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Elliptic functions and transcendence, by David Masser, Lecture Notes in Math., vol. 437, Springer-Verlag, Berlin, Heidelberg, New York, xii + 143 pp., $7.80.

I. A history. The original proof of the transcendental nature of the number e by Hermite in 1873 was based on a delicate scheme of rational approximations which seemed to be applicable only to the exponential function. In this light one may view with a sympathetic eye Hermite's pessimism toward the problem of the transcendental nature of π, as he openly states in a letter to Borchardt (Crelle, vol. 76, p. 342): "Que d'autres tentent l'entreprise, nul ne sera plus heureux que moi de leur succès, mais croyez-m'en, mon cher ami, il ne laissera pas que de leur en coûter quelques efforts." A few years later in 1882 Hermite would be amazed by the remarkable simplicity of Lindemann's proof of the transcendentality of π based on Euler's identity $e^{\pi i} = -1$ and on Hermite's earlier ideas. This episode marks the exalting birth of the theory of transcendental numbers and was to represent the only significant contribution for some time. What followed in the next quarter of a century was no more than a generalization of ideas and a simplification of methods, first in the hand of Weierstrass who saw that the method of Hermite and Lindemann could be made to yield a proof of the algebraic independence of the values of the exponential function at distinct algebraic points; this was followed by technical simplifications by Gordan, Hilbert and Hurwitz.

By the end of the nineteenth century it was generally believed that the main arithmetical properties of the exponential function were well understood; there were good reasons for this. For one, the work of Kummer on cyclotomic extensions had been around for more than half a century, even though his methods were beginning to be forgotten; the work of Kronnecker on complex multiplication was being brought to completion. One knew well that the values taken by the exponential function $e^{2\pi i z}$ at the rational points on the projective line $\mathbf{P}^1(Q)$ were values at *special points*, i.e. they generate abelian extensions of the rationals and all such extensions arose in this way. One may surmise that in 1900 Hilbert, being thoroughly familiar with these properties of the exponential function after the manner of his Bericht, would have present in the back of his mind these results when formulating his Seventh Problem on the arithmetical nature of numbers of the form α^{β} and in particular of $2^{\sqrt{2}}$, and in his Twelfth Problem concerning the search for automorphic forms whose values at *special points* of certain moduli varieties would generate algebraic extensions of number fields with special Galois properties.

By including a major question of transcendental number theory among his famous list of 23 problems, Hilbert, whose authoritative standing in the mathematical world was not small in comparison to that of the influential F. Klein, would provide the theory with a vitality which it was lacking after the work of Hermite and Lindemann and would at the same time indicate avenues for future work. The only obstacle which seemed to blear Hilbert's prophetic vision was his pessimism; which unlike that of Hermite, could only be justified by his ignorance of the inherent difficulties involved with some of his problems; in fact, as Siegel recounts [3, p. 243]: Hilbert thought (circa 1919) that he would live to see the Riemann hypothesis proved but that the establishment of the transcendence of $2^{\sqrt{2}}$ would be beyond the then rising generation of mathematicians.

Not more than ten years would pass when in 1929 Gel'fond and independently Siegel would develop methods strong enough to establish the transcendentality of $2^{\sqrt{2}}$. The degree of generality of Siegel's methods made them more attractive since they were also applicable to the study of numbers which, like

$$\frac{\pi}{2} = \int_0^1 \frac{dx}{\sqrt{1 - x^2}},$$

are the periods of differentials of curves defined by equations with algebraic coefficients. The schools of Siegel in Germany and of Gel'fond in Russia made, from about 1930 to the early sixties, significant contributions by extending the range of applicability of these methods, by simplifying some technical points, and by bringing to the surface problems which appeared to be intractable by the traditional tools.

In the decade of the sixties, A. Baker introduced and exploited his many variables interpolation techniques. Almost single handed he developed applications of his methods to a circle of ideas as wide ranging as the theory of diophantine equations, class number problems, linear forms of logarithms. In 1970, when A. Baker was awarded the Fields Medal,the protean face of transcendental number theory was no longer what it had been before the sixties.

In 1970 there was reason again to be optimistic that the new methods would elucidate certain difficult diophantine questions. It had been known for some time that the work of Siegel and Schneider on the transcendental nature of the periods of differentials of algebraic curves had applications to the study of rational and integral points on such curves. The simplest case, that of elliptic curves, had been treated successfully by Schneider who had shown that if ω_1 and ω_2 are a pair of fundamental periods for an elliptic curve with algebraic invariants, then the ration $\tau = \omega_1/\omega_2$ would be a *special point* in $\mathbf{P}^1(\mathbf{C})$, i.e. the value of the j-invariant at τ generates an abelian extension of an imaginary quadratic extension of Q, if and only if the elliptic curve had complex multiplication. Baker was the first to apply the newly discovered methods to obtain a complement to Schneider's theorem by now involving two arbitrary elliptic curves; he proved that if ω_1 and ω_2 are respectively the periods of two elliptic curves with algebraic invariants, then a linear combination $\alpha\omega_1 + \beta\omega_2$ with α and β algebraic would be either 0 or transcendental. Since 1970 the school of Baker, consisting mostly of Coates and Masser have made important contributions to this area of research which is perhaps considered one of the most difficult parts of number theory. The monograph of Masser, which we now proceed to review, contains an up to date presentation of their developments.

II. The state of affairs as of 1975. Let $E: y^2 = 4x^3 - g_2x - g_3$ be an elliptic curve with algebraic invariants g_2, g_3. Let ω_1 and ω_2 be two basic periods. Let $H^1(E)$ be the space of differentials on E of the second kind modulo the exact differentials. A basis for $H^1(E)$ is $\omega = dx/y$ and $\eta = xdx/y$, where $x = \wp(z)$, $y = \wp'(z)$ and $\wp(z)$ is the Weierstrass p-function which uniformizes E with periods ω_1 and ω_2. The quasiperiods of E are $\eta_i = \int_0^{\omega_i}\eta$, $i = 1, 2$. The height of an algebraic number α is defined to be the maximum of the absolute values of the relatively prime integral coefficients of the irreducible equation satisfied by α. As usual one says that E has complex multiplication if $\tau = \omega_2/\omega_1$ is a complex quadratic irrationality. A complex number $u \in \mathbf{C}$ is called an algebraic point of E if its image under the Weierstrass uniformization map

$$\mathbf{C}/\Omega \to E, \qquad \Omega = \mathbf{Z}\omega_1 \oplus \mathbf{Z}\omega_2,$$

has algebraic coordinates.

Chapter I deals exclusively with elliptic curves without complex multiplication. The main result established is a measure of how far away $\tau = \omega_2/\omega_1$ is from being algebraic. In fact it is shown (Theorem I) that the distance of τ from an algebraic point of height H is effectively bounded from below by a multiple of $c(\varepsilon)\exp(-(\log H)^{3+\varepsilon})$ for any $\varepsilon > 0$.

Chapters II *and* III deal with the vector space spanned by the six numbers $1, \omega_1, \omega_2, \eta_1, \eta_2, 2\pi i$ over the field of algebraic numbers. It is shown (Theorem II) that in the case of no complex multiplication the dimension of the vector space is six and in the case of complex multiplication (Theorem III) the dimension is four. An explanation for the drop in dimension in the latter case is to be found in the classical Legendre relation $\eta_1\omega_2 - \eta_2\omega_1 = 2\pi i$ and in the rather unexpected fact (Lemma 3.1) that in the case of complex multiplication the three numbers η_1, η_2 and ω_2 are linearly dependent over the algebraic numbers. As a corollary it is shown that the ratio of the quasiperiods is transcendental if and only if neither of the algebraic invariants g_2 and g_3 vanish. This is perhaps not an isolated fact, since these are precisely the *special points* that play a singular role in the complex projective line when viewed as a parameter space for the isomorphism classes of generalized elliptic curves.

An interesting development in recent years of the theory of elliptic curves has been Serre's generalization of the classical theory of complex multiplication. One of the main results is the fact that for an elliptic with rational invariants and without complex multiplication, the coordinates of the l-division points generate a Galois extension of the rationals which has a group of order as large as it could possibly be, in other words the elliptic curve behaves as if it were a specialization of the generic fibre, whose l-division points generate extensions with well-known Galois groups.

Results of Serre had been previously used by Coates to show that under the assumption that the elliptic curve has no complex multiplication, the numbers ω_1, ω_2 and $2\pi i$ are linearly independent over the field of algebraic numbers. In *Chapter* IV the author gives an effective proof of Coates' theorem which avoids the use of Serre's theorem.

In *Chapter* V one finds a transcendence measure for the linear form

$$\Lambda = \alpha_0 + \alpha_1\omega_1 + \alpha_2\omega_2 + \beta_1\eta_1 + \beta_2\eta_2 + \gamma \cdot 2\pi i,$$

where α_0, α_1, α_2, β_1, β_2 and γ are algebraic numbers of heights at most H and degrees at most d. It is shown (Theorem IV) that for any $\varepsilon > 0$ there is a constant c depending only on ω_1, ω_2, d and ε such that

$$|\Lambda| \geqslant c \exp\left(-\log H (\log H)^{7+\varepsilon}\right).$$

(If $\alpha_2 = \beta_1 = \beta_2 = 0$, then the exponent can be replaced by $4 + \varepsilon$.) Among other things this gives a transcendence measure for $\pi + \omega_1$.

Chapter VI develops various estimates for the coefficients of the l-division equation of $\wp(z)$; it also contains a generalization, which appears to be of independent interest, of Kronnecker's theorem on diophantine approximation applied to the lattice of periods.

Chapter VII may be considered the climax of the whole development. It is assumed here that the Weierstrass p-function $\wp(z)$ admits complex multiplication over the imaginary quadratic number field K and that $u_1, u_2, \ldots, u_n$ are n algebraic points of $\wp(z)$, i.e. $(\wp(u_i), \wp'(u_i), 1) \in \mathbf{P}^2(A)$, which are linearly independent over K. The main result proved (Theorem V) states that for any positive $\varepsilon > 0$ and any positive integer d there is an effectively computable constant $c = c(g_2, g_3, u_1, \ldots, u_n, d, \varepsilon)$ such that

$$|\alpha_1 u_1 + \cdots + \alpha_n u_n| > c \exp(-H^{\varepsilon}),$$

for all algebraic numbers $\alpha_1, \ldots, \alpha_n$ not all zero with degrees at most d and heights at most H. The proof of this theorem involves a delicate argument by induction on the number of algebraic points $u_1, \ldots, u_n$ and the author does not hesitate to use all the arsenal at his disposal: notably the construction of auxiliary functions à la Baker.

Masser's monograph ends with four appendices. In the first of these the values of a certain real-analytic modular function which is connected with the numbers η_1, η_2, ω_2 is studied with the help of the well-known theory of the multiplication equation. The second appendix contains an application of a result of Lelong in the theory of holomorphic functions of several complex variables to give an estimate for the degree of real separation of the zero regions of a polynomial in several complex variables. Such results provide estimates for the size of the coefficients in terms of the values of the polynomial on certain tubular neighborhoods of a real ball. In the third appendix the previous ideas are elaborated to give a proof of the fact that any linear combination of algebraic points with algebraic coefficients is either zero or transcendental. The last appendix contains a proof of a partial refinement of Siegel's theorem on the finiteness of the number of integral points on an elliptic curve. It is shown that if E is an elliptic curve with complex multiplication, then for any $\varepsilon > 0$ and any rational point $(x, y) \in E(Q)$ whose denominator is at most q one has $|x| \ll \exp((\log q)^{\varepsilon})$, where the implied constant depends only on E and ε and is ineffective.

III. Looking ahead. Although the body of knowledge concerning the arithmetic nature of the periods of elliptic curves has been steadily increasing since the appearance of Siegel's first paper on the subject, as may be observed in Masser's monograph, it may fairly be said that only its surface has been touched. It would appear that present techniques should be capable of throwing light on Grothendieck's question [**2**, p. 101] concerning the algebraic nature of the $4g$ periods and quasiperiods of an algebraic curve of genus g associated with the cohomology group $H^1(X)$ of differentials of the second kind modulo exact differentials. In another direction, the recent work of Shimura and Deligne [**1**] on Hilbert's Twelfth Problem and their theory of *special points* on certain moduli varieties which arise as homogeneous quotients of adelized reductive groups defined over number fields suggests that there may be relations between the coordinates of the *special points*, the periods and quasiperiods of the fibres over the special points, and the values at integral points of certain Euler products connected with global zeta functions of the fibres. Despite a strong belief in the pre-established harmony of mathematics, the situation beyond curves and over field extensions of the rationals baffles the imagination (of the reviewer), perhaps in accordance with Vico's Principle.

A topic which has received much attention but which is hardly even mentioned in Masser's monograph concerns p-adic analogues of the results proved. It may be pointed out that the age old question of the arithmetic nature of the values of the zeta functions at integral points appears to have a transcendental component, not entirely unrelated to the periods of geometric objects, and an arithmetical component which affords some kind of p-adic interpolation.

The overall organization of the material in Masser's monograph into small separate chapters makes it easy reading. The list of references contains 30 items and is fairly comprehensive. The monograph could perhaps be used for a one semester topics course at the graduate level. The difficult nature of the analysis, which is the distinguishing mark of transcendental number theory, could have been smoothed out a little by the inclusion of numerical examples. To remedy this situation, at least for the reader of this review, we include here a well-known example [**4**]. Let $l = 2g + 1$ be a prime number and let C_a be the curve $y^l = x^a(1 - x)$. Let T be that part of the group $(\mathbf{Z}/l\mathbf{Z})^*$ consisting of elements of the group such that $\langle at/l\rangle + \langle t/l\rangle < 1$, where $\langle x\rangle$ denotes the fractional part of x. Let $\zeta = e^{2\pi i/l}$, and put

$$p_t = (\zeta^{at} - 1)B(\langle at/l\rangle, \langle t/l\rangle),$$

where $B(x, y)$ is Euler's beta function. Let $\xi \in \mathbf{Z}[\zeta]$ and let $\xi \to \xi^{\sigma(t)}$ be the obvious extension of the automorphism $\zeta \to \zeta^t$. Then the lattice of periods consists of the vectors

$$g(\xi) = \left(\xi^{\sigma(t)}p_t\right)_{t \in T}, \qquad \xi \in \mathbf{Z}[\zeta].$$

The numbers p_t are all transcendental and the Jacobian of the curve C_a admits "complex" multiplication by $\mathbf{Z}[\zeta]$. The question of the nature of the quasiperiods of C_a and the relation of these quasiperiods to the values of certain Hecke L-functions associated with the zeta function of C_a and how these things should fit into a general framework are challenges that should be given serious consideration by any student of transcendental number theory.

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Combinatorial optimization: networks and matroids, by Eugene L. Lawler, Holt, Rinehart and Winston, New York, 1976, x + 374 pp.

This is a well-written introduction to an attractive area of modern mathematics. It is highly recommended.

Some problems in this area are:

1. Find the shortest path through a finite network.
2. Find the kth shortest path through a finite network.
3. Find the path of shortest length through all points of a finite network ("the travelling salesman" problem or technically a Hamiltonian circuit.)
4. How does one process m items on n machines?
5. How does one calculate 2^n with a minimum number of multiplications?
6. How does one compute a polynomial in many variables with a minimum number of multiplications?
7. How does one find m defective coins among n coins?

The fourth, fifth, sixth, and seventh problems are not treated in this book. The fourth problem is very important in many industrial applications and in operating a computer installation. Nabeshima has written a book in Japanese on this problem, which he is translating into English. Many other mathematicians have worked on this problem. Branch and bound techniques have been used by many. The fifth problem has no applications that the reviewer knows of. It is like many problems in number theory, simply stated and intractable. The sixth problem has many applications in a number of algorithms. In this case of polynomials of one variable, the problem is solved. Ostrowski treated the case of polynomials up to degree four, and the general case was treated by Pan. They showed that the well-known technique of Horner was best.

In problem 7 the case $m = 1$ is a well-known puzzle which may be solved using many methods. The case $m = 2$ was treated in [**1**]. The case of general m is part of a mathematical theory of experimentation which does not yet exist.

Since these are finite problems and we have a digital computer at our