16. G. C. Rota, *An "Alternieven de Verfahren" for general positive operators*, Bull. Amer. Math. Soc. **68** (1962), 95–102.
17. D. Spencer, *A function theoretic identity*, Amer. J. Math. **65** (1943), 147–160.
18. E. M. Stein, *On the functions of Littlewood-Paley, Lusin and Marcinkiewicz*, Trans. Amer. Math. Soc. **88** (1958), 430–466.
19. E. M. Stein and Guido Weiss, *On the theory of harmonic functions of several variables*, Acta Math. **103** (1960), 25–62.
20. E. M. Stein, *Classes H^p, multiplicateurs et fonctions de Littlewood-Paley*, C. R. Acad. Sci. Paris **263** (1966), 716–719.
21. ———, *Singular integrals and differentiability properties of functions*, Princeton Univ. Press, Princeton, N.J., 1970.
22. ———, *Topics in harmonic analysis related to the Littlewood-Paley theory*, Ann. Math. Studies, no. 63, Princeton Univ. Press, Princeton, N.J., 1970.
23. ———, *Maximal functions: Spherical means*, Proc. Nat. Acad. Sci. U.S.A. **73** (1976), 2174–2175.
24. M. H. Taibleson, *On the theory of Lipschitz spaces of distributions on Euclidean n-spaces*. I, J. Appl. Math. Mech. **13** (1964), 407–480; II (ibid) **14** (1965), 821–840.
25. ———, *Fourier analysis on local fields*, Math. Notes, no. 15, Princeton Univ. Press, Princeton, N.J., 1975.
26. A. Zygmund, *Une remarque sur un théorème de M. Kaczmarz*, Math. Z. **25** (1926), 297–298.
27. ———, *Trigonometric series*, Cambridge Univ. Press, New York, 1959.

R. R. COIFMAN
GUIDO WEISS

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Trigonometrical sums in number theory, by I. M. Vinogradov, Statistical Publishing Society, first published: October 1975, Calcutta, viii + 151 pp., $15.00.

1. Introduction. The theory of trigonometric (sometimes called exponential) sums is so intimately associated with I. M. Vinogradov that a book by this master is a noteworthy event.

The book under review is a translation from the Russian edition of 1970 which, in its turn, is described as a revised edition of Vinogradov's 1947 book of the same title. "Revised edition", however, is a misnomer, since a comparison with the 1947 edition shows that the present work is a complete rewriting and incorporates new refinements and improvements–results mostly due to the author himself.

Trigonometric sums have been used in one form or another in number theory since Gauss's solution of the cyclotomic equation in which he introduced "Gaussian" sums. These led to a highly interesting and somewhat unexpected proof of the quadratic law of reciprocity. There are many other examples and applications. The applications given in this book deal with the Waring problem, the distribution of the fractional parts of polynomials and with the Waring-Goldbach problem. Reference is also made to the applications to the zeta function.

2. The problem of trigonometric sums. The integers are naturally embedded in the complex numbers but a useful point of view for number theory-

–especially additive number theory–is to look at the character group of $Z(+)$, the group of integers under addition. For the benefit of readers not acquainted with analytic number theory, we make the following observations: if X is a character of $Z(+)$, then it is well known that there exists a real number α $(0 \leqslant \alpha < 1)$ such that if $n \in Z$;

$$X(n) = e^{2\pi\alpha n},$$

further if $d\alpha$ is the measure on the character group, then

$$\frac{1}{2\pi} \int_X X(n) d\alpha = \begin{cases} 0 & \text{if } n \neq 0, \\ 1 & \text{if } n = 0. \end{cases}$$

Let X be such a character, $P > 0$ a real number, and f: $Z \to Z$ a function and

$$S = S(X, f, P) = \sum_{n \leqslant P} X(f(n)).$$

We should like to evaluate S but such an objective is unreasonable and we must rest content with estimates. It is clear at the outset that $|S| \leqslant P$ and this leads to the

Basic problem. Can we give conditions on X, f, P so that

$$S = O(PQ)$$

where $Q \to 0$ as $P \to \infty$?

The problem is made more complicated by the fact that in applications we may need to specify the rate at which $Q \to 0$. Moreover, we may have more than one character or function or range. And finally, we may, and do frequently require an estimate which holds uniformly for a large class of characters. Three cases arise naturally

(i) The character is of order p–a prime;

(ii) The character is of finite order;

(iii) The character is of infinite order

and different techniques are used to get the best results in each case.

In case (i) A. Weil, using methods of algebraic geometry gave a complete and spectacular solution for a large class of functions. These beautiful results have, in recent years, been obtained with a minimal use of algebraic geometry in work initiated by S. A. Stepanoff.

In the third case H. Weyl was the first to develop a method for handling the problem and deduce significant results. By a simple and ingenious argument, he showed that if $f(x) = \alpha_n X^n + \cdots + \alpha_1 x$ then for every α

$$\sum_{n \leqslant x} e^{2\pi\alpha f(n)} = o(x)$$

and in fact gave an explicit rate at which the ratio tends to zero.

He then concluded via a reduction to trigonometric sums that the fractional values of $f(x)$ were uniformly distributed mod 1. [We have dropped the language of characters since it is not particularly useful.] To honor the author of this outstanding paper, Vinogradov named the sums in case (iii) "Weyl sums." [We would observe parenthetically that the translator of this book confuses Weil and Weyl. It is a fortuitous situation where, I feel, neither would object to being mistaken for the other!]

Vinogradov's contributions to estimations of trigonometric sums date from about 1934. His methods are characterized by great ingenuity and, at times, extreme intricacy. This, however, appears to be in the nature of the problem. To get a nontrivial estimate, some cancellation must occur and there is no reason, a priori, to expect this to happen in general–on the contrary, there are easily formulated instances when the terms reinforce each other. Roughly speaking, there will be cancellation if there is some "regularity" in the distribution of the values of $f(n)$. It is a tribute to Vinogradov that, by giving about 1937 a suitably nontrivial estimate of the sum

$$\sum_{p < x} e^{2\pi i\alpha p},$$

the sum taken over primes, he achieved one of his triumphs–a solution of the ternary Goldbach problem. There is a price to be paid however. The argument necessitates pages of delicate and intricate arguments whose motivation is decidely clouded. The effect on a newcomer to the subject is that he does not know where he is going until long after he has been there. Moreover, the research worker in this area often works with a variety of parameters which are chosen only after the work has been completed. But when stating the theorem, the path to the choice of parameters has been obliterated. Hence the mysterious looking hypotheses that frequently occur. This, however, is a minor hazard when compared with the intricacy of the arguments.

3. The topics. Although Vinogradov does not treat the ternary Goldbach problem in this book (he does in the earlier edition), it is an excellent setting in which to show how trigonometric sums intervene in additive number theory. The method is a general one, was initiated by Hardy and Littlewood, and goes under the name of the "circle method" for reasons given below. In its original form, and in the modifications introduced by Vinogradov, it has proven to be an exceedingly fruitful idea and has led to the solution of numerous problems in number theory. In particular, Hardy and Littlewood gave a new solution of the Waring problem going further than Hilbert by giving an asymptotic formula for the number of solutions. They also gave a provisional solution of the ternary Goldbach problem subject to a weakened form of the generalized Riemann hypothesis.

Suppose then that we wish to show that every odd integer is the sum of three primes, i.e., that if n is odd, the diophantine equation

$$n = p_1 + p_2 + p_3 \tag{1}$$

is always solvable. We consider the sum

$$S(\alpha) = \sum_{2 < p < n} e^{2\pi\alpha p} \tag{2}$$

the summation being over primes only, and denote by $r(n)$ the number of solutions of 3.1, that is, we replace the original problem by the seemingly more difficult one of counting the number of solutions and showing that $r(n) > 0$. We give the barest sketch omitting large numbers of steps. We hope specialists will not consider the sketch misleading. Using 1.1, it is readily seen that

$$r(n) = I = \frac{1}{2\pi}\int_0^1 S^3(\alpha)e^{-2\pi i\alpha n}\,d\alpha. \tag{3}$$

Hardy and Littlewood originally used power series, and integrated around a circle inside the unit circle using Cauchy's theorem. It was Vinogradov who observed that it is technically much simpler to work with finite sums. We hasten to add, however, that this change by no means eliminates the difficulty–that lies much deeper. To evaluate the integral, we first write α in the form

$$\alpha = \frac{a}{q} + z, \qquad (a, q) = 1 \qquad (a, q \in Z)$$

and z is a parameter which is a function of n. The interval [0, 1] (or properly speaking a certain translation of it) is divided into classes of nonoverlapping intervals. In class M, we put all intervals with $0 < q \leqslant s$, $|z| \leqslant t$, s, t being parameters which depend on n. In m, we put the rest of the unit interval. [In Hardy and Littlewood's original work, the intervals were arcs of circles determined by Farey fractions–the "Farey dissection"–hence the name "circle method".]

We get $I = I_M + I_m$. If $\alpha = a/q + z$ and q is "small", then it is possible to approximate $S(\alpha)$ by $S(a/q)$ with a controllable error. By a sequence of estimations, and invoking a deep theorem of C. L. Siegel, Vinogradov obtains the principal term

$$I_M = \frac{n^2}{2\log^3 n}\,\mathfrak{S}(n) + \text{error}, \tag{4}$$

where $\mathfrak{S}(n)$, the so-called singular series, is a kind of measure of the local number of solutions of 3.1 and is shown to be > 0 if n is odd.

It remains to prove that $I_m = o(n^2/\log^3 n)$. While the previous step is difficult, the principal obstacle lies here. Note that the parameters s, t are at our disposal–the more we put in M the larger will be the error in 3.4 and indeed may dominate the principal term, but then the easier will I_m be to handle. We have a delicate balancing act–to minimize the estimate for I_m and minimize the error in 3.4. It is at this stage that Vinogradov invokes his nontrivial estimate of $S(\alpha)$ to achieve the famous result. This general procedure with appropriate modifications naturally, is fairly typical.

We turn now to some of the results in Vinogradov's book. As we have seen, to deal with problems in number theory by this method (as well as innumerable other applications), we must have suitable estimates for trigonometric sums. In Chapter 5, Vinogradov proves the following remarkable and general result on Weyl sums.

THEOREM. *Let* $f(x) = \alpha_n x^n + \alpha_{n-1}x^{n-1} + \cdots + \alpha_1 x$, α_i *real*, $P \in Z$, $P > 1$; *let* π_n *denote a hypercube of unit side and unit volume. Let*

$$T = T(\alpha_n, \ldots, \alpha_1) = \sum_{0<x<P} e^{2\pi i f(x)}$$

and let $\alpha_i = a_i/q_i + z_i$; $(a_i, q_i) = 1$, l.c.m. $q_i = Q < P^{1/n}$.

The points of π_n *are divided into two classes. In the first class, we put the points for which* $|z_i| < P^{-i+1/n}$.

In the second class we put points not in the first class. Then, for points of the second class, we have $T = O(P^{1-1/e})$ *where*

$$e = 8n^2\left(\log n + \frac{1}{2}\log\log n + 3\right).$$

An inequality is given for points of the first class but it turns out that these points comprise a "negligible" fraction of π_n. This is a remarkable achievement since it shows that for a "large" proportion of characters, there is a cancellation. This result has the following implications to Waring's problem. If $r(N)$ is the number of solutions of $N = x_1^k + \cdots + x_s^k$, and $k > 12$, then the asymptotic formula

$$r(N) \sim f(s, k)N^{s/k-1}$$

holds if

$$s > 2\left[n^2(2\log n + \log\log n + 3)\right]. \tag{5}$$

Here $f(s, k)$ is a certain function of s and k.

In the previous edition, we have $s > [10n^2 \log n]$, although (5) is given in the latest edition of L.-K. Hua's book. The method can also be applied to sums in which a polynomial is replaced by a differentiable function with suitably bounded derivatives. It leads to an estimate for $S = \sum e^{2\pi i F(n)}$ where $F(u) = -t \log u/2\pi$.

Specialists will recognize the importance of such estimates to the Riemann zeta function and will be interested to know that Vinogradov asserts that his result implies that if

$$\pi(N) = Li(N) + R(N)$$

then

$$R(N) = O(Ne^{-c(\log n)^{3/5}}). \tag{6}$$

Other writers, in particular A. Walfisz, have observed that they are able to deduce only that

$$R(N) = O(Ne^{-c(\log n)^{3/5}(\log\log n)^{-1/5}}).$$

Meanwhile nonspecialists will politely suppress a smile as they ponder the problem of dancing elves.

It would be impossible to convey here any but the vaguest idea of Vinogradov's methods. Suffice it to say that the problem of estimating S is reduced to the problem of estimating a multiple integral which in its turn reduces to the purely arithmetic problem of counting the number of solutions of a system of diophantine equations. This oversimplification scarcely does justice to the extreme ingenuity and insight needed to effect the result. It is a noteworthy fact that Vinogradov's estimates have scarcely ever been improved upon except by Vinogradov himself.

In Chapter 7, Vinogradov considers the problem of estimating sums of the type $\sum_{p<P} e^{2\pi i m f(p)}$ where the summation is over primes p only. Once again the result is quite general and supercedes previous ones. It is highly interesting to note that, despite the different range of summation, the result differs but little from that given above.

An application is made in Chapter 9 to the Goldbach-Waring problem, i.e., to the representation of an integer in the form $N = p_1^k + \cdots + p_s^k$ where the summands are primes and $k \geqslant 12$. The asymptotic formula is shown to hold for

$$s \geqslant 2\left[n^2(\log n + \log\log n + 3)\right].$$

The investigation of this problem is due originally to L.-K. Hua and is also worked out in detail in his book including the case $n < 12$. Vinogradov's estimates are, however, relied upon.

In Chapters 5 and 8, applications are made to the problem of the distribution of the fractional points of a polynomial when the argument ranges over all integers or primes only.

This together with an introduction and preliminary results comprise the content of the book. It is, unfortunately, difficult to read and is not for the fainthearted. It does not benefit, as the earlier edition did, from the very helpful comments of the translators. It is relentless, and must be read "avec la plume dans la main." But it will certainly be rewarding to students of number theory who, in the words of Gauss, "ont le courage de l'approfondir."

4. Quo vadis? Does this book represent the end of a chapter or the beginning of a new one? The reviewer does not feel competent to speculate on the future but some observations may be pertinent.

In the case of the Waring problem, the conjectured value for the range of validity of the asymptotic formula is $s \geqslant 4k$. This contrasts with the best value given above currently available by these methods.

In the case of the binary Goldbach problem, the methods fail but do yield results in which prime numbers enter linearly. The most spectacular result in this connection is the beautiful and profound asymptotic formula of Linnik for the number of representations of an integer in the form $N = p + x^2 + y^2$–a result which had been conjectured by Hardy and Littlewood. It is worth observing that the frequency of the set of integers of the form $x^2 + y^2$ is given by the asymptotic value $cn/\sqrt{\log n}$ while that of the primes by $n/\log n$. By contrast, applications of the deepest sieve methods of A. Selberg have enabled C.-J. Chen to prove that every even integer is the sum of a prime and an integer containing at most two prime factors.

In the case of the zeta function and its application to the distribution of primes we note the following:

$R(N) = O(Ne^{-c(\log N)^{1/2}})$	de la Vallée Poussin, 1895;
$R(N) = O(Ne^{-c(\log N \log\log N)^{1/2}})$	H. Weyl, 1919;
$R(N) = O(Ne^{-c(\log N)^{3/5}})$	Vinogradov, 1966.

These are to be contrasted with the result obtainable on the assumption of the Riemann hypothesis

$$R(N) = O(N^{1/2}\log N).$$

The above results are not even of the form $R(N) = O(N^{1-\varepsilon})$. It might appear to some that years of herculean effort have yielded limited progress, but after all the antagonist is a formidable foe.

R. G. AYOUB

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The Selberg trace formula for PSL (2, **R**), Volume I, by Dennis A. Hejhal, Lecture Notes in Mathematics, Springer-Verlag, Berlin, Heidelberg, New York, 1976, iv + 516 pp., \$ 15.20.

For the last twenty-five years or so the Selberg trace formula has had, in the general mathematical community, an aura of mystery which is only slowly dissipating. This circumstance makes it necessary for us to look a little at the history and nature of this formula in order to understand properly the position of this new book.

First of all, the Selberg trace formula has precedents some of which are very old indeed. The underlying technical ideas have been in common currency amongst applied mathematicians since the turn of the century; these arose in the study of Laplace's equation and we would now associate them with groups like $O(3, \mathbf{R})$. Furthermore, various versions are to be found in earlier investigations concerning automorphic forms. These were mostly number-theoretical and hinged around the class-number formulae discovered by Kronecker and studied further by Fricke, Mordell, Hecke and Eichler. But also from the differential-geometric point of view both J. Delsarte and H. Huber came very close to an explicit trace formula (for $PSL(2, \mathbf{R})$).

Yet, nevertheless, Selberg's discovery of this formula in the early 1950's was a revolutionary event and its impact is far from spent. This lies in the nature of the formula. Although I have continually referred to it as a *formula* it is much more a *method*; a method, that is, for probing more deeply into the nature of discontinuous groups and their function theory. In broad terms, the Selberg trace formula arises when one learns to think functional-analytically about automorphic functions and forms. This has been the *pons asinorum*; it forces one to shed preferences for complex-analytic functions and prejudices against 'soft analysis'. Once this has been done a new land, full of promise, opens up.

There are two approaches to the trace formula; that due to Selberg which uses differential and integral operators–and in fact the differential operators can be eliminated–and that due to Gelfand and his collaborators which uses representation theory. The latter is now almost indispensable for general, especially number-theoretic questions, whereas for the study of Fuchsian groups the former is more flexible. It is this that is used in this book and we shall first look at it a little more closely.

The basic idea is the following. Let S be a 'good' topological space and m a measure on S. Let A be a commutative family of compact integral operators on $L^2(S, m)$ and we suppose that the adjoint of any operator in A is also in A. Then, from spectral theory, we know that A can be 'diagonalised' and under our assumptions there exists a countable orthonormal basis $\{v_n;\ n \in \mathbf{N}\}$ of