an important class of rings R called *semifirs*, it is not hard to show that the set $\mathfrak{P}_0$ of square matrices A over R which can be factored $A = BC$, where B has fewer columns than A (equivalently, C has fewer rows than A) forms a prime matrix ideal of R. But clearly, such matrices A have singular image under any homomorphism of R into any field D. Hence in this situation $\mathfrak{P}_0$ does give us a smallest prime matrix ideal of R, and we get a universal R-field. Particular examples of semifirs are *free associative algebras*, and *coproducts* of fields with amalgamation of a common subfield. The universal field of the first gives us a concept of *free field* (isomorphic, by virtue of having the same universal property, to Amitsur's 1966 construction). The second gives what Cohn calls a *field coproduct* of the given fields.

Let us pause to note a peculiarity of this concept of "field coproduct". If U is the field coproduct of D_1 and D_2 over a subfield D_0, then we know that for any field E and homomorphisms $D_1 \to E$, $D_2 \to E$ agreeing on D_0, we get a specialization $U \to E$ making the appropriate triangles commute. What we surely *ought* to be able to say is that given *specializations* $D_1 \to E$, $D_2 \to E$ agreeing on D_0 such a specialization $U \to E$ is induced. But no one has yet been able to prove (or disprove) this.

Let us now turn to the book at hand. The first three chapters consider "classical" topics, in particular *Ore rings*, which are the largest class of noncommutative rings for which a field of fractions consisting of elements of the simple form $a^{-1}b$ *does* exist, and noncommutative Galois theory.

Chapter 4 discusses the construction of R-fields from prime matrix ideals of R. These results were obtained and inserted in [2] shortly before that book was completed, and I had hoped that the author might have been able to work out a simpler and perhaps more conceptual proof of the basic existence theorem in the interim. However, the reader is referred to [2] for the proof of this result. (For a subsequent simpler construction see Malcolmson [4].)

Chapter 5 develops the properties of coproducts that allow one to conclude that free algebras and coproducts of fields are semifirs, and hence have universal fields. With the help of free fields the author completes here a construction begun in Chapter 3, of a field extension L/K with finite right degree and infinite left degree.

In Chapter 6 it is observed that the field coproduct construction shows that the class of fields has what model-theorists call the *amalgamation property*, and that this leads to the construction of *existentially closed* fields. Existential closure is the model-theoretic concept which in commutative field theory characterizes algebraically closed fields. In the noncommutative theory it resembles that concept in some ways, and differs in many others. Solvability and nonsolvability for word problems in certain fields are also obtained.

Chapter 7 presents work of the reviewer on rational relations and rational identities in skew fields [1]. Let R be a ring and consider two R-fields D_0 and D_1, each generated as a field ("rationally generated") by the image of R.

Within the direct product ring $D_0 \times D_1$, let S_{01} denote the smallest subring containing the "diagonal" image of R, and closed under taking inverses of all invertible elements (elements with both coordinates nonzero) which it contains. What will S_{01} look like? There turn out to be four cases. S_{01} may be a field; this will be so if and only if $D_0 \cong D_1$ as R-fields in which case S_{01} is the graph of the isomorphism. A second possibility is that S_{01} will be a *local ring*, which embeds in D_0 and has residue field D_1. In this case it is the graph of a specialization map $D_0 \rightarrow D_1$; this happens if and only if the singular kernel of D_0 is strictly contained in that of D_1, i.e. every rational relation in elements of R satisfied in D_0 holds in D_1 but not vice versa. A third case corresponds to the above but with the roles of D_0 and D_1 reversed. Finally, if neither singular kernel contains the other, equivalently if each R-field satisfies a rational relation which fails in the other, then S_{01} is the full direct product $D_0 \times D_1$.

So far this is just another way of approaching concepts we have seen before. However, suppose we start with three R-sfields D_0, D_1 and D_2. Then there turn out to be more types of behavior than one would expect. For instance, the subrings $S_{ij} \subseteq D_i \times D_j$ $(0 \leqslant i < j \leqslant 3)$ may each be the full direct product $D_i \times D_j$, yet the analogously defined subring $S_{012} \subseteq D_0 \times D_1 \times D_2$ may be a *semilocal* ring which embeds in D_0 and has *two* residue fields D_1 and D_2. This happens when every rational relation satisfied in D_0 is satisfied *either* in D_1 *or* in D_2, but when neither D_1 nor D_2 accounts for all such rational relations.

These concepts arose in the study of the case where $D_0, \ldots, D_r$ were "generic matrix fields"–skew fields that are "free" subject to certain polynomial identities. This case is studied in detail. The treatment follows that of [1] closely.

The last chapter discusses the solution of equations in fields, with the ultimate aim of setting up a noncommutative algebraic geometry. Though there are some nice reductions (e.g. from arbitrary equations in an unknown x to the problem of finding x making the matrix $A + Bx$ singular, where A and B are matrices over the given field), actual results are fairly weak. The author's conjecture that for every square matrix A over a field, there exists an element α in some extension field such that $A - \alpha I$ is singular, remains open.

Passing beyond the text proper, it is worth mentioning that the functions of bibliography and name-index are combined nicely by listing, after each bibliography entry, the pages (if any) of the text where the item is referred to. Mentions of an individual that do not refer to a particular paper are listed directly after the person's name. Unfortunately, the bibliography does not cross-reference second coauthors of papers.

The book contains one important false theorem (6.3.6) and several incomplete or incorrect proofs (notably those of 3.4.3, 3.4.4 and 5.6.1), but these and more minor errors and omissions are corrected in a set of *Corrigenda and Addenda* which may be obtained by writing the author.

In general, the book is a useful introduction to large areas of current research in skew fields. It does not emphasize open problems but it should leave the reader prepared to read current articles at the cutting edge of the field, such as [3].

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G. M. Bergman

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Approximation of vector valued functions, João B. Prolla, Mathematics Studies 25, North-Holland Publishing Company, Amsterdam, New York, Oxford, 1977, xiii + 219 pp., $22.50.

During the thirty years which have elapsed since the publication in *Mathematics Magazine* of a justly famous paper by M. H. Stone [12], the Stone-Weierstrass theorem, as the contents of this article have come to be collectively known, has exercised a pervasive influence on the development of modern analysis. Aside from the effect of Stone's pioneering work with what can be termed an algebraic approach to analysis, the theorem itself has direct applications in areas ranging from spectral theory to group representations. Subsequent generalizations, moreover, have served to extend the power and utility of Stone's theorem. One striking example is the well-known extension due to Errett Bishop [2]; obtained in response to a question raised by Šilov [11], Bishop's result has some of its deepest implications in the theory of uniform algebras (e.g., see Burckel's monograph [4]).

Stone's generalization of the Weierstrass approximation theorem is, of course, primarily an assertion about the (real or complex) algebra $C(X)$ of all scalar valued continuous functions on a compact Hausdorff space X, where $C(X)$ comes equipped with the topology of uniform convergence on X. From the very outset, however, further generalizations which relax the conditions on X, or otherwise broaden the domain of application to include this or that topological algebra of scalar or vector valued continuous functions, have been of interest. As noted by Stone himself (op. cit.), if X is locally compact, then utilization of the one-point compactification of X yields a variation phrased in terms of the subalgebra $C_0(X)$ consisting of those $f \in C(X)$ which vanish at infinity, where the topology on $C_0(X)$ is still that of uniform convergence. On the other hand, even if X is only assumed to be, say, a completely regular Hausdorff space, endowing $C(X)$ with the compact-open topology also leads directly to an instance (cf. [9]). Then there is R. C. Buck's version for the strict topology on the subalgebra $C_b(X)$ of all bounded continuous functions, where X is again taken to be locally compact [3], and the list could go on. But rather than considering each individual case as the need arises, or the whim may strike, it becomes natural to ask whether there isn't some way to unify this growing body of results and put it, so to speak, all under one roof.